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THE BIRTH OF A CUSP IN THE TWO-DIMENSIONAL, UNDERCOOLED STEFAN PROBLEM

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Abstract. This paper deals with the one-phase, undercooled Stefan problem, in space dimension $N = 2$. We show herein that planar, one-dimensional blow-up behaviours corresponding to the undercooling parameter $\Delta = 1$ are unstable with respect to small, transversal perturbations. The solutions thus produced are shown to generically generate cusps in finite time, when they exhibit an undercooling $\Delta = 1 - O(\epsilon) < 1$, where $0 < \epsilon << 1$, and ϵ is a parameter that measures the strength of the perturbation. The asymptotic behaviour of solutions and interfaces near their cusps is also obtained. All results are derived by means of matched asymptotic expansions techniques.

1. Introduction. This work is concerned with the following problem: To find a function $T(x_1, x_2, t)$ and a planar region $\Omega(t)$ such that

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} \quad \text{when } (x_1, x_2) \in \Omega(t), \quad t > 0, \tag{1.1}$$

$$\begin{aligned} &T(x_1, x_2, t) = 0 \quad \text{when } (x_1, x_2) \quad \text{lies on } \partial\Omega(t) \quad \text{(the boundary of} \\ &\Omega(t)) \quad \text{for } t > 0, \quad \text{and} \quad T(x_1, x_2, t) = 0 \quad \text{for } (x_1, x_2) \text{ outside } \Omega(t). \end{aligned} \tag{1.2}$$

$$\begin{aligned} &\tfrac{\partial T}{\partial n}(x_1, x_2, t) = -v_n(x_1, x_2, t) \quad \text{for } (x_1, x_2) \in \partial\Omega(t), \quad t > 0, \quad \text{where } v_n \\ &\text{denotes the velocity of } \partial\Omega(t) \text{ along the outer normal } n \text{ to it.} \end{aligned} \tag{1.3}$$

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Equations (1.1)-(1.3) are usually referred to as the one-phase, two-dimensional Stefan problem. They describe the evolution in time of an initial datum:

$$T(x_1, x_2, 0) = T_0(x_1, x_2), \tag{1.4}$$

such that

$$T_0(x_1, x_2) \not\equiv 0 \quad \text{for } (x_1, x_2) \in \Omega(0); \quad T_0(x_1, x_2) = 0 \quad \text{for } (x_1, x_2) \notin \Omega(0), \tag{1.5}$$

where $\Omega(0)$ is a given region in the plane. A typical situation modelled by (1.1)-(1.5) is the melting or growth of a plane ice crystal in water. Here $T(x_1, x_2, t)$ denotes the temperature of the medium, ice is assumed to remain at zero temperature, and water occupies the region $\Omega(t)$ at time $t \geq 0$. The boundary $\partial\Omega(t)$ then represents the interface (or free boundary) between liquid and crystal. For simplicity, all physical parameters in (1.1)-(1.3) (specific and latent heat, thermal diffusivity, ...) have been set equal to one after a suitable rescaling of the variables.

When analysing the behaviour of solutions of (1.1)-(1.5), a crucial role is played by the sign of $T_0(x_1, x_2)$ in $\Omega(0)$. More precisely, when

$$T_0(x_1, x_2) < 0 \quad \text{in } \Omega(0), \tag{1.6}$$

the problem is said to be undercooled, and the crystal advances into the liquid phase (cf. [9], [3] for a description of the corresponding mechanism). Solutions of problem (1.1)-(1.6), which will be henceforth referred to as the undercooled Stefan problem (USP), are known to develop a number of instabilities even when initial values of $T_0(x, y)$ and $\Omega(0)$ are smooth. In particular, initially regular interfaces $\partial\Omega(0)$ may generate singularities in finite time, a fact usually termed as blow-up (cf. [10] and [4]). The form of these singularities strongly depends on the space dimension N. For instance, when $N = 1$, it was proved in [5] that there exist solutions of (USP) such that $T(x, t)$ and $s(t) = \partial\Omega(t)$ remain regular for $t < t_0$, where $t_0 > 0$ is given. Then, at $t = t_0$, the interface velocity $|v_n|$ becomes infinite at some point $x = x_0$, and one has that

$$T(x, t_0) = -1 - \frac{1}{2\log|\log(x - x_0)|}(1 + o(1)) \quad \text{as } x \to x_0 \text{ with } x > x_0, \tag{1.7}$$

$$s(t) = x_0 + 2\left((t_0 - t)\log|\log(t_0 - t)|\right)^{\frac{1}{2}}(1 + o(1)) \quad \text{as } t \to t_0 \text{ with } t < t_0. \tag{1.8}$$

Note that (1.7) yields that $\lim_{\substack{x \to x_0 \\ x > x_0}} T(x, t_0) = -1$. It is then said that the solutions under consideration have undercooling parameter $\Delta = 1$ (see [9],[3] for a discussion of the role played by this parameter in the theory of (USP)). No singularities at all will develop in one space dimension with undercooling $0 < \Delta < 1$.

It was conjectured, however, that finite-time blow-up might occur, with undercooling $\Delta < 1$, in the case of two or three space dimensions (cf. [7]). This fact was recently shown in [11] (see also [6] in this context), where asymptotic formulae akin to (1.7), (1.8) were derived for $N = 2$ and $N = 3$ by means of matched asymptotic expansion techniques. In particular, when $N = 2$ and symmetry around the x_1-axis is assumed, one has that for any $\Delta \in (0, 1)$ there exist solutions of (USP) such that a cusp unfolds at the interface at, say, $(x_1, x_2) = (0, 0)$ and $t = t_0$. Moreover, the following estimates hold:

$$T(x_1, x_2, t_0) \to -\Delta \quad \text{as } (x_1, x_2) \to (0, 0) \quad \text{from within the liquid phase,} \tag{1.9}$$

$$\begin{aligned}&\text{If we denote the interface near } (0,0) \text{ at } t=t_0 \text{ by } x_2=x_2(x_1),\\&\text{one then has that } \quad |x_2| \sim C(\Delta)\frac{x_1}{\sqrt{\log|\log x_1|}}\\&\text{as } x_1 \to 0 \text{ with } x_1>0, \text{ where } C=C(\Delta)>0\\&\text{is such that } \lim_{\Delta\to 0} C(\Delta)=0 \text{ and } \lim_{\Delta\to 1} C(\Delta)=+\infty.\end{aligned} \tag{1.10}$$

On the other hand, solutions exhibiting singularities with undercooling $\Delta = 1$ are readily seen to exist for $N = 2$. One merely takes a one-dimensional solution satisfying (1.7), (1.8), and allows for dependence on an extra space variable, along which the solution remains constant.

The purpose of this paper consists in showing that, in two space dimensions, an arbitrarily small perturbation of the one-dimensional planar fronts recalled above will evolve towards the formation of a cusp satisfying (1.9), (1.10). In particular, the undercooling parameter of such solutions will be strictly less than one, although it will remain close to that value. An immediate consequence of this result is the instability of the one-dimensional blow-up mechanism described in [5] when small perturbations that are transversal to the front are allowed. However, our analysis will show that the solutions under consideration will remain very close to the one-dimensional, planar ones, everywhere except at times t close to the singularity formation, when a transversal component will grow in a small space region, to eventually develop a cusp. In a sense, this result was to be expected, since in the solutions obtained in [5] that satisfy the asymptotics (1.7), (1.8), the temperature of the region near the planar interface is locally described by a travelling planar front that increases its velocity as $t \to t_0$ with $t < t_0$. It is a well-known fact that such fronts are unstable under small transversal perturbations, a fact related to the so-called Mullins-Sekerka instability (cf. [8]).

To describe our results in a more precise way, it will be convenient to introduce a suitable transformation. Suppose that we are given a solution $T(x_1, x_2, t)$ of (USP) which is defined for $0 < t < t^*$ and some $t^* > 0$. Suppose also that the corresponding interface is given by a curve $t = \Lambda(x_1, x_2)$ for $0 < t < t^*$, where by assumption $\Lambda(x_1, x_2) = 0$ if $T(x_1, x_2, 0) = 0$. Then, on setting

$$u(x_1, x_2, t) = -\int_t^{\Lambda(x_1,x_2)} T(x_1, x_2, \xi)d\xi, \tag{1.11}$$

one readily sees that u satisfies

$$u_t = \Delta u - \chi_{\Omega(t)} \quad \text{for } (x_1, x_2) \in \mathbb{R}^2, \quad 0 < t < t^*, \tag{1.12}$$

where $\chi_{\Omega(t)} = 1$ when $(x_1, x_2) \in \Omega(t)$ and $\chi_{\Omega(t)} = 0$ otherwise. The integral transformation (1.11) is usually termed as the Baiocchi transformation. It turns out that it is possible to obtain solutions of (USP), starting from solutions of (1.12), by means of (1.11). As a matter of fact, we will restrict ourselves to this class of solutions in the analysis of perturbations of one-dimensional fronts that follows. That will be enough for our purposes, since we are just interested in showing an instability result for these fronts.

Let now $\overline{T}(x_1, x_2, t)$ be a solution of (USP) that is independent of x_2 and satisfies (1.7), (1.8). Let us denote by $\overline{u}(x_1, x_2, t)$ the function obtained from $\overline{T}$ by (1.11). For

$0 < \epsilon << 1$, we consider a small perturbation of $\overline{u}(x_1, x_2, 0)$ given by

$$u(x_1, x_2, 0) = \overline{u}(x_1, x_2, 0) + \epsilon v(x_1, x_2, 0), \quad \text{with } v = O(1). \tag{1.13}$$

Clearly, this gives rise to a new solution $u(x_1, x_2, t)$ of (1.12) corresponding to a slight change in the initial value $\overline{u}(x_1, x_2, 0)$ and in the original interface $x_1 = s(t)$. Such perturbation will change the blow-up time of u (denoted by t_ϵ) with respect to that of $\overline{u}$ (denoted by t_0). As a matter of fact, we will show in Sec. 2 below that

$$t_\epsilon = t_0 + O(\epsilon) \quad \text{as } \epsilon \to 0. \tag{1.14}$$

Concerning the point of formation of the cusp, we remark that since $\overline{u}$ blows up along the whole line $x_1 = 0$, the singularity of u could in principle appear anywhere close to this line, depending on the particular perturbation of $\overline{u}(x_1, x_2, 0)$ that is being made. We shall see in Sec. 2 that blow-up may be triggered around any point $(0, x_0)$, with x_0 depending on the nature of the perturbation made in (1.13). Then the blow-up point $a = (a_1, a_2)$ will be such that

$$|a_1| + |a_2 - x_0| = O(\epsilon) \quad \text{as } \epsilon \to 0. \tag{1.15}$$

Function $u(x_1, x_2, t)$ will then develop a cusp at $t = t_0$, whose profile is given by

$$|\tilde{x}_2 - a_2| \sim \frac{1}{\sqrt{\epsilon C}} \cdot \frac{\tilde{x}_1 - a_1}{\sqrt{\log|\log(\tilde{x}_1 - a_1)|}} \quad \text{as } (\tilde{x}_1, \tilde{x}_2) \to (a_1, a_2),$$

where the new coordinates $(\tilde{x}_1, \tilde{x}_2)$ are obtained from (x_1, x_2) by means of a rotation of angle $\theta = O(\epsilon)$, $\tilde{x}_1 > a_1$, and $C > 0$ is a constant depending on the perturbation made in (1.13). (1.16)

Notice that the coefficient $(\epsilon C)^{-\frac{1}{2}}$ goes to infinity as $\epsilon \to 0$, as can be expected from the fact that the initial front was planar. The temperature profile near the cusp, when approached from within the water phase, will be as follows:

$$T(\tilde{x}_1, \tilde{x}_2, t) = -1 + A\epsilon + \dots$$

as $(\tilde{x}_1, \tilde{x}_2) \to (a_1, a_2)$ with $\tilde{x}_1 > a_1$, where $A > 0$ is a constant depending on the perturbation made in (1.13). (1.17)

Comparing (1.17) with (1.7), we see that this profile is indeed a small perturbation of that obtained for $N = 1$. The undercooling parameter is now $\Delta = 1 - A\epsilon < 1$.

We conclude this section by describing the plan of the paper. We have already observed that formulae (1.14) and (1.15) will be derived in the following Sec. 2. The asymptotic expansions (1.16) and (1.17) will in turn be obtained in Sec. 3.

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2. Preliminaries. Dominance of planar behaviour. In this section we shall describe the behaviour of the solutions of the problem under consideration, during the stages of evolution where departure from the one-dimensional behaviour is rather small. To this end, we begin by briefly recalling the results obtained in [5] in the case $N = 1$, in a way better suited for the purposes of this paper.

2.1. *Blow-up behaviour for the one-dimensional problem.* Let $(\overline{u}(x,t), s(t))$ be a solution of Eq. (1.12) with $N = 1$, and assume that a singularity unfolds at the interface at a time $t_0 > 0$, located at the point $x = 0$. To describe the manner of blow-up, it is convenient to introduce self-similar variables as follows:

$$\begin{aligned} &\overline{u}(x,t) = (t_0 - t)\Phi(y,\tau), \quad y = (t_0 - t)^{-\frac{1}{2}}x, \quad \tau = -\log(t_0 - t), \\ &\lambda(\tau) = (t_0 - t)^{-\frac{1}{2}}s(t). \end{aligned} \tag{2.1}$$

Then the rescaled function Φ satisfies

$$\Phi_\tau = \Phi_{yy} - \tfrac{1}{2}y\Phi_y + \Phi - \chi_{\lambda(\tau)} \equiv A\Phi - \chi_{\lambda(\tau)}, \tag{2.2}$$

where $\chi_{\lambda(\tau)} = 1$ when $y < \lambda(\tau)$ and $\chi_{\lambda(\tau)} = 0$ otherwise. We are now led to study the asymptotic behaviour of solutions of (2.2) as $\tau \to \infty$. The kind of result that one looks for is

$$\Phi(y,\tau) \to \Phi^*(y) \quad \text{as } \tau \to \infty, \tag{2.3a}$$

where $\Phi^*(y)$ is a stationary solution of (2.2). However, determining such a function requires deriving suitable information on the behaviour of $\lambda(\tau)$ as $\tau \to \infty$. As a matter of fact, it has been shown in [5] that

$$\lim_{\tau\to\infty} \lambda(\tau) = \infty, \quad \Phi^*(y) = 1. \tag{2.3b}$$

As it turns out, unraveling the nature of the blow-up patterns requires more detailed information on the behaviour of $\Phi(y,\tau)$ as $\tau \to \infty$ than that provided in (2.3). We then set

$$\Phi(y,\tau) = 1 + \psi(y,\tau),$$

and observe that ψ satisfies

$$\psi_\tau = A\psi + (1 - \chi_{\lambda(\tau)}). \tag{2.4}$$

We now exploit the fact that Eqs. (2.2) and (2.4) are semilinear, and that operator A is selfadjoint in a suitable function space, namely $L^2_w(\mathbb{R}) = \{f \in L^2_{loc}(\mathbb{R}) : \int_{-\infty}^{\infty} |f(r)|^2 e^{-\frac{r^2}{4}} dr \equiv ||f||^2 < +\infty\}$. The eigenvalues of A are given by

$$\lambda_n = 1 - \frac{n}{2}; \quad n = 0, 1, 2, \ldots$$

and the corresponding eigenfunctions are of the form

$$H_n(y) = (-1)^n 2^n e^{\frac{y^2}{4}} \frac{d^n}{dy^n}(e^{-\frac{y^2}{4}}), \tag{2.5}$$

so that for $n = 0, 1, 2, \ldots$, $H_n(y)$ is a Hermite polynomial, which satisfies the normalization condition $H_n(y) \sim y^n$ for $y >> 1$. In particular, one has that

$$H_0(y) = 1, \quad H_1(y) = y, \quad H_2(y) = y^2 - 2, \ldots.$$

It then makes sense to expand $\psi(y,\tau)$ in Fourier series as follows:

$$\psi(y,\tau) = \sum_{k=0}^{\infty} b_k(\tau) H_k(y),$$

and by analogy with classical ODE theory, it is reasonable to expect that only one of the modes in this equation will eventually dominate there. Actually, it has been shown in [5] that there exist solutions whose asymptotics is driven by the neutral mode, i.e., such that

$$\Phi(y,\tau) = 1 + \psi(y,\tau) \sim 1 + b_2(\tau) H_2(y) + \cdots \tag{2.6}$$

as $\tau \to \infty$. Notice that no behaviour induced by the modes $n = 0, 1$ is compatible with (2.3b). On the other hand, modes $n \geq 3$ correspond to space profiles $H_n(y)$ that are unstable compared with $H_2(y)$. The behaviour described in (2.6) is therefore expected to be the generic blow-up mechanism in one space dimension.

It was shown in [5] that the following estimates hold:

$$\lambda(\tau) \sim 2(\log\tau)^{\frac{1}{2}} \quad \text{as } \tau \to \infty; \quad b_2(\tau) \sim -\frac{1}{4\log\tau} \quad \text{as } \tau \to \infty \tag{2.7}$$

and that (2.6) reads:

$$\begin{aligned} &\Phi(y,\tau) = 1 + b_2(\tau) H_2(y) + \cdots = 1 - \frac{1}{4\log\tau}(y^2 - 2) + \cdots \\ &\text{as } \tau \to \infty, \quad \text{uniformly on regions where } |y| = O(1). \end{aligned} \tag{2.8}$$

We point out, however, that (2.8) cannot be uniformly valid up to the interface $y = \lambda(\tau)$. Indeed, (2.8) yields that $\Phi(\lambda(\tau),\tau) = 0 = 1 + b_2(\tau)(\lambda(\tau)^2 - 2) + \cdots \sim 1 + b_2(\tau)\lambda(\tau)^2$ for large τ. Therefore, $b_2(\tau)\lambda(\tau)^2 \sim -1$ as $\tau \to \infty$. While this is compatible with (2.7), it then turns out that, should such behaviour hold near $y = \lambda(\tau)$, we would have that $\Phi_y(\lambda(\tau),\tau) = 2\lambda(\tau) b_2(\tau) \sim -\frac{2}{\lambda(\tau)} \neq 0$ there, which is incompatible with the fact that solutions of (2.2) are globally C^1 by classical parabolic theory. Actually, a boundary layer has to be introduced near $y = \lambda(\tau)$, where the asymptotic profile will be driven by a travelling wave. To check this point, we introduce new variables as follows:

$$y = \lambda(\tau) + \frac{\xi}{\lambda(\tau)}, \quad \Phi(y,\tau) = \frac{1}{\lambda(\tau)^2} G(\xi,\tau), \tag{2.9}$$

in which case, and since $|\dot{\lambda}| << \lambda$ and $\lambda >> 1$ for large times τ, retaining only dominant terms, (2.2) is transformed into

$$\frac{1}{\lambda^2} G_\tau = G_{\xi\xi} - \tfrac{1}{2} G_\xi - H(G),$$

where $H(G)$ is the standard Heaviside function, i.e., $H(G) = 1$ for $G > 0$ and $H(G) = 0$ otherwise. On defining a new time scale by setting $ds = \lambda(\tau)^2 d\tau$, we may rewrite the equation above in the form

$$G_s = G_{\xi\xi} - \tfrac{1}{2} G_\xi - H(G), \tag{2.10}$$

and expansion (2.8) provides now the matching condition:

$$G(\xi,s) \sim -2(\xi - 1) \quad \text{as} \quad \xi \to -\infty, \quad \text{for any} \quad s \geq 0. \tag{2.11}$$

We now claim that, as $s \to \infty$, any solution of (2.10) that satisfies (2.11) and is such that

$$G(\xi, 0) \leq M \quad \text{for some} \quad M > 0, \quad \text{when} \quad \xi \geq 0,$$

will converge, as $s \to \infty$, to the unique solution of:

$$G'' - \tfrac{1}{2}G' - H(G) = 0, \quad -\infty < \xi < +\infty, \tag{2.12a}$$

$$G(\xi) \sim -2(\xi - 1) \quad \text{as } \xi \to -\infty. \tag{2.12b}$$

Notice that an elementary analysis reveals that the solution of (2.12) is explicit, and given by

$$\begin{cases} G(\xi) = -2(\xi - 1) + 4e^{\frac{\xi - 3}{2}} & \text{for } \xi < 3, \\ G(\xi) = 0 \quad \text{for } \xi > 3. \end{cases} \tag{2.13}$$

This is the travelling wave referred to above that describes the behaviour at an inner layer near the interface tip.

To show our convergence claim, we first observe that (2.12a) has a monoparametric family of solutions given by

$$\begin{aligned} &G_\theta(\xi) = -2(\xi + \theta - 1) + 4e^{\frac{\xi + \theta - 3}{2}} \quad \text{for } \xi \leq 3 - \theta, \\ &G_\theta(\xi) = 0 \quad \text{for } \xi > 3 - \theta, \end{aligned}$$

where θ is an arbitrary real constant. Let us fix now $\delta > 0$. Then there exists $M_1 > 0$ such that

$$G_\delta(\xi) - M_1 \leq G(\xi, 0) \leq G_{-\delta}(\xi) + M_1.$$

Set now $W = |G - G_{-\delta}|$. Recalling (2.10) and (2.12a), a quick check reveals that

$$W_s \leq W_{\xi\xi} - \tfrac{1}{2}W_\xi \quad \text{for} \quad -\infty < \xi < \infty, \quad s > 0, \tag{2.14a}$$

$$W(\xi, 0) \leq M_1 \quad \text{for all } \xi, \quad W(\xi, 0) \leq 2\delta \quad \text{as } \xi \to -\infty. \tag{2.14b}$$

We shall compare now W with the solution $\overline{W}(\xi, s)$ of

$$\overline{W}_s = \overline{W}_{\xi\xi} - \tfrac{1}{2}\overline{W}_\xi \quad \text{when } -\infty < \xi < \infty, \quad t > 0,$$

such that $\overline{W}(\xi, 0) = W(\xi, 0)$. On setting $\eta = \xi - \frac{s}{2}$, we eventually obtain the linear heat equation:

$$\overline{W}_s = \overline{W}_{\eta\eta} \quad \text{when } -\infty < \eta < \infty, \quad s > 0.$$

Moreover, since we are interested in deriving an upper bound for W, we may assume (after replacing perhaps η by $\eta + \eta_0$ with $\eta_0 = O(1)$) that $W(\eta, 0) = \delta$ for $\eta < 0$ and $W(\eta, 0) = M_1$ for $\eta > 0$. The solution of the heat equation with such initial condition is explicit, and given by

$$\overline{W}(\xi, s) = \delta + (M_1 - \delta)\mathrm{erf}\left(\frac{\xi - \frac{s}{2}}{\sqrt{s}}\right), \quad \text{where} \quad \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

Back to the original variables, this means that, in the region close to $\xi = 0$, $W(\xi, s)$ stabilizes towards $G_{-\delta}$ in times $0 < s << 1$ (which corresponds to times $\tau >> 1$ in the former scale). A similar result applies when we replace $G_{-\delta}$ by G_δ in the previous argument, and letting $\delta \to 0$ we obtain the desired result.

2.2. *Early stages of the singularity formation.* From now on, we consider a solution $(\overline{u}(x,t),\ s(t))$ of a one-dimensional problem that blows up at $x_0 = 0$ and a time $t_0 > 0$, in such a way that (2.8) holds. Let us write

$$\overline{u}(x_1, x_2, t) = \overline{u}(x_1, t). \tag{2.15}$$

We then obtain at once a solution of (1.12) in two dimensions, whose interface is located along a curve $x_1 = s(t)$, where we are assuming for simplicity that $\lim_{t\to t_0} s(t) = 0$. Suppose now that we slightly perturb the initial value $\overline{u}(x_1, x_2, 0)$, by replacing it, for instance, by $\overline{u}(x_1, x_2, 0) + \epsilon v(x_1, x_2, 0)$ with $v = O(1)$ and $0 < \epsilon << 1$. We may now look for a solution of (1.12) in the form

$$u(x_1, x_2, t) = \overline{u}(x_1, x_2, t) + \epsilon v(x_1, x_2, t) + \cdots . \tag{2.16}$$

Then, to the first order, v satisfies

$$v_t = \Delta v \quad \text{when } x_1 < s(t), \tag{2.17a}$$

$$v(s(t), x_2, t) = 0, \tag{2.17b}$$

for any time $t > 0$ that is far away from the blow-up time $t_0 > 0$ corresponding to the one-dimensional interface $s(t)$ of $\overline{u}$. While (2.17a) is easily arrived at, obtaining (2.17b) requires some analysis as follows. Let us write

$$u(x_1, x_2, t) = \epsilon^2 W\left(\frac{x_1 - s(t)}{\epsilon}, x_2, t\right); \quad \xi_1 = \frac{x_1 - s(t)}{\epsilon}.$$

Then, to the first order, (1.12) yields

$$\frac{\partial^2 W}{\partial \xi_1^2} = H(W),$$

whence

$$W(\xi_1, x_2, t) = \frac{\xi_1^2}{2} + a(x_2, t)\xi_1 + b(x_2, t),$$

or in other words:

$$u(x_1, x_2, t) = \frac{(x_1 - s(t))^2}{2} + \epsilon a(x_2, t)(x_1 - s(t)) + \epsilon^2 b(x_2, t) + \cdots , \tag{2.18}$$

and comparing (2.16) with (2.18), (2.17b) follows.

Expansion (2.16), (2.17) cannot be expected to hold uniformly near the singularity as $t \to t_0$. To understand what happens as t approaches the former blow-up time, we need to obtain a detailed asymptotic expansion of function v in (2.17). To this end, we shall make use of self-similar variables. Namely, for any real number x_0, we set

$$\begin{aligned} &v(x_1, x_2, t) = (t_0 - t)F(y_1, y_2, \tau); \\ &y_1 = (t_0 - t)^{-\frac{1}{2}} x_1, \quad y_2 = (t_0 - t)^{-\frac{1}{2}}(x_2 - x_0), \quad y = (y_1, y_2), \\ &\tau = -\log(t_0 - t), \quad \lambda(\tau) = (t_0 - t)^{-\frac{1}{2}} s(t), \end{aligned} \tag{2.19}$$

where x_0 will be left as a free parameter for the time being. Equation (2.17a) transforms them into

$$F_\tau = \Delta F - \frac{y\nabla F}{2} + F \equiv AF \quad \text{for } y_1 < \lambda(\tau). \tag{2.20}$$

The spectral properties of operator A in (2.20) are natural extensions of those already recalled for the one-dimensional case in paragraph 2.1. A is selfadjoint in $L^2_w(\mathbb{R}^2) = \{f : \int_{\mathbb{R}^2} |f|^2 d\mu(y) \equiv ||f||^2 < +\infty\}$, where $d\mu(y) = e^{-\frac{|y|^2}{4}} dy$ and $|y|^2 = y_1^2 + y_2^2$. The spectrum of A consists of the eigenvalues $\lambda_\alpha = 1 - \frac{\alpha_1+\alpha_2}{2}$, where $\alpha = (\alpha_1, \alpha_2)$ and α_1, α_2 are integer nonnegative numbers. The corresponding eigenfunctions are of the form $H_\alpha(y) \equiv H_{\alpha_1\alpha_2}(y) = H_{\alpha_1}(y_1)H_{\alpha_2}(y_2)$, where $H_n(y)$ has been defined in (2.5). The space $L^2_w(\mathbb{R}^2)$ can be endowed with a Hilbert space structure corresponding to a scalar product given by

$$\langle f, g\rangle = \int_{\mathbb{R}^2} f(y)g(y)e^{-\frac{|y|^2}{4}} dy.$$

We now point out an important difference with respect to the case $N = 1$ and $N = 2, 3$ considered respectively in [5] and [11]. If we had to solve (2.20) in the whole space $\mathbb{R}^2$, it would be natural to try for F an expansion in eigenfunctions:

$$F(y, \tau) = \sum_\alpha c_\alpha(\tau)H_\alpha(y) \quad \text{with} \quad |\alpha| = \alpha_1 + \alpha_2 = 0, 1, 2, \ldots. \tag{2.21}$$

However, (2.20) is satisfied in the region $y_1 < \lambda(\tau)$, rather than in all $\mathbb{R}^2$, and it is not a priori clear whether a representation like (2.21) will hold there. To ascertain this point, let us assume initially that

$$F(y, \tau) = a(\tau)H_0(y) + o(a(\tau)) \equiv a(\tau)(1 + o(1)) \quad \text{as} \quad \tau \to \infty, \tag{2.22}$$

for some function $a(\tau)$ to be determined. To analyse the region where $y_1 \sim \lambda(\tau)$, we rescale the first space coordinate as in (2.9). Namely, we set

$$\eta_1 = \lambda(\tau)(y_1 - \lambda(\tau)).$$

Since $\lambda(\tau) \to \infty$ as $\tau \to \infty$ (cf. (2.7)), it turns out that, to the first order, $F(\eta_1, y_2, \tau)$ satisfies

$$\frac{\partial^2 F}{\partial \eta_1^2} - \frac{1}{2}\frac{\partial F}{\partial \eta_1} = 0 \quad \text{for} \quad \eta_1 < 0, \tag{2.23a}$$

$$F(0, y_2, \tau) = 0, \tag{2.23b}$$

and the matching condition arising from (2.22), namely,

$$F(\eta_1, y_2, \tau) \sim a(\tau) \quad \text{as} \quad \eta_1 \to -\infty. \tag{2.24}$$

Solving (2.23) and (2.24), we obtain the following asymptotics of F near $y_1 = \lambda(\tau)$:

$$F(y, \tau) \sim a(\tau)\left(1 - e^{\frac{\lambda(\tau)(y_1 - \lambda(\tau))}{2}}\right) \quad \text{for } y_1 < \lambda(\tau), \quad y_1 \sim \lambda(\tau). \tag{2.25}$$

To compute $a(\tau)$, we shall extend $F(y,\tau)$ by zero when $y_1 > \lambda(\tau)$. It then follows that F satisfies

$$F_\tau = \Delta F - \frac{y\nabla F}{2} + F + \frac{\partial F}{\partial y_1}(\lambda(\tau)^-, y_2, \tau)\delta(y_1 - \lambda(\tau)) \quad \text{in } \mathbb{R}^2, \tag{2.26}$$

where, as usual, $\lambda(\tau)^- = \lim_{s\to\tau}\lambda(s)$ with $s < \tau$, and $\delta(y_1 - \lambda(\tau))$ denotes a unit Dirac mass that charges along the moving boundary $y_1 = \lambda(\tau)$. From (2.25) we obtain that

$$\frac{\partial F}{\partial y_1}(\lambda(\tau)^-, y_2, \tau) \sim -\frac{a(\tau)\lambda(\tau)}{2} \quad \text{for} \quad \tau >> 1,$$

whereupon (2.26) can be approximated by setting

$$F_\tau = \Delta F - \frac{y\nabla F}{2} + F - \frac{a(\tau)\lambda(\tau)}{2}\delta(y_1 - \lambda(\tau)) \quad \text{in} \quad \mathbb{R}^2, \tag{2.27}$$

provided that $\tau >> 1$. Recalling (2.22) and taking the scalar product of both sides of (2.27) with H_0, we obtain

$$\begin{aligned} \frac{da}{d\tau} &\sim a(\tau) - \frac{a(\tau)\lambda(\tau)}{2}\cdot\frac{\langle H_0, \delta(y_1 - \lambda(\tau))\rangle}{\langle H_0, H_0\rangle} \\ &= a(\tau) - \frac{\sqrt{\pi}a(\tau)\lambda(\tau)}{\langle H_0, H_0\rangle}e^{-\frac{\lambda^2}{4}} \quad \text{as } \tau \to \infty. \end{aligned} \tag{2.28}$$

We now observe that the results recalled in §2.1 can be translated, word by word, to the case of the function $\overline{u}(x_1, x_2, t)$ defined in (2.15). In particular, we have that

$$\Phi(y,\tau) \sim 1 + b_2(\tau)H_2(y) \quad \text{as } \tau \to \infty, \tag{2.29a}$$

where we recall that $H_2(y) \equiv H_{20}(y) = H_2(y_1)H_0(y_2)$, and

$$\frac{db_2}{d\tau} \sim \frac{4\sqrt{\pi}\lambda e^{-\frac{\lambda^2}{4}}}{\langle H_{20}, H_{20}\rangle} \quad \text{as} \quad \tau \to \infty. \tag{2.29b}$$

We may thus write (2.28) in the form

$$\frac{da(\tau)}{d\tau} \sim \left(1 - \frac{\langle H_{20}, H_{20}\rangle}{4\langle H_0, H_0\rangle}\cdot\frac{db_2(\tau)}{d\tau}\right)a(\tau) \quad \text{as } \tau \to \infty. \tag{2.30}$$

In view of (2.7), integrating (2.30) gives

$$a(\tau) \sim C_0(x_0)e^\tau \quad \text{when} \quad |y| = O(1) \text{ and } \tau >> 1, \tag{2.31}$$

for some real constant $C_0(x_0)$. It will be relevant for our purposes to compute higher order terms in the asymptotics of F when $\tau >> 1$. To this end, we set

$$\begin{aligned} &F = F_0 + F_1 + \cdots, \quad \text{where } F_0 \sim C_0 e^\tau \\ &\text{and } F_1 = o(e^\tau) \quad \text{as } \tau \to \infty. \end{aligned} \tag{2.32}$$

Since (2.20) is a linear equation, $F_1(y,\tau)$ is also a solution of that equation, and by assumption $F_1 = o(e^\tau)$ as $\tau \to \infty$. We thus expect F_1 to behave in the form

$$F_1 \sim a_{01}(\tau)H_{01}(y) + a_{10}(\tau)H_{10}(y), \quad \text{when} \quad |y| = O(1) \quad \text{and} \quad \tau >> 1. \tag{2.33}$$

Arguing as before, we readily obtain that $a_{01}(\tau)$ and $a_{10}(\tau)$ are such that

$$\frac{da_{01}}{d\tau} \sim \frac{a_{01}}{2} - \frac{2\sqrt{\pi}\lambda e^{-\frac{\lambda^2}{4}}}{\langle H_{01}, H_{01}\rangle}a_{01} \sim \frac{1}{2}\left(1 - \frac{\langle H_{20}, H_{20}\rangle}{\langle H_{01}, H_{01}\rangle}\frac{db_2}{d\tau}\right)a_{01}, \tag{2.34}$$

$$\frac{da_{10}}{d\tau} \sim \frac{a_{10}}{2} - \frac{\sqrt{\pi}\lambda^3 e^{-\frac{\lambda^2}{4}}}{\langle H_{10}, H_{10}\rangle} a_{10} \sim \frac{1}{2}\left(1 - \frac{\langle H_{20}, H_{20}\rangle}{2\langle H_{10}, H_{10}\rangle}\lambda^2 \frac{db_2}{d\tau}\right) a_{10}. \tag{2.35}$$

From (2.34), we readily obtain that

$$a_{01}(\tau) \sim C_{01} e^{\frac{\tau}{2}} \quad \text{as} \quad \tau \to \infty \quad \text{for some} \quad C_{01} = C_{01}(x_0). \tag{2.36}$$

To estimate $a_{10}(\tau)$, we notice that, by (2.7), we have that

$$\frac{da_{10}}{d\tau} \sim \frac{a_{10}}{2} + \frac{a_{10}}{b_2}\frac{db_2}{d\tau} \quad \text{for} \quad \tau >> 1.$$

Hence,

$$a_{10}(\tau) \sim C_{10} e^{\frac{\tau}{2}} b_2(\tau) \quad \text{as} \quad \tau \to \infty \text{ for some } C_{10} = C_{10}(x_0). \tag{2.37}$$

It then follows from (2.32)-(2.37) that

$$F(y,\tau) = C_0 e^{\tau} + (C_{01} H_{01}(y) + C_{10} H_{10}(y) b_2(\tau)) e^{\frac{\tau}{2}} + o(e^{\frac{\tau}{2}}) \text{ as } \tau \to \infty. \tag{2.38}$$

We need yet to compute an extra term in the expansion of $F(y,\tau)$ for $\tau >> 1$. To this end, we write $F_2 = F - F_0 - F_1$, and try in (2.20) an expansion of the form

$$F_2(y,\tau) \sim a_{20}(\tau) H_{20}(y) + a_{11}(\tau) H_{11}(y) + a_{02}(\tau) H_{02}(y), \tag{2.39}$$

when $|y| = O(1)$ and $\tau >> 1$. Arguing as in our previous cases, we now obtain

$$\frac{da_{20}}{d\tau} \sim -\frac{\lambda^3 a_{20}}{2} \cdot \frac{\langle H_{20}, \delta(y_1 - \lambda)\rangle}{\langle H_{20}, H_{20}\rangle} = -\frac{\lambda^5 a_0}{2} \cdot \frac{2\sqrt{\pi} e^{-\frac{\lambda^2}{4}}}{\langle H_{20}, H_{20}\rangle}.$$

Since $\langle H_{20}, H_{20}\rangle = 2\sqrt{\pi}\int_{\mathbb{R}} (y_1^2 - 2)^2 e^{-\frac{y_1^2}{4}} dy_1 = 32\pi$, upon recalling the asymptotic formula (2.29b) we eventually obtain

$$\frac{da_{20}}{d\tau} \sim -\frac{\lambda^5 a_{20}}{32\sqrt{\pi}} e^{-\frac{\lambda^2}{4}} \sim -\frac{1}{4}\lambda^4 a_{20}\frac{db_2}{d\tau} \quad \text{for } \tau >> 1.$$

On the other hand, using the facts that $\langle H_{11}, H_{11}\rangle = (\int_{\mathbb{R}} y_1^2 e^{-\frac{y_1^2}{4}} dy_1)^2 = 16\pi$, and $\langle H_{11}, y_2\delta(y_1 - \lambda)\rangle = 4\sqrt{\pi}\lambda e^{-\frac{\lambda^2}{4}}$, we derive in a similar way that

$$\frac{da_{11}}{d\tau} \sim -\frac{\lambda^2 a_{11}}{2} \cdot \frac{\langle H_{11}, y_2\delta(y_1 - \lambda)\rangle}{\langle H_{11}, H_{11}\rangle} \sim -\frac{\lambda^3 a_{11}}{8\sqrt{\pi}} e^{-\frac{\lambda^2}{4}} \sim -\lambda^2 a_{11}\frac{db_2}{d\tau}.$$

Finally, one readily sees now that

$$\frac{da_{02}}{d\tau} \sim -\frac{\lambda a_{02}}{2} \cdot \frac{\langle H_{02}, (y_2^2 - 2)\delta(y_1 - \lambda)\rangle}{\langle H_{02}, H_{02}\rangle} \sim -2a_{02}\frac{db_2}{d\tau},$$

whereupon we derive the following estimates:

$$a_{20}(\tau) \sim C_{20} e^{-\frac{\lambda^2}{4}}, \quad a_{11}(\tau) \sim C_{11} b_2(\tau), \quad a_{02} \sim C_{02} \tag{2.40}$$

for large times τ, where C_{20}, C_{11}, and C_{02} are some real constants depending on x_0.

Having obtained an expansion for $F(y,\tau)$, we now observe that, by (2.16), (2.19), and (2.38)-(2.40), we have that

$$\begin{aligned} u(x_1,x_2,t) &= (t_0-t)(1+b_2(\tau)H_{20}(y)+\epsilon C_0 e^{\tau} \\ &\quad + \epsilon e^{\frac{\tau}{2}}(C_{01}H_{01}(y)+C_{10}H_{10}(y)b_2(\tau)) \\ &\quad + \epsilon(C_{20}e^{-\frac{\lambda^2}{4}}H_{20}(y)+C_{11}b_2(\tau)H_{11}(y)+C_{02}H_{02}(y))+\cdots), \end{aligned} \tag{2.41a}$$

which is valid for $|y| = O(1)$ and τ large but not too much. Actually, the third term on the right of (2.41a) becomes of order unity when $\epsilon e^{\tau} \sim 1$, i.e., when $(t_0 - t) = O(\epsilon)$. Therefore, (2.41a) holds provided that

$$|y| = O(1), \quad \epsilon << t_0 - t << 1. \tag{2.41b}$$

2.3. *Estimating the new blow-up parameters.* To proceed further, we now observe that it is natural to expect the blow-up time associated to $u(x_1,x_2,t)$ (henceforth denoted by t_ϵ) to be different from t_0, the blow-up time corresponding to $\overline{u}$. Moreover, blow-up for $\overline{u}$ occurs along the whole line $x_1 = 0$, but we want the interface of u to develop a singularity at some point $a = (a_1, a_2)$ such that $|a| = |a_1| + |a_2|$ is small, but whose precise location has yet to be determined. As a matter of fact, these changes in values of blow-up time and blow-up points are the reason behind the onset of exponentially growing terms in (2.41a). We shall describe next how these terms can be eliminated by means of a suitable choice of t_ϵ and a defined above. To this end, we introduce new self-similar variables as follows:

$$\begin{aligned} &u(x,t) = (t_\epsilon - t)\tilde{\Phi}(\tilde{y},\tilde{\tau}), \quad \text{where} \quad \tilde{y} = (\tilde{y}_1,\tilde{y}_2), \\ &\tilde{y}_1 = (x_1 - a_1)(t_\epsilon - t)^{-\frac{1}{2}}, \quad \tilde{y}_2 = (x_2 - a_2)(t_\epsilon - t)^{-\frac{1}{2}}, \\ &\text{and} \quad \tilde{\tau} = -\log(t_\epsilon - t). \end{aligned} \tag{2.42}$$

At this juncture, it is worth recalling that coefficient C_0 in (2.41) is a function of x_0 (cf. (2.19) and (2.31)). Actually, $C_0 = C_0(x_0)$ is a differentiable function of x_0, and moreover $\frac{dC_0}{dx_0} = C_{01}$. To check this, we notice that, on replacing x_0 by $x_0 + \delta x_0$ (δx_0 being a small perturbation) in (2.19), we are led to replace y_2 by $y_2 + \delta x_0 e^{\frac{\tau}{2}}$ in (2.38), whence we obtain after a simple computation that

$$C_0(x_0 + \delta x_0) = C_0(x_0) + C_{01}(x_0)\delta x_0 + o(\delta x_0) \quad \text{as} \quad \delta x_0 \to 0,$$

whereupon the result follows.

Let us select now an initial value for $u(x_1,x_2,t)$ such that $C_0 = C_0(x_0)$ achieves a single absolute minimum at a point $\overline{x}_0$, and let us pick $a_2 = \overline{x}_0$ in (2.42), and $x_0 = \overline{x}_0$ in (2.19) and (2.41a). Moreover, in (2.42) let us take

$$t_\epsilon = t_0 + C_0(\overline{x}_0)\epsilon.$$

It is then readily seen that

$$\begin{cases} \dfrac{t_0 - t}{t_\epsilon - t} = e^{\tilde{\tau} - \tau} = 1 - C_0(\overline{x}_0)\epsilon e^{\tilde{\tau}}, \\ \\ y_1 = a_1 e^{\frac{\tau}{2}} + \tilde{y}_1 e^{\frac{\tau - \tilde{\tau}}{2}}, \quad y_2 = \tilde{y}_2 e^{\frac{\tau - \tilde{\tau}}{2}}, \\ \\ \tilde{\Phi}(\tilde{y}, \tilde{\tau}) = e^{\tilde{\tau} - \tau}\Phi(y, \tau). \end{cases} \tag{2.43}$$

From (2.41a) and (2.43), we obtain after some simple computations that

$$\begin{aligned} \tilde{\Phi}(\tilde{y}, \tilde{\tau}) &= 1 + b_2(\tilde{\tau})H_{20}(\tilde{y}) + (2a_1 + C_{10}\epsilon)e^{\frac{\tilde{\tau}}{2}}H_{10}(\tilde{y}) \\ &\quad + \epsilon C_{20}e^{-\frac{\lambda^2}{4}}H_{20}(\tilde{y}) + \epsilon C_{11}b_2(\tilde{\tau})H_{11}(\tilde{y}) + \epsilon C_{02}H_{02}(\tilde{y}) + \cdots . \end{aligned} \tag{2.44}$$

We can cancel the only exponentially growing term remaining in (2.44) by taking

$$a_1 = -\frac{\epsilon C_{10}}{2}.$$

On the other hand, the term containing $H_{11}(\tilde{y})$ can be dispensed with by means of a small rotation of the coordinate axes. More precisely, we define

$$\tilde{y}_1 = \hat{y}_1 - \epsilon C_{11}\hat{y}_2, \quad \tilde{y}_2 = \hat{y}_2 + \epsilon C_{11}\hat{y}_1.$$

In this new set of variables, (2.44) reads now

$$\tilde{\Phi}(\hat{y}, \tilde{\tau}) = 1 + b_2(\tilde{\tau})H_{20}(\hat{y}) + \epsilon C_{02}H_{02}(\hat{y}) + \cdots . \tag{2.45}$$

Expansion (2.45) will be the starting point for all further analysis to be made in this article. Notice that (2.45) holds for times $\epsilon << t_\epsilon - t << 1$. Our next goal, to be done in Sec. 3 below, will be to analyse the asymptotics of Φ when $(t_\epsilon - t) \to 0$.

We conclude this section with a short discussion on the choice made of $x_0 = \overline{x}_0$. The reason for having selected it as the point where $C_0(x)$ reaches an absolute minimum is that, by doing so, we can ensure that no singularity appears at a different point, before the one we are looking at may develop. To see this, let us take any point $x_0 \neq \overline{x}_0$ such that $C_0(x_0) > C_0(\overline{x}_0)$, and let us check that in a neighbourhood of $x_2 = x_0$ there are no singularities for times $t \leq t_\epsilon$. To this end we define a "local blow-up time" $\hat{t}_\epsilon(x_0) = t_0 + C_0(x_0)\epsilon$, which is the time when blow-up would occur at x_0, assuming that the time of formation of the singularity is no longer $t = t_\epsilon$. Then we still can drop the term proportional to $H_{10}(y)$ in (2.41a) and (2.44) by taking $a_1 = -\frac{C_{10}(x_0)}{2}$. Let us write now $\hat{t}_\epsilon(x_0) - t = e^{-\tilde{\tau}}$, and let $\hat{\Phi}$ be given by (2.42) with t_ϵ replaced by $\hat{t}_\epsilon$. Then, we eventually arrive at

$$\begin{aligned} \hat{\Phi}(\hat{y}, \hat{\tau}) &= 1 + b_2(\tau)H_{20}(\hat{y}) + \epsilon C_{01}e^{\frac{\hat{\tau}}{2}}H_{01}(\hat{y}) \\ &\quad + \epsilon C_{02}H_{02}(\hat{y}) + \cdots , \end{aligned} \tag{2.46}$$

where we have discarded the term containing $H_{11}(\hat{y})$ by means of a suitable rotation as before. Notice that, by assumption,

$$t_\epsilon - \hat{t}_\epsilon(x_0) = (C_0(\overline{x}_0) - C_0(x_0))\epsilon < 0.$$

Since we are supposing that blow-up occurs at $t = t_\epsilon$, it suffices to consider the evolution of $\hat{\Phi}(\hat{y}, \hat{\tau})$ for times such that $e^{\frac{\hat{\tau}}{2}} \leq (\hat{t}_\epsilon(x_0) - t_\epsilon)^{-\frac{1}{2}} = ((C_0(x_0) - C_0(\overline{x}_0))\epsilon)^{-\frac{1}{2}}$, and for

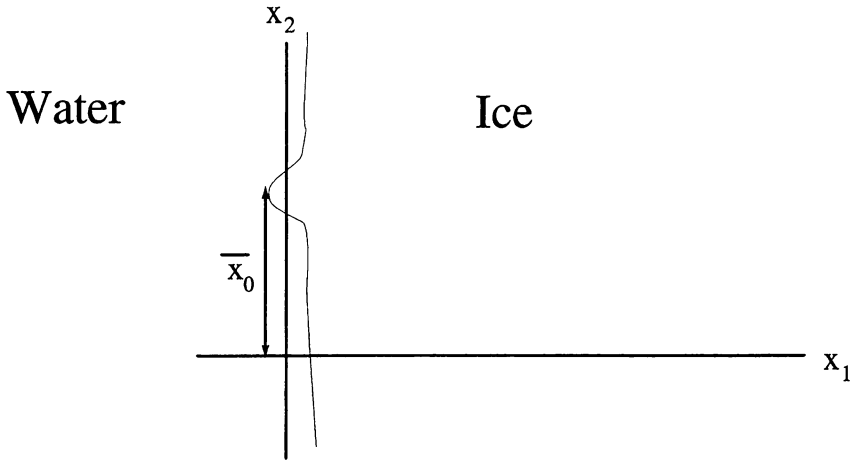

FIG. 1. Perturbing a flat profile as indicated in (2.44)

such range of times, the third term in the right of (2.46) can be bounded by $K\sqrt{\epsilon}$, where $K = K(x_0) > 0$, and it is then negligible compared with $b_2(\hat{\tau})H_{20}(\hat{y})$. It then turns out that the evolution of $\hat{\Phi}(\hat{y}, \hat{\tau})$ is basically one-dimensional, and this also applies for the boundary layer arising in the region where $y_1 \sim \lambda(\tau)$ (see the discussion at the end of subsection 2.1).

The conclusion that we obtain is that, near to any point $x_0 \neq \overline{x}_0$, the interface remains almost one-dimensional. In that case, it is natural to expect that the interface will be smooth close to such points. Unfortunately, such type of result has been rigorously proved only under the assumption that $u_t \geq 0$ (i.e., in the absence of undercooling; see [2]). However, the fact that an almost one-dimensional interface should remain smooth

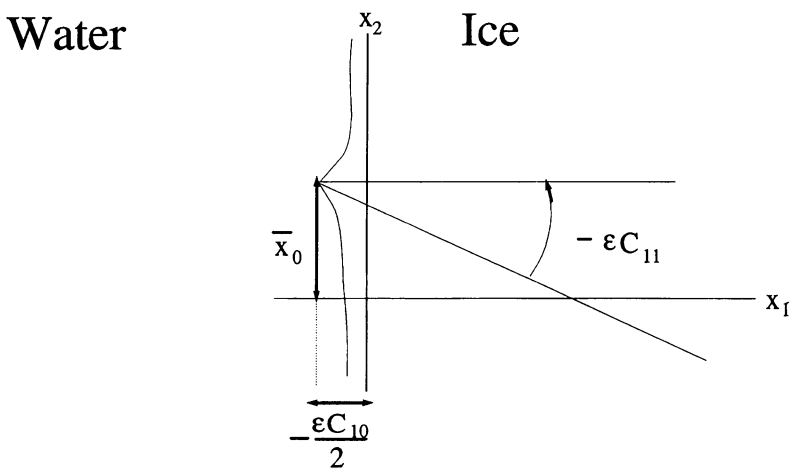

FIG. 2. Early stage of the formation of a cusp

is strongly suggested by the asymptotics analysis made in [1], which includes the case $u_t \leq 0$.

The results obtained in this section are summarized in Figures 1 and 2.

3. The unfolding of a cusp. In this section we shall describe the evolution of the rescaled function $\tilde{\Phi}$ given in (2.42), as well as that of the corresponding interface, as t approaches the blow-up time t_ϵ. For the ease of notation, we shall replace in the sequel variables $\hat{y}$, $\tilde{\tau}$ and $\tilde{\Phi}$ in (2.45) by y, τ and Φ respectively.

3.1. *The initial departure from a planar profile.* We have obtained in the previous section the following expansion:

$$\begin{aligned} &\Phi(y,\tau) = 1 + b_2(\tau)H_{20}(y) + \epsilon C_{02}H_{02}(y) + \cdots \\ &\text{when} \quad |y| = O(1) \quad \text{and} \quad \epsilon << e^{-\tau} << 1 \end{aligned} \tag{3.1}$$

(cf. (2.45)). It turns out that, at such a preliminary stage, we may approximate the moving interface by means of the equation

$$1 + b_2(\tau)H_{20}(y) + \epsilon C_{02}H_{02}(y) = 0, \tag{3.2a}$$

where we may assume without loss of generality that

$$C_{02} > 0. \tag{3.2b}$$

Indeed, on repeating the analysis leading to the differentiability of function $C_0(x_0)$, we readily see that $C_0''(\overline{x}_0) = 2C_{02}$, whence $C_{02} > 0$ generically. Since $b_2(\tau) < 0$ (cf. (2.7)), Eq. (3.2) corresponds to a hyperbola. This last is rather flat (and close to the vertical line $y_1 = |b_2(\tau)|^{-\frac{1}{2}}$ in the range $\epsilon << t_\epsilon - t << 1$), since by (2.7) one has that $|b_2(\tau)| >> \epsilon$ there.

It is readily seen that function $\Phi(y,\tau)$ satisfies

$$\Phi_\tau = \Delta\Phi - \frac{y\nabla\Phi}{2} + \Phi - H(\Phi), \tag{3.3}$$

where $H(\Phi) = 1$ when $\Phi > 0$ and $H(\Phi) = 0$ when $\Phi = 0$. On setting

$$\psi = \Phi - 1, \tag{3.4a}$$

function ψ solves

$$\psi_\tau = \Delta\psi - \frac{y\nabla\psi}{2} + \psi + \chi_{D(\tau)}, \tag{3.4b}$$

where $D(\tau) = \{y : \Phi(y,\tau) = 0\}$, and $\chi_{D(\tau)} = 1$ for $y \in D(\tau)$ and is zero otherwise. A key point in the forthcoming analysis is played by the following remark. Bearing in mind (3.1), we expect that, as $\tau \to \infty$ (i.e., as $t \to t_\epsilon$), $\Phi(y,\tau)$ will have an asymptotic behaviour of the form

$$\Phi(y,\tau) \sim 1 + c_1(\tau)H_{20}(y) + c_2(\tau)H_{02}(y), \tag{3.5}$$

where, by (3.1),

$$c_1(\tau) \sim b_2(\tau), \quad c_2(\tau) \sim \epsilon C_{02} \quad \text{for} \quad \epsilon << e^{-\tau} << 1. \tag{3.6}$$

The moving interface will approximately correspond to $\partial D(\tau)$, the boundary of $D(\tau)$, which is given by the following expression:

$$1 + c_1(\tau)H_{20}(y) + c_2(\tau)H_{02}(y) = 0. \tag{3.7}$$

We are thus led to determining the evolution of $c_1(\tau)$ and $c_2(\tau)$ as $\tau \to \infty$. To examine this question, we first observe that, on taking the scalar product of both sides of (3.4) with respect to $H_{20}(y)$ (respectively, $H_{02}(y)$), the following pair of differential equations is obtained:

$$\frac{dc_1}{d\tau} = \frac{1}{32\pi} \iint_{D(\tau)} H_{20}(y) e^{-\frac{|y|^2}{4}} dy, \tag{3.8}$$

$$\frac{dc_2}{d\tau} = \frac{1}{32\pi} \iint_{D(\tau)} H_{02}(y) e^{-\frac{|y|^2}{4}} dy, \tag{3.9}$$

where we have used the fact that, for $i = 1, 2$, $\langle y_i^2 - 2, y_i^2 - 2\rangle = 32\pi$. Equations (3.8) and (3.9) are to be considered together with the matching conditions (3.6) at $e^{-\tau} \sim \epsilon$. Notice that in the one-dimensional case (i.e., when u is replaced by $\overline{u}$), $D(\tau)$ coincides with the half-plane $\{y_1 > \lambda(\tau) : \lambda(\tau)$ as in (2.7)$\}$, and the analysis in [5] yields that $c_1(\tau) = b_2(\tau)$, in agreement with (3.6). In this case one also has that $\iint_{D(\tau)} H_{02}(y) e^{-\frac{|y|^2}{4}} dy = 0$, and $c_2(\tau) \equiv 0$, as expected.

To describe the behaviour of $c_1(\tau)$ and $c_2(\tau)$, it will be useful to rewrite (3.7) in the form

$$y_2 = \pm \left(2 - \frac{1 + c_1(y_1^2 - 2)}{c_2}\right)^{\frac{1}{2}} \equiv \pm g(y_1; c_1, c_2). \tag{3.10}$$

We shall also denote by $\lambda(\tau)$ the value of y_1 corresponding to the intersection of the hyperbola (3.8) with the line $y_2 = 0$, i.e.,

$$\lambda(\tau)^2 = -\frac{(1 - 2c_2)}{c_1} + 2. \tag{3.11}$$

Notice that this function $\lambda(\tau)$ reduces to that in (2.7) if $c_2 = 0$. The integral terms on the right of (3.8), (3.9) can now be written in the form

$$I_1 \equiv \iint_{D(\tau)} H_{20}(y) e^{-\frac{|y|^2}{4}} dy = 2 \int_{\lambda(\tau)}^{\infty} dy_1 \int_0^{g(y_1; c_1, c_2)} (y_1^2 - 2) e^{-\frac{y_1^2 + y_2^2}{4}} dy_2, \tag{3.12}$$

$$I_2 \equiv \iint_{D(\tau)} H_{02}(y) e^{-\frac{|y|^2}{4}} dy = 2 \int_{\lambda(\tau)}^{\infty} dy_1 \int_0^{g(y_1; c_1, c_2)} (y_2^2 - 2) e^{-\frac{y_1^2 + y_2^2}{4}} dy_2. \tag{3.13}$$

We point out that, so long as we have that

$$y_1 \geq \lambda >> 1, \quad |c_2| << 1, \tag{3.14a}$$

we may approximate (3.10) and (3.11) by

$$g(y_1; c_1, c_2) \sim \left(-\frac{(c_1 y_1^2 + 1)}{c_2}\right)^{\frac{1}{2}} \equiv \overline{g}(y_1; c_1, c_2), \tag{3.14b}$$

$$\lambda^2 \sim -\frac{1}{c_1}. \tag{3.14c}$$

Making use of (3.14), and setting $y_1 = \lambda + \xi$, (3.12) gives

$$\begin{aligned} I_1 &\sim 2\int_\lambda^\infty dy_1 \int_0^{\overline{g}} y_1^2 e^{-\frac{y_1^2+y_2^2}{4}} dy_2 \\ &= 2\int_0^\infty d\xi \int_0^{(-\frac{c_1}{c_2}(\xi^2+2\lambda\xi))^{\frac{1}{2}}} (\lambda+\xi)^2 e^{-\frac{\lambda^2}{4}} e^{-\frac{2\lambda\xi+\xi^2+y_2^2}{4}} dy_2, \end{aligned} \tag{3.15}$$

and in a similar way we obtain

$$I_2 \sim 2\int_0^\infty d\xi \int_0^{(-\frac{c_1}{c_2}(\xi^2+2\lambda\xi))^{\frac{1}{2}}} (y_2^2-2) e^{-\frac{\lambda^2}{4}} e^{-\frac{2\lambda\xi+\xi^2+y_2^2}{4}} dy_2. \tag{3.16}$$

Note that the right-hand sides of (3.15) and (3.16) depend only on $\lambda(\tau)$ and on the ratio

$$\chi \equiv -\frac{c_1}{c_2}.$$

We now set $\eta = \chi(2\lambda\xi + \xi^2)$ (whence $\xi = -\lambda + \left(\lambda^2 + \frac{\eta}{\chi}\right)^{\frac{1}{2}}$). Then (3.15) and (3.16) become

$$I_1 \sim \frac{e^{-\frac{\lambda^2}{4}}}{\chi^{\frac{3}{2}}} \int_0^\infty (\lambda^2\chi+\eta)^{\frac{1}{2}} e^{-\frac{\eta}{4\chi}} d\eta \int_0^{\sqrt{\eta}} e^{-\frac{y_2^2}{4}} dy_2, \tag{3.17}$$

$$I_2 \sim \frac{e^{-\frac{\lambda^2}{4}}}{\chi^{\frac{1}{2}}} \int_0^\infty (\lambda^2\chi+\eta)^{-\frac{1}{2}} e^{-\frac{\eta}{4\chi}} d\eta \int_0^{\sqrt{\eta}} (y_2^2-2) e^{-\frac{y_2^2}{4}} dy_2. \tag{3.18}$$

At the initial departure from a flat profile, $|c_1| >> |c_2|$, and thus $\chi >> 1$. On the other hand, by the orthogonality properties of Hermite polynomials, one readily sees that $\int_0^{\sqrt{\eta}} (y_2^2-2)e^{-\frac{y_2^2}{4}} dy_2 = -\int_{\sqrt{\eta}}^\infty (y_2^2-2)e^{-\frac{y_2^2}{4}} dy_2$. Since $\lambda(\tau) >> 1$, it follows from (3.18) that

$$I_2 \sim \frac{e^{-\frac{\lambda^2}{4}}}{\lambda\chi} \int_0^\infty d\eta \int_0^{\sqrt{\eta}} (y_2^2-2)e^{-\frac{y_2^2}{4}} dy_2 = -\frac{8\sqrt{\pi}}{\lambda\chi} e^{-\frac{\lambda^2}{4}} \tag{3.19}$$

provided that $\chi >> 1$.

To describe the asymptotics of I_1 for large χ, we set $z = \frac{\eta}{\chi}$ in (3.17) and notice that

$$I_1 \sim \frac{e^{-\frac{\lambda^2}{4}}}{\chi^{\frac{1}{2}}} \int_0^\infty (\lambda^2\chi+\chi z)^{\frac{1}{2}} e^{-\frac{z}{4}} dz \int_0^{\sqrt{z\chi}} e^{-\frac{y_2^2}{4}} dy_2,$$

whence

$$I_1 \sim 4\sqrt{\pi}\lambda e^{-\frac{\lambda^2}{4}} \quad \text{when} \quad \chi >> 1. \tag{3.20}$$

Putting together (3.8), (3.9), (3.19), and (3.20), we have obtained that, whenever $\chi >> 1$, the evolution of c_1 and c_2 is governed by

$$\frac{dc_1}{d\tau} = \frac{1}{8\sqrt{\pi}}\lambda e^{-\frac{\lambda^2}{4}}, \tag{3.21}$$

$$\frac{dc_2}{d\tau} = \frac{e^{-\frac{\lambda^2}{4}}}{4\sqrt{\pi}\lambda}\frac{c_2}{c_1}. \tag{3.22}$$

It then turns out that

$$c_1(\tau) \sim b_2(\tau) \sim -\frac{1}{4\log\tau} \quad \text{when} \quad \chi >> 1. \tag{3.23}$$

Notice that (3.21) agrees with the result obtained in [5] for the case $N = 1$ (see formula (2.18) there). On the other hand, from (3.21) and (3.22) we obtain

$$\frac{dc_2}{dc_1} = \frac{2c_2}{\lambda^2 c_1}. \tag{3.24}$$

It follows from (3.23), (3.24) that $c_2 e^{2c_1}$ is approximately constant. Using again (3.23), it follows that c_2 experiences only minor changes in the range considered, whence

$$c_2 \sim \epsilon C_{02} \quad \text{whenever} \quad \chi >> 1. \tag{3.25}$$

3.2. *The onset of the final stage.* In view of our previous discussion, the asymptotic behaviours described by (3.23) and (3.25) will countinue to hold at least until

$$|c_1| \sim |c_2|, \tag{3.26a}$$

which corresponds to times t such that

$$\log|\log(t_\epsilon - t)| \sim \frac{1}{\epsilon}. \tag{3.26b}$$

Therefore, the one-dimensional asymptotics described in [5] is valid up to times very close to the appearance of the singularity. As a matter of fact, we have that

$$\begin{aligned} &c_2 \sim \epsilon C_{02} \quad \text{for arbitrarily large times,} \\ &\text{provided that (3.15) and (3.16) are satisfied.} \end{aligned} \tag{3.27}$$

Actually, for any $A > 0$ one has that

$$\int_0^A (y_2^2 - 2)e^{-\frac{y_2^2}{4}}\,dy_2 \le C\int_0^A e^{-\frac{y_2^2}{4}}\,dy_2 \quad \text{for some} \quad C > 0,$$

and thus, whenever (3.15) and (3.16) hold true, we readily see that

$$|I_2| \le \frac{CI_1}{\lambda^2},$$

and by (3.8) and (3.9), this in turn implies that

$$\left|\frac{dc_2}{dc_1}\right| \le \frac{C}{\lambda^2} \sim C|c_1|, \tag{3.28}$$

where we have made use of (3.14c) (which is satisfied if $|c_2| << 1$) to derive the last statement above. If we now integrate (3.28) starting from an initial time where (3.26) holds, we deduce at once that, as $\tau \to \infty$, variations of $|c_2|$ are negligible compared with $|c_2|$ itself, and (3.27) holds. We now take advantage of (3.27) to evaluate $c_1(\tau)$ as follows.

Since $\frac{d}{d\tau}\chi \equiv \frac{d}{d\tau}(-\frac{c_1}{c_2}) \sim -\frac{1}{c_2}\frac{dc_1}{d\tau} \sim -\frac{1}{\epsilon C_{02}}\frac{dc_1}{d\tau}$ as long as (3.27) remains true, we have in that case that, by (3.8) and (3.17),

$$\frac{d\chi}{d\tau} \sim -\frac{e^{-\frac{\lambda^2}{4}}}{32\pi\epsilon C_{02}\chi^{\frac{3}{2}}}\int_0^\infty (\lambda^2\chi+\eta)^{\frac{1}{2}}e^{-\frac{\eta}{4\chi}}d\eta\int_0^{\sqrt{\eta}} e^{-\frac{y_2^2}{4}}dy_2.$$

Recalling that $\frac{\lambda^2}{4} = \frac{1}{2}+\frac{c_2}{2c_1}-\frac{1}{4c_1} = \frac{1}{2}-\frac{1}{2\chi}+\frac{1}{4c_2\chi}$ (cf. (3.11)) and that $\lambda^2\chi \sim \frac{1}{\epsilon C_{02}}$ (by (3.11) and (3.27)), we see that, since $0<\epsilon<<1$:

$$\begin{aligned}\frac{d\chi}{d\tau} &\sim -\frac{e^{-\frac{1}{2}}e^{-\frac{1}{4\epsilon C_{02}\chi}+\frac{1}{2\chi}}}{32\pi\epsilon C_{02}\chi^{\frac{3}{2}}}\int_0^\infty\left(\frac{1}{\epsilon C_{02}}+\eta\right)^{\frac{1}{2}}e^{-\frac{\eta}{4\chi}}d\eta\int_0^{\sqrt{\eta}}e^{-\frac{y_2^2}{4}}dy_2\\ &\sim -\frac{e^{-\frac{1}{2}}e^{-\frac{1}{4\epsilon C_{02}\chi}+\frac{1}{2\chi}}}{32\pi(\epsilon C_{02})^{\frac{3}{2}}\chi^{\frac{3}{2}}}\int_0^\infty e^{-\frac{\eta}{4\chi}}d\eta\int_0^{\sqrt{\eta}}e^{-\frac{y_2^2}{4}}dy_2,\end{aligned}$$

so that integrating by parts in the last expression gives

$$\frac{d\chi}{d\tau} \sim -\frac{e^{-\frac{1}{2}}e^{-\frac{1}{4\epsilon C_{02}\chi}+\frac{1}{2\chi}}}{16\pi(\epsilon C_{02})^{\frac{3}{2}}\chi^{\frac{3}{2}}}\int_0^\infty e^{-\frac{\eta}{4\chi}-\frac{\eta}{4}}\eta^{-\frac{1}{2}}d\eta, \tag{3.29}$$

provided that (3.15) and (3.16) are satisfied.

Equation (3.29) shows that, as τ increases, χ (and hence c_1) will decrease. As this happens, the hyperbola (3.7), that can be written in the form

$$y_2^2 = \chi y_1^2 + \left(2(1-\chi)-\frac{1}{c_2}\right), \tag{3.30}$$

will bend sharply, and we need to check whether the approximations made to derive (3.27) and (3.29) will continue to hold. To this end, we introduce new variables as follows:

$$y_1 = \lambda + \frac{\xi_1}{\lambda}, \quad y_2 = \frac{\xi_2}{\lambda\sqrt{\epsilon}}, \quad \Phi = \frac{Q}{\lambda^2}.$$

In these new variables, when $\tau >> 1$, (3.3) becomes to the first order:

$$\frac{\partial^2 Q}{\partial\xi_1^2} - \frac{1}{2}\frac{\partial Q}{\partial\xi_1} - H(Q) = 0, \tag{3.31}$$

whereas by (3.6) and (3.11) the following matching condition has to be satisfied:

$$Q(\xi_1,\xi_2) \sim (-2\xi_1 + C_{02}\xi_2^2) \quad \text{as} \quad |\xi_1|+|\xi_2| \to \infty \quad \text{with} \quad \xi_1 < 0. \tag{3.32}$$

The solution of (3.31), (3.32) is given by

$$\begin{cases} Q(\xi_1,\xi_2) = -2\xi_1 + C_{02}\xi_2^2 + 4e^{\frac{\xi_1}{2}-(1+\frac{C_{02}}{4}\xi_2^2)} \quad \text{when} \quad \xi_1 < 2\left(1+\dfrac{C_{02}\xi_2^2}{4}\right), \\ Q(\xi_1,\xi_2) = 0 \quad \text{for} \quad \xi_1 \geq 2\left(1+\frac{C_{02}\xi_2^2}{4}\right). \end{cases} \tag{3.33}$$

Notice that, in this new set of variables, the free boundary becomes a fixed parabola, namely,

$$\xi_1 = 2\left(1 + \frac{C_{02}\xi_2^2}{4}\right). \tag{3.34}$$

In the original variables, the curvature of this parabola is of order unity when $\lambda \sim \frac{1}{\sqrt{\epsilon}}$, or equivalently (by (3.14c) and (3.29)) when $\chi \sim \epsilon$. It is interesting to remark that in the cusp mechanism described in [11], the asymptotics near the tip of the cusp was described by an Ivantsov parabola (cf. [9]), which satisfies the equation

$$\frac{\partial^2 Q}{\partial \xi_1^2} + \frac{\partial^2 Q}{\partial \xi_2^2} - \frac{1}{2}\frac{\partial Q}{\partial \xi_1} - H(Q) = 0,$$

instead of (3.31). Actually, the Ivantsov profiles obtained in [11] vanish along parabolae of the type $\xi_1 = \mu\xi_2^2 + \gamma$, for some positive numbers μ and $\gamma = \gamma(\mu)$. In the case of very wide profiles (i.e., when $0 < \mu << 1$) we may rescale the length of the coordinate axes in the form $\bar{\xi}_1 = \xi_1$, $\bar{\xi}_2 = \sqrt{\mu}\,\xi_2$, and we would then obtain Eq. (3.31) and the interface shape (3.34) in the limit $\mu \to 0$ (cf. Figure 3).

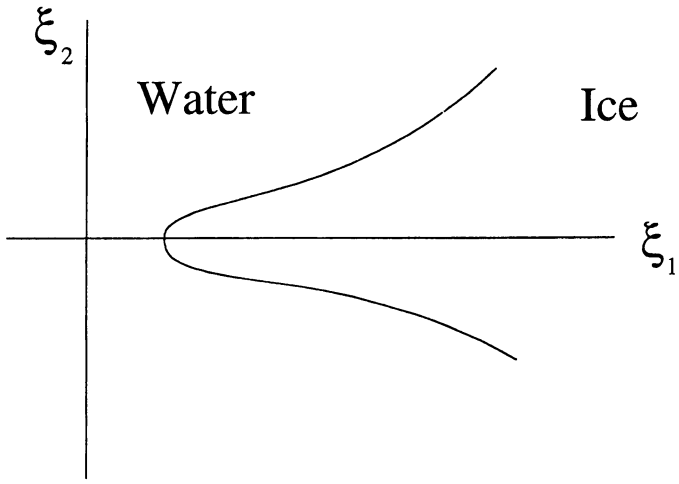

FIG. 3. The blow-up pattern near the unfolding cusp. At an inner region near the point where the singularity is formed, the rescaled function Φ approaches asymptotically towards a wide Ivantsov parabola.

We conclude this paragraph by observing that (3.27) and (3.29) are valid for all times $\tau >> 1$. This is due to the fact that (3.15) and (3.16) remain valid approximations when $\tau \to \infty$. Indeed, our basic assumption was that the major contribution to I_1 in (3.15) is that coming from the region where $y_1 \sim \lambda(\tau)$, and this continues to be true regardless of the bending of the interface (3.30).

3.3. *Computing the final profiles.* We conclude this section by deriving the final profile of the cusp, as well as the distribution of temperatures near the singularity. For this purpose, we introduce new variables, given by

$$z_1 = \frac{y_1}{\lambda}, \quad z_2 = \sqrt{\epsilon}\, y_2.$$

In these variables, (3.3) reads to the first order as follows:

$$-\frac{z_1}{2}\frac{\partial \Phi}{\partial z_1} - \frac{z_2}{2}\frac{\partial \Phi}{\partial z_2} + \Phi - H(\Phi) = 0, \tag{3.35}$$

and (3.5), (3.6) provide the following matching condition:

$$\Phi \sim 1 - z_1^2 + C_{02} z_2^2 \quad \text{as} \quad |z| \equiv |z_1| + |z_2| \to 0. \tag{3.36}$$

The solution of (3.35), (3.36) is explicit, and given by

$$\Phi(z_1, z_2) = 1 - z_1^2 + C_{02} z_2^2. \tag{3.37}$$

This function encodes the final profile of the temperature, as well as that of the interface, in a neighbourhood of the cusp. To show this, we first remark that setting $\Phi = 0$ in (3.37), we obtain the hyperbola

$$z_1^2 = C_{02} z_2^2 + 1.$$

When $|z| \to \infty$, such a curve is approximately given by

$$|z_1| \sim \sqrt{C_{02}}\, |z_2|. \tag{3.38}$$

In terms of the original variables (x_1, x_2), (3.38) reads

$$|x_2 - a_2| \sim \frac{1}{\lambda\sqrt{\epsilon C_{02}}} |y_1| \sim \sqrt{\chi}\,(x_1 - a_1) \tag{3.39}$$

(cf. (3.14)). We next recall that, by (3.27), (3.23), and (3.29),

$$\chi = -\frac{c_1}{c_2} \sim \frac{1}{4\epsilon C_{02} \log \tau} \quad \text{as } \tau \to \infty. \tag{3.40}$$

Let us fix now $\bar{t}$ such that $\bar{t}$ is close to (but strictly less than) t_ϵ, and select $x_1 = \overline{x}_1$ given by $x_1 - a_1 = A(t_\epsilon - \bar{t})^{\frac{1}{2}}$, with $A = O(1)$ and positive. Then $\overline{\tau} = -\log(t_\epsilon - \bar{t}) \sim -2\log(x_1 - a_1)$. Substituting this into (3.40), plugging the resulting formula in (3.39), and letting then $\bar{t} \to t_\epsilon$, we eventually obtain

$$\begin{aligned} &|x_2 - a_2| \sim \tfrac{1}{2\sqrt{\epsilon C_{02}}} \cdot \tfrac{(x_1 - a_1)}{\sqrt{\log|\log(x_1 - a_1)|}} \\ &\text{as} \quad (x_1, x_2) \to (a_1, a_2) \quad \text{with} \quad x_1 > a_1. \end{aligned} \tag{3.41}$$

The final profile of $u(x, t_\epsilon)$ can be computed in a similar way to obtain

$$\begin{aligned} u(x_1, x_2, t_\epsilon) &\sim -\epsilon\, C_{02}\, \chi (x_1 - a_1)^2 + \epsilon C_{02}(x_2 - a_2)^2 + \cdots \\ &= -\frac{(x_1 - a_1)^2}{4\log|\log(x_1 - a_1)|} + \epsilon C_{02}(x_2 - a_2)^2 + \cdots, \end{aligned} \tag{3.42}$$

provided that $(x_1, x_2) \to (a_1, a_2)$. Finally, taking into account that $T = u_t = \Delta u - H(u)$, we derive from (3.42) the following asymptotic expansion for the temperature near the tip of the cusp:

$$\begin{aligned} &T(x_1, x_2, t_\epsilon) = -1 + 2\epsilon C_{02} - \frac{1}{2\log|\log(x_1 - a_1)|} + \cdots, \\ &\text{as} \quad (x_1, x_2) \to (a_1, a_2) \quad \text{with} \quad x_1 > a_1. \end{aligned} \tag{3.43}$$

Note that the undercooling of $T(x, t_\epsilon)$ is strictly less than one, and equal to $1 - 2\epsilon C_{02}$.

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