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EXISTENCE AND MULTIPLICITY OF SOLUTIONS OF AN EQUATION FROM POOL BOILING ON WIRES

By

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Dedicated to Professor Hwai-Chiuan Wang on his 60th birthday

Abstract. We investigate the existence and multiplicity of steady states of the equation

$$\frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x^2} + \sigma q(\theta) - a\lambda(1 + \alpha\theta) = 0, \quad 0 < x < 1,$$

with Dirichlet boundary conditions and initial conditions. This equation was derived and studied by Joly, Kernevez, and Llory [7] and Joly [8] in studying thermal effects from pool boiling, in which wires are heated by the Joule effect and are cooled in a bath of boiling water at constant pressure. They studied the steady-state problem for two kinds of heat flux density $q(\theta)$ (corresponding to whether or not the radiation is taken into account) and for $\alpha \neq 0$ or $\alpha = 0$. (A) In the case with radiation and $\alpha = 0$, for given specific function $q(\theta)$ and constants $\sigma > 0$, $a > 0$, by numerical methods, they found an S-shaped bifurcation diagram and three solutions for some parameter values. We prove this rigorously, for a specific range of parameters of physical interest. Specifically, we show that, for specific values of α, σ, and a, there exist two positive numbers $\underline{\lambda} < \bar{\lambda}$ such that the steady-state problem has at least three solutions for $\underline{\lambda} < \lambda < \bar{\lambda}$, at least two solutions for $\lambda = \underline{\lambda}$ or $\lambda = \bar{\lambda}$, and exactly one solution for $0 \leq \lambda < \underline{\lambda}$ or $\lambda > \bar{\lambda}$. Moreover, we give lower and upper bounds for $\underline{\lambda}$ and $\bar{\lambda}$. (B) In the case without radiation and $\alpha \neq 0$, we show that there exist two positive numbers $\underline{\lambda} < \bar{\lambda}$ such that the steady-state problem has at least two solutions for $\underline{\lambda} < \lambda < \bar{\lambda}$, at least one solution for $0 \leq \lambda \leq \underline{\lambda}$ or $\lambda = \bar{\lambda}$, exactly one solution for $0 < \lambda < \underline{\lambda}$ and λ small enough, and no solution for $\lambda > \bar{\lambda}$. Moreover, we give upper and lower bounds for $\underline{\lambda}$ and $\bar{\lambda}$. Also, we find and correct two mistakes in [7, Proposition 2.6, (i), (ii)].

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1. Introduction. We investigate the existence and multiplicity of steady states of the equation

$$\frac{\partial\theta}{\partial t} - \frac{\partial^2\theta}{\partial x^2} + \sigma q(\theta) - a\lambda(1+\alpha\theta) = 0, \quad 0 < x < 1, \tag{1.1}$$

with Dirichlet boundary and initial conditions

$$\theta(0,t) = \theta(1,t) = 0, \tag{1.2}$$

$$\theta(x,0) = \theta_0(x). \tag{1.3}$$

This reaction-diffusion model equation in (1.1)-(1.3) was derived and studied by Joly, Kernevez, and Llory [7] and Joly [8] in studying thermal effects from pool boiling, in which wires are heated by the Joule effect and are cooled in a bath of boiling water at constant pressure. They assumed that the heat flux density q only depends upon the temperature difference θ between the heat-transfer temperature and the two-phase flow temperature. In Eq. (1.1), constant $\alpha \geq 0$ is the temperature coefficient of electrical resistance. Positive constants σ, a and positive bifurcation parameter λ are as follows:

$$\sigma = \frac{2l^2}{Dr}, \quad a = \frac{l^2\rho_0}{DS^2}, \quad \lambda = i^2, \tag{1.4}$$

where l is the wire length, D its thermal conductivity, r its radius, ρ_0 its specific electrical resistance, S its cross-sectional area, and i is the current intensity along the electric wire [7].

The characteristic boiling curve $q(\theta)$ satisfies that $q(0) = 0$, q increases on an interval $(0, \theta_M)$, then decreases on an interval (θ_M, θ_m) and last increases for $\theta > \theta_m$; see Figs. 1 and 2. These intervals correspond to three heat-transfer regimes: natural convection and nucleate boiling for $0 < \theta < \theta_M$, transition boiling for $\theta_M < \theta < \theta_m$ (in this stage $dq/d\theta < 0$), and film boiling for $\theta > \theta_m$ [7, p. 1294]. In particular, the transition boiling region (θ_M, θ_m) is inherently unstable in experiments in which only the heat flux is controlled, due to the negative slope in the transition region [1, p. 985]. Due to experimental difficulties, the form of the curve representing $q(\theta)$ for $\theta > \theta_m$ is not exactly known [7, p. 1294]. Experimentally, the temperature at point $d > \theta_m$ with $q(d) = q(\theta_M)$ is in the order of magnitude of the melting point of the metal wires, e.g., the platinum wire. Only when using an extremely thin wire for the heat-transfer surface would the point $(d, q(d))$ really exist [11, p. 1421].

Yanagida [15] proposed a simple coupled map lattice model for two-dimensional pool boiling, which exhibits the transition from nucleate to film boiling.

The steady-state equation of (1.1)-(1.3) takes the form

$$\begin{cases} -\frac{\partial^2\theta}{\partial x^2} + \sigma q(\theta) - a\lambda(1+\alpha\theta) = 0, \quad 0 < x < 1, \\ \theta(0) = \theta(1) = 0. \end{cases} \tag{1.5}$$

Define

$$g(\theta) = \sigma q(\theta) - a\lambda(1+\alpha\theta), \tag{1.6}$$

$$G(\theta) = \int_0^\theta g(u)du. \tag{1.7}$$

Equation (1.5) then can be written as

$$\begin{cases} -\frac{\partial^2\theta}{\partial x^2} + g(\theta) = 0, & 0 < x < 1, \\ \theta(0) = \theta(1) = 0. \end{cases} \tag{1.8}$$

Clearly, when $\lambda = 0$, problem (1.8) has a unique solution $\theta \equiv 0$. Moreover, it is easy to show that, for every $\lambda > 0$, the solution θ of (1.8) is positive; that is,

$$\theta(x) \geq 0 \quad \text{for every } x \in [0,1];$$

see [7, Proposition 2.2]. We study the existence and multiplicity of solutions of (1.5) as the bifurcation parameter λ varies.

Joly, Kernevez, and Llory [7] and Joly [8] studied (1.5) for the next two cases:

(i) with radiation and $\alpha = 0$, as $q(\theta)$ given in Fig. 1 whose data are due to Berenson [1],

(ii) without radiation and with $\alpha \neq 0$, as $q(\theta)$ given in Fig. 2 whose data are due to Madsen [10, Fig. 1].

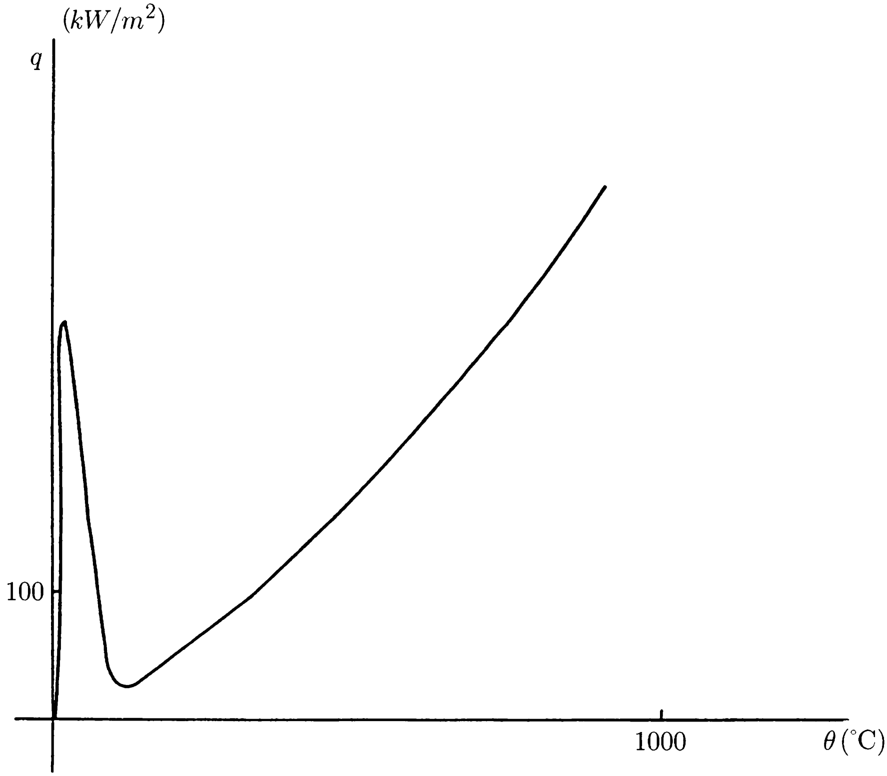

FIG. 1. ([7, Fig. 2] and [8, Fig. 2]). Pool-boiling characteristic in the case with radiation

They found an S-shaped bifurcation diagram and three solutions for (i). While they showed that (1.8) has at least two solutions for a certain range of λ and has no solution for λ large enough for (ii). However, the work of Joly, Kernevez, and Llory was mainly numerical and, in particular, they only studied particular parameter values, $\alpha = 0$, $\sigma = 465$, $a = 11560$ for (i), and $\alpha = 0.0039$, $\sigma = 172.7$, $a = 1286$ for (ii); see [8, Figs. 1 and 2].

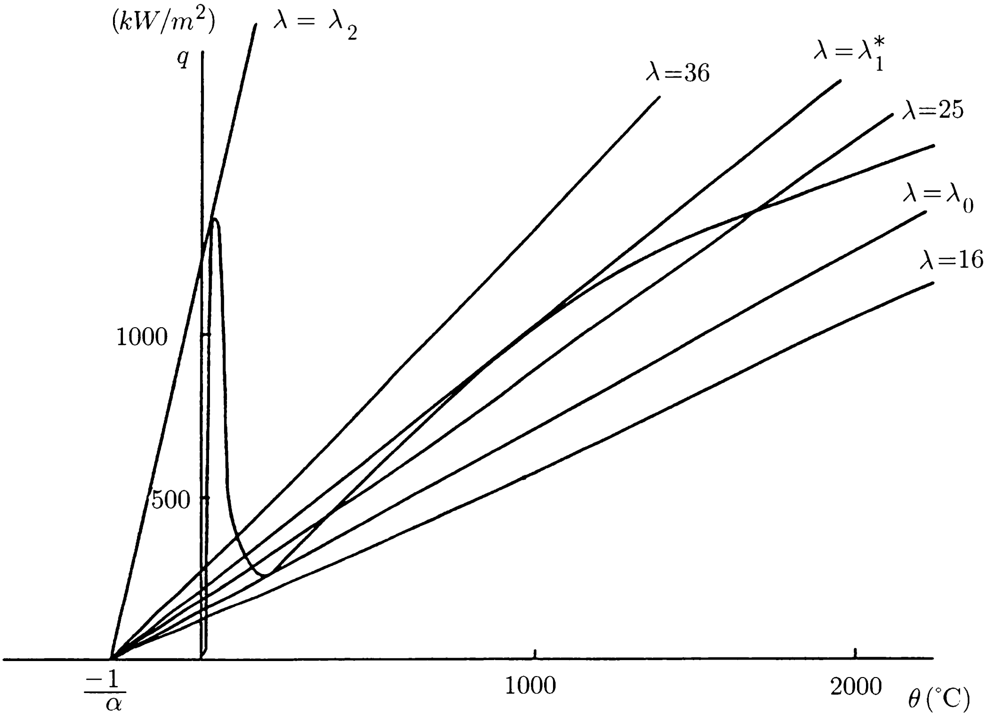

FIG. 2. ([7, Fig. 1] and [8, Fig. 1]). Pool-boiling characteristic in the case without radiation

1.1. *Theorems for the case with radiation and* $\alpha = 0$. In this case radiation is taken into account. The pool-boiling characteristic $q(\theta)$ is given in Fig. 1. The function $q(\theta) \in C^1[0, \infty)$ and satisfies

(H1) $q(0) = 0$,

(H2) there exist two positive numbers $\theta_M < \theta_m$ such that

$$\begin{aligned} q'(\theta) &> 0 \quad \text{on} \quad (0, \theta_M), \\ q'(\theta) &< 0 \quad \text{on} \quad (\theta_M, \theta_m), \\ q'(\theta) &> 0 \quad \text{on} \quad (\theta_m, \infty), \end{aligned}$$

(H3) $\lim_{\theta \to \infty} q(\theta) = \infty$ and $q(\theta) = O(\theta^4)$ as $\theta \to \infty$ (Boltzmann's fourth-power law).

We use positive numbers $\lambda_1 < \lambda_2$ as introduced in [7, p. 1300]. λ_1 is chosen such that $g(\theta) = \sigma q(\theta) - a\lambda_1 = 0$ admits three positive roots $\theta_1 < \theta_2 < \theta_3$ such that

$$\int_{\theta_1}^{\theta_3} g(\theta) d\theta = 0. \tag{1.9}$$

λ_2 is chosen such that

$$\sigma q(\theta_M) - a\lambda_2 = 0. \tag{1.10}$$

THEOREM 1.1. Let constants $\alpha = 0$, $\sigma > 0$, $a > 0$. Assume that $q(\theta) \in C^1[0, \infty)$ satisfies (H1)-(H3). For $\lambda = \lambda_2$, suppose that there exist numbers n, p, r, s, t, and m satisfying (3.3)-(3.6) stated below, where $f(\theta)$ is defined in (3.4). Then there exist two positive numbers $\underline{\lambda}$ and $\bar{\lambda}$ with $\lambda_1 < \underline{\lambda} < \lambda_2 < \bar{\lambda}$ such that problem (1.8) has at least three solutions for $\underline{\lambda} < \lambda < \bar{\lambda}$, at least two solutions for $\lambda = \underline{\lambda}$ or $\lambda = \bar{\lambda}$, and exactly one solution for $0 \leq \lambda < \underline{\lambda}$ or $\lambda > \bar{\lambda}$.

For $q(\theta)$ given in Fig. 1, $\alpha = 0$, $\sigma = 465$ and $a = 11560$, it can be seen that

$$3.3 < \lambda_1 < 4.3, \tag{1.11}$$

$$12 < \lambda_2 < 13; \tag{1.12}$$

cf. [7, Figs. 2 and 6].

In the case with radiation and $\alpha = 0$, for given specific function $q(\theta)$ and constants $\sigma > 0$, $a > 0$, by numerical methods, Joly, Kernevez, and Llory [7] and Joly [8] found an S-shaped bifurcation diagram and three solutions for some parameter values; see [7, Proposition 2.3 and section 6]. We prove this rigorously, for a specific range of parameters of physical interest as in the next Theorems 1.2 and 1.3. Theorem 1.2 is an application of Theorem 1.1, in which, with $q(\theta)$ given in Fig. 1, the hypotheses of Theorem 1.1 are verified to hold for parameters $\alpha = 0$, $\sigma = 465$, and $a = 11560$. Theorem 1.3 is an extension of Theorem 1.2, which is valid for parameters $\alpha = 0$, σ and a satisfying (1.15) and (1.16).

THEOREM 1.2. For $q(\theta)$ given in Fig. 1, $\alpha = 0, \sigma = 465, a = 11560$, there exist two positive numbers $\underline{\lambda}$ and $\bar{\lambda}$ with $\lambda_1 < \underline{\lambda} < \lambda_2 < \bar{\lambda}$ such that problem (1.8) has at least three solutions for $\underline{\lambda} < \lambda < \bar{\lambda}$, at least two solutions for $\lambda = \underline{\lambda}$ or $\lambda = \bar{\lambda}$, and exactly one solution for $0 \leq \lambda < \underline{\lambda}$ or $\lambda > \bar{\lambda}$. Moreover,

$$3.3 < \underline{\lambda} < 5.5, \tag{1.13}$$

$$12 < \bar{\lambda} < 13.526. \tag{1.14}$$

Hence, for $q(\theta)$ given in Fig. 1, $\alpha = 0$, $\sigma = 465$ and $a = 11560$, problem (1.8) has at least three solutions for $5.5 \leq \lambda \leq 12$.

Theorem 1.2 is consistent with a numerical result obtained by Joly, Kernevez, and Llory [7] in which, for $q(\theta)$ given in Fig. 1, $\alpha = 0$, $\sigma = 465$ and $a = 11560$, they showed numerically that there exist numbers

$$\underline{\lambda} \approx 3.8, \quad \bar{\lambda} \approx 13.3,$$

such that problem (1.8) has exactly three solutions for $\underline{\lambda} < \lambda < \bar{\lambda}$; see [7, section 6] for details.

THEOREM 1.3. Let $\underline{\lambda}$ and λ_2 be numbers defined in Theorem 1.2 and (1.10) when $\alpha = 0$, $\sigma = 465$, $a = 1156$. Let $q(\theta)$ be given in Fig. 1, $\alpha = 0$, and constants σ and a satisfy

$$\sigma \geq 465, \tag{1.15}$$

$$(11.3943 \approx) \quad \frac{3179}{279} < \frac{a}{\sigma} \leq \frac{11560}{465} \ (\approx 24.8602). \tag{1.16}$$

Then

(i) problem (1.8) has at least three solutions for $\lambda \in (\frac{11560\sigma}{465a}\underline{\lambda}, \lambda_2)$,

(ii) problem (1.8) has exactly one solution for positive λ large or small enough.

1.2. *Theorems for the case without radiation and with $\alpha \neq 0$.* In this case the pool-boiling characteristic $q(\theta)$ is given in Fig. 2, it exists for θ from 0 up to some finite number $\theta_\infty \approx 2200$; see [7, Fig. 1] and [10, Fig. 1]. The function $q(\theta) \in C^1[0, \theta_\infty)$ and satisfies

(Q1) $q(0) = 0$,

(Q2) there exist three positive numbers $\theta_M < \theta_m < \theta_\infty$ $(< \infty)$ such that

$$\begin{aligned} q'(\theta) &> 0 \text{ on } (0, \theta_M), \\ q'(\theta) &< 0 \text{ on } (\theta_M, \theta_m), \\ q'(\theta) &> 0 \text{ on } (\theta_m, \theta_\infty), \end{aligned}$$

(Q3) there exist exactly three solution-pairs $(\lambda_0, \theta_{\lambda_0}), (\lambda_{1^*}, \theta_{\lambda_1^*}), (\lambda_2, \theta_{\lambda_2})$ of

$$\begin{cases} g(\theta) = \sigma q(\theta) - a\lambda(1 + \alpha\theta) = 0, \\ g'(\theta) = \sigma q'(\theta) - a\lambda\alpha = 0 \end{cases}$$

satisfying

$$0 < \lambda_0 < \lambda_{1^*} < \lambda_2,$$

$$0 < \theta_{\lambda_2} < \theta_M < \theta_m < \theta_{\lambda_0} < \theta_{\lambda_{1^*}} < \theta_\infty.$$

First it is obvious to see from Fig. 2 that

$$16 < \lambda_0 < 25, \tag{1.17}$$

$$25 < \lambda_{1^*} < 36, \tag{1.18}$$

$$140 < \lambda_2 < 180. \tag{1.19}$$

In particular, (1.19) is obtained by evaluating the q intercepts on the (θ, q)-plane of functions $a\lambda(1 + \alpha\theta)$ for $\lambda = 36$ and $\lambda = \lambda_2$. In this case we look for experimentally meaningful solutions θ of (1.8) satisfying

$$0 < \|\theta\|_\infty < \theta_\infty (\approx 2200). \tag{1.20}$$

Analytically, we may consider (1.8) for an arbitrary function $q(\theta)$ defined for all $\theta \in [0, \infty)$ under additional reasonable asymptotic behavior hypotheses on $q(\theta)$ at infinity. We assume $q(\theta) \in C^1[0, \infty)$ and satisfies

(H1) $q(0) = 0$,

(H2) there exist two positive numbers $\theta_M < \theta_m$ $(< \infty)$ such that

$$\begin{aligned} q'(\theta) &> 0 \quad \text{on} \quad (0, \theta_M), \\ q'(\theta) &< 0 \quad \text{on} \quad (\theta_M, \theta_m), \\ q'(\theta) &> 0 \quad \text{on} \quad (\theta_m, \infty), \end{aligned}$$

(H3′) there exist exactly three solution-pairs $(\lambda_0, \theta_{\lambda_0}), (\lambda_{1^*}, \theta_{\lambda_{1^*}}), (\lambda_2, \theta_{\lambda_2})$ of

$$\begin{cases} g(\theta) = \sigma q(\theta) - a\lambda(1 + \alpha\theta) = 0, \\ g'(\theta) = \sigma q'(\theta) - a\lambda\alpha = 0 \end{cases}$$

satisfying

$$0 < \lambda_0 < \lambda_{1^*} < \lambda_2,$$

$$0 < \theta_{\lambda_2} < \theta_M < \theta_m < \theta_{\lambda_0} < \theta_{\lambda_{1^*}} < \infty,$$

(H4) $\lim_{\theta\to\infty} \frac{q(\theta)}{\theta} = 0$, $\lim_{\theta\to\infty}(q(\theta) - \theta q'(\theta)) = \infty$.

NOTE. Functions $q(\theta)$ behave as θ^β with $0 < \beta < 1$ or as $\theta(\ln\theta)^{-\delta}$ with $\delta > 0$ at infinity satisfying (H4).

THEOREM 1.4. Let constants $\alpha > 0, \sigma > 0$ and $a > 0$. Assume that $q(\theta) \in C^1[0,\infty)$ satisfies (H1), (H2), (H3′), and (H4). Then there exist two numbers $\underline{\lambda} < \bar{\lambda}$ with

$$\underline{\lambda} = \frac{\pi^2}{a\alpha} \quad \text{and} \quad \lambda_2 < \bar{\lambda} < \lambda_2 + \frac{\pi^2}{a\alpha}$$

such that

(i) for $0 \le \lambda \le \underline{\lambda}$, problem (1.8) has at least one solution. Moreover, for $0 < \lambda < \underline{\lambda}$ and λ small enough, problem (1.8) has a unique solution,
(ii) for $\underline{\lambda} < \lambda < \bar{\lambda}$, problem (1.8) has at least two solutions,
(iii) for $\lambda = \bar{\lambda}$, problem (1.8) has at least one solution,
(iv) for $\lambda > \bar{\lambda}$, problem (1.8) has no solution.

It is important to note that, for $\lambda_0 < \lambda < \lambda_{1^*}$,

(A) the equation $g(\theta) = \sigma q(\theta) - a\lambda(1+\alpha\theta) = 0$ has four (but not three) positive roots $\theta_1 < \theta_2 < \theta_3 < \theta_4$ ($\theta_1, \theta_2, \theta_3, \theta_4$ depending upon λ),
(B) there may not exist a number $\lambda = \lambda_1 \in (\lambda_0, \lambda_{1^*})$ such that

$$\int_{\theta_1}^{\theta_3} g(\theta)d\theta = 0 \quad \text{for} \quad \lambda = \lambda_1 \tag{1.21}$$

(see [7, p. 1302]) since

$$\int_{\theta_1}^{\theta_3} g(\theta)d\theta > 0 \quad \text{for} \quad \lambda = \lambda_{1^*}. \tag{1.22}$$

In particular, for function $q(\theta)$ given in Fig. 2 (see also [10, Fig. 1]) and constants $\alpha = 0.0039, \sigma = 172.7$, and $a = 1286$, (1.22) holds and hence there does not exist a number $\lambda = \lambda_1$ with $\lambda_0 < \lambda_1 < \lambda_{1^*}$ $(< \lambda_2)$ such that (1.21) holds. Thus [7, Proposition 2.6, (i), (ii)] are false. Nevertheless, they can be corrected as in the next theorem.

THEOREM 1.5. Let function $q(\theta)$ be given in Fig. 2 and constants $\alpha = 0.0039, \sigma = 172.7, a = 1286$. Then

(i) for $0 \le \lambda \le 25$, problem (1.8) has a unique solution satisfying (1.20),

(ii) for $40 \le \lambda \le 140$ ($< \lambda_2$), problem (1.8) has at least two solutions satisfying (1.20),

(iii) for $\lambda \ge 182$ ($> \lambda_2 + (\pi^2/(a\alpha))$), problem (1.8) has no solution satisfying (1.20).

Theorem 1.5 is consistent with a numerical result of Joly, Kernevez, and Llory [7, section 6]; see [7, Fig. 5] in which the solution branch of solutions θ of problem (1.8) satisfying $0 < \|\theta\|_\infty < 500$ is plotted. They found that problem (1.8) has at least two solutions satisfying $0 < \|\theta\|_\infty < 500$ for $40 \le \lambda \le 160$.

We finally remark that it is also of interest to consider problem (1.8) for the next two other cases:

(iii) with radiation and $\alpha \neq 0$ for function $q(\theta)$ satisfying (H1)-(H3) and constants $\alpha > 0, \sigma > 0, a > 0$,

(iv) without radiation and with $\alpha = 0$ for function $q(\theta)$ satisfying (H1), (H2), (H3′), (H4) and constants $\sigma > 0, a > 0$.

In either case (iii) or (iv), it can be shown that problem (1.8) has exactly one solution for positive λ small or large enough. The exact shape of the solution branch of problem (1.8), similarly as in the two previous cases discussed in Theorems 1.1-1.5, depends significantly upon the function $q(\theta)$ and values of the constants $\alpha \ge 0, \sigma > 0$, $a > 0$.

This paper is organized as follows: In Sec. 2, we first give some preliminary lemmas. In Sec. 3, we consider the case with radiation and $\alpha = 0$ and prove Theorems 1.1-1.3. Finally, in Sec. 4, we consider the case without radiation and with $\alpha \neq 0$ and prove Theorems 1.4-1.5.

2. Preliminary lemmas.

In [7, 8], problem (1.8) was studied by analysis of a quadrature formula $k(\gamma)$.

LEMMA 2.1. ([7, Lemma 2.2]) There are as many solutions to (1.8) as one can find values of γ such that

$$k(\gamma) := \int_0^\gamma \frac{du}{\{2[G(u) - G(\gamma)]\}^{1/2}} = \frac{1}{2}, \tag{2.1}$$

where $G(u)$ is defined in (1.7).

Sometimes, to indicate the dependence more precisely on the bifurcation parameter λ, we write g_λ instead of g, G_λ instead of G, and k_λ instead of k in (1.6)-(1.8) and (2.1), respectively, when necessary. Moreover, to indicate the dependence more precisely on the function g, we sometimes write k^g (or k^g_λ) instead of k in (2.1) when necessary.

In order for $k(\gamma)$ to be defined we must have

$$G(u) > G(\gamma), \quad 0 < u < \gamma. \tag{2.2}$$

For $\lambda > 0$, the set S where $k(\gamma)$ can be defined is generally not an interval. Nevertheless we can show that the set $S\backslash\{0\}$ is an open set; see [12, Proposition 4.1]. Moreover, it is well known that $k \in C^{(n)}$ if $g \in C^{(n)}$, $n = 0, 1, 2, \ldots$.

The next two lemmas are similar to results in [9], which are useful in studying $k(\gamma)$ and hence the existence and multiplicity of problem (1.8). First, continuous dependence of the time map $k(\gamma)$ on the bifurcation parameter λ can be obtained from the representation in (2.1).

LEMMA 2.2. Suppose that $k_\lambda(\gamma)$ exists at $\gamma = \hat{\gamma} > 0, \lambda = \hat{\lambda} > 0$. Then $k_\lambda(\hat{\gamma})$, considered as a function of λ, is continuous at $\lambda = \hat{\lambda}$.

The next comparison theorem can be shown easily by a generalized mean-value theorem [4, p. 118, Theorem 4]. We omit the proof.

LEMMA 2.3. Suppose that both

$$k^g(\gamma) := \int_0^\gamma \frac{du}{\{2[G(u) - G(\gamma)]\}^{1/2}}$$

and

$$k^f(\gamma) := \int_0^\gamma \frac{du}{\{2[F(u) - F(\gamma)]\}^{1/2}}$$

exist at $\gamma > 0$, where $G(\theta) = \int_0^\theta g(u)du$ and $F(\theta) = \int_0^\theta f(u)du$. Suppose that $g(\theta) \le f(\theta)$ for $0 < \theta < \gamma$. Then $0 < k^g(\gamma) \le k^f(\gamma)$. If, additionally, $g(\hat{\theta}) < f(\hat{\theta})$ for some $\hat{\theta} \in [0, \gamma)$, then $0 < k^g(\gamma) < k^f(\gamma)$. In particular, let $\alpha \ge 0$, $\sigma > 0, a > 0, q(0) = 0, q(\theta) > 0$ for $\theta > 0$. Then

$$g_{\lambda^*}(\theta) = \sigma q(\theta) - a\lambda^*(1 + \alpha\theta), \qquad g_{\lambda^{**}}(\theta) = \sigma q(\theta) - a\lambda^{**}(1 + \alpha\theta),$$

$$G_{\lambda^*}(\theta) = \int_0^\theta g_{\lambda^*}(u)du, \quad G_{\lambda^{**}}(\theta) = \int_0^\theta g_{\lambda^{**}}(u)du.$$

Assume that both

$$k_{\lambda^*}(\gamma) := \int_0^\gamma \frac{du}{\{2[G_{\lambda^*}(u) - G_{\lambda^*}(\gamma)]\}^{1/2}}$$

and

$$k_{\lambda^{**}}(\gamma) := \int_0^\gamma \frac{du}{\{2[G_{\lambda^{**}}(u) - G_{\lambda^{**}}(\gamma)]\}^{1/2}}$$

exist at $\gamma > 0$. If $0 < \lambda^* < \lambda^{**}$, then $0 < k_{\lambda^{**}}(\gamma) < k_{\lambda^*}(\gamma)$.

3. Proofs of Theorems 1.1-1.3.

3.1. *Proof of Theorem 1.1.* First, it is easy to obtain the next lemma. We omit the proof.

LEMMA 3.1. Let constants $\sigma > 0, a > 0$. Let $\lambda \ge 0$, $g(\theta) = \sigma q(\theta) - a\lambda$, where $q(\theta) \in C^1[0, \infty)$ and satisfies (H1)-(H3). Then for any fixed number $\gamma = \hat{\gamma} > 0$, there exists a smallest positive number $\lambda = \hat{\lambda} > 0$ such that $k_\lambda(\hat{\gamma})$ defined by (2.1) exists for $\lambda \in (\hat{\lambda}, \infty)$. Moreover,

$$\lim_{\lambda \to \hat{\lambda}+} k_\lambda(\hat{\gamma}) = \infty \quad \text{and} \quad \lim_{\lambda \to \infty} k_\lambda(\hat{\gamma}) = 0.$$

Let $\lambda \ge 0, g(\theta) = \sigma q(\theta) - a\lambda$, where $q(\theta) \in C^1[0, \infty)$ and satisfies (H1)-(H3). Recall the positive number $\lambda_1 < \lambda_2$ introduced in (1.9) and (1.10). λ_1 is chosen such that $g(\theta) = \sigma q(\theta) - a\lambda_1 = 0$ admits three positive roots $\theta_1 < \theta_2 < \theta_3$ such that $\int_{\theta_1}^{\theta_3} g(\theta)d\theta = 0$.

λ_2 is chosen such that $\sigma q(\theta_M) - a\lambda_2 = 0$. For $\lambda_1 < \lambda < \lambda_2$, $g(\theta) = \sigma q(\theta) - a\lambda = 0$ admits three positive roots $\theta_1 < \theta_2 < \theta_3$ (θ_1, θ_2, and θ_3 depending upon λ) such that

$$\int_{\theta_1}^{\theta_3} g(\theta)d\theta < 0.$$

So there exists a number $\theta_5 \in (\theta_2, \theta_3)$ such that

$$\int_{\theta_1}^{\theta_5} g(\theta)d\theta = 0. \tag{3.1}$$

For $\lambda = \lambda_2, g(\theta) = \sigma q(\theta) - a\lambda_2 = 0$ admits two positive roots $\theta_1 < \theta_2$. For $\lambda > \lambda_2$, $g(\theta) = \sigma q(\theta) - a\lambda = 0$ admits one positive root θ_1. The next lemma is well known.

LEMMA 3.2. (Cf. [13, Fig. 4(b)]) For $\lambda_1 < \lambda < \lambda_2$, $k_\lambda(\gamma)$ exists for every $\gamma \in [0, \theta_1) \cup (\theta_5, \theta_3)$. Moreover,

(i) on $[0, \theta_1), k_\lambda(\gamma)$ is strictly increasing from 0 to infinity,

(ii) on $(\theta_5, \theta_3), \lim_{\gamma \to \theta_5^+} k_\lambda(\gamma) = \lim_{\gamma \to \theta_3^-} k_\lambda(\gamma) = \infty$.

For $\lambda = \lambda_2, k_\lambda(\gamma)$ exists for every $\gamma \in [0, \theta_1) \cup (\theta_1, \theta_2)$. Moreover,

(iii) on $[0, \theta_1), k_\lambda(\gamma)$ is strictly increasing from 0 to infinity,

(iv) on $(\theta_1, \theta_2), \lim_{\gamma \to \theta_1^+} k_\lambda(\gamma) = \lim_{\gamma \to \theta_2^-} k_\lambda(\gamma) = \infty$.

For $\lambda > \lambda_2, k_\lambda(\gamma)$ exists for every $\gamma \in [0, \theta_1)$. Moreover,

(v) on $[0, \theta_1), k_\lambda(0) = 0$, $\lim_{\gamma \to \theta_1^-} k_\lambda(\gamma) = \infty$.

For $\lambda_1 < \lambda < \lambda_2$, by Lemma 3.2, there exists a number $\tilde{\gamma} \in (0, \theta_1)$ such that $k_\lambda(\tilde{\gamma}) = 1/2$. For $q(\theta)$ given in Fig. 1 and constants $\alpha = 0, \sigma > 0$ and $a > 0$, suppose that, for some $\lambda = \tilde{\lambda} \in (\lambda_1, \lambda_2)$,

$$\min_{\gamma \in (\theta_5, \theta_3)} k_{\tilde{\lambda}}(\gamma) < \tfrac{1}{2}; \tag{3.2}$$

see [7, p. 1301, lines 27-29]. Then by Lemmas 2.1 and 3.2, problem (1.8) has at least three solutions for $\lambda = \tilde{\lambda}$. Moreover, by Lemmas 2.2 and 2.3, problem (1.8) has at least three solutions for $\tilde{\lambda} \le \lambda \le \lambda_2$. (Similarly, for $\lambda = \lambda_2$, it can be proved that problem (1.8) has at least three solutions.)

For $\lambda = \lambda_2$, it is assumed in Theorem 1.1 that there exist numbers n, p, r, s, t, and m satisfying

$$n < 0, \quad t < 0, \quad 0 < p < \theta_1 < r < s < m < \theta_2 \tag{3.3}$$

such that the curve $S : f(\theta) = 0$ which consists of four line segments connecting points $(0, n), (p, 0), (r, 0), (s, t)$ and $(m, 0)$ point by point is above the curve $g_{\lambda_2}(\theta) = 0$ for $\theta \in [0, m]$ except for $\theta = \theta_1$. That is,

$$f(\theta) := \begin{cases} \frac{n(p-\theta)}{p} & \text{for} \quad 0 \le \theta < p, \\ 0 & \text{for} \quad p \le \theta < r, \\ \frac{t(\theta - r)}{(s-r)} & \text{for} \quad r \le \theta < s, \\ \frac{t(\theta - m)}{(s-m)} & \text{for} \quad s \le \theta \le m, \end{cases} \tag{3.4}$$

$$g_{\lambda_2}(\theta_1) = f(\theta_1) = 0 \text{ and } g_{\lambda_2}(\theta) = \sigma q(\theta) - a\lambda_2 < f(\theta) \text{ for } \theta \in [0, m]\backslash\{\theta_1\}; \tag{3.5}$$

see Fig. 3.

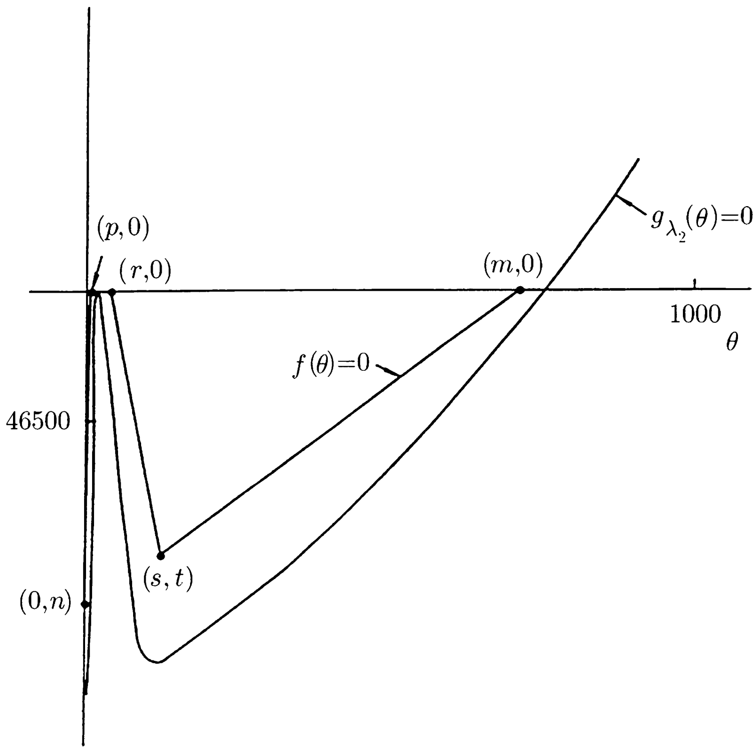

FIG. 3. Curves $f(\theta) = 0$ and $g_{\lambda_2}(\theta) = 0$

Denote

$$k^g_{\lambda_2}(\gamma) := \int_0^\gamma \frac{du}{\{2[G_{\lambda_2}(u) - G_{\lambda_2}(\gamma)]\}^{1/2}}$$

and

$$k^f(\gamma) := \int_0^\gamma \frac{du}{\{2[F(u) - F(\gamma)]\}^{1/2}},$$

where $G_{\lambda_2}(\theta) = \int_0^\theta g_{\lambda_2}(u)du$ and $F(\theta) = \int_0^\theta f(u)du$. It is clear that $k^g_{\lambda_2}(\gamma)$ and $k^f(\gamma)$ exist on $[r, m]$ and on (r, m) respectively. By (3.5) and Lemma 2.3,

$$0 < k^g_{\lambda_2}(\gamma) < k^f(\gamma) \quad \text{for} \quad \gamma \in (r, m).$$

It is important to note that $k^f(\gamma)$ can be calculated explicitly since f is piecewisely a line segment. (The calculation for $k^f(\gamma)$ is straightforward but tedious.) It is assumed in Theorem 1.1 that

$$\inf_{\gamma \in (r,m)} k^f(\gamma) < \tfrac{1}{2}. \tag{3.6}$$

Thus

$$\min_{\gamma \in [r,m]} k^g_{\lambda_2}(\gamma) < \tfrac{1}{2}.$$

In these circumstances, let $\lambda = \underline{\lambda} \in (\lambda_1, \lambda_2)$ be the number such that

$$\min_{\gamma \in (\theta_5, \theta_3)} k^g_{\underline{\lambda}}(\gamma) = \tfrac{1}{2} \quad (\theta_5 \text{ is defined in (3.1)}) \tag{3.7}$$

for $g_{\underline{\lambda}}(\theta) = \sigma q(\theta) - a\underline{\lambda}$, and let $\bar{\lambda} > \lambda_2$ be the largest number such that $k^g_{\bar{\lambda}}(\gamma)$ defined on $(0, \theta_1)$ has a local maximum with value 1/2. ($\underline{\lambda}$ and $\bar{\lambda}$ exist by Lemmas 2.2, 2.3, 3.1, and 3.2.) By Lemmas 2.2, 2.3, 3.1, and 3.2, Theorem 1.1 follows. □

3.2. *Proof of Theorem 1.2.* (i) For the function $q(\theta)$ given in Fig. 1, it satisfies (H1)-(H3). Let $\alpha = 0, \sigma = 465, a = 11560$, and $\lambda = \lambda_2$. We choose

$$n = -120000, \quad p = 10, \quad r = 35, \quad s = 120, \quad t = -100000, \quad m = 720.$$

Then (3.3)-(3.5) hold. Moreover, it can be computed that (3.6) holds; see Fig. 4 which is obtained using the symbolic manipulator *Mathematica*. Thus, by Theorem 1.1, the first part of Theorem 1.2 follows. We next show (1.13) and (1.14) in (ii) and (iii).

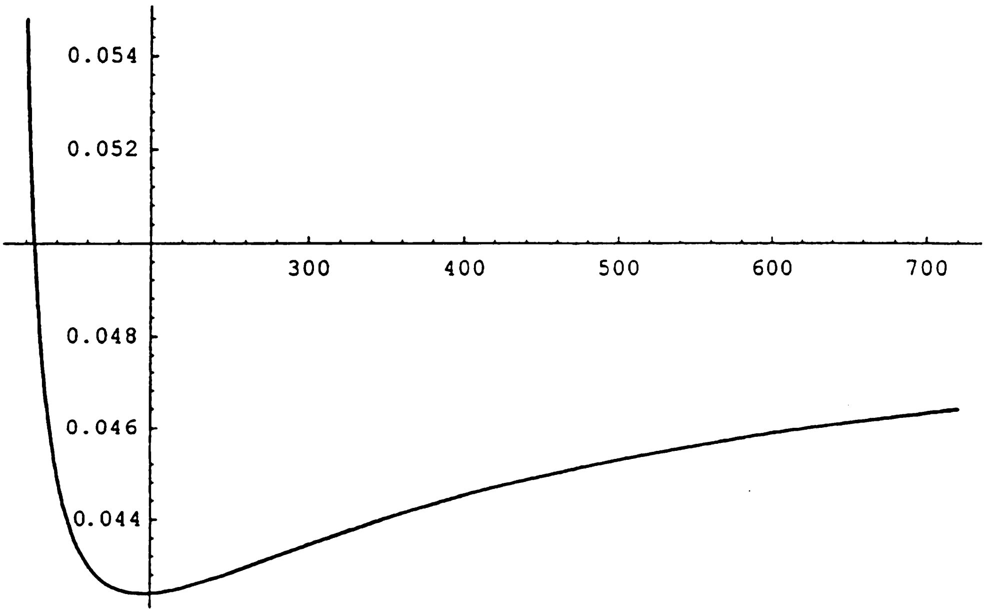

FIG. 4. Curve $k^f(\gamma)$ on (s, m)

(ii) For $\lambda = \hat{\lambda} \in (\lambda_1, \lambda_2)$, $g_{\hat{\lambda}}(\theta) = \sigma q(\theta) - a\hat{\lambda} = 0$ admits three positive roots $\theta_1 < \theta_2 < \theta_3$ such that

$$\int_{\theta_1}^{\theta_3} g_{\hat{\lambda}}(\theta) d\theta < 0.$$

So there exists a number $\theta_5 \in (\theta_2, \theta_3)$ such that

$$\int_{\theta_1}^{\theta_5} g_{\hat{\lambda}}(\theta) d\theta = 0.$$

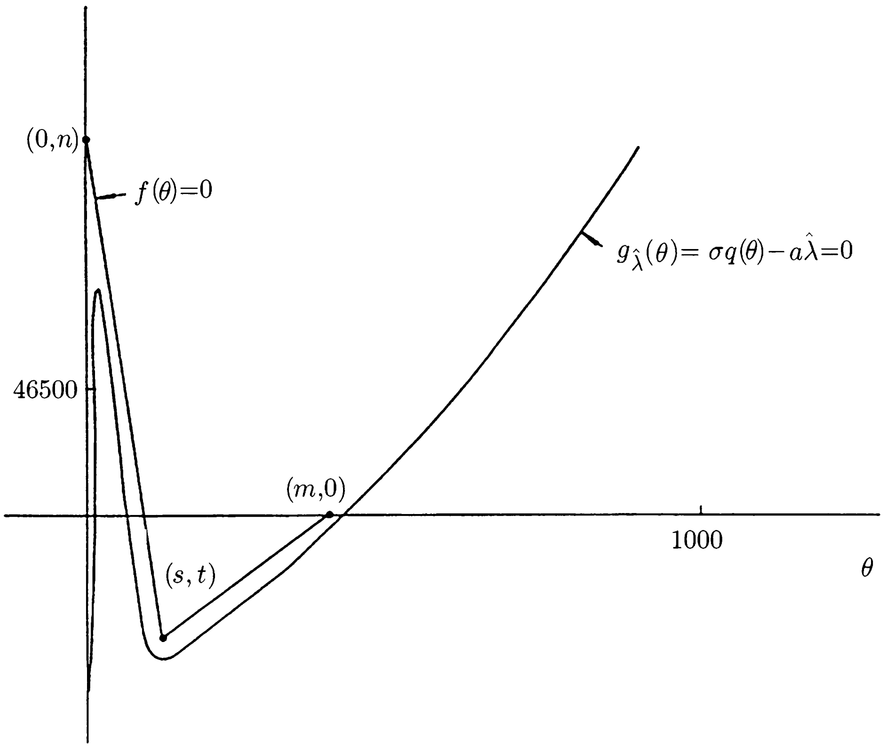

FIG. 5. Curves $f(\theta) = 0$ and $g_{\hat{\lambda}}(\theta) = 0$

Assume that there exist numbers n, s, t, and m satisfying

$$n > 0, \quad t < 0, \quad \theta_2 < s < m < \theta_3 \tag{3.8}$$

such that the curve $V : f(\theta) = 0$ which consists of two line segments connecting points $(0, n), (s, t)$ and $(m, 0)$ point by point is above the curve $g_{\hat{\lambda}}(\theta) = 0$ on $[0, m]$ for $\lambda = \hat{\lambda}$, i.e.,

$$g_{\hat{\lambda}}(\theta) = \sigma q(\theta) - a\hat{\lambda} < f(\theta) := \begin{cases} \frac{(t-n)\theta}{s} + n & \text{for} \quad 0 \le \theta < s, \\ \frac{t(\theta - m)}{(s-m)} & \text{for} \quad s \le \theta \le m, \end{cases} \tag{3.9}$$

and f satisfies

$$\int_0^m f(\theta)d\theta < 0; \tag{3.10}$$

see Fig. 5. So there exists a number $r \in (\theta_5, m)$ such that

$$\int_0^r f(\theta)d\theta = 0. \tag{3.11}$$

Similarly as before, we let

$$k_{\hat{\lambda}}^g(\gamma) := \int_0^\gamma \frac{du}{\{2[G_{\hat{\lambda}}(u) - G_{\hat{\lambda}}(\gamma)]\}^{1/2}}$$

and

$$k^f(\gamma) := \int_0^\gamma \frac{du}{\{2[F(u) - F(\gamma)]\}^{1/2}},$$

where $G_{\hat\lambda}(\theta) = \int_0^\theta g_{\hat\lambda}(u)du$ and $F(\theta) = \int_0^\theta f(u)du$. It is clear that $k^g_{\hat\lambda}(\gamma)$ and $k^f(\gamma)$ exist on $[r, m]$ and on (r, m) respectively. By (3.9) and Lemma 2.3,

$$0 < k^g_{\hat\lambda}(\gamma) < k^f(\gamma) \quad \text{for } \gamma \in (r, m).$$

$k^f(\gamma)$ can be calculated explicitly since f is piecewisely a line segment. (Similarly as before, the calculation for $k^f(\gamma)$ is straightforward but tedious.) Hence suppose that

$$\inf_{\gamma\in(r,m)} k^f(\gamma) < \tfrac{1}{2}. \tag{3.12}$$

Then

$$\min_{\gamma\in[r,m]} k^g_{\hat\lambda}(\gamma) < \tfrac{1}{2}$$

for $\lambda = \hat\lambda \in (\lambda_1, \lambda_2)$. Thus we obtain the next lemma.

LEMMA 3.3. For $\lambda = \hat\lambda \in (\lambda_1, \lambda_2)$, suppose that there exist numbers n, s, t, and m satisfying (3.8)-(3.12). Then problem (1.8) has at least three solutions for $\hat\lambda \leq \lambda \leq \lambda_2$. Moreover, let $\underline{\lambda} \in (\lambda_1, \lambda_2)$ be defined by (3.7). Then

$$(\lambda_1 <)\, \underline{\lambda} < \hat\lambda.$$

For $q(\theta)$ given in Fig. 1, $\alpha = 0$, $\sigma = 465$, and $a = 11560$, we choose

$$\hat\lambda = 5.5, \quad n = 140000, \quad s = 126, \quad t = -44500, \quad m = 400. \tag{3.13}$$

Then $r \approx 356.943$ and (3.8)-(3.11) hold. Moreover, it can be computed that

$$\inf_{\gamma\in(r,m)} k^f(\gamma) < 0.3 < \tfrac{1}{2};$$

see Fig. 6 which is obtained using the symbolic manipulator *Mathematica*. Thus (3.12) holds. So it follows by Lemma 3.3, (1.11) and (3.13) that

$$3.3 < \underline{\lambda} < 5.5$$

and hence (1.13) holds.

(iii) In [7, p. 1301, lines 38-44], for

$$\lambda > \lambda_2 + \frac{8\theta_p}{a},$$

it was shown that (1.8) has a unique solution, where $\theta_p > \theta_m$ is the number satisfying $a\lambda_2 = \sigma q(\theta_p)$. For $q(\theta)$ given in Fig. 1, $\sigma = 465$, and $a = 11560$, it is easy to see that

$$730 < \theta_p < 760.$$

So by Theorem 1.1 and (1.12), we obtain that

$$12 < \lambda_2 < \bar\lambda < \lambda_2 + \frac{8\theta_p}{a} < 13 + \frac{8 \cdot 760}{11560} < 13.526.$$

This proves (1.14).

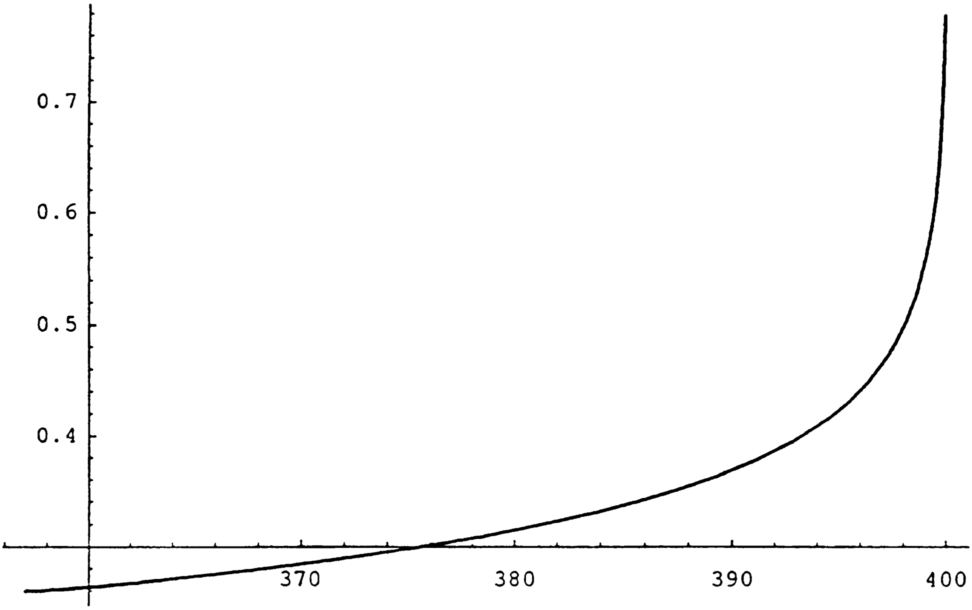

FIG. 6. Curve $k^f(\gamma)$ on (r, m)

By the above, for $q(\theta)$ given in Fig. 1, $\alpha = 0$, $\sigma = 465$ and $a = 11560$, problem (1.8) has at least three solutions for $5.5 \le \lambda \le 12$. The proof of Theorem 1.2 is now complete. □

3.3. *Proof of Theorem 1.3.* Let λ_1, $\underline{\lambda}$ and λ_2, with $\lambda_1 < \underline{\lambda} < \lambda_2$, be positive numbers defined in (1.9), Theorem 1.2 and (1.10) when $\alpha = 0$, $\sigma = 465, a = 1156$. Let constants $\sigma = \sigma^*$ and $a = a^*$ satisfy (1.15) and (1.16). By (1.16), we obtain

$$1 \le \frac{11560\sigma^*}{465a^*},$$

and hence

$$\underline{\lambda} \le \frac{11560\sigma^*}{465a^*}\underline{\lambda}. \tag{3.14}$$

In addition, by (1.16), (1.12) and (1.13),

$$\frac{11560\underline{\lambda}}{465\lambda_2} < \frac{11560 \cdot 5.5}{465 \cdot 12} = \frac{3179}{279} < \frac{a^*}{\sigma^*},$$

and hence

$$\frac{11560\sigma^*}{465a^*}\underline{\lambda} < \lambda_2. \tag{3.15}$$

(3.14) and (3.15) imply that

$$\underline{\lambda} \le \frac{11560\sigma^*}{465a^*}\underline{\lambda} < \lambda_2.$$

It follows that

$$\left(\frac{11560\sigma^*}{465a^*}\underline{\lambda}, \frac{11560\sigma^*}{465a^*}\lambda_2\right) \cap (\underline{\lambda}, \lambda_2) = \left(\frac{11560\sigma^*}{465a^*}\underline{\lambda}, \lambda_2\right) \neq \emptyset. \tag{3.16}$$

By (3.16), for any number $\lambda = \lambda^* \in (\frac{11560\sigma^*}{465a^*}\underline{\lambda}, \lambda_2) \subset (\underline{\lambda}, \lambda_2) \subset (\lambda_1, \lambda_2)$, we have

$$\lambda^* = \frac{11560\sigma^*}{465a^*}\tilde{\lambda} \tag{3.17}$$

for some $\tilde{\lambda} \in (\underline{\lambda}, \lambda_2) \subset (\lambda_1, \lambda_2)$. For constants $\sigma > 0, a > 0, \lambda > 0$, define the function

$$g_{\sigma,a,\lambda}(\theta) = \sigma q(\theta) - a\lambda, \quad \theta \geq 0.$$

Then

$$\begin{aligned} g_{\sigma^*,a^*,\lambda^*}(\theta) &= \sigma^* q(\theta) - a^*\lambda^* \\ &= \frac{\sigma^*}{465}\left(465q(\theta) - \frac{465a^*}{\sigma^*}\lambda^*\right) \\ &= \frac{\sigma^*}{465}(465q(\theta) - 11560\tilde{\lambda}) \quad \text{(by (3.17))} \\ &= \frac{\sigma^*}{465}g_{465,11560,\tilde{\lambda}}(\theta). \end{aligned} \tag{3.18}$$

It is important to note that $g_{\sigma^*,a^*,\lambda^*}(\theta)$ is a constant multiple of $g_{465,11560,\tilde{\lambda}}(\theta)$ and hence functions $g_{\sigma^*,a^*,\lambda^*}(\theta)$ and $g_{465,11560,\tilde{\lambda}}(\theta)$ have the same positive zeros $\theta_1 < \theta_2 < \theta_3$ and the same value of θ_5 defined by (3.1) since $\tilde{\lambda} \in (\lambda_1, \lambda_2)$. Moreover, functions $k_{\lambda^*}(\gamma, \sigma = \sigma^*, a = a^*)$ and $k_{\tilde{\lambda}}(\gamma, \sigma = 465, a = 11560)$ defined by (2.1) have the same interval of definition for $\gamma \in [0, \theta_1) \cup (\theta_5, \theta_3)$.

By Theorem 1.2 and Lemmas 2.1 and 3.2, for $\lambda = \tilde{\lambda} \in (\underline{\lambda}, \lambda_2) \subset (\lambda_1, \lambda_2)$ and $g(\theta) = g_{465,11560,\tilde{\lambda}}(\theta)$,

$$\min_{\gamma \in (\theta_5, \theta_3)} k_{\tilde{\lambda}}(\gamma, \sigma = 465, a = 11560) = \tilde{t} < \tfrac{1}{2} \tag{3.19}$$

for some positive $\tilde{t}$. By (1.15), (3.18), (3.19) and the formula of k_λ in (2.1), for $\lambda = \lambda^*$ and

$$g(\theta) = g_{\sigma^*,a^*,\lambda^*}(\theta) = \sigma^* q(\theta) - a^*\lambda^* = \frac{\sigma^*}{465}g_{465,11560,\tilde{\lambda}}(\theta),$$

we obtain that

$$\begin{aligned} \min_{\gamma \in (\theta_5, \theta_3)} k_{\lambda^*}(\gamma, \sigma = \sigma^*, a = a^*) &= (\frac{\sigma^*}{465})^{-1/2} \min_{\gamma \in (\theta_5, \theta_3)} k_{\tilde{\lambda}}(\gamma, \sigma = 465, a = 11560) \\ &\leq \min_{\gamma \in (\theta_5, \theta_3)} k_{\tilde{\lambda}}(\gamma, \sigma = 465, a = 11560) \\ &= \tilde{t} \\ &< \tfrac{1}{2}. \end{aligned}$$

Hence, if (1.15) and (1.16) are satisfied, by Lemmas 2.1 and 3.2, problem (1.8) has at least three solutions for $\lambda = \lambda^* \in (\frac{11560\sigma^*}{465a^*}\underline{\lambda}, \lambda_2)$. This proves (i).

The proof of part (ii) of Theorem 1.3 is easy. We omit it. The proof of Theorem 1.3 is now complete. □

3.3.1. *A remark to Theorems 1.1-1.3.* For an arbitrary function $q(\theta)$ satisfying (H1)-(H3), $\alpha = 0$, and any constants $\sigma > 0$, $a > 0$, it is not necessary that problem (1.8) has at least three solutions for some $\lambda \in (\lambda_1, \lambda_2)$. This can be shown by the next lemma, which follows easily by the definition of $k(\gamma)$ in (2.1).

LEMMA 3.4. Let the function $q(\theta) \in C^1[0,\infty)$ and satisfy (H1)-(H3) and $g(\theta) = \sigma q(\theta) - a\lambda$. Suppose that for $\sigma = \tilde{\sigma} > 0$, $a = \tilde{a} > 0$, and $\lambda = \tilde{\lambda} > 0$, $k_{\tilde{\lambda}}(\gamma; \sigma = \tilde{\sigma}, a = \tilde{a})$ defined by (2.1) exists and

$$k_{\tilde{\lambda}}(\gamma; \sigma = \tilde{\sigma}, a = \tilde{a}) = t > 0.$$

Then for $\sigma = c\tilde{\sigma}, a = c\tilde{a}$, and $\lambda = \tilde{\lambda}$, $k_{\tilde{\lambda}}(\gamma; \sigma = c\tilde{\sigma}, a = c\tilde{a})$ exists and

$$k_{\tilde{\lambda}}(\gamma; \sigma = c\tilde{\sigma}, a = c\tilde{a}) = c^{-1/2} k_{\tilde{\lambda}}(\gamma; \sigma = \tilde{\sigma}, a = \tilde{a}) = c^{-1/2} t$$

for any constant $c > 0$.

In Lemma 3.4, functions $\tilde{\sigma}q(\theta) - \tilde{a}\lambda$ and $(c\tilde{\sigma})q(\theta) - (c\tilde{a})\lambda$ have the same zeros and the same values of λ_1 and λ_2. In particular, they have the same zeros (i) $\theta_1 < \theta_2 < \theta_3$ for $\lambda_1 < \lambda < \lambda_2$ and (ii) $\theta_1 < \theta_2$ for $\lambda = \lambda_2$. Define constants $\sigma = \tilde{\sigma} > 0$, $a = \tilde{a} > 0$. Suppose that, for some $\lambda = \tilde{\lambda} \in (\lambda_1, \lambda_2)$, $k_{\tilde{\lambda}}(\gamma; \sigma = \tilde{\sigma}, a = \tilde{a})$ has its minimum on (θ_5, θ_3) at $\gamma = \tilde{\gamma} > 0$, and

$$k_{\tilde{\lambda}}(\tilde{\gamma}; \sigma = \tilde{\sigma}, a = \tilde{a}) = \tilde{t} \le \tfrac{1}{2} \quad (\tilde{t} > 0); \tag{3.20}$$

see Lemma 3.2. Then $k_{\tilde{\lambda}}(\gamma; \sigma = c\tilde{\sigma}, a = c\tilde{a})$ has the same interval of definition as that of $k_{\tilde{\lambda}}(\gamma; \sigma = \tilde{\sigma}, a = \tilde{a})$. Moreover, $k_{\tilde{\lambda}}(\gamma; \sigma = c\tilde{\sigma}, a = c\tilde{a})$ has its minimum on (θ_5, θ_3) at $\gamma = \tilde{\gamma}$, and

$$k_{\tilde{\lambda}}(\tilde{\gamma}; \sigma = c\tilde{\sigma}, a = c\tilde{a}) = c^{-1/2} k_{\tilde{\lambda}}(\tilde{\gamma}; \sigma = \tilde{\sigma}, a = \tilde{a}) = c^{-1/2}\tilde{t} > \tfrac{1}{2}, \tag{3.21}$$

if the positive number c is small enough. (3.20)-(3.21) and Lemmas 2.2-2.3 imply: In order that the results of Theorems 1.1-1.3 hold, in addition to the given function $q(\theta)$, it significantly depends upon the values of σ and a. In the case that, for an arbitrary function $q(\theta)$ satisfying (H1)-(H3) and any constants $\sigma > 0$, $a > 0$,

$$k_{\lambda_2}(\gamma) > \tfrac{1}{2} \quad \text{for any} \quad \gamma \in (\theta_1, \theta_2),$$

and thus problem (1.8) has exactly one solution for $0 \le \lambda \le \lambda_2$. It is still possible that, for some $\lambda = \hat{\lambda} > \lambda_2$, the equality

$$k_{\hat{\lambda}}(\gamma) = \tfrac{1}{2}$$

has at least three solutions for $\gamma > 0$, and hence problem (1.8) has at least three solutions. In this case, to study the global bifurcation diagram for problem (1.8) on the $(\lambda, \|\theta\|_\infty)$-plane, further understanding on the quadrature formula $k_\lambda(\gamma)$ for $\lambda > \lambda_2$ is necessary.

4. Proofs of Theorems 1.4-1.5.

4.1. *Proof of Theorem 1.4.* In Theorem 1.4 we look for solutions θ of (1.8) satisfying

$$0 < \|\theta\|_\infty < \infty. \tag{4.1}$$

For any given function $q(\theta)$ satisfying (H1), (H2), (H3′), and (H4) and any given positive constants σ, a, α, the equation

$$g(\theta) = \sigma q(\theta) - a\lambda(1 + \alpha\theta) = 0 \quad (\lambda > 0)$$

has at most 4 positive roots $\theta_1 \le \theta_2 \le \theta_3 \le \theta_4$ which can be the same. More precisely,

(i) for $0 < \lambda < \lambda_0$, $g(\theta) = 0$ has two distinct positive roots $\theta_1 < \theta_4$,
(ii) for $\lambda = \lambda_0$, $g(\theta) = 0$ has three distinct positive roots $\theta_1 < \theta_2 < \theta_4$,
(iii) for $\lambda_0 < \lambda < \lambda_{1^*}$, $g(\theta) = 0$ has four distinct positive roots $\theta_1 < \theta_2 < \theta_3 < \theta_4$,
(iv) for $\lambda = \lambda_{1^*}$, $g(\theta) = 0$ has three distinct positive roots $\theta_1 < \theta_2 < \theta_3$,
(v) for $\lambda_{1^*} < \lambda < \lambda_2$, $g(\theta) = 0$ has two distinct positive roots $\theta_1 < \theta_2$,
(vi) for $\lambda = \lambda_2$, $g(\theta) = 0$ has one positive root θ_1,
(vii) for $\lambda > \lambda_2$, $g(\theta) = 0$ has no positive root,

cf. Fig. 2. By the above,

(i) for $0 < \lambda < \lambda_0$, there exists a number $\theta_5 > \theta_4$ satisfying $\int_{\theta_1}^{\theta_5} g(\theta)d\theta = 0$ such that $k(\gamma)$ exists for $\gamma \in [0, \theta_1) \cup (\theta_5, \infty)$,
(ii) for $\lambda = \lambda_0$, there exists a number $\theta_5 > \theta_4$ satisfying $\int_{\theta_1}^{\theta_5} g(\theta)d\theta = 0$ such that $k(\gamma)$ exists for $\gamma \in [0, \theta_1) \cup (\theta_5, \infty)$,
(iii) for $\lambda_0 < \lambda < \lambda_{1^*}$, if $\int_{\theta_1}^{\theta_3} g(\theta)d\theta \ge 0$, then there exists a number $\theta_5 > \theta_4$ satisfying $\int_{\theta_1}^{\theta_5} g(\theta)d\theta = 0$ such that $k(\gamma)$ exists for $\gamma \in [0, \theta_1) \cup (\theta_5, \infty)$; if $\int_{\theta_1}^{\theta_3} g(\theta)d\theta < 0$, then there exist two numbers θ_5, θ_6 with $\theta_2 < \theta_6 < \theta_3 < \theta_4 < \theta_5$ satisfying $\int_{\theta_1}^{\theta_6} g(\theta)d\theta = 0$ and $\int_{\theta_3}^{\theta_5} g(\theta)d\theta = 0$ such that $k(\gamma)$ exists for $\gamma \in [0, \theta_1) \cup (\theta_6, \theta_3) \cup (\theta_5, \infty)$,
(iv) for $\lambda = \lambda_{1^*}$, if $\int_{\theta_1}^{\theta_3} g(\theta)d\theta \ge 0$, then there exists a number $\theta_5 > \theta_3$ satisfying $\int_{\theta_1}^{\theta_5} g(\theta)d\theta = 0$ such that $k(\gamma)$ exists for $\gamma \in [0, \theta_1) \cup (\theta_5, \infty)$; if $\int_{\theta_1}^{\theta_3} g(\theta)d\theta < 0$, then there exist two numbers θ_5, θ_6 with $\theta_2 < \theta_6 < \theta_3 < \theta_5$ satisfying $\int_{\theta_1}^{\theta_6} g(\theta)d\theta = 0$ and $\int_{\theta_3}^{\theta_5} g(\theta)d\theta = 0$ such that $k(\gamma)$ exists for $\gamma \in [0, \theta_1) \cup (\theta_6, \theta_3) \cup (\theta_5, \infty)$,
(v) for $\lambda_{1^*} < \lambda < \lambda_2$, there exists a number $\theta_5 > \theta_2$ satisfying $\int_{\theta_1}^{\theta_5} g(\theta)d\theta = 0$ such that $k(\gamma)$ exists for $\gamma \in [0, \theta_1) \cup (\theta_5, \infty)$,
(vi) for $\lambda = \lambda_2$, $k(\gamma)$ exists for $\gamma \in [0, \theta_1) \cup (\theta_1, \infty)$,
(vii) for $\lambda > \lambda_2$, $k(\gamma)$ exists for $\gamma \in [0, \infty)$.

First, it is easy to obtain the next two lemmas. We omit the proofs.

Lemma 4.1. (cf. Lemma 3.1) Let constants $\alpha > 0$, $\sigma > 0$ and $a > 0$. Let $g(\theta) = \sigma q(\theta) - a\lambda(1 + \alpha\theta)$, where $q(\theta) \in C^1[0, \infty)$ satisfies (H1), (H2), (H3′) and (H4). Then, for any fixed number $\gamma = \hat{\gamma} > 0$, there exists a smallest positive number $\lambda = \hat{\lambda} > 0$ such that k_λ $(\hat{\gamma})$ defined by (2.1) exists for $\lambda \in (\hat{\lambda}, \infty)$. Moreover,

$$\lim_{\lambda \to \hat{\lambda}^+} k_\lambda(\hat{\gamma}) = \infty \quad \text{and} \quad \lim_{\lambda \to \infty} k_\lambda(\hat{\gamma}) = 0.$$

LEMMA 4.2. (cf. [7, p. 1301, Lemma 2.4]) Let constants $\alpha > 0, \sigma > 0$ and $a > 0$. Let $g(\theta) = \sigma q(\theta) - a\lambda(1 + \alpha\theta)$, where $q(\theta) \in C^1[0, \infty)$ satisfies (H1), (H2), (H3′) and (H4). Then, for $0 < \lambda \leq \lambda_2$,

(i) $k(\gamma)$ exists for every $\gamma \in [0, \theta_1)$, $k(0) = 0$ and $\lim_{\gamma \to \theta_1^-} k(\gamma) = \infty$,

(ii) the derivative $k'(\gamma)$ exists and $k'(\gamma) > 0$ on $(0, \theta_1)$.

Moreover, we have

LEMMA 4.3. Let constants $\alpha > 0$, $\sigma > 0$ and $a > 0$. Let $\lambda > 0$, $g(\theta) = \sigma q(\theta) - a\lambda(1+\alpha\theta)$, where $q(\theta) \in C^1[0, \infty)$ satisfies (H1), (H2), (H3′) and (H4). Then

(i) $\lim_{\gamma \to \infty} k(\gamma) = (\pi/2)(a\lambda\alpha)^{-1/2}$,

(ii) $k'(\gamma) < 0$ if γ is large enough.

Proof. (i) Suppose that $q(\theta)$ satisfies (H1), (H2), (H3′) and (H4). Then

$$\lim_{\theta \to \infty} \frac{q(\theta)}{\theta} = a\lambda\alpha.$$

Hence, it is well known that

$$\lim_{\gamma \to \infty} k(\gamma) = (\pi/2)(a\lambda\alpha)^{-1/2}; \tag{4.2}$$

see [9, Theorem 3.1].

(ii) It can be calculated that

$$k'(\gamma) = \int_0^\gamma \frac{H(u) - H(\gamma)}{\{2[G(u) - G(\gamma)]\}^{3/2}} \frac{du}{\gamma}, \tag{4.3}$$

where

$$\begin{aligned} H(x) &= 2G(x) - xg(x) \\ &= 2\sigma Q(x) - a\lambda(2x + \alpha x^2) - x[\sigma q(x) - a\lambda(1 + \alpha x)] \quad (Q(x) := \textstyle\int_0^x q(s)ds) \\ &= \sigma[2Q(x) - xq(x)] - a\lambda x; \end{aligned}$$

see, e.g., [13, (1.8)]. Hence by (H4),

$$\lim_{x \to \infty} H'(x) = \lim_{x \to \infty} \{\sigma[q(x) - xq'(x)] - a\lambda\} = \sigma \lim_{x \to \infty} [q(x) - xq'(x)] - a\lambda = \infty.$$

Thus for $k'(\gamma)$ in (4.3), it is easy to see that $k'(\gamma) < 0$ if γ is large enough. This completes the proof of Lemma 4.3. □

LEMMA 4.4. Let constants $\alpha > 0, \sigma > 0$ and $a > 0$. Let $\lambda > 0, g(\theta) = \sigma q(\theta) - a\lambda(1+\alpha\theta)$, where $q(\theta) \in C^1[0, \infty)$ satisfies (H1), (H2), (H3′) and (H4).

(i) For $0 < \lambda \leq \lambda_2$, problem (1.8) has at least one solution. Moreover, for $0 < \lambda < \lambda_2$ and λ small enough, problem (1.8) has a unique solution.

(ii) For $\lambda > \lambda_2 + (\pi^2/(a\alpha))$, problem (1.8) has no solution.

Lemma 4.4, (ii) improves [7, Proposition 2.6, (iii)] slightly.

Proof. Let

$$k^g(\gamma) := \int_0^\gamma \frac{du}{\{2[G(u) - G(\gamma)]\}^{1/2}} \quad (G(\theta) = \int_0^\theta g(u)du).$$

(i) For $0 < \lambda \leq \lambda_2$, by Lemma 4.2, problem (1.8) has at least one solution satisfying (4.1). Moreover, for $0 < \lambda < \lambda_0$, $g(\theta) = \sigma q(\theta) - a\lambda(1 + \alpha\theta) = 0$ has two positive roots $\theta_1 < \theta_4$, and there exists a number $\theta_5 > \theta_4$ such that

$$\int_{\theta_1}^{\theta_5} g(u)du = 0.$$

θ_5 depends on λ and is a strictly decreasing function of $\lambda < \lambda_0$. Furthermore,

$$\lim_{\lambda \to 0} \theta_5 = \infty. \tag{4.4}$$

For $0 < \lambda \leq \lambda_2$, $k^g(\gamma)$ exists for $\gamma \in [0, \theta_1) \cup (\theta_5, \infty)$. It is known in Lemma 4.2 that

(A) $k^g(0) = 0$,

(B) $\lim_{\gamma \to \theta_1^-} k^g(\gamma) = \infty$,

(C) $(k^g)'(\gamma) > 0$ on $(0, \theta_1)$.

In addition, for $0 < \lambda \leq \pi^2/(2a\alpha)$,

$$g(\theta) = \sigma q(\theta) - a\lambda(1 + \alpha\theta) > -a\lambda(1 + \alpha\theta) := f(\theta) \quad \text{for} \quad \theta > 0.$$

By Lemma 2.3,

$$k^g(\gamma) > k^f(\gamma) > 0.$$

For $f(\theta) = -a\lambda(1 + \alpha\theta)$, it is well known that $k^f(\gamma)$ exists on $[0, \infty)$. Moreover, for $\lambda = \pi^2/(2a\alpha)$ and $f(\theta) = -\pi^2(1 + \alpha\theta)/(2\alpha)$, it is well known that

(D) $k^f(0) = 0$,

(E) $\lim_{\gamma \to \infty} k^f(\gamma) = (\pi/2)(a\lambda\alpha)^{-1/2} = \sqrt{2}/2 > 1/2$,

(F) $k^f(\gamma)$ is a strictly increasing function on $(0, \infty)$.

Hence, for $\lambda = \pi^2/(2a\alpha)$, there exists a fixed number $\theta_7 > 0$ satisfying

$$k^f(\gamma = \theta_7) = \tfrac{1}{2},$$

and

$$k^f(\gamma) > \tfrac{1}{2} \quad \text{for} \quad \gamma > \theta_7.$$

So by (4.4), for $0 < \lambda < \pi^2/(2a\alpha)$ and λ small enough,

$$\theta_5 > \theta_7,$$

and hence it follows that

$$k^g(\gamma) > k^f(\gamma) > \tfrac{1}{2} \quad \text{for } \gamma > \theta_5 \ (> \theta_7).$$

Thus, by above, problem (1.8) has a unique solution if $0 < \lambda < \lambda_2$ and λ is small enough.

(ii) For $\lambda \geq \lambda_2 + (\pi^2/(a\alpha))$,

$$\begin{aligned} g(\theta) &= \sigma q(\theta) - a\lambda(1 + \alpha\theta) \\ &\leq [\sigma q(\theta) - a\lambda_2(1 + \alpha\theta)] - (\pi^2/\alpha)(1 + \alpha\theta) \\ &< -\pi^2\theta := j(\theta) \quad \text{on } (0, \infty). \end{aligned}$$

So by Lemma 2.3, we obtain that

$$0 < k^g(\gamma) < k^j(\gamma) = \tfrac{1}{2} \quad \text{for all } \gamma \in (0,\infty).$$

Hence problem (1.8) has no solution for $\lambda \geq \lambda_2 + (\pi^2/(a\alpha))$. This completes the proof of Lemma 4.4. □

By Lemmas 2.1-2.3 and 4.1-4.4, for $g_\lambda(\theta) = \sigma q(\theta) - a\lambda(1+\alpha\theta)$, there exist two positive numbers $\underline{\lambda} < \bar{\lambda}$ with

$$\underline{\lambda} = \frac{\pi^2}{a\alpha} \quad \text{and} \quad \lambda_2 < \bar{\lambda} < \lambda_2 + \frac{\pi^2}{a\alpha}$$

satisfying

$$\lim_{\gamma\to\infty} k^g_{\underline{\lambda}}(\gamma) = (\pi/2)(a\underline{\lambda}\alpha)^{-1/2} = \tfrac{1}{2},$$

$$\max_{\gamma\in(0,\infty)} k^g_{\bar{\lambda}}(\gamma) = \tfrac{1}{2},$$

such that Theorem 1.4 follows. □

4.2. *Proof of Theorem 1.5.* We first recall from (1.17)-(1.19) that

$$16 < \lambda_0 < 25,$$

$$25 < \lambda_{1^*} < 36,$$

$$140 < \lambda_2 < 180.$$

(i) $(0 \leq \lambda \leq 25)$ For $\lambda = 0$, problem (1.8) has a unique solution $\theta \equiv 0$. For $0 < \lambda \leq 25$ $(< \lambda_2)$, by Lemma 4.2, it is known that there exists a unique solution θ of problem (1.8) satisfying $0 < \|\theta\|_\infty < \theta_1$. Moreover, for $q(\theta)$ given in Fig. 2, it is clear to see that

(A) For $0 < \lambda < \lambda_0, g(\theta) = 0$ has two positive roots $\theta_1 < \theta_4$ $(< \theta_\infty \approx 2200)$, with

$$\int_{\theta_1}^{\theta_\infty} g(\theta)d\theta > 0.$$

(B) For $\lambda = \lambda_0$, $g(\theta) = 0$ has three positive roots $\theta_1 < \theta_2 < \theta_4$ $(< \theta_\infty)$, with

$$\int_{\theta_1}^{\theta_\infty} g(\theta)d\theta > 0.$$

(C) For $\lambda_0 < \lambda \leq 25$ $(< \lambda_{1^*})$, $g(\theta) = 0$ has four positive roots $\theta_1 < \theta_2 < \theta_3 < \theta_4$ $(< \theta_\infty)$, with

$$\int_{\theta_1}^{\theta_3} g(\theta)d\theta > 0, \qquad \int_{\theta_3}^{\theta_\infty} g(\theta)d\theta > 0.$$

So in either case of (A)-(C), there exists no solution θ of problem (1.8) satisfying $\theta_1 \leq \|\theta\|_\infty < \theta_\infty$. So problem (1.8) has a unique solution satisfying (1.20) for $0 \leq \lambda \leq 25$.

(ii) $(40 \leq \lambda \leq 140$ $(< \lambda_2))$ For $40 \leq \lambda \leq 140 < \lambda_2$, $g(\theta) = 0$ has exactly two positive roots $\theta_1 < \theta_2$ $(< \theta_\infty)$. Moreover,

$$\int_{\theta_1}^{\theta_\infty} g(\theta)d\theta < 0; \tag{4.5}$$

see Fig. 2. So there exists a number θ_5 with $\theta_2 < \theta_5 < \theta_\infty$ satisfying $\int_{\theta_1}^{\theta_5} g(\theta)d\theta = 0$ such that $k(\gamma)$ is defined for $\gamma \in [0,\theta_1) \cup (\theta_5, \theta_\infty)$. To show that problem (1.8) has at least two solutions satisfying (1.20) for $40 \le \lambda \le 140$, by Lemmas 2.1-2.3 and 4.1-4.3, it suffices to show that for $\lambda = 40$ and $g_{40}(\theta) = \sigma q(\theta) - 40a(1+\alpha\theta)$,

$$\inf_{\gamma \in (\theta_5, \theta_\infty)} k_{40}^g(\gamma) < \tfrac{1}{2}, \tag{4.6}$$

where

$$k_{40}^g(\gamma) := \int_0^\gamma \frac{du}{\{2[G_{40}(u) - G_{40}(\gamma)]\}^{1/2}} \quad (G_{40}(\theta) = \int_0^\theta g_{40}(u)du).$$

Similarly as before, we show (4.6) by Lemma 2.3. For $\lambda = 40$, assume that there exist numbers n, s, t, m, and p satisfying

$$n < 0, \quad t > 0, \quad p < 0, \quad \theta_1 < s < \theta_2 < m < \theta_\infty \tag{4.7}$$

such that the curve $\Lambda : f(\theta) = 0$ which consists of two line segments connecting points $(0,n)$, (s,t), and (m,p) point by point is below the curve $g_{40}(\theta) = 0$ on $[0,m]$, i.e.,

$$g_{40}(\theta) = \sigma q(\theta) - 40a(1+\alpha\theta) < f(\theta) := \begin{cases} \frac{(t-n)\theta}{s} + n & \text{for} \quad 0 \le \theta < s, \\ \frac{(p-t)(\theta-s)}{(m-s)} + t & \text{for} \quad s \le \theta \le m, \end{cases} \tag{4.8}$$

and f satisfies

$$\int_{r_1}^m f(\theta)d\theta < 0, \tag{4.9}$$

where $r_1 = ns/(n-t)$ is the solution of $f(\theta) = 0$ on $(0,s)$. Let $r_2 = (ps - mt)/(p-t) + s$ be the solution of $f(\theta) = 0$ on (s,m). By (4.9), there exists a number $r \in (r_2, m)$ such that

$$\int_{r_1}^r f(\theta)d\theta = 0;$$

see Fig. 7.

For $\lambda = 40$, let

$$k^f(\gamma) := \int_0^\gamma \frac{du}{\{2[F(u) - F(\gamma)]\}^{1/2}} \quad (F(\theta) = \int_0^\theta f(u)du).$$

It is clear that, in particular, $k_{40}^g(\gamma)$ and $k^f(\gamma)$ exist on $(\theta_5, \theta_\infty)$ and (r,m) respectively. (Note that $\theta_5 < r < m < \theta_\infty$.) By (4.8) and Lemma 2.3,

$$0 < k_{40}^g(\gamma) < k^f(\gamma) \quad \text{for } \gamma \in (r,m). \tag{4.10}$$

$k^f(\gamma)$ can be calculated explicitly since f is piecewisely a line segment. Now for function $q(\theta)$ given in Fig. 2 and constants $\alpha = 0.0039$, $\sigma = 172.7$, $a = 1286$, we choose

$$n = -45000, \quad s = 45, \quad t = 190000, \quad m = 2100, \quad p = -204000. \tag{4.11}$$

Then $r_1 = 405/47 \approx 8.6170$, $r_2 = 204090/197 \approx 1035.99$, $r \approx 2045.01$ and (4.7)-(4.9) hold. Moreover, it can be computed that

$$\inf_{\gamma \in (r,m)} k^f(\gamma) < 0.22 < \tfrac{1}{2};$$

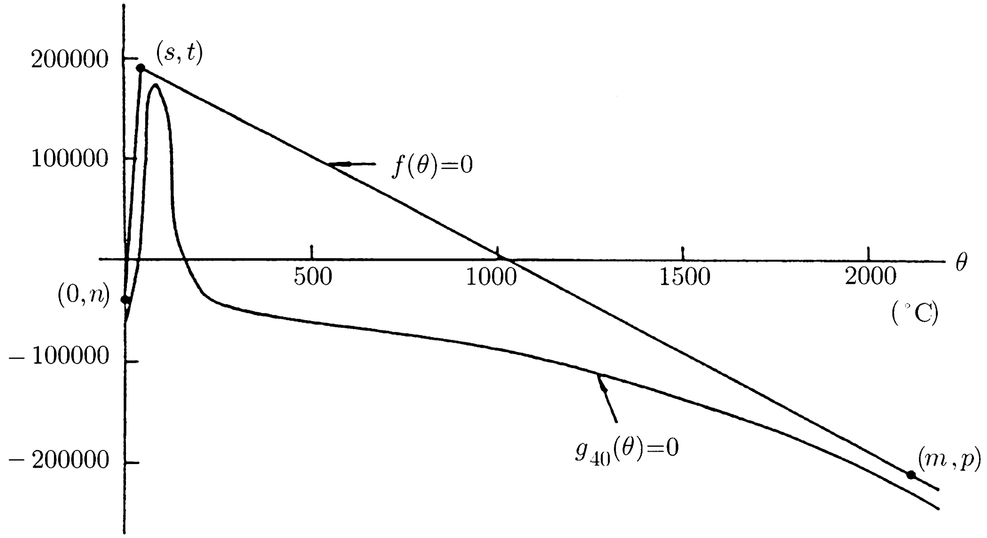

FIG. 7. Curves $f(\theta) = 0$ and $g_{40}(\theta) = 0$

see Fig. 8 which is obtained using the symbolic manipulator *Mathematica.* Thus (4.6) holds. It follows that problem (1.8) has at least two solutions satisfying (1.20) for $40 \leq \lambda \leq 140$.

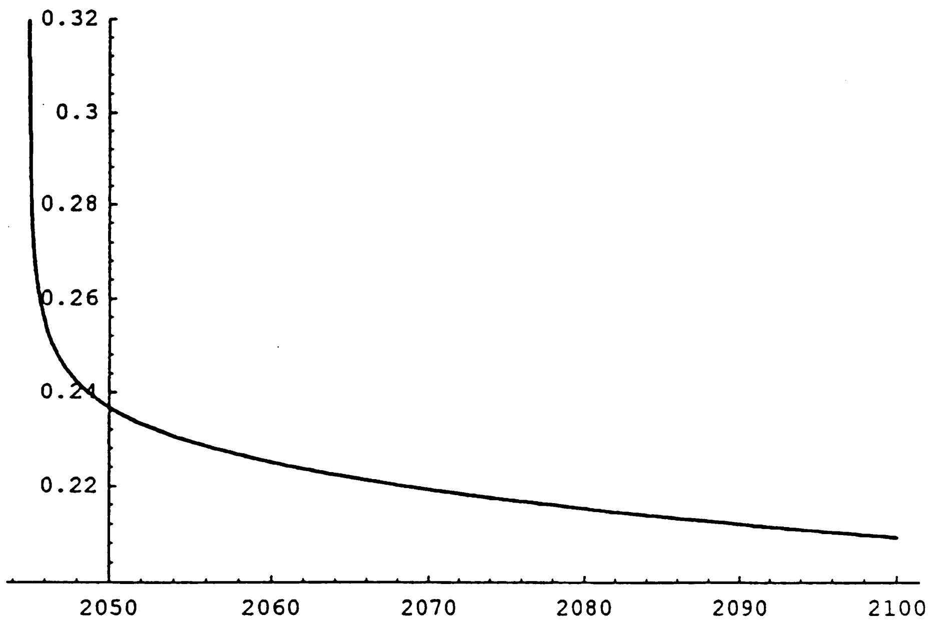

FIG. 8. Curves $k^f(\gamma)$ on (r, m)

(iii) ($\lambda \geq 182$ ($> \lambda_2 + (\pi^2/(a\alpha))$)) It follows from Lemma 4.4 and (1.19) that problem (1.8) has no solution satisfying (1.20) for

$$\lambda \geq 182 > 180 + \frac{\pi^2}{1286 \cdot 0.0039} > \lambda_2 + \frac{\pi^2}{a\alpha}. \tag{4.12}$$

(Note that $\pi^2/(1286 \cdot 0.0039) \approx 1.9679$.) This completes the proof of Theorem 1.5. □

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