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# ASYMPTOTIC BEHAVIOUR OF THE ENERGY IN PARTIALLY VISCOELASTIC MATERIALS

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**Abstract.** In this paper we study models of materials consisting of an elastic part (without memory) and a viscoelastic part, where the dissipation given by the memory is effective. We show that the solutions of the corresponding partial viscoelastic model decay exponentially to zero, provided the relaxation function also decays exponentially, no matter how small is the viscoelastic part of the material

**1. Introduction.** Let us consider an $n$-dimensional body which in its reference configuration is homogeneous and occupies the open bounded set $\Omega \subset \mathbb{R}^n$ with smooth boundary $\Gamma$. Let $x \mapsto u(x,t)$ be the position of the material particle $x$ at time $t$. Then the viscoelastic equation of motion is given by

$$\rho u_{tt} - \kappa\Delta u + \int_0^\infty g(s)\,\mathrm{div}\{a(x)\nabla u(\cdot, t-s)\}\,ds = f \quad \text{in } \Omega\times\,]0,\infty[,$$
$$u(x,t) = 0 \quad \text{on } \partial\Omega\times\,]0,\infty[=\Gamma\times\,]0,\infty[,$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_t(x) \quad \text{in } \Omega,$$

where $\rho$ is the mass density function, $g$ is the relaxation function and $f$ denotes the body force. Here we are mainly interested in the asymptotic behaviour of the solution $u$ when $t$ tends to infinity. Note that the above model is dissipative, and the dissipation is given by the memory term, where $a \geq 0$. The memory is effective only in a part of the body $\Omega$ where $a > 0$.

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Concerning the exponential stability of dissipative systems, it is well known by now that the solution of the wave equation with frictional damping

$$u_{tt} - \Delta u + a(x)u_t = 0 \quad \text{in } \Omega\times]0,\infty[,$$
$$u(x,t) = 0 \quad \text{on } \Gamma\times\, ]0,\infty[,$$

decays exponentially to zero as time goes to infinity provided $a(x) \geq a_0 > 0$ a.e. in $\Omega$. In this case, since $a(x)$ is a positive function, the dissipative effect is working in the whole domain $\Omega$. In this direction we may ask whether the damping term $a(x)u_t$ continues to be effective when the function $a$ satisfies only $a(x) \geq 0$. That is, suppose that the function $a(x)$ vanishes outside an open subset $\omega$ of $\Omega$. Is the dissipative term $a(x)u_t$ strong enough to produce the exponential decay of the solution? Under what conditions on $\omega$ may we expect a uniform rate of decay for the solution? A first answer to these questions was given in [6], [14], [18], [27]. In those works the authors proved that the solution of the wave equation with "local" damping decays exponentially to zero provided $\omega$ is a neighborhood of the boundary $\Gamma$. This means that a dissipation mechanism that is effective over a strategic part of the material is strong enough to produce exponential rate of decay of the total energy. Such results are connected with Control Theory. It was proved by Lions [13] that the solution of the wave equation can be controlled acting only in a neighborhood of the boundary. Roughly speaking, any time that we can control the solution acting on a part of the material, it is possible to introduce a dissipative mechanism at the same place producing uniform decay in time of the energy. The methods in Bardos et al. [1] allow us to give a necessary and sufficient condition for the uniform decay of the solutions of the dissipative wave equation. Namely, this condition requests that every ray of geometric optics that propagates in $\Omega$ and is reflected on the boundary enters the region $a > 0$ in a uniform time. In fact, the necessity of this condition is due to the construction of Ralston [22, 23] of the Gaussian beam solutions of the wave equation. Other examples of localized damping are given by introducing dissipative boundary conditions acting in a part of the boundary. In this direction there is an extensive literature; see for example [4], [5], [6], [7], [8], [10], [11], [12], [17], [19], [20], [21], [24], [26], [28] among others.

In this paper we also consider locally distributed dissipation; but this dissipation does not appear by the introduction of any artificial mechanism. On the contrary, it arises because of the mixed structure of the material. That is, we consider a body consisting of an elastic and a viscoelastic part. So, the dissipation is due to the memory effect which works only over a portion of the material. The one-dimensional case is represented in Fig. 1, but our analysis applies without restriction on the dimension.

Denoting by $\sigma$ the stress and by $*$ the convolution product $g * f = \int_0^t g(t-\tau)f(\tau)\,d\tau$, the constitutive law we use in this paper is given by

$$\sigma = \kappa\nabla u + a(x)g * \nabla u,$$

so that the corresponding motion equation for $\kappa = 1$ may be written as

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\operatorname{div}\{a(x)\nabla u\}\,d\tau = 0 \quad \text{in } \Omega \times 0,\infty[ \tag{1.1}$$

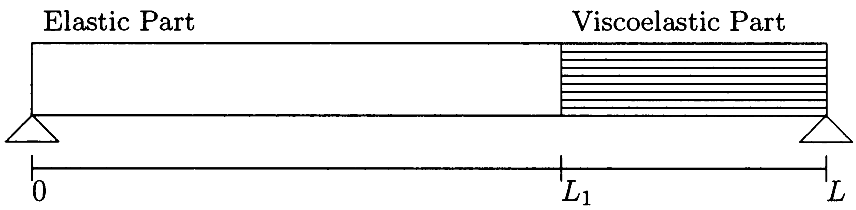


FIG. 1

with initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \tag{1.2}$$

and Dirichlet's boundary condition

$$u(x,t) = 0 \quad \text{on } \Gamma \times ]0,\infty[. \tag{1.3}$$

For materials with memory the stress depends not only on the present values but also on the entire temporal history of the motion. Therefore, we have also to prescribe the history of $u$ prior to 0 as the initial data. Here we assume that $u$ vanishes identically for $t < 0$, that is,

$$u(x,t) = 0, \quad \text{for } t < 0.$$

We do not assume that $u(\cdot, 0^+) = u(\cdot, 0^-)$ or $u_t(\cdot, 0^+) = u_t(\cdot, 0^-)$.

Let $x \mapsto a(x)$ be a nonnegative $C^2$-function defined over $\Omega$ and let us denote by $\omega_\varepsilon$ the set

$$\omega_\varepsilon = \left( \bigcup_{x \in \Gamma_0} B_\varepsilon(x) \right) \cap \Omega,$$

where $B_\varepsilon(x) = \{x \in \mathbb{R}^n;\ \|x\| \le \varepsilon\}$ and $\Gamma_0$ is given by

$$\Gamma_0 = \{x \in \Gamma;\ (x - x_0) \cdot \nu \ge 0\},$$

where $\nu$ is the unitary external normal defined over $\Gamma$ and $x_0$ is any point of $\mathbb{R}^n$. For the one-dimensional case we have that $\omega_\varepsilon = ]L - \varepsilon, L[$. The hypotheses we use on $a$ are the following:

$$x \mapsto a(x) \in C^2(\overline{\Omega}); \quad a(x) = \begin{cases} 0 & \text{on } \Omega \backslash \omega_\varepsilon, \\ 1 & \text{on } \omega_{\varepsilon/2}, \end{cases} \tag{1.4}$$

$$|\nabla a(x)|^2 \le c|a(x)|. \tag{1.5}$$

We can build a function $a$ satisfying condition (1.5) by taking $a = b^2$, with $b$ as in (1.4). In Fig. 2, $\Omega$ is a rectangle and the set $\omega_\varepsilon$ denotes the viscoelastic part of the body. Note that $\omega_\varepsilon$ is behind the part that an observer can see when situated on $x_0$. This is a particular case in which the geometric requirements of Bardos et al. [1] and Ralston [22, 23] are satisfied.

Let us mention some other papers related to the problems we address. Dafermos in [2] proved that the solution to a viscoelastic system goes to zero as time goes to infinity, but without giving the explicit rate of decay. Lagnese in [9] considered the linear

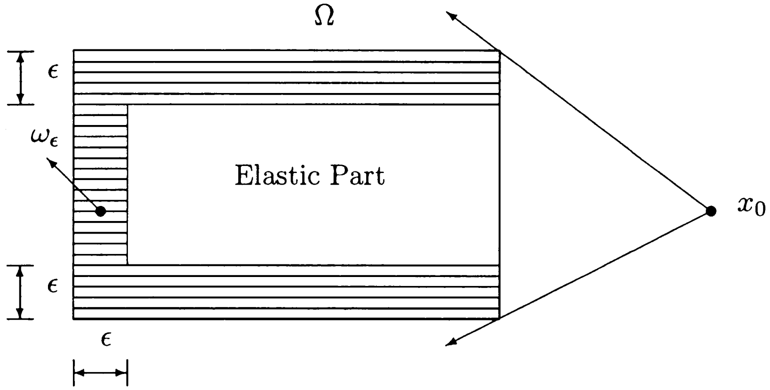


FIG. 2

viscoelastic plate equation, obtaining uniform rates of decay but introducing additional damping terms acting on the boundary. Uniform rates of decay for the solutions of a linear viscoelastic system with memory were obtained recently by M. Rivera et al [15]. Unfortunately, the methods used to achieve uniform rates of decay in those works are based on second-order estimates, which are time-dependent in our problem. Thus, the methods that have been used for establishing uniform rates of decay fail in the case of a partially viscoelastic equation. Therefore, a new asymptotic technique has to be devised. For nonlinear models see [16].

The aim of this paper is to show that in the geometrical setting above, the energy decays exponentially provided the kernel $g$ also decays exponentially. More specifically, we assume that $g$ satisfies

$$g \in C^3(\,]0,\infty[\,), \quad g(t) > 0, \quad |g''(t)| \le cg(t), \quad |g'''(t)| \le cg(t), \tag{1.6}$$

$$-\kappa_0 g(t) \le g'(t) \le \kappa_1 g(t), \tag{1.7}$$

$$\alpha := 1 - \int_0^\infty g(\tau)\, d\tau < 1. \tag{1.8}$$

To facilitate our analysis, we introduce the following binary operators:

$$g\Box\nabla u = \int_0^t g(t-\tau) \int_\Omega a(x)|\nabla u(x,t) - \nabla u(x,\tau)|^2\, dx\, d\tau,$$

$$g\Box u = \int_0^t g(t-\tau) \int_\Omega a(x)|u(x,t) - u(x,\tau)|^2\, dx\, d\tau.$$

Under the above conditions the main result of this paper is given by

THEOREM 1.1. Under the above assumptions on $\Omega$, $\omega$, $a$, and with the kernel $g$ satisfying (1.6)–(1.8), the weak solution of the viscoelastic equation (1.1)–(1.3) decays exponentially as time goes to infinity. That is, there exist positive constants $c$ and $\gamma$ that do not depend on the initial data such that

$$E(t) \le cE(0)e^{-\gamma t}$$

where by $E(t)$ we are denoting the first-order energy

$$E(t) = \frac{1}{2}\int_\Omega |u_t|^2 + \left\{1 - a(x)\int_0^t g(\tau)\,d\tau\right\}|\nabla u|^2\,dx + \frac{1}{2}g\Box\nabla u.$$

The method we use here is based on the construction of a functional $\mathcal{L}$ for which an inequality of the form

$$\frac{d}{dt}\mathcal{L}(t) \le -c\mathcal{L}(t)$$

holds, with $c > 0$. To construct such a functional $\mathcal{L}$ we start from the energy identity. Then, we look for other functions whose derivatives introduce negative terms such as: $-\int a(x)|u_t|^2 - \int a(x)|\nabla u|^2$, etc. until we are able to construct the whole energy in the right-hand side of the energy identity. Finally, we take $\mathcal{L}$ as the summation of such functions. Unfortunately, the above process also introduces terms without definite sign. To overcome this difficulty, we introduce a new multiplier, which allows us to get the appropriate estimates. Finally, we choose carefully the coefficients of each term of $\mathcal{L}$, such that the resulting summation satisfies the required inequality.

The remaining part of this article is organized as follows. In Sec. 2, we establish the existence and regularity result to equations (1.1)–(1.3) as well as the energy identity. Section 3 deals with the regularity of the convolution term. Finally, in Sec. 4 we show the exponential decay of the solution of the equations (1.1)–(1.3).

## 2. Existence results and preliminaries.

Our starting point is given by the following lemma.

LEMMA 2.1. For any $v \in C^1(0, T; H^1(0, L))$ we get

$$\int_\Omega\int_0^t g(t-\tau)a(x)\nabla v\,d\tau\cdot\nabla v_t\,dx = -\frac{1}{2}g(t)\int_\Omega a(x)|\nabla v|^2\,dx + \frac{1}{2}g'\Box\nabla v - \frac{1}{2}\frac{d}{dt}\left\{g\Box\nabla v - \left(\int_0^t g\,d\tau\right)\int_\Omega a(x)|\nabla v|^2\,dx\right\}$$

$$\int_\Omega\int_0^t g(t-\tau)a(x)v\,d\tau\cdot v_t\,dx = -\frac{1}{2}g(t)\int_\Omega a(x)|v|^2\,dx + \frac{1}{2}g'\Box v - \frac{1}{2}\frac{d}{dt}\left\{g\Box v - \left(\int_0^t g\,d\tau\right)\int_\Omega a(x)|v|^2\,dx\right\}.$$

*Proof.* It is easy to see that

$$\begin{aligned}\frac{d}{dt}\{g\Box\nabla v\} &= g'\Box\nabla v - 2\int_\Omega\int_0^t g(t-\tau)a(x)\nabla v(\tau)\,d\tau\cdot\nabla v_t(x,t)\,d\tau\,dx \\ &\quad + 2\int_0^t g(t-\tau)\,d\tau\int_\Omega a(x)\nabla v\nabla v_t\,dx \\ &= g'\Box\nabla v - 2\int_\Omega\int_0^t g(t-\tau)a(x)\nabla v(\tau)\,d\tau\cdot\nabla v_t\,dx \\ &\quad + \frac{d}{dt}\left\{\int_0^t g(\tau)\,d\tau\int_\Omega a(x)|\nabla v|^2\,dx\right\} - g(t)\int_\Omega a(x)|\nabla v|^2\,dx.\end{aligned}$$

This shows our result. The proof of the other identity is similar. □

It is not difficult to show that there exists only one solution to equations (1.1)–(1.3). We summarize the existence result in the following theorem.

THEOREM 2.1. Let us suppose that $g$ is a $C^0$-function and that the initial data satisfies

$$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega).$$

Then there exists only one weak solution $u$ to equations (1.1)–(1.3) with the following regularity:

$$u \in L^\infty(0, \infty; H_0^1(\Omega)), \qquad u_t \in L^\infty(0, \infty; L^2(\Omega)).$$

In addition, if $g \in C^1$ and

$$(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega),$$

then there exists only one strong solution $u$ of equations (1.1)–(1.3) satisfying

$$u \in C^{2-i}(0, \infty; H_0^1(\Omega) \cap H^i(\Omega)), \quad i = 1, 2, \quad u \in C^2(0, \infty; L^2(\Omega)).$$

□

The dissipative property of the viscoelastic equation is summarized in the following lemma.

LEMMA 2.2. Any strong solution of (1.1)–(1.3) satisfies

$$\frac{d}{dt} E(t) = \frac{1}{2} g' \Box \nabla u - \frac{1}{2} g(t) \int_\Omega a(x) |\nabla u|^2 \, dx.$$

*Proof.* Multiplying Eq. (1.1) by $u_t$ and integrating over $\Omega$ yields

$$\frac{d}{dt} \int_\Omega (|u_t|^2 + |\nabla u|^2) \, dx = \int_\Omega a(x) g * \nabla u \cdot \nabla u_t \, dx.$$

From Lemma 2.1 our conclusion follows. □

Lemma 2.2 tells us that the dissipation given by the memory term is effective only on the support of the function $a$. We will show in Sec. 4 that such dissipation is enough to produce the exponential decay of the energy, as time goes to infinity.

**3. Regularity of the convolution.** Let us denote by $\| \cdot \|_{C^0}$ the norm in $C^0(\overline{\Omega})$. The following lemma will play an important role in the sequel.

LEMMA 3.1. Let us suppose that $g$ is a positive function satisfying condition (1.8), $a \in C^0(\overline{\Omega})$ is such that $\|a\|_{C^0} \leq 1$ and finally let us take $f \in L^p(0, T; L^2(\Omega))$ with $1 \leq p < \infty$. Under these conditions we have that there exists only one solution $v$ of Volterra's equation

$$v(x, t) - \int_0^t g(t - \tau) a(x) v(\cdot, \tau) = f(x, t) \quad \text{a.e. } (x, t) \in \Omega \times ]0, T[$$

satisfying

$$v \in L^p(0, T; L^2(\Omega)).$$

Besides, there exists a positive constant $c$ independent of $T$ such that

$$\|v\|_{L^p(0,T;L^2)} \leq c\|f\|_{L^p(0,T;L^2)}.$$

*Proof.* We will use Picard's method. Let us denote by

$$v_0 := f, \qquad v_{-1} = 0,$$

and consider the iterative equation

$$v_\mu(x,t) = \int_0^t g(t-\tau)a(x)v_{\mu-1}(\cdot,\tau)\,d\tau + f(x,t).$$

Denoting by

$$w_\mu = v_\mu - v_{\mu-1},$$

we have that

$$w_\mu(x,t) = \int_0^t g(t-\tau)a(x)w_{\mu-1}(\cdot,\tau)\,d\tau,$$

from which it follows that

$$\|w_\mu(t)\|_{L^2} \leq \int_0^t g(t-\tau)\|a\|_{C^0}\|w_{\mu-1}(\cdot,\tau)\|_{L^2}\,d\tau.$$

Using Young's inequality we get

$$\|w_\mu\|_{L^p(0,T;L^2)} \leq \left(\int_0^T g(\tau)\,d\tau\right)\|a\|_{C^0}\|w_{\mu-1}\|_{L^p(0,T;L^2)},$$

from which it follows that

$$\begin{aligned}
\|w_\mu\|_{L^p(0,T;L^2)} &\leq \left[\left(\int_0^T g\,d\tau\right)\|a\|_{C^0}\right]^2 \|w_{\mu-2}\|_{L^p(0,T;L^2)} \\
&\leq \left[\left(\int_0^T g\,d\tau\right)\|a\|_{C^0}\right]^\mu \|w_0\|_{L^p(0,T;L^2)} \\
&\leq \left[\left(\int_0^T g\,d\tau\right)\|a\|_{C^0}\right]^\mu \|f\|_{L^p(0,T;L^2)}.
\end{aligned}$$

Since $(\int_0^T g\,d\tau)\|a\|_{C^0} < 1$, it follows that

$$\sum_{\mu=1}^\infty \|w_\mu\|_{L^p(0,T;L^2)} \leq \frac{\|f\|_{L^p(0,T;L^2)}}{1-\int_0^T g\,d\tau\|a\|_{C^0}} \leq \frac{\|f\|_{L^p(0,T;L^2)}}{1-(1-\alpha)\|a\|_{C^0}},$$

where $\alpha$ was defined in (1.8). Recalling that

$$v_\mu - v_0 = \sum_{i=1}^\mu w_i,$$

we conclude that the sequence $v_\mu$ is convergent. So, there exists a function $v \in L^p(0,T;L^2(\Omega))$ for which we have

$$v_\mu \to v \quad \text{strong in } L^p(0,T;L^2(\Omega)).$$

Besides, we have that

$$\|v\|_{L^p(0,T;L^2)} \leq \underbrace{\left\{ \frac{1}{1-(1-\alpha)\|a\|_{C^0}} + 1 \right\}}_{:=C} \|f\|_{L^p(0,T;L^2)},$$

where $C$ is a positive constant that does not depend on $T$. To show the uniqueness, let us suppose that there exists another solution $\overline{v}$. Denoting by $V = v - \overline{v}$, we have that

$$V = \int_0^t g(t-\tau)a(x)V(x,\tau)\,d\tau.$$

In particular,

$$|V| \leq \int_0^t g(t-\tau)|a(x)V(x,\tau)|\,d\tau.$$

From Gronwall's inequality we conclude that $V = 0$. The proof is now complete. □

In the next lemma we will show that integration in time is equivalent to removing spatial derivatives.

LEMMA 3.2. Let us suppose that $0 \leq a(x) \leq 1$ satisfies conditions (1.6)–(1.8) and that $g$ is a positive function satisfying (1.8). If $u$ is a weak solution of (1.1)–(1.3) satisfying

$$u_t \in L^\infty(0,T;L^2(\Omega)), \qquad u \in L^\infty(0,T;H_0^1(\Omega)),$$

then we have that

$$g * u \in L^2(0,T;H^2(\Omega))$$

and

$$\|g*u\|_{L^2(0,T;H^2)} \leq C\int_0^T E(t)\,dt + CE(0), \tag{3.1}$$

where $C$ is a positive constant independent of $T$.

*Proof.* Applying convolution to Eq. (1.1), we have

$$-\Delta g * u + g * g * \operatorname{div}\{a(x)\nabla u\} = -g * u_{tt}.$$

Performing an integration by parts over $]0,t[$, we get

$$g * u_{tt} = g(0)u_t - g(t)u_t(\cdot,0) + \int_0^t g'(t-\tau)u_t\,d\tau := -F.$$

From the hypotheses we conclude that $F \in L^2(0,T;L^2(\Omega))$. Denoting by $v = g * \Delta u$, we have that

$$-v + a(x)g * v = G, \tag{3.2}$$

where

$$G = F - g * \nabla a(x) \cdot \nabla u. \tag{3.3}$$

Applying Lemma 3.1 for $p = 2$ and since $\|a\|_{C^0} \leq 1$ we conclude that

$$v \in L^2(0,T;L^2(\Omega)).$$

Since

$$\int_0^T \int_\Omega |F|^2\,dx \le 2g(0)^2 \int_0^T \int_\Omega |u_t|^2\,dx\,dt + \int_0^T g(t) \int_\Omega |u_1|^2\,dx\,dt + \underbrace{\int_0^T \int_\Omega |g' * u_t|^2\,dx\,dt}_{\le \int_0^T |g'|\,dt \int_0^T \int_\Omega |u_t|^2\,dx}$$

we have that

$$\|v\|_{L^2(0,T;L^2(\Omega))} \le c\|G\|_{L^2(0,T;L^2(\Omega))} \le C\int_0^T E(t)\,dt + C\int_0^T g(t)\,dt E(0),$$

which implies that $g * \Delta u \in L^2(0,T;L^2(\Omega))$, from which we obtain

$$g * u \in L^2(0,T;H^2(\Omega)).$$

Finally, using elliptic regularity, inequality (3.1) follows. The proof is now complete. □

**4. Exponential decay.** To show the exponential decay of the solution let us introduce the following functional:

$$I(t) := \int_\Omega a(x)\left\{u_t(g*u)_t - \frac{1}{2}g(0)|u|^2 - \int_0^t g\,d\tau|u|^2\,dx\right\}dx - \frac{1}{2}g''\Box u + \frac{1}{2}\int_\Omega a^2(x)|g*\nabla u|^2\,dx.$$

In these conditions we have

LEMMA 4.1. Under the above conditions and for $g \in C^3$, satisfying conditions (1.6)–(1.8), we have that for any $\delta > 0$ there exists $C_\delta$ satisfying

$$\frac{d}{dt}I(t) \le -g(0)\int_\Omega a(x)|u_t|^2\,dx + \delta\int_\Omega a(x)|\nabla u|^2\,dx + C_\delta g\Box\nabla u + C_\delta g(t)\int_\Omega a(x)|\nabla u|^2\,dx + C_\delta\int_{\omega_\varepsilon}\int_0^t g(t-\tau)|u(x,t)-u(x,\tau)|^2\,dx\,dt + C_\delta\int_{\omega_\varepsilon}|u(x,t)|^2\,dx.$$

Here $C_\delta \to \infty$ when $\delta \to 0$.

*Proof.* Multiplying Eq. (1.1) by $a(x)(g*u)_t$ we get

$$\underbrace{\int_\Omega u_{tt}a(x)(g*u)_t\,dx}_{:=I_1} - \underbrace{\int_\Omega \Delta u a(x)(g*u)_t\,dx}_{:=I_2} - \underbrace{\int_\Omega a(x)g*\nabla u\cdot\nabla\{a(x)(g*u)_t\}\,dx}_{:=I_3} = 0,$$

from which we have

$$\begin{aligned} I_1 &= \frac{d}{dt}\int_\Omega u_t a(x)(g*u)_t\,dx - \int_\Omega u_t a(x)(g*u)_{tt}\,dx \\ &= \frac{d}{dt}\int_\Omega u_t a(x)(g*u)_t\,dx - \int_\Omega u_t a(x)(g(0)u + g'*u)_t\,dx \\ &= \frac{d}{dt}\int_\Omega u_t a(x)(g*u)_t\,dx - g(0)\int_\Omega a(x)|u_t|^2\,dx \\ &\quad - \int_\Omega a(x)u_t\{g'(0)u + g''*u\}\,dx \\ &= \frac{d}{dt}\int_\Omega u_t a(x)(g*u)_t\,dx - g(0)\int_\Omega a(x)|u_t|^2\,dx - \frac{g'(0)}{2}\frac{d}{dt}\int_\Omega a(x)|u|^2\,dx \\ &\quad - \int_\Omega a(x)u_t g''*u\,dx. \end{aligned}$$

Using Lemma 2.1 we get that

$$\begin{aligned} \int_\Omega a(x)u_t g''*u\,dx = \frac{1}{2}g'''\Box u - \frac{1}{2}\int_\Omega a(x)|u|^2\,dx \\ - \frac{1}{2}\frac{d}{dt}\left\{g''\Box u - \int_0^t g\,d\tau\int_\Omega a(x)|u|^2\,dx\right\}, \end{aligned}$$

from which it follows that

$$I_1(t) = \frac{d}{dt}I_0(t) - g(0)\int_\Omega a(x)|u_t|^2\,dx + \frac{1}{2}g''(t)\int_\Omega a(x)|u|^2\,dx - \frac{1}{2}g'''\Box u,$$

where by $I_0$ we are denoting

$$I_0 = \int_\Omega a(x)\left\{u_t(g*u)_t - \frac{g'(0)}{2}|u|^2\right\}dx - \frac{1}{2}g''\Box u - \int_0^t g\,d\tau\int_\Omega a(x)|u|^2\,dx.$$

On the other hand,

$$\begin{aligned} I_2 &= -\int_\Omega a(x)\Delta u\{g(0)u + g'*u\}\,dx \\ &= \int_\Omega \nabla u\cdot\nabla\{a(x)g(0)u + a(x)g'*u\}\,dx \\ &= \int_\Omega[\nabla u\cdot\nabla a(x)]g(0)u + g(0)\int_\Omega a(x)|\nabla u|^2\,dx + \int_\Omega[\nabla u\cdot\nabla a]g'*u\,dx \\ &\quad + \int_\Omega a(x)\nabla u\cdot g'*\nabla u\,dx. \end{aligned}$$

Note that

$$\begin{aligned} &\int_\Omega[\nabla u\cdot\nabla a]g(0)u + \int_\Omega[\nabla u\cdot\nabla a]g'*u\,dx \\ &\qquad = g(t)\int_\Omega[\nabla u\cdot\nabla a]u + \int_\Omega \nabla u\cdot\nabla a\int_0^t g'(t-\tau)\{u(\cdot,\tau) - u(\cdot,t)\}\,dx. \end{aligned}$$

Similarly, we have

$$g(0)\int_\Omega a(x)|\nabla u|^2\,dx+\int_\Omega a(x)\nabla u\cdot\nabla g'*u\,dx$$
$$=g(t)\int_\Omega a(x)|\nabla u|^2\,dx+\int_\Omega a(x)\nabla u\cdot\int_0^t g'(t-\tau)\{\nabla u(\cdot,\tau)-\nabla u(\cdot,t)\}\,dx.$$

Using the above formulas we conclude that $I_2$ may be written as

$$I_2(t)=g(t)\int_\Omega[\nabla u\cdot\nabla a]u+\int_\Omega[\nabla u\cdot\nabla a]\int_0^t g'(t-\tau)\{u(\cdot,\tau)-u(\cdot,t)\}\,dx$$
$$+g(t)\int_\Omega a(x)|\nabla u|^2\,dx+\int_\Omega a(x)\nabla u\cdot\int_0^t g'(t-\tau)\{\nabla u(\cdot,\tau)-\nabla u(\cdot,t)\}\,dx.$$

From hypotheses (1.5), (1.7)–(1.8) we have that

$$\int_\Omega[\nabla u\cdot\nabla a]\int_0^t g'(t-\tau)\{u(\cdot,\tau)-u(\cdot,t)\}\,d\tau\,dx$$
$$\le c\int_\Omega\sqrt{a}|\nabla u|\left\{\int_0^t g\,d\tau\right\}^{1/2}\left\{\int_0^t g(t-\tau)|u(\cdot,t)-u(\cdot,\tau)|^2\,d\tau\right\}^{1/2}dx$$
$$\le\delta\int_\Omega a(x)|\nabla u|^2\,dx+C_\delta\int_0^t\int_{\omega_\varepsilon}|u(\cdot,t)-u(\cdot,\tau)|^2\,dx\,dt.$$

Using similar ideas to estimate the term

$$\int_\Omega a(x)\nabla u\cdot\int_0^t g'(t-\tau)\{\nabla u(\cdot,\tau)-\nabla u(\cdot,t)\}\,dx\,dt,$$

we arrive at

$$I_2(t)\le C_\delta g(t)\int_\Omega a(x)|\nabla u|^2\,dx+C_\delta g\Box\nabla u+\delta\int_\Omega a(x)|\nabla u|^2\,dx$$
$$+C_\delta\int_0^t\int_{\omega_\varepsilon}g'(t-\tau)|u(x,t)-u(x,\tau)|^2\,dx\,d\tau,$$

where $\delta$ is a small parameter to be fixed later and $C_\delta\to\infty$ as $\delta\to 0$. Finally,

$$I_3(t)=-\int_\Omega ag*\nabla u\cdot\nabla(ag*u)_t\,dx$$
$$=-\int_\Omega a(x)\int_0^t g(t-\tau)[\{\nabla u(\cdot,\tau)-\nabla u(\cdot,t)\}\cdot\nabla a](g*u)_t$$
$$-\int_0^t g\,d\tau\int_\Omega a(x)[\nabla u\cdot\nabla a](g*u)_t\,dx-\frac{1}{2}\frac{d}{dt}\int_\Omega a^2|g*\nabla u|^2\,dx$$
$$=-\int_\Omega a(x)\int_0^t g(t-\tau)\{\nabla u(\cdot,\tau)-\nabla u(\cdot,t)\}\cdot\nabla a(x)\{g(0)u+g'*u\}\,dx$$
$$-\int_0^t g\,d\tau\int_\Omega a(x)\nabla u\cdot\nabla a(x)\{g(0)u+g'*u\}-\frac{1}{2}\frac{d}{dt}\int_\Omega a^2|g*\nabla u|^2\,dx.$$

Using the identity

$$g(0)u+g'*u=g(t)+\int_0^t g'(t-\tau)\{u(\cdot,\tau)-u(\cdot,t)\}\,d\tau$$

and since $a(x) \leq 1$, with the same arguments we used to estimate $I_2$ we have

$$-\int_\Omega a(x)\int_0^t g(t-\tau)[\{\nabla u(\cdot,\tau)-\nabla u(\cdot,t)\}\cdot\nabla a]\,d\tau\{g(0)u+g'*u\}\,dx$$
$$\leq C\left\{g\Box\nabla u+g(t)\int_\Omega a(x)|\nabla u|^2\,dx\right\}+\delta\int_\Omega a(x)|\nabla u|^2\,dx+C_\delta\int_{\omega_\varepsilon}|u|^2\,dx$$
$$+C_\delta\int_0^t\int_{\omega_\varepsilon} g(t-\tau)|u(\cdot,t)-u(\cdot,\tau)|^2\,dx\,dt$$

and

$$\int_0^t g\,d\tau\int_\Omega a(x)\nabla u\cdot\nabla a(x)\{g(0)u+g'*u\}$$
$$\leq \delta\int_\Omega a(x)|\nabla u|^2\,dx+C_\delta\int_{\omega_\varepsilon}|u|^2\,dx$$
$$+C_\delta\int_0^t\int_{\omega_\varepsilon} g(t-\tau)|u(\cdot,t)-u(\cdot,\tau)|^2\,dx\,dt,$$

from which it follows that

$$I_3(t)\leq -\frac{1}{2}\frac{d}{dt}\int_\Omega a^2|g*\nabla u|^2\,dx+Cg\Box\nabla u+\delta\int_\Omega a(x)|\nabla u|^2\,dx$$
$$+C_\delta\int_{\omega_\varepsilon}|u|^2\,dx+C_\delta\int_0^t\int_{\omega_\varepsilon} g(t-\tau)|u(\cdot,t)-u(\cdot,\tau)|^2\,dx\,dt.$$

Since

$$-I_1=I_2+I_3$$

our conclusion follows by substitution of the relations for $I_i$, $i=1,2,3$, into the above identity. The proof is not complete □

LEMMA 4.2. With the same hypotheses as Lemma 4.1 we have that the solution of Eqs. (1.1)–(1.3) satisfies

$$\frac{d}{dt}\int_\Omega auu_t\,dx\leq\int_\Omega a(x)|u_t|^2\,dx-c_0\int_\Omega a(x)|\nabla u|^2\,dx+c\int_{\omega_\varepsilon}|u|^2\,dx$$
$$+C_\delta\int_0^t\int_{\omega_\varepsilon} g(t-\tau)|u(\cdot,t)-u(\cdot,\tau)|^2\,dx\,dt.$$

*Proof.* Let us multiply Eq. (1.1) by $a(x)u(x,t)$ to get

$$\underbrace{\int_\Omega au_{tt}u\,dx}_{:=I_1}-\underbrace{\int_\Omega a(x)u\Delta u\,dx}_{:=I_2}+\underbrace{\int_\Omega \operatorname{div}\{ag*\nabla u\}au\,dx}_{:=I_3}=0.$$

Note that

$$I_1(t)=\frac{d}{dt}\int_\Omega au_tu\,dx-\int_\Omega a(x)|u_t|^2\,dx.$$

On the other hand,

$$I_2(t) = \int_\Omega \nabla u \nabla\{au\}\,dx$$
$$= \int_\Omega u\nabla u\nabla a\,dx + \int_\Omega a(x)|\nabla u|^2\,dx$$
$$= -\frac{1}{2}\int_\Omega \Delta a(x)|u|^2\,dx + \int_\Omega a(x)|\nabla u|^2\,dx.$$

Finally,

$$I_3(t) = -\int_\Omega a(x)g * \nabla u \cdot \nabla\{au\}\,dx$$
$$= -\int_\Omega a(x)[g*\nabla u\cdot\nabla a]u\,dx - \int_\Omega a^2 g*\nabla u\cdot\nabla u\,dx$$
$$= -\int_\Omega a(x)[g*\nabla u\cdot\nabla a]u\,dx + \int_0^t g\,d\tau\int_\Omega a^2|\nabla u|^2\,dx$$
$$-\int_\Omega a^2\int_0^t g(t-\tau)\{\nabla u(\cdot,\tau) - u(\cdot,t)\}\cdot\nabla u\,dx.$$

Summing up $I_1$, $I_2$, and $I_3$ we get

$$\frac{d}{dt}\int_\Omega auu_t\,dx = \int_\Omega a(x)|u_t|^2\,dx - \int_\Omega a\left(1 - a(x)\int_0^t g\,d\tau\right)|\nabla u|^2\,dx + \int_\Omega(\Delta a)|u|^2\,dx$$
$$+\int_\Omega a(x)\int_0^t g(t-\tau)\{\nabla u(\cdot,\tau)-\nabla u(\cdot,t)\}\nabla a(x)u\,d\tau\,dx$$
$$+\int_\Omega a^2\int_0^t g(t-\tau)\{\nabla u(\cdot,\tau)-\nabla u(\cdot,t)\}\,d\tau\nabla u\,dx.$$

Using $|\Delta a(x)| \le C$ our conclusion follows. □

The following lemma is proved by Lions [13]. For convenience we rewrite it here.

LEMMA 4.3. Let us denote by $q_k$ a $C^1$-function. Then any strong solution ($u \in C^i(0,T;$ $H^{2-i}(\Omega))$ for $i = 0,1,2$) of the wave equation

$$u_{tt} - \Delta u = f,$$
$$u(x,t) = 0 \quad \text{on } \Gamma\times\,]0\infty[, \tag{4.1}$$

satisfies the following identity:

$$\frac{d}{dt}\int_\Omega u_t q_k\frac{\partial u}{\partial x_k}\,dx = \int_\Omega f q_k\frac{\partial u}{\partial x_k}\,dx + \frac{1}{2}\int_\Gamma q_k\nu_k\left|\frac{\partial u}{\partial\nu}\right|^2 d\Gamma + \int_\Gamma\frac{\partial u}{\partial\nu}q_k\frac{\partial u}{\partial x_k}\,d\Gamma$$
$$+\frac{1}{2}\int_\Omega\frac{\partial q_k}{\partial x_k}\{|u_t|^2 + |\nabla u|^2\}\,dx + \int_\Omega\nabla u\cdot\nabla q_k\frac{\partial u}{\partial x_k}\,dx.$$

*Proof.* Let us multiply Eq. (4.1) by $q_k\frac{\partial u}{\partial x_k}$ to get

$$\int_\Omega\{u_{tt}-\Delta u\}q_k\frac{\partial u}{\partial x_k}\,dx = \int_\Omega f q_k\frac{\partial u}{\partial x_k}\,dx. \tag{4.2}$$

It is easy to see that

$$\begin{aligned}\int_\Omega u_{tt}q_k\frac{\partial u}{\partial x_k}\,dx &= \frac{d}{dt}\int_\Omega u_tq_k\frac{\partial u}{\partial x_k}\,dx - \int_\Omega u_tq_k\frac{\partial u_t}{\partial x_k}\,dx\\ &= \frac{d}{dt}\int_\Omega u_tq_k\frac{\partial u}{\partial x_k}\,dx - \frac{1}{2}\int_\Omega q_k\frac{\partial |u_t|^2}{\partial x_k}\,dx\\ &= \frac{d}{dt}\int_\Omega u_tq_k\frac{\partial u}{\partial x_k}\,dx + \frac{1}{2}\int_\Omega \frac{\partial q_k}{\partial x_k}|u_t|^2\,dx.\end{aligned}$$

On the other hand,

$$\begin{aligned}-\int_\Omega \Delta u q_k\frac{\partial u}{\partial x_k}\,dx &= -\int_\Gamma \frac{\partial u}{\partial \nu}q_k\frac{\partial u}{\partial x_k}\,d\Gamma + \int_\Omega \nabla u\cdot\nabla q\frac{\partial u}{\partial x_k}\,dx + \int_\Omega \nabla u\cdot q\frac{\partial \nabla u}{\partial x_k}\,dx\\ &= -\int_\Gamma \frac{\partial u}{\partial \nu}q_k\frac{\partial u}{\partial x_k}\,d\Gamma + \int_\Omega \nabla u\cdot\nabla q\frac{\partial u}{\partial x_k}\,dx + \frac{1}{2}\int_\Omega q_k\frac{\partial |\nabla u|^2}{\partial x_k}\,dx\\ &= -\int_\Gamma \frac{\partial u}{\partial \nu}q_k\frac{\partial u}{\partial x_k}\,d\Gamma + \int_\Omega \nabla u\cdot\nabla q\frac{\partial u}{\partial x_k}\,dx + \frac{1}{2}\int_\Gamma q_k\nu_k|\nabla u|^2\,d\Gamma\\ &\quad - \frac{1}{2}\int_\Omega \frac{\partial q_k}{\partial x_k}|\nabla u|^2\,dx.\end{aligned}$$

Since $u(x,t) = 0$ on $\Gamma$, we have that

$$\frac{\partial u}{\partial x_k} = \nu_k\frac{\partial u}{\partial \nu},$$

from which our conclusion follows. □

LEMMA 4.4. Let us take $q_k = a^2(x)h_k$, where $h_k \in C^2(\overline{\Omega})$ is such that $h_k = \nu_k$ on $\Gamma$. Then we have that

$$\begin{aligned}-\frac{d}{dt}\int_\Omega a^2(x)u_th_k\frac{\partial u}{\partial x_k}\,dx \le -\frac{1}{2}\int_{\Gamma_0}\left|\frac{\partial u}{\partial \nu}\right|^2\,d\Gamma + C\int_\Omega a(x)\{|u_t|^2 + |\nabla u|^2\}\,dx\\ + C\int_\Omega a(x)|h(x)\operatorname{div}\{ag*\nabla u\}|^2\,dx,\end{aligned}$$

for any solution of Eqs. (1.1)–(1.3).

*Proof.* From Lemma 4.3 applied to $q_k = a^2(x)h_k$ we have

$$\begin{aligned}-\frac{d}{dt}\int_\Omega a^2h_ku_t\frac{\partial u}{\partial x_k}\,dx = -\int_\Omega -fa^2(x)h_k\frac{\partial u}{\partial x_k}\,dx = \int_\Gamma a^2(x)h_k\nu_k\left|\frac{\partial u}{\partial \nu}\right|^2\,dx\\ + \frac{1}{2}\int_\Omega \frac{\partial\{a^2(x)h_k\}}{\partial x_k}\{|u_t|^2 - |\nabla u|^2\}\,dx + \int_\Omega \nabla u\cdot\nabla a^2h_k\frac{\partial u}{\partial x_k}\,dx.\end{aligned}$$

Since $a = 1$ on $\Gamma_0$ and $h_k = \nu_k$ on $\Gamma$, using the Cauchy-Schwarz inequality, our conclusion follows. □

Let us denote by $\mathcal{L}_N(t)$ the functional

$$\mathcal{L}_N(t) = NE(t) + I(t) + \frac{g(0)}{2}\int_\Omega a(x)uu_t\,dx - \delta_0\int_\Omega a^2(x)u_th_k\frac{\partial u}{\partial x_k}\,dx.$$

Under these conditions we get

LEMMA 4.5. Under the above notation we have

$$\frac{d}{dt}\mathcal{L}_N(t) \le -\kappa_0 \int_\Omega a(x)\{|u_t|^2 + |\nabla u|^2\}\, dx - \frac{N}{2}\left\{g\Box\nabla u + g(t)\int_\Omega a(x)|\nabla u|^2\, dx\right\}$$
$$- \frac{\delta_0}{2}\int_\Gamma \left|\frac{\partial u}{\partial \nu}\right|^2 d\Gamma + C\delta_0 \int_\Omega a(x)|h(x)\operatorname{div}\{ag * \nabla u\}^2\, dx + C\int_{\omega_\varepsilon} |u|^2\, dx \quad (4.3)$$
$$+ C\int_0^t \int_{\omega_\varepsilon} g(t-\tau)|u(x,t) - u(x,\tau)|^2\, dx\, d\tau.$$

*Proof.* From Lemma 4.1 and Lemma 4.2 we get that

$$\frac{d}{dt}\left\{I(t) + \frac{g(0)}{2}\int_\Omega a(x)uu_t\, dx\right\} \le -\frac{g(0)}{2}\int_\Omega a(x)|u_t|^2\, dx - (C-\delta)\int_\Omega a(x)|\nabla u|^2\, dx$$
$$+ C\int_{\omega_\varepsilon} |u|^2\, dx + C_\delta\left\{g\Box\nabla u + g(t)\int_\Omega a(x)|\nabla u|^2\, dx\right\}$$
$$+ C\int_0^t \int_{\omega_\varepsilon} g(t-\tau)|u(x,t) - u(x,\tau)|^2\, dx\, d\tau.$$

So, taking $\delta$ small enough, we get that there exists a positive constant $k_0$ such that

$$\frac{d}{dt}\left\{I(t) + \frac{g(0)}{2}\int_\Omega a(x)uu_t\, dx\right\} \le -2\kappa_0\left\{\int_\Omega a(x)|u_t|^2\, dx + \int_\Omega a(x)|\nabla u|^2\, dx\right\}$$
$$+ C\int_{\omega_\varepsilon} |u|^2\, dx + C_\delta\left\{g\Box\nabla u + g(t)\int_\Omega a(x)|\nabla u|^2\, dx\right\}$$
$$+ C\int_\Omega^t \int_{\omega_\varepsilon} g(t-\tau)|u(x,t) - u(x,\tau)|^2\, dx\, d\tau.$$

Using Lemma 2.2, Lemma 4.4, and the above inequality we arrive at

$$\frac{d}{dt}\mathcal{L}_N(t) \le -\kappa_0\left\{\int_\Omega a(x)|u_t|^2\, dx + \int_\Omega a(x)|\nabla u|^2\, dx\right\} - \frac{\varepsilon}{2}\int_\Gamma \left|\frac{\partial u}{\partial \nu}\right|^2 d\Gamma$$
$$- (N - C_\delta)\left\{g\Box\nabla u + g(t)\int_\Omega a(x)|\nabla u|^2\, dx\right\} + C\int_{\omega_\varepsilon} |u|^2\, dx$$
$$+ C\varepsilon\int_\Omega a(x)|h(x)\operatorname{div}\{ag * \nabla u\}|^2\, dx$$
$$+ C\int_0^t \int_{\omega_\varepsilon} g(t-\tau)|u(x,t) - u(x,\tau)|^2\, dx\, d\tau.$$

Therefore, taking $N > 2C_\delta$, we get

$$\frac{d}{dt}\mathcal{L}_N(t) \le -\kappa_0\left\{\int_\Omega a(x)|u_t|^2\, dx + \int_\Omega a(x)|\nabla u|^2\, dx\right\} - \frac{\varepsilon}{2}\int_\Gamma \left|\frac{\partial u}{\partial \nu}\right|^2 d\Gamma$$
$$- \frac{N}{2}\left\{g\Box\nabla u + g(t)\int_\Omega a(x)|\nabla u|^2\, dx\right\} + C\int_{\omega_\varepsilon} |u|^2\, dx$$
$$+ C\delta_0\int_\Omega a(x)|h(x)\operatorname{div}\{ag * \nabla u\}|^2\, dx$$
$$+ \int_0^t \int_{\omega_\varepsilon} g(t-\tau)|u(x,t) = u(x,\tau)^2\, dx\, d\tau.$$

This proof is now complete. □

LEMMA 4.6. Let us suppose that $u$ is the weak solution of (1.1)–(1.3). Then there exists a positive constant $C$, independent of $T$, such that

$$\int_0^T \int_\Omega a(x)|\operatorname{div}\{ag*\nabla u\}|^2\,dx\,dt \le C\int_0^T\int_\Omega a(x)\{|u_t|^2+|\nabla u|^2\}\,dx\,dt + C\int_\Omega^T g(t)\,dt E(0), \tag{4.4}$$

$$\int_0^T \int_\Omega |\operatorname{div}\{ag*\nabla u\}|^2\,dx\,dt \le C\int_0^T E(t)\,dt + C\int_0^T g(t)\,dt E(0). \tag{4.5}$$

*Proof.* Note that

$$\operatorname{div}\{ag*\nabla u\} = \nabla a\cdot\nabla u + ag*\Delta u.$$

As in the proof of Lemma 3.2, $v=\sqrt{a}g*\Delta u$ satisfies

$$-v + a(x)g*v = \sqrt{a}G,$$

where $G$ is given by (3.3). Using similar arguments, we conclude that

$$\begin{aligned}\|v\|^2_{L^2(0,T;L^2)} &\le \int_0^T\int_\Omega a(x)|G|^2\,dx\,dt\\ &\le \int_0^T\int_\Omega a(x)\{|u_t|^2+|\nabla u|^2\}\,dx\,dt + C\int_0^T g\,dt E(0).\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}&\int_0^T\int_\Omega a(x)|\operatorname{div}\{ag*\nabla u\}|^2\,dx\,dt\\ &\quad\le C\int_0^T\int_\Omega a(x)\{|u_t|^2+|\nabla u|^2\}\,dx\,dt + C\int_0^T g(t)\,dt E(0),\end{aligned}$$

for a positive constant $C$. The proof is now complete. □

LEMMA 4.7. Let us suppose that $\varphi$ is a weak solution of the wave equation

$$\begin{gathered}\varphi_{tt} - \Delta\varphi = 0,\\ \varphi(x,0)=\varphi_0,\quad \varphi_t = \varphi_1,\\ \varphi(x,t) = 0 \quad \text{on } \Sigma = \Gamma\times\,]0,\infty[.\end{gathered}$$

for any $(\varphi_0,\varphi_1)\in H_0^1(\Omega)\times L^2(\Omega)$ where

$$R(x_0) = \max_{x\in\overline{\Omega}}\left|\sum_{k=1}^n (x_k - x_k^0)^2\right|^{1/2}.$$

*Proof.* See [13], Lemma 2.3, Chapter VIII, p. 411. □

Our next step is to estimate the term $\int_\omega |u|^2\,dx$. To do this we will use the following lemma.

LEMMA 4.8. Let us suppose that $u$ is a weak solution of (1.1)–(1.3). Then for any $\varepsilon > 0$ there exists a positive constant $C_\varepsilon$ for which we have

$$\int_0^T \int_\Omega |u|^2 \, dx \, dt \leq C_\varepsilon \left\{ \int_0^T g(t) \int_\Omega a(x)|\nabla u|^2 \, dx \, dt + \int_0^T g\Box\nabla u \, dt \right\} + \varepsilon \int_0^T \int_\Omega a(x)\{|\nabla u|^2 + |u_t|^2 + |\operatorname{div} ag * \nabla u|^2\} \, dx \, dt,$$

$$\int_0^T \int_\Omega |g * \nabla u|^2 \, dx \, dt \leq C_\varepsilon \left\{ \int_0^T g(t) \int_\Omega a(x)|\nabla u|^2 \, dx \, dt + \int_0^T g\Box\nabla u \, dt \right\} + \varepsilon \int_0^T \int_\Omega a(x)\{|\nabla u|^2 + |u_t|^2 + \operatorname{div} ag * \nabla u|^2\} \, dx \, dt,$$

and

$$\int_0^T \int_0^t \int_\Omega g(\sigma - t)|u(x,\sigma) - u(x,t)|^2 \, dx \, d\sigma \, dt \leq C_\varepsilon \left\{ \int_0^T g(t) \int_\Omega a(x)|\nabla u|^2 \, dx \, dt + \int_0^T g\Box\nabla u \, dt \right\} + \varepsilon \int_0^T \int_\Omega a(x)\{|\nabla u|^2 + |u_t|^2 + |\operatorname{div} ag * \nabla u|^2\} \, dx \, dt,$$

provided $T$ is large enough.

*Proof.* We argue by contradiction. Suppose that there exists $\varepsilon_0 > 0$ and a sequence of functions such that

$$\int_0^T \int_\Omega |u^\nu|^2 \, dx \, dt \geq \nu \left\{ \int_0^T g(t) \int_\Omega a(x)|\nabla u^\nu|^2 \, dx \, dt + \int_0^T g\Box\nabla u^\nu \, dt \right\} + \varepsilon_0 \int_0^T \int_\Omega a(x)\{|\nabla u^\nu|^2 + |u_t^\nu|^2 + |\operatorname{div} ag * \nabla u|^2\} \, dx \, dt, \tag{4.6}$$

for $\nu \to \infty$. By the linearity of the problem we may suppose that

$$\int_0^T \int_\Omega |u^\nu|^2 \, dx \, dt = 1 \quad \forall \nu \in \mathbb{N}. \tag{4.7}$$

So, we get that

$$g(t)a(x)|\nabla u^\nu|^2 + \int_0^t a(\cdot)g(t-\tau)|u^\nu(\cdot,\tau) - u^\nu(\cdot,t)|^2 \, d\tau \to 0 \quad \text{strongly in } L^1(\,]0,\infty[\,\times\Omega). \tag{4.8}$$

Let us decompose $u^\nu$ into

$$u^\nu = w^\nu + v^\nu,$$

where

$$w_{tt}^\nu - \Delta w^\nu = -\operatorname{div}\{ag * \nabla u^\nu\} \quad (\text{bounded in } L^2(0,T;L^2(\Omega))),$$
$$w^\nu(x,0) = 0, \quad w_t^\nu(x,0) = 0 \quad \text{in } \Omega,$$
$$w^\nu(x,t) = 0 \quad \text{on } \Gamma\times\,]0,\infty[,$$

and

$$v_{tt}^{\nu} - \Delta v^{\nu} = 0,$$
$$v^{\nu}(x,0) = u^{\nu}(x,0), \quad v_t^{\nu}(x,0) = u_t^{\nu}(x,0) \quad \text{in } \Omega,$$
$$v^{\nu}(x,t) = 0 \quad \text{on } \Gamma\times ]0,\infty[.$$

From (4.6) and (4.7) it follows that $u^{\nu}$ is bounded in

$$W^{1,\infty}(0,T;L^2(\omega)) \cap L^{\infty}(0,T;H^1(\omega)).$$

Note that $w^{\nu}$ is also bounded in

$$W^{1,\infty}(0,T;L^2(\Omega)) \cap L^{\infty}(0,T;H_0^1(\Omega)).$$

Thereby, we conclude that $v^{\nu} = u^{\nu} - w^{\nu}$ satisfies

$$v_t^{\nu} \text{ is bounded in } L^2(0,T;L^2(\omega)),$$
$$v^{\nu} \text{ is bounded in } L^2(0,T;H^1(\omega)).$$

Using Lemma 4.7 we have

$$(u^{\nu}(\cdot,0), u_t^{\nu}(\cdot,0)) = (v^{\nu}(\cdot,0), v_t^{\nu}(\cdot,0)) \text{ is bounded in } H_0^1(\Omega) \times L^2(\Omega),$$

which implies that

$$v^{\nu} \text{ is bounded in } W^{1,\infty}(0,T;L^2(\Omega)) \cap L^{\infty}(0,T;H_0^1(\Omega)).$$

Hence

$$u^{\nu} = w^{\nu} + v^{\nu} \text{ is bounded in } W^{1,\infty}(0,T;L^2(\Omega)) \cap L^{\infty}(0,T;H_0^1(\Omega)).$$

Therefore, there exists a subsequence (which we still denote in the same way) and a function $u \in W^{1,\infty}(0,T;L^2(\Omega))$ such that

$$u^{\nu} \to u \quad \text{weak * in } W^{1,\infty}(0,T;L^2(\Omega))$$

and satisfying

$$u_{tt} - \Delta u = 0,$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega,$$
$$u(x,t) = 0 \quad \text{on } \Gamma\times ]0,T[.$$

From (4.8) we conclude that

$$u = 0 \quad \text{on } \omega_{\varepsilon}\times ]0,T[.$$

Using Holmgren's Theorem for $T > 2\,\mathrm{diam}(\Omega\backslash\omega_{\varepsilon})$, we get that $u = 0$ on $\Omega\times ]0,T[$. But this is contradictory with (4.7) since, due to the compactness of the embedding $H^1(\Omega \times\ 0,T[\,) \subset L^2(\Omega\times ]0,T[\,)$, the sequence $u^{\nu}$ converges strongly in $L^2(\Omega\times ]0,T[\,)$. This contradiction proves the first inequality. To prove the other we use similar arguments. Thereby, our conclusion follows. □

Using the inequalities (4.3), (4.5), Lemma 4.8 and taking $\varepsilon > 0$ small enough we arrive at

$$\mathcal{L}_N(T) - \mathcal{L}_N(0) \leq -\kappa_0 \int_0^T \mathcal{M}(t)\,dt + C\varepsilon E(0) + C\varepsilon \int_0^T E(t)\,dt \tag{4.9}$$

for $N > 2C$, where by $\mathcal{M}$ we are denoting

$$\mathcal{M}(t) = \int_\Omega a(x)\{|u_t|^2 + |\nabla u|^2\}\,dx + g\Box\nabla u + \int_{\Gamma_0} \left|\frac{\partial u}{\partial \nu}\right|^2 d\Gamma.$$

Now we have the conditions to prove the main result of this paper.

*Proof of Theorem 1.1.* We will suppose that the initial data belongs to $H^2(\Omega)\cap H_0^1(\Omega) \times H_0^1(\Omega)$. Our conclusion will follow using standard density arguments. Using Lemma 4.3 for $q = x - x^0$ we conclude that

$$-\frac{d}{dt}\int_\Omega u_t q_k \frac{\partial u}{\partial x_k}\,dx = -\int_\Omega f q_k \frac{\partial u}{\partial x_k}\,dx + \frac{1}{2}\int_\Omega \frac{\partial q_k}{\partial x_k}\{|u_t|^2 - |\nabla u|^2\}\,dx + \int_\Omega \nabla u \cdot \nabla q_k \frac{\partial u}{\partial x_k}\,dx - \frac{1}{2}\int_\Gamma q_k \nu_k \left|\frac{\partial u}{\partial \nu}\right|^2,$$

from which it follows that

$$-\frac{d}{dt}\int_\Omega u_t q_k \frac{\partial u}{\partial x_k}\,dx = -\int_\Omega f q_k \frac{\partial u}{\partial x_k}\,dx + \frac{n}{2}\int_\Omega \{|u_t|^2 - |\nabla u|^2\}\,dx + \int_\Omega |\nabla u|^2\,dx - \frac{1}{2}\int_\Gamma q_k \nu_k \left|\frac{\partial u}{\partial \nu}\right|^2,$$

which implies that

$$\begin{aligned} -\frac{d}{dt}\int_\Omega u_t q_k \frac{\partial u}{\partial x_k}\,dx = &-\int_\Omega f q_k \frac{\partial u}{\partial x_k}\,dx + \frac{n-1}{2}\int_\Omega \{|u_t|^2 - |\nabla u|^2\}\,dx \\ &+ \frac{1}{2}\int_\Omega \{|u_t|^2 + |\nabla u|^2\}\,dx - \frac{1}{2}\int_\Gamma qk \nu_k \left|\frac{\partial u}{\partial \nu}\right|^2. \end{aligned} \tag{4.10}$$

Multiplying Eq. (1.1) by $u$ we get

$$\frac{d}{dt}\int_\Omega u u_t\,dx = \int_\Omega \{|u_t|^2 - |\nabla u|^2\}\,dx + \int_\Omega a g * \nabla u \cdot \nabla u\,dx.$$

Inserting this identity into (4.10) we have

$$\begin{aligned} -\frac{d}{dt}\int_\Omega u_t q_k \frac{\partial u}{\partial x_k}\,dx = &-\int_\Omega f q_k \frac{\partial u}{\partial x_k}\,dx + \frac{n-1}{2}\frac{d}{dt}\int_\Omega u u_t\,dx \\ &- \frac{n-1}{2}\int_\Omega a g * \nabla u \cdot \nabla u\,dx + \frac{1}{2}\int_\Omega \{|u_t|^2 + |\nabla u|^2\}\,dx \\ &- \frac{1}{2}\int_\Gamma q_k \nu_k \left|\frac{\partial u}{\partial \nu}\right|^2, \end{aligned}$$

from which we have

$$\begin{aligned}\frac{d}{dt}\underbrace{\left\{-\int_\Omega u_t q_k \frac{\partial u}{\partial x_k}\,dx-\frac{n-1}{2}\int_\Omega u u_t\,dx\right\}}_{:=X(t)}\\ =-\int_\Omega f q_k\frac{\partial u}{\partial x_k}\,dx-\frac{n-1}{2}\int_\Omega ag*\nabla u\cdot\nabla u\,dx\\ +\frac{1}{2}\int_\Omega\{|u_t|^2+|\nabla u|^2\}\,dx-\frac{1}{2}\int_\Gamma q_k\nu_k\left|\frac{\partial u}{\partial\nu}\right|^2.\end{aligned}$$

Integrating over $[0,T]$ we get

$$\begin{aligned}X(T)-X(0)=-\int_0^T\int_\Omega fq_k\frac{\partial u}{\partial x_k}\,dx\,dt-\frac{n-1}{2}\int_0^T\int_\Omega ag*\nabla u\cdot\nabla u\,dx\,dt\\ +\frac{1}{2}\int_0^T\int_\Omega\{|u_t|^2+|\nabla u|^2\}\,dx\,dt-\frac{1}{2}\int_0^T\int_\Gamma qk\nu_k\left|\frac{\partial u}{\partial\nu}\right|^2\,dt.\end{aligned}\tag{4.11}$$

Since

$$X(T)\le CE(T),\qquad X(0)\le CE(0),$$

and using

$$\int_0^T E(t)\,dt\le C\left\{\int_0^T\int_\Omega\{|u_t|^2+|\nabla u|^2\}\,dx\,dt+\int_0^T g\Box\nabla u\,dt\right\},$$

together with inequality (4.11) we conclude that

$$\int_0^T E(t)\,dt\le C\int_0^T\mathcal{M}(t)\,dt+C\{E(T)+E(0)\}.$$

From the energy identity we get

$$E(0)\le E(T)+\int_0^T\mathcal{M}(t)\,dt.\tag{4.12}$$

Therefore, there exists a positive constant $C_1$ such that

$$\int_0^T E(t)\,dt\le C_1\int_0^T\mathcal{M}(t)\,dt+C_1E(T).\tag{4.13}$$

Since $E(t)$ is a decreasing function we have that

$$E(T)\le\frac{1}{T}\int_0^T E(t)\,dt.$$

Inserting the above inequality into (4.13) we get

$$\left(1-\frac{C}{T}\right)\int_0^T E(t)\,dt\le C_1\int_0^T\mathcal{M}(t)\,dt.\tag{4.14}$$

On the other hand, it is not difficult to see that

$$c_0E(t)\le\mathcal{L}(t)\le c_1E(t).\tag{4.15}$$

Therefore, using (4.9), (4.14), and (4.15) we conclude that

$$\begin{aligned}\mathcal{L}(T)-\mathcal{L}(0) &\le -\kappa_0\int_0^T \mathcal{M}(t)\,dt + C\varepsilon E(0) + C\varepsilon\int_0^T E(t)\,dt \quad \text{(using (4.12))}\\ &\le -\kappa_0\int_0^T \mathcal{M}(t)\,dt + C\varepsilon\left\{E(T)+\int_0^T \mathcal{M}(t)\,dt\right\} + C\varepsilon\int_0^T E(t)\,dt\\ &\le -\kappa_1\int_0^T \mathcal{L}(t)\,dt,\end{aligned}$$

provided $\varepsilon$ is small enough. Using inequality (4.15) we can establish that

$$\int_0^T \mathcal{L}(t)\,dt \ge c\int_0^T E(t)\,dt \ge cTE(T) \ge c_2 T\mathcal{L}(T),$$

which implies that

$$\mathcal{L}(T)-\mathcal{L}(0) \le -\kappa_1 cT\mathcal{L}(T).$$

This is equivalent to

$$\mathcal{L}(T) \le \frac{1}{1+CT}\mathcal{L}(0).$$

Repeating the above process from $T$ to $2T$ we get that

$$\mathcal{L}(2T) \le \frac{1}{1+CT}\mathcal{L}(T) \le \frac{1}{(1+CT)^2}\mathcal{L}(0).$$

In general we have that

$$\mathcal{L}(nT) \le \frac{1}{(1+CT)^n}\mathcal{L}(0).$$

Since any number $t$ can be written as $t = nT + r$ where $r < T$ and $E(t)$ is a decreasing function, from (4.15) we arrive at

$$\mathcal{L}(t) \le c\mathcal{L}(t-r) \le \frac{C}{(1+CT)^{(t-r)/T}}\mathcal{L}(0) \le c_0 e^{-\gamma t}\mathcal{L}(0),$$

where $\gamma = \frac{\ln(1+CT)}{T}$, from which the exponential decay follows. □

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