QUARTERLY OF APPLIED MATHEMATICS
VOLUME LIX, NUMBER 4
DECEMBER 2001, PAGES 701–736

WELL-POSEDNESS AND LONGTIME BEHAVIOR OF THE PHASE-FIELD MODEL WITH MEMORY IN A HISTORY SPACE SETTING

By

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Abstract. A thermodynamically consistent phase-field model with memory, based on the linearized version of the Gurtin-Pipkin heat conduction law, is considered. The formulation of an initial and boundary value problem for the phase-field evolution system is framed in a history space setting. Namely, the summed past history of the temperature is regarded itself as a variable along with the temperature and the phase-field. Well-posedness results are discussed, as well as longtime behavior of solutions. Under suitable conditions, the existence of an absorbing set can be achieved.

1. Introduction. We consider a phase-field model of some temperature-dependent transition in a rigid heat conductor occupying a given bounded domain $\Omega \subset \mathbb{R}^3$. We assume that, at each point $x \in \Omega$ and any time $t \in \mathbb{R}$, the state of the material is described by the triplet $(\vartheta(t), \vartheta^t, \chi(t))$, where $\vartheta(x,t)$ is the *temperature variation field* from a reference value, $\vartheta^t(x,s) = \vartheta(x,t-s)$, $s \geq 0$, is the *past history* of ϑ up to time t, and $\chi(x,t)$ is the *phase variable*, which accounts for the kinetics of the solid-liquid transition.

The evolution of the temperature-dependent phase change phenomenon is governed by the *energy balance equation*

$$\partial_t e + \operatorname{div} \mathbf{q} = f \quad \text{in } \Omega \times \mathbb{R}$$

Received November 30, 1998.
2000 *Mathematics Subject Classification.* Primary 35B40, 45K05, 80A22.
Key words and phrases. Phase-field models, materials with memory, existence and uniqueness, continuous dependence, absorbing set.
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where f is the external heat source.

If we consider only small variations of ϑ and $\nabla\vartheta$, we may suppose that the *heat flux vector* $\mathbf{q} : \Omega \times \mathbb{R} \to \mathbb{R}^3$ and the *internal energy* $e : \Omega \times \mathbb{R} \to \mathbb{R}$ are described by the following constitutive equations (cf. (A.17)-(A.18) in the Appendix):

$$e(x,t) = e_c + c_v\theta_c\vartheta(x,t) + \int_0^\infty a(\sigma)\vartheta(x,t-\sigma)\,d\sigma + \theta_c\lambda(\chi(x,t)),$$
$$\mathbf{q}(x,t) = -\int_0^\infty k(\sigma)\nabla\vartheta(x,t-\sigma)\,d\sigma$$

for $(x,t) \in \Omega \times \mathbb{R}$, where c_v, θ_c, and e_c are positive constants denoting the *specific heat*, the *critical value* of the temperature (corresponding to the phase transition), and the internal energy at the critical temperature, respectively. We assume that the memory kernel k is smooth enough, nonincreasing and summable along with its first derivative on $[0,\infty)$. As far as the (smooth) kernel a is concerned, we always suppose that its first and second derivatives are summable on $[0,\infty)$ and $a(0) > 0$. Then we have two possible choices, both of which are thermodynamically consistent (see Appendix). We suppose that either

$$a \text{ is bounded, nondecreasing, and concave} \tag{1.1}$$

or

$$a \text{ is summable, nonincreasing, and convex.} \tag{1.2}$$

We shall discuss the importance of these assumptions below.

Going back to the constitutive laws and making suitable assumptions on the behavior of $\vartheta(t)$ as $t \to -\infty$, the energy balance yields

$$c_v\theta_c\partial_t\vartheta + a(0)\vartheta + \int_0^\infty a'(\sigma)\vartheta(t-\sigma)\,d\sigma + \theta_c\lambda'(\chi)\partial_t\chi - \int_0^\infty k(\sigma)\Delta\vartheta(t-\sigma)\,d\sigma = f$$

in $\Omega \times \mathbb{R}$.

The variable χ, which appears in the internal energy, can be regarded as an internal state variable. Consequently, a constitutive equation for χ is in order. As shown in the Appendix (cf. (A.22)), we assume that

$$m\partial_t\chi - m_0\Delta\chi + \beta(\chi) \ni \gamma(\chi) + \vartheta\lambda'(\chi)$$

where β is a maximal monotone graph, γ is a Lipschitz function, and m, m_0 are positive parameters.

Consider now a given initial time $\tau \in \mathbb{R}$. To specify the initial conditions, besides the values of ϑ and χ at τ, the whole past history of ϑ up to τ must be given, namely,

$$\begin{aligned} \vartheta(\tau) &= \vartheta_0 && \text{in } \Omega, \\ \chi(\tau) &= \chi_0 && \text{in } \Omega, \\ \vartheta(\tau - s) &= \vartheta_0(s) && \text{in } \Omega,\ \forall s > 0, \end{aligned}$$

where $\vartheta_0(s)$ is the *initial past history* of ϑ.

Concerning boundary conditions, a quite natural one for χ is given by

$$\partial_{\mathbf{n}}\chi = 0 \quad \text{on } \partial\Omega \times (\tau, +\infty)$$

where $\partial_{\mathbf{n}}$ denotes the outward normal derivative. As far as ϑ is concerned, we suppose that the adiabatic boundary condition is satisfied on $\Gamma_N \subseteq \partial\Omega$, that is,

$$\int_0^\infty k(\sigma)\partial_{\mathbf{n}}\vartheta(t-\sigma)\,d\sigma = 0 \quad \text{on } \Gamma_N \times (\tau, +\infty)$$

while the Dirichlet homogeneous boundary condition on $\Gamma_D = \partial\Omega \setminus \Gamma_N$ holds for any time, that is,

$$\vartheta = 0 \quad \text{on } \Gamma_D \times \mathbb{R}.$$

We are now dealing with an initial and boundary value problem for the phase-field system under consideration. To formulate it in a history space setting, we follow [21] (see also [17]) and we introduce a new variable, namely, the *summed past history* of ϑ, which is defined by

$$\eta^t(x,s) = \int_0^s \vartheta^t(x,y)\,dy = \int_{t-s}^t \vartheta(x,y)\,dy \quad x \in \Omega,\ s \geq 0.$$

One can easily check that η satisfies the first-order linear evolution equation

$$\partial_t\eta^t(s) + \partial_s\eta^t(s) = \vartheta(t) \quad \text{in } \Omega, \quad (t,s) \in (\tau,+\infty)\times(0,+\infty)$$

along with the initial condition

$$\eta^\tau = \eta_0 \quad \text{in } \Omega \times (0,+\infty)$$

where

$$\eta_0(s) = \int_0^s \vartheta_0(y)\,dy \quad \text{in } \Omega,\ s \geq 0,$$

is the *initial summed past history* of ϑ.

Then, we observe that a formal integration by parts yields (see [21])

$$\int_0^\infty k(\sigma)\nabla\vartheta(t-\sigma)\,d\sigma = -\int_0^\infty k'(\sigma)\nabla\eta^t(\sigma)\,d\sigma \quad \text{in } \Omega,\ t > \tau,$$

and

$$\int_0^\infty a'(\sigma)\vartheta(t-\sigma)\,d\sigma = -\int_0^\infty a''(\sigma)\eta^t(\sigma)\,d\sigma \quad \text{in } \Omega,\ t \geq \tau.$$

Thus, on account of (1.1) and (1.2), we set

$$\mu(s) = -\frac{k'(s)}{a(0)} \quad \text{and} \quad \nu_0\nu(s) = -\frac{a''(s)}{a(0)}$$

for any $s > 0$, where $\nu_0 = 1$ or $\nu_0 = -1$ whenever (1.1) or (1.2) holds, respectively.

Taking for simplicity all the constants equal to 1, our previous considerations and the above choice of variables lead us to formulate the following initial and boundary value problem.

Problem P. Find a (ϑ, χ, η) solution to the system

$$\begin{aligned}
&\partial_t(\vartheta(t) + \lambda(\chi(t))) + \vartheta(t) + \int_0^\infty \nu_0\nu(\sigma)\eta^t(\sigma)\,d\sigma - \int_0^\infty \mu(\sigma)\Delta\eta^t(\sigma)\,d\sigma = f(t),\\
&\partial_t\chi(t) - \Delta\chi(t) + \beta(\chi(t)) \ni \gamma(\chi(t)) + \lambda'(\chi(t))\vartheta(t),\\
&\partial_t\eta^t(s) + \partial_s\eta^t(s) = \vartheta(t)
\end{aligned}$$

in Ω, for any $t > \tau$ and any $s > 0$, that satisfies the initial and boundary conditions

$$\begin{aligned}
\int_0^\infty \mu(\sigma)\partial_{\mathbf{n}}\eta^t(\sigma)\,d\sigma &= 0 && \text{on } \Gamma_N \times (\tau, +\infty),\\
\eta^t(s) &= 0 && \text{on } \Gamma_D \times (\tau, +\infty) \times (0, +\infty),\\
\eta^t(0) &= 0 && \text{in } \Omega \times (\tau, +\infty),\\
\partial_{\mathbf{n}}\chi &= 0 && \text{on } \partial\Omega \times (\tau, +\infty),\\
\vartheta(\tau) &= \vartheta_0 && \text{in } \Omega,\\
\chi(\tau) &= \chi_0 && \text{in } \Omega,\\
\eta^\tau &= \eta_0 && \text{in } \Omega \times (0, +\infty).
\end{aligned}$$

Problems like **P** have been studied firstly in [1] when $a \equiv 0, \beta(r) = r^3$ and γ, λ are both linear. Using a semigroup approach, existence, uniqueness, and longtime results have there been proved. Quite general well-posedness results have then been obtained in [14, 15] via energy methods (see also [16]). Recently, in [11, 12] a thorough investigation along the lines of [1] has been carried out. In particular, existence and uniqueness when λ is a quadratic nonlinearity as well as a detailed characterization of the ω-limit set have been shown. In this framework, it is also worth quoting [5, 6], which are devoted to studying a phase-field model with memory based on a constitutive law for the heat flux, proposed by Coleman and Gurtin, where $\mathbf{q}$ depends both on the present value of the temperature gradient and on its past history. Regarding the longtime behavior and existence of a maximal attractor for other phase transition models without memory effects, the reader is referred, e.g., to [28, 32] and the references therein.

In all the mentioned papers about phase-field models with memory, the summed past history of ϑ (and, possibly, of χ) is simply incorporated in the source term f and, sometimes, in the boundary data. However, this approach seems not suitable for studying the longtime behavior of solutions from a more general point of view, namely, the stability of sets of trajectories. Instead, a formulation in the history space setting, which regards η as a variable of the evolution phenomenon, has been proved effective in analyzing such issues (cf. [20-22], see also [4]). This fact has led us to formulate **P**.

Here we prove some well-posedness theorems for **P**. In these results the role of a is marginal and we could also take $a \equiv 0$. On the contrary, our result about the longterm behavior is based on the fact that $a(0) > 0$, that is, on the presence of a memory term in the internal energy as well. To be more precise, in this case the assumptions on the memory kernels (in particular, (1.1)–(1.2)) as well as the boundary conditions play a crucial role. Going a bit into the details, we are able to obtain some uniform-in-time estimates which imply the existence of an absorbing set, provided that β fulfills suitable growth conditions. Nevertheless, this result strongly depends on the sign of ν_0, that is, on the assumptions (1.1)–(1.2). Indeed, the case $\nu_0 = 1$ turns out to be quite nice since the related memory term has a dissipative behavior which is similar to the heat flux term, provided that ν and μ decay exponentially. However, in the literature (see, e.g., [19] and the references therein) it is often assumed that $\nu_0 = -1$, that is, (1.2). This is a much more delicate case, since the related memory term has an antidissipative behavior which has to be counterbalanced by the dissipation associated with the heat flux (see Section

6 below). We are able to do that when the homogeneous Dirichlet boundary condition holds on a portion Γ_D of positive measure, provided that ν is dominated by μ. These restrictions allow us to take advantage of the Poincaré inequality.

A strictly related model is analyzed in [20], the only difference being that the heat flux law also contains an instantaneous temperature gradient. This helps of course. There, under the assumption (1.1), we also prove the existence of a uniform attractor of finite fractal dimension. It is worth noting that the argument used here to deal with assumption (1.2) can also be adapted to the setting of [20].

The plan of the paper goes as follows. In the next section we introduce the functional setting and some technical lemmas. In Section 3 we state our main results whose proofs are carried out in Sections 4–9. The Appendix is devoted to presenting the construction of the model starting from a general nonlinear setting. Using the Clausius-Duhem inequality and linearizing with respect to ϑ, we deduce that our evolution system is thermodynamically consistent.

2. Notation and basic tools. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. We set

$$H = L^2(\Omega), \quad V = H^1(\Omega), \quad V_0 = \{v \in V : v|_{\partial\Omega} = 0 \text{ a.e. on } \Gamma_D\}$$

where $v|_{\partial\Omega}$ stands for the trace of v, and

$$W = \{w \in H^2(\Omega) : \partial_{\mathbf{n}} w = 0 \text{ a.e. on } \partial\Omega\},$$
$$W_0 = \{w \in V_0 : \Delta w \in H, \partial_{\mathbf{n}} w = 0 \text{ in } (H_{00}^{1/2}(\Gamma_N))^*\},$$

$(H_{00}^{1/2}(\Gamma_N))^*$ being the dual space of (cf., for instance, [29])

$$H_{00}^{1/2}(\Gamma_N) = \{w \in L^2(\Gamma_N) : \exists \tilde{w} \in V_0 : \tilde{w}|_{\partial\Omega} = w \text{ on } \Gamma_N\}.$$

Clearly, when $\Gamma_D = \varnothing$, then $V_0 \equiv V$ and $W_0 \equiv W$. Then, denote by V^*, V_0^*, W^*, and W_0^* the dual spaces of V, V_0, W, and W_0, respectively. As usual, we identify H with its dual space H^*. We recall the continuous and dense embeddings

$$W_0 \hookrightarrow V_0 \hookrightarrow H \equiv H^* \hookrightarrow V_0^* \hookrightarrow W_0^*,$$
$$V_0 \hookrightarrow V \hookrightarrow H \equiv H^* \hookrightarrow V^* \hookrightarrow V_0^*$$

and, in particular, the inequalities

$$\|v\|_H \leq \|v\|_V \quad \forall v \in V, \quad \text{and} \quad \|u\|_{V^*} \leq \|u\|_H \quad \forall u \in H. \tag{2.1}$$

We also need the Poincaré inequality

$$\|v\|_H \leq C_P \|\nabla v\|_{H^3} \quad \forall v \in V_0, \tag{2.2}$$

which holds whenever the Lebesgue measure $|\Gamma_D|$ is positive (see, e.g., [33, Ch. II, 1.4]).

To avoid confusion, we will always denote the norm and the inner product on a Hilbert space X by $\langle \cdot, \cdot \rangle_X$ and $\| \cdot \|_X$, respectively. In particular, due to the Poincaré inequality, if $|\Gamma_D| > 0$, then we can take $\|v\|_{V_0} = \|\nabla v\|_{H^3}$. Besides, the symbol $\langle\langle \cdot, \cdot \rangle\rangle_X$ will stand for the duality pairing between X and its dual space X^*, whenever X is a real Banach space.

Given a positive function α defined on $\mathbb{R}^+ = (0, +\infty)$, and a real Hilbert space X, let $L^2_\alpha(\mathbb{R}^+, X)$ be the Hilbert space of X-valued functions on $\mathbb{R}^+$, endowed with the inner product

$$\langle \varphi, \psi \rangle_{L^2_\alpha(\mathbb{R}^+, X)} = \int_0^\infty \alpha(\sigma) \langle \varphi(\sigma), \psi(\sigma) \rangle_X \, d\sigma.$$

It is worth recalling that, given two Hilbert spaces X and Y, the space $X \cap Y$ turns out to be a Hilbert space endowed with the inner product

$$\langle \cdot, \cdot \rangle_{X \cap Y} = \langle \cdot, \cdot \rangle_X + \langle \cdot, \cdot \rangle_Y.$$

In addition, given a (possibly unbounded) interval $I \subset \mathbb{R}$ and a Hilbert space X, we indicate by $\mathcal{D}(I, X)$ the space of infinitely differentiable X-valued functions with compact support in I.

In order to describe the longtime behavior of the solutions of our system we also need to introduce the Banach space $\mathcal{T}$ of L^1_{loc}-*translation bounded* functions with values in H, namely

$$\mathcal{T} = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}, H) : \|f\|_{\mathcal{T}} = \sup_{r \in \mathbb{R}} \left(\int_r^{r+1} \|f(y)\|_H \, dy \right) < \infty \right\}.$$

Finally, for the reader's convenience, we report here below some technical results which will be useful in the course of the investigation.

LEMMA 2.1 (Generalized Young inequality). Let $a, b \geq 0$ be given. Then for every $\kappa > 0$, and $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, it follows that

$$ab \leq \kappa a^p + \frac{1}{q(\kappa p)^{q/p}} b^q. \tag{2.3}$$

LEMMA 2.2 (Gagliardo-Nirenberg). Let $2 < p \leq 6$. Then there exists $k > 0$ such that the inequality

$$\|v\|_{L^p(\Omega)} \leq k \|v\|_H^\rho \|v\|_V^{1-\rho} \tag{2.4}$$

holds for all $v \in V$, with $\rho = \frac{6-p}{2p}$.

The following Gronwall-type lemmas subsume Lemma A.5 in [7] and some results in [30].

LEMMA 2.3. Let Φ be an absolutely continuous function on an interval $[\tau, T] \subset \mathbb{R}$, and let r_1 and r_2 be positive summable functions on $[\tau, T]$. Then the differential inequality

$$\frac{d}{dt} \Phi^2(t) \leq c\Phi^2(t) + r_1(t) + r_2(t)\Phi(t) \quad \text{for a.e. } t \in [\tau, T]$$

and for some $c \geq 0$ implies

$$\Phi^2(t) \leq 2e^{c(t-\tau)} \Phi^2(\tau) + 2e^{c(t-\tau)} \int_\tau^t r_1(y) \, dy + e^{2c(t-\tau)} \left(\int_\tau^t r_2(y) \, dy \right)^2$$

for any $t \in [\tau, T]$.

LEMMA 2.4. Let Φ be a nonnegative, absolutely continuous function on $[\tau, +\infty)$ that satisfies for some $\varepsilon > 0$ the differential inequality

$$\frac{d}{dt}\Phi^2(t) + \varepsilon\Phi^2(t) \leq \Lambda + \|f(t)\|_H \Phi(t) + r_3(t) \quad \text{for a.e. } t \in [\tau, +\infty)$$

where $\Lambda \geq 0$, r_3 is a nonnegative locally summable function, and $f \in \mathcal{T}$. Then

$$\Phi^2(t) \leq 2\Phi^2(\tau)e^{-\varepsilon(t-\tau)} + \frac{2\Lambda}{\varepsilon} + \frac{e^\varepsilon}{(1-e^{-\varepsilon/2})^2}\|f\|_{\mathcal{T}}^2 + \int_\tau^t r_3(y)e^{-\varepsilon(t-y)}\,dy$$

for any $t \in [\tau, +\infty)$.

3. Main results. Before stating the main results, we have to introduce a rigorous formulation of problem **P**. First of all, some assumptions on the memory kernels and on the data are in order. As far as ν and μ are concerned, we suppose

$$\nu, \mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \tag{K1}$$

$$\nu(s) \geq 0, \mu(s) \geq 0 \quad \forall s \in \mathbb{R}^+, \tag{K2}$$

$$\nu'(s) \leq 0, \mu'(s) \leq 0 \quad \forall s \in \mathbb{R}^+, \tag{K3}$$

$$\nu'(s) + \delta\nu(s) \leq 0, \mu'(s) + \delta\mu(s) \leq 0 \quad \text{for some } \delta > 0,\ \forall s \in \mathbb{R}^+. \tag{K4}$$

We recall that assumption (K4), which basically implies the exponential decay of the memory kernel (see, e.g., [17]) is used only to prove the existence of an absorbing set.

Then, we set

$$k_0 = \int_0^\infty \mu(\sigma)\,d\sigma \geq 0 \quad \text{and} \quad a_0 = \int_0^\infty \nu(\sigma)\,d\sigma \geq 0 \tag{K5}$$

and, in view of (K1)–(K2), we introduce the Hilbert spaces

$$\mathcal{M} = L^2_\nu(\mathbb{R}^+, H) \cap L^2_\mu(\mathbb{R}^+, V_0), \qquad \widetilde{\mathcal{M}} = L^2_\mu(\mathbb{R}^+, V_0).$$

Furthermore, we assume

$$\phi : \mathbb{R} \to [0, +\infty] \text{ proper convex and lower semicontinuous with } \phi(0) = 0, \tag{H1}$$

$$\beta = \partial\phi \subset \mathbb{R} \times \mathbb{R} \text{ such that } \beta(0) \ni 0, \tag{H2}$$

$$\exists\, \Gamma > 0 \text{ such that } |\gamma(r_1) - \gamma(r_2)| \leq \Gamma|r_1 - r_2|, \tag{H3}$$

$$\lambda \in C^2(\mathbb{R}), \tag{H4}$$

$$\lambda', \lambda'' \in L^\infty(\mathbb{R}), \tag{H5}$$

$$f \in L^1_{\mathrm{loc}}(\mathbb{R}, H), \tag{H6}$$

$$\vartheta_0 \in H, \tag{H7}$$

$$\chi_0 \in V, \tag{H8}$$

$$\eta_0 \in \mathcal{M}, \tag{H9}$$

$$\phi(\chi_0) \in L^1(\Omega). \tag{H10}$$

We now introduce our definition of a solution to **P**.

DEFINITION 3.1. Let $\tau, T \in \mathbb{R}$, $T > \tau$, and set $I = [\tau, T]$. A quadruplet $(\vartheta, \chi, \xi, \eta)$ that fulfills

$$\vartheta \in C^0(I, H), \tag{3.1}$$
$$\partial_t \vartheta \in L^\infty(I, V_0^*) + L^1(I, H), \tag{3.2}$$
$$\chi \in C^0(I, V) \cap H^1(I, H) \cap L^2(I, W), \tag{3.3}$$
$$\xi \in L^2(I, H), \tag{3.4}$$
$$\eta \in C^0(I, \mathcal{M}), \tag{3.5}$$
$$\partial_t \eta + \partial_s \eta \in C^0(I, L^2_\nu(\mathbb{R}^+, H) \cap L^2_\mu(\mathbb{R}^+, H)), \tag{3.6}$$
$$\xi \in \beta(\chi) \quad \text{a.e. in } \Omega \times I \tag{3.7}$$

is a solution to problem **P** in the time interval I provided that

$$\langle\langle \partial_t(\vartheta + \lambda(\chi)), v \rangle\rangle_{V_0} + \langle \vartheta, v \rangle_H + \int_0^\infty \nu_0 \nu(\sigma) \langle \eta(\sigma), v \rangle_H \, d\sigma \\ + \int_0^\infty \mu(\sigma) \langle \nabla \eta(\sigma), \nabla v \rangle_{H^3} \, d\sigma = \langle f, v \rangle_H \quad \forall v \in V_0, \text{ a.e. in } I, \tag{3.8}$$

$$\partial_t \chi - \Delta \chi + \xi = \gamma(\chi) + \lambda'(\chi) \vartheta \quad \text{a.e. in } \Omega \times I, \tag{3.9}$$

$$\int_0^\infty \nu(\sigma) \langle \partial_t \eta(\sigma) + \partial_s \eta(\sigma), \zeta(\sigma) \rangle_H \, d\sigma + \int_0^\infty \mu(\sigma) \langle \partial_t \eta(\sigma) + \partial_s \eta(\sigma), \zeta(\sigma) \rangle_H \, d\sigma \\ - \int_0^\infty \mu(\sigma) \langle \partial_t \eta(\sigma) + \partial_s \eta(\sigma), \Delta \zeta(\sigma) \rangle_H \, d\sigma \\ = \int_0^\infty \nu(\sigma) \langle \vartheta, \zeta(\sigma) \rangle_H \, d\sigma + \int_0^\infty \mu(\sigma) \langle \vartheta, \zeta(\sigma) \rangle_H \, d\sigma - \int_0^\infty \mu(\sigma) \langle \vartheta, \Delta \zeta(\sigma) \rangle_H \, d\sigma \\ \forall \zeta \in L^2_\nu(\mathbb{R}^+, H) \cap L^2_\mu(\mathbb{R}^+, W_0), \text{ a.e. in } I, \tag{3.10}$$

$$\vartheta(\tau) = \vartheta_0 \quad \text{a.e. in } \Omega, \tag{3.11}$$
$$\chi(\tau) = \chi_0 \quad \text{a.e. in } \Omega, \tag{3.12}$$
$$\eta^\tau = \eta_0 \quad \text{a.e. in } \Omega \times \mathbb{R}^+. \tag{3.13}$$

Here $-\partial_s$ has to be understood as the infinitesimal generator of the right-translation semigroup on $\mathcal{M}$.

Problem **P** is well posed according to Definition 3.1. Indeed, we can prove the following results.

THEOREM 3.2. *Assume that* (K1)–(K3) *and* (H1)–(H5) *hold. Let* $\{f_i, \vartheta_{0i}, \chi_{0i}, \eta_{0i}\}$, $i = 1, 2$, *be two sets of data satisfying* (H6)–(H10), *denote by* $\{\vartheta_i, \chi_i, \xi_i, \eta_i\}$ *two corresponding solutions to Problem* **P**, *and fix* $C_0 > 0$ *so that*

$$\max_{i=1,2} \{ \|\vartheta_i\|_{L^\infty(I,H)}, \|\chi_i\|_{L^\infty(I,H)}, \|\chi_i\|_{L^2(I,W)} \} \leq C_0. \tag{3.14}$$

Then there exists a positive constant $C = C(C_0, T)$ such that

$$\|\vartheta_1(t) - \vartheta_2(t)\|_H^2 + \|\chi_1(t) - \chi_2(t)\|_H^2 + \int_\tau^t \|\chi_1(y) - \chi_2(y)\|_V^2 \, dy + \|\eta_1^t - \eta_2^t\|_{\mathcal{M}}^2$$
$$\leq C(\|\vartheta_{01} - \vartheta_{02}\|_H^2 + \|\chi_{01} - \chi_{02}\|_H^2 + \|\eta_{01} - \eta_{02}\|_{\mathcal{M}}^2 + \|f_1 - f_2\|_{L^1(I,H)}^2) \quad (3.15)$$

for any $t \in I$. In particular, Problem $\mathbf{P}$ has a unique solution.

THEOREM 3.3. Let (K1)–(K3) and (H1)–(H10) hold. Then, given any $\tau \in \mathbb{R}$ and any $T > \tau$, problem $\mathbf{P}$ has a unique solution $(\vartheta, \chi, \xi, \eta)$ in the time interval $I = [\tau, T]$, with initial data $(\vartheta_0, \chi_0, \eta_0)$. Moreover, $\phi(\chi) \in W^{1,1}(I, L^1(\Omega))$.

Set now

$$\mathcal{H} = H \times V \times \mathcal{M}$$

and consider the space

$$\mathcal{H}_\phi = \{z = (\vartheta, \chi, \eta) \in \mathcal{H} : \phi(\chi) \in L^1(\Omega)\}.$$

It is readily seen that $\mathcal{H}_\phi$ is a convex subset of $\mathcal{H}$ that can be endowed with the metric

$$\mathrm{dist}_\phi(z_1, z_2) = \|z_1 - z_2\|_{\mathcal{H}} + \|\phi(\chi_1) - \phi(\chi_2)\|_{L^1(\Omega)}.$$

In addition, $B_\phi(0, R)$ stands for the ball of $\mathcal{H}_\phi$ centered at zero of radius R.

We agree to denote by $U_f(t, \tau)z_0$ the solution (ϑ, χ, η) to problem $\mathbf{P}$ at time t with source term f and initial data z_0 given at time τ. Note that ξ is an auxiliary variable which is automatically determined by (ϑ, χ, η).

REMARK 3.4. For any fixed $f \in L^1_{\mathrm{loc}}(\mathbb{R}, H)$, the two-parameter family $U_f(t, \tau)$, with $t \geq \tau$, $\tau \in \mathbb{R}$, satisfies the following properties:

(i) $U_f(t, \tau) : \mathcal{H}_\phi \to \mathcal{H}_\phi$ for any $t \geq \tau$, $\tau \in \mathbb{R}$;

(ii) $U_f(\tau, \tau)$ is the identity map on $\mathcal{H}_\phi$ for any $\tau \in \mathbb{R}$;

(iii) $U_f(t, s)U_f(s, \tau) = U_f(t, \tau)$ for any $t \geq s \geq \tau$, $\tau \in \mathbb{R}$;

(iv) $U_f(t, \tau)z \to z$ as $t \downarrow \tau$ for any $z \in \mathcal{H}_\phi$, $\tau \in \mathbb{R}$.

These properties are consequences of Theorems 3.2 and 3.3.

As we pointed out in the Introduction, the asymptotic behavior of solutions strongly depends on the choice of parameter ν_0. Let us consider first the case $\nu_0 = 1$, which corresponds to the supposition that a satisfies (1.1). Besides the decay properties (K4), we need to require that the dissipation given by β be prevailing on the contribution given by the term γ. More precisely, we set

$$\hat{\gamma}(r) = \int_0^r \gamma(y) \, dy$$

and we assume that there exists $\varepsilon_0 \in (0, 1)$ such that

$$-\xi r + \gamma(r) r \leq -\varepsilon(\phi(r) - \hat{\gamma}(r)) - m_1 r^2 + m_2$$
$$\text{for any } \varepsilon \in (0, \varepsilon_0] \text{ and } m_1, m_2 > 0, \quad \text{(H11)}$$

$$\phi(r) - \hat{\gamma}(r) + L \geq 0 \quad \text{for some } L > 0 \quad \text{(H12)}$$

for every $r \in D(\beta)$ and every $\xi \in \beta(r)$, where $D(\beta)$ denotes the effective domain of β. The constant m_2 is allowed to depend on ε.

It is easily seen that (H11)–(H12) are satisfied if $D(\beta)$ is bounded. In the case when $\beta : \mathbb{R} \to \mathbb{R}$, a sufficient condition that ensures (H11)–(H12) is

$$\liminf_{|r|\to\infty} \frac{\beta(r)}{r} > \Gamma.$$

Indeed, in this case there exist $0 < \varepsilon_1 < 1$ and $\rho > 0$ such that

$$\liminf_{|r|\to\infty} \frac{(1-\varepsilon)\beta(r)}{r} \geq \Gamma + 2\rho \quad \forall \varepsilon \leq \varepsilon_1. \tag{3.16}$$

It is also clear from (H3) and (2.3) that

$$|\hat{\gamma}(r)| \leq \Gamma r^2 + \frac{|\gamma(0)|^2}{2\Gamma}. \tag{3.17}$$

Hence (H1)–(H2) and (3.17) entail

$$\varepsilon(\phi(r) - \hat{\gamma}(r)) \leq \beta(r)r - \gamma(r)r - [(1-\varepsilon)\beta(r)r - \gamma(r)r - \varepsilon\Gamma r^2 - \rho r^2] - \rho r^2 + \frac{\varepsilon|\gamma(0)|^2}{2\Gamma}.$$

Setting $\varepsilon_0 < \min\{\varepsilon_1, \rho/\Gamma\}$, it is straightforward to see that (H3) and (3.16) yield, for any $\varepsilon \in (0, \varepsilon_0]$,

$$\sup_{r\in\mathbb{R}} -[(1-\varepsilon)\beta(r)r - \gamma(r)r - \varepsilon\Gamma r^2 - \rho r^2] < +\infty$$

and (H11) follows at once. The proof of (H12) is straightforward. Notice that (H11)–(H12) allow, for instance, the significant case $\beta(r) - \gamma(r) = r^3 - r$ examined in [1] (cf. also [8]).

To deal with the more common assumption (1.2), which corresponds to the case $\nu_0 = -1$, we require $|\Gamma_D| > 0$ in order to take advantage of the Poincaré inequality. Also, we need that ν is suitably dominated by μ, while we still need that μ satisfies (K4). Of course, even in this case, the above assumptions (H11) and (H12) are essential.

Thanks to Theorems 3.4 and 3.5, we can state our result about the longtime behavior of the solutions to **P**.

THEOREM 3.5. Let (K1)–(K3), (H1)–(H5), and (H11)–(H12) hold, and let $\mathcal{F} \subset \mathcal{T}$ be a bounded set. If, in addition, $\nu_0 = 1$ and (K4) holds, then there exists $R_0 > 0$, depending only on $\sup_{f\in\mathcal{F}} \|f\|_{\mathcal{T}}$, such that, given any $R > 0$ and any $\tau \in \mathbb{R}$, there exists $t^* = t^*(R)$ such that

$$\sup_{z_0\in B_\phi(0,R)} \sup_{f\in\mathcal{F}} \|U_f(t,\tau)z_0\|_{\mathcal{H}} \leq R_0 \quad \forall t \geq \tau + t^*. \tag{3.18}$$

Moreover, if $\mathcal{F}$ is a bounded subset of $L^1(\mathbb{R}, H)$, for any $\tau \in \mathbb{R}$,

$$\sup_{z_0\in B_\phi(0,R)} \sup_{f\in\mathcal{F}} \int_\tau^\infty \|\partial_t \chi(y)\|_H^2 \, dy \leq C \tag{3.19}$$

where $\chi(y)$ is the second component of $U_f(y,\tau)z_0$, and the constant C depends on R. Suppose now that $\nu_0 = -1$. If $|\Gamma_D| > 0$, μ enjoys (K4) and ν fulfills the following condition:

$$a_0\nu(s) \leq \frac{2}{C_P^2}\delta_0\mu(s) \quad \text{for some } \delta_0 \in (0,\delta),\ \forall s \in \mathbb{R}^+, \tag{K6}$$

then estimates (3.18)–(3.19) still hold.

REMARK 3.6. If (K6) holds, then note that $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ coincide and have equivalent norms. Moreover, recalling the positions made in the Introduction, it is not difficult to realize that condition (K6) can be rewritten in terms of the original kernels a and k as follows:

$$-\frac{a'(0)}{a(0)}a''(s) \leq -\frac{2}{C_P^2}\delta_0 k'(s) \quad \text{for some } \delta_0 \in (0,\delta),\ \forall s \in \mathbb{R}^+.$$

In this form, it can be compared with [4, (2.4)].

REMARK 3.7. Due to the continuous embedding $V \hookrightarrow L^6(\Omega)$, if there is a positive constant c such that

$$|\xi| \leq c(1+|r|^q) \quad \forall r \in D(\beta), \quad \forall \xi \in \beta(r) \tag{H13}$$

for some $q \in [0,5]$, it is apparent that $\mathcal{H}_\phi \equiv \mathcal{H}$. Then inequality (3.18) may be replaced by

$$\sup_{\|z_0\|_{\mathcal{H}} \leq R}\ \sup_{f \in \mathcal{F}} \|U_f(t,\tau)z_0\|_{\mathcal{H}} \leq R_0 \quad \forall t \geq \tau + t^*. \tag{3.20}$$

In this case, the ball of radius R_0 in $\mathcal{H}$ is called a *uniform absorbing set* (uniform with respect to $\tau \in \mathbb{R}$ and $f \in \mathcal{F}$) associated with problem $\mathbf{P}$.

In the particular, but significant, case $\beta(r) = r^3$ and λ linear, we can also prove that

(v) $U_f(t,\tau) \in C^0(\mathcal{H},\mathcal{H})$ for any $\tau \in \mathbb{R}$, $t \geq \tau$.

Therefore, $U_f(t,\tau)$ is a process with symbol f according to the usual definition (see, e.g., [25], Chapter 6). This fact is ensured by (3.15) and

PROPOSITION 3.8. Let (K1)–(K3), (H3), (H6)–(H10) hold. Assume moreover that

$$\beta(r) = r^3 \qquad \forall r \in \mathbb{R}, \tag{H14}$$

$$\lambda(r) = \lambda_0 r \quad \forall r \in \mathbb{R}, \lambda_0 \in \mathbb{R}. \tag{H15}$$

Then, if we consider two sets of data $\{f_i, \vartheta_{0i}, \chi_{0i}, \eta_{0i}\}$, $i = 1,2$, fulfilling (H6)–(H10) and we denote by $\{\vartheta_i, \chi_i, \xi_i, \eta_i\}$ two corresponding solutions to problem $\mathbf{P}$, we can find a positive constant C such that, for any $t \in I$,

$$\|\chi_1(t) - \chi_2(t)\|_V^2 + \int_\tau^t \|\chi_1(y) - \chi_2(y)\|_W^2\, dy$$
$$\leq C(\|\vartheta_{01} - \vartheta_{02}\|_H^2 + \|\chi_{01} - \chi_{02}\|_V^2 + \|\eta_{01} - \eta_{02}\|_{\mathcal{M}}^2 + \|f_1 - f_2\|_{L^1(I,H)}). \tag{3.21}$$

In a slightly different functional setting, an existence and uniqueness result can be proved when λ is a quadratic nonlinearity (cf. [11, 12]). In this case, it is worth noting that our model may describe not only solid-liquid phase transitions, but also ferromagnetic transformations (see, e.g., [24]).

THEOREM 3.9. Let (K1)–(K3), (H1)–(H4), and (H6)–(H10) hold. Moreover, suppose that

$$\lambda'' \in L^\infty(\mathbb{R}). \tag{H16}$$

Then, given any $\tau \in \mathbb{R}$ and any $T > \tau$, there exists $(\vartheta, \chi, \xi, \eta)$ in the time interval $I = [\tau, T]$, satisfying (3.2), (3.7)–(3.12), and

$$\vartheta \in L^\infty(I, H) \cap C^0(I, V_0^*), \tag{3.22}$$

$$\chi \in L^\infty(I, V) \cap H^1(I, H) \cap L^2(I, W^{2,3/2}(\Omega)), \tag{3.23}$$

$$\partial_{\mathbf{n}} \chi = 0 \quad \text{a.e. on } \partial\Omega \times I, \tag{3.24}$$

$$\eta \in L^\infty(I, \mathcal{M}) \cap C^0(I, L^2_\nu(\mathbb{R}^+, V_0^*) \cap L^2_\mu(\mathbb{R}^+, V_0^*)), \tag{3.25}$$

$$\xi \in L^2(I, L^{3/2}(\Omega)), \tag{3.26}$$

$$\phi(\chi) \in L^\infty(I, L^1(\Omega)). \tag{3.27}$$

Moreover, the initial condition (3.13) holds in V_0^*, almost everywhere in $\mathbb{R}^+$. If, in addition, $\chi_0 \in L^\infty(\Omega)$, then the above solution is unique and it fulfills (3.1), (3.4)–(3.5), (3.13), and

$$\chi \in L^\infty(\Omega \times I) \cap L^2(I, H^2(\Omega)). \tag{3.28}$$

We can still prove the existence of a solution to problem $\mathbf{P}$ when χ_0 is only in H. In this case, the equation for χ has to be understood in V^*. Uniqueness also holds provided that λ is linear.

THEOREM 3.10. Let (K1)–(K3), (H1)–(H7), and (H9)–(H10) hold. Moreover, assume

$$\chi_0 \in H. \tag{H17}$$

Then, given any $\tau \in \mathbb{R}$ and any $T > \tau$, there exists a quadruplet $(\vartheta, \chi, \xi, \eta)$ that fulfills, in the time interval $I = [\tau, T]$, (3.4), (3.7), (3.10), (3.12), (3.27), and

$$\vartheta \in L^\infty(I, H) \cap C^0(I, V_0^*), \tag{3.29}$$

$$\partial_t(\vartheta + \lambda(\chi)) \in L^2(I, V_0^*) + L^1(I, H), \tag{3.30}$$

$$\chi \in C^0(I, H) \cap L^2(I, V), \tag{3.31}$$

$$\partial_t \chi \in L^2(I, V^*), \tag{3.32}$$

$$\eta \in L^\infty(I, \mathcal{M}) \cap C^0(I, L^2_\nu(\mathbb{R}^+, V_0^*) \cap L^2_\mu(\mathbb{R}^+, V_0^*)), \tag{3.33}$$

$$\begin{aligned} &\langle\langle \partial_t(\vartheta + \lambda(\chi)), v\rangle\rangle_{V_0} + \langle \vartheta, v\rangle_H + \int_0^\infty \nu_0 \nu(\sigma) \langle \eta(\sigma), v\rangle_H \, d\sigma \\ &\qquad + \int_0^\infty \mu(\sigma) \langle \nabla\eta(\sigma), \nabla v\rangle_{H^3} \, d\sigma = \langle f, v\rangle_H \quad \forall v \in V_0, \text{ a.e. in } I, \end{aligned} \tag{3.34}$$

$$\begin{aligned} \langle\langle \partial_t \chi, v\rangle\rangle_V + \langle \nabla\chi, \nabla v\rangle_{H^3} + \langle\langle \xi, v\rangle\rangle_V = \langle \gamma(\chi), v\rangle_H + \langle \lambda'(\chi)\vartheta, v\rangle_H& \\ \forall v \in V, \text{ a.e. in } I,& \end{aligned} \tag{3.35}$$

$$\vartheta(\tau) = \vartheta_0 \quad \text{in } V_0^*, \tag{3.36}$$

$$\eta^\tau = \eta_0 \quad \text{in } V_0^*, \text{ a.e. in } \mathbb{R}^+. \tag{3.37}$$

If λ satisfies (H15), then the continuous dependence estimate (3.15) holds with a constant C independent of C_0. Thus the solution is unique and (3.1), (3.5)–(3.6), (3.11), and (3.13) are fulfilled.

4. Proof of Theorem 3.2. For the sake of simplicity, we suppose $\Gamma_D \equiv \varnothing$ so that $V_0 \equiv V$ and $W_0 \equiv W$. However, the arguments can be easily adapted to the other cases.

Let $\{\vartheta_i, \chi_i, \xi_i, \eta_i\}$, $i = 1, 2$, be two solutions to problem **P** corresponding to the source terms and initial data $\{f_i, \vartheta_{0i}, \chi_{0i}, \eta_{0i}\}$, and denote their differences by $\{\vartheta, \chi, \xi, \eta\}$ and $\{f, \vartheta_0, \chi_0, \eta_0\}$, respectively. Introduce the new variables $\omega_i = \vartheta_i + \lambda(\chi_i)$, $i = 1, 2$, and set $\omega = \omega_1 - \omega_2$ and $\tilde{\lambda} = \lambda(\chi_1) - \lambda(\chi_2)$. Then, according to Definition 3.1, the quadruplet $(\vartheta, \chi, \xi, \eta)$ fulfills the system

$$\begin{aligned}\langle\langle \partial_t \omega, v\rangle\rangle_V + \langle \omega, v\rangle_H = &-\int_0^\infty \nu_0 \nu(\sigma)\langle \eta(\sigma), v\rangle_H \, d\sigma \\ &- \int_0^\infty \mu(\sigma)\langle \nabla\eta(\sigma), \nabla v\rangle_{H^3}\, d\sigma + \langle \tilde{\lambda}, v\rangle_H + \langle f, v\rangle_H \quad \forall v \in V, \text{ a.e. in } I, \end{aligned} \tag{4.1}$$

$$\begin{aligned}\partial_t \chi - \Delta\chi + \xi = {}&\gamma(\chi_1) - \gamma(\chi_2) \\ &+ \lambda'(\chi_1)(\omega_1 - \lambda(\chi_1)) - \lambda'(\chi_2)(\omega_2 - \lambda(\chi_2)) \quad \text{a.e. in } \Omega \times I,\end{aligned} \tag{4.2}$$

$$\begin{aligned}&\int_0^\infty \nu(\sigma)\langle \partial_t\eta(\sigma) + \partial_s\eta(\sigma), \zeta(\sigma)\rangle_H \, d\sigma + \int_0^\infty \mu(\sigma)\langle \partial_t\eta(\sigma) + \partial_s\eta(\sigma), \zeta(\sigma)\rangle_H \, d\sigma \\ &\qquad - \int_0^\infty \mu(\sigma)\langle \partial_t\eta(\sigma) + \partial_s\eta(\sigma), \Delta\zeta(\sigma)\rangle_H \, d\sigma \\ &= \int_0^\infty \nu(\sigma)\langle \vartheta, \zeta(\sigma)\rangle_H \, d\sigma + \int_0^\infty \mu(\sigma)\langle \vartheta, \zeta(\sigma)\rangle_H \, d\sigma - \int_0^\infty \mu(\sigma)\langle \vartheta, \Delta\zeta(\sigma)\rangle_H \, d\sigma \\ &\qquad \forall \zeta \in L^2_\nu(\mathbb{R}^+, H) \cap L^2_\mu(\mathbb{R}^+, W), \text{ a.e. in } I\end{aligned} \tag{4.3}$$

with initial conditions (3.11)–(3.13).

Multiplying (4.2) by χ and integrating over Ω, we deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\chi\|_H^2 = {}&-\|\nabla\chi\|^2_{H^3} - \langle \xi, \chi\rangle_H + \langle \gamma(\chi_1) - \gamma(\chi_2), \chi\rangle_H + \langle \lambda'(\chi_2)\omega, \chi\rangle_H \\ &- \langle \lambda'(\chi_2)\tilde{\lambda}, \chi\rangle_H + \langle (\lambda'(\chi_1) - \lambda'(\chi_2))(\omega_1 - \lambda(\chi_1)), \chi\rangle_H.\end{aligned} \tag{4.4}$$

Using (H2)–(H3) we get at once that

$$-\langle \xi, \chi\rangle_H \le 0 \tag{4.5}$$

and

$$\langle \gamma(\chi_1) - \gamma(\chi_2), \chi\rangle_H \le \Gamma\|\chi\|_H^2 \tag{4.6}$$

whereas (H4)–(H5) and (2.3) imply that

$$\langle \lambda'(\chi_2)\omega, \chi\rangle_H - \langle \lambda'(\chi_2)\tilde{\lambda}, \chi\rangle_H \le \frac{\Lambda_0}{2}\|\omega\|_H^2 + \left(\frac{\Lambda_0}{2} + \Lambda_0^2\right)\|\chi\|_H^2. \tag{4.7}$$

To handle the last term in (4.4), notice that for every $\kappa > 0$ there exists $K(\kappa) > 0$ such that

$$\|v\|_{L^4(\Omega)}^2 \leq \kappa \|v\|_V^2 + K(\kappa)\|v\|_H^2 \quad \forall v \in V.$$

Indeed, the above relation follows quite directly from the Gagliardo-Nirenberg inequality (2.4) with $\rho = 1/4$, and the generalized Young inequality (2.3) with $p = 4/3, q = 4$, and $K(\kappa) = \kappa + \frac{27k^8}{256}\frac{1}{\kappa^3}$. Thus, using (H5), the Hölder inequality, and the fact that

$$\sup_{t\in I}(\|\omega_1(t)\|_H + \|\lambda(\chi_1(t))\|_H) = C_1 = C_1(C_0) < \infty$$

by force of (3.14), we obtain

$$\begin{aligned}\langle(\lambda'(\chi_1) - \lambda'(\chi_2))(\omega_1 - \lambda(\chi_1)), \chi\rangle_H &\leq \Lambda_1(\|\omega_1\|_H + \|\lambda(\chi_1)\|_H)\|\chi\|_{L^4(\Omega)}^2 \\ &\leq \kappa\Lambda_1 C_1\|\chi\|_V^2 + K(\kappa)\Lambda_1 C_1\|\chi\|_H^2 \\ &= \frac{1}{2}\|\nabla\chi\|_{H^3}^2 + (1 + 2K(\kappa_0)\Lambda_1 C_1)\|\chi\|_H^2\end{aligned} \tag{4.8}$$

upon choosing $\kappa_0 = (2\Lambda_1 C_1)^{-1}$ with $\Lambda_1 = \|\lambda''\|_{L^\infty(\mathbb{R})}$. Therefore, a view of (4.5)–(4.8), equality (4.4) leads to

$$\frac{d}{dt}\|\chi\|_H^2 + \|\nabla\chi\|_{H^3}^2 \leq \Lambda_0^2\|\omega\|_H^2 + 2(1 + \Gamma + \Lambda_0 + 2\Lambda_0^2 + 2K(\kappa_0)\Lambda_1 C_1)\|\chi\|_H^2. \tag{4.9}$$

Hence, picking $C_2 = C_2(C_0) > 0$ large enough, and integrating in time (4.9) from τ to $t \leq T$, we end up with

$$\|\chi(t)\|_H^2 + \int_\tau^t \|\chi(y)\|_V^2\, dy \leq \|\chi(\tau)\|_H^2 + C_2\int_\tau^t \|\omega(y)\|_H^2\, dy + C_2\int_\tau^t \|\chi(y)\|_H^2\, dy. \tag{4.10}$$

We now turn attention to the other two equations. For the moment, let us assume that ω has more space regularity, precisely, $\omega \in L^\infty(I, V)$. Set then $v = \omega$ in (4.1) to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega\|_H^2 + \|\omega\|_H^2 = \int_0^\infty \mu(\sigma)\langle\eta(\sigma), \omega\rangle_H\, d\sigma - \int_0^\infty \nu_0\nu(\sigma)\langle\eta(\sigma), \omega\rangle_H\, d\sigma \\ - \int_0^\infty \mu(\sigma)\langle\eta(\sigma), \omega\rangle_V\, d\sigma + \langle\tilde{\lambda}, \omega\rangle_H + \langle f, \omega\rangle_H.\end{aligned}$$

Recalling (K5) and using repeatedly (2.1), (2.3), and (H4)–(H5), the above elation entails

$$\begin{aligned}\frac{d}{dt}\|\omega\|_H^2 \leq (k_0 + a_0 - 1)\|\omega\|_H^2 + \Lambda_0^2\|\chi\|_H^2 + \|\eta\|_{\mathcal{M}}^2 \\ - 2\int_0^\infty \mu(\sigma)\langle\eta(\sigma), \omega\rangle_V\, d\sigma + 2\|f\|_H\|\omega\|_H\end{aligned} \tag{4.11}$$

where $\Lambda_0 = \|\lambda'\|_{L^\infty(\mathbb{R})}$. From (4.3) and the assumed regularity of ω, we read that $\partial_t\eta + \partial_s\eta \in L^\infty(I, \mathcal{M})$. Let then $\rho_\varepsilon \in \mathcal{D}((0,\varepsilon), \mathbb{R}^+)$ of L^1-norm equal to one, and introduce the mollified approximation of η

$$\eta_\varepsilon^t(s) = (\eta^t * \rho_\varepsilon)(s) = \int_0^s \eta^t(y)\rho_\varepsilon(s - y)\, dy.$$

Notice that $\partial_s\eta_\varepsilon \in \mathcal{M}$ and $\eta_\varepsilon(0) = 0$. Furthermore,

$$\|\eta_\varepsilon^t - \eta^t\|_{\mathcal{M}} = h_\varepsilon(t) \to 0 \quad \text{as } \varepsilon \to 0$$

for every fixed $t \in I$. Moreover, since $\eta \in L^\infty(I, \mathcal{M})$, h_ε turns out to be uniformly bounded. Integration by parts and (K3) yield (see [21] for the details)

$$\begin{aligned}\langle \partial_t \eta_\varepsilon + \partial_s \eta_\varepsilon, \eta_\varepsilon \rangle_{\mathcal{M}} &= \frac{1}{2}\frac{d}{dt}\|\eta_\varepsilon\|^2_{\mathcal{M}} - \frac{1}{2}\int_0^\infty \mu'(\sigma)\|\eta_\varepsilon(\sigma)\|^2_V\, d\sigma - \frac{1}{2}\int_0^\infty \nu'(\sigma)\|\eta_\varepsilon(\sigma)\|^2_H\, d\sigma \\ &\geq \frac{1}{2}\frac{d}{dt}\|\eta_\varepsilon\|^2_{\mathcal{M}}.\end{aligned}$$

Hence, the regularized version of (4.3) reads

$$\langle \partial_t \eta_\varepsilon + \partial_s \eta_\varepsilon, \zeta \rangle_{\mathcal{M}} = \langle (\omega - \tilde{\lambda}) * \rho_\varepsilon, \zeta \rangle_{\mathcal{M}} \quad \forall \zeta \in \mathcal{M}, \text{ a.e. in } I$$

and taking $\zeta = \eta_\varepsilon$, we are led to

$$\begin{aligned}\frac{d}{dt}\|\eta_\varepsilon\|^2_{\mathcal{M}} &\leq 2\langle (\omega - \tilde{\lambda}) * \rho_\varepsilon, \eta_\varepsilon \rangle_{\mathcal{M}} \\ &= 2\langle (\omega - \tilde{\lambda}), \eta_\varepsilon \rangle_{\mathcal{M}} \\ &\leq 2\int_0^\infty \mu(\sigma)\langle \eta(\sigma), \omega \rangle_V\, d\sigma - 2\int_0^\infty \mu(\sigma)\langle \eta_\varepsilon(\sigma), \tilde{\lambda} \rangle_V\, d\sigma \\ &\quad + 2\int_0^\infty \nu(\sigma)\langle \eta_\varepsilon(\sigma), \omega \rangle_H\, d\sigma - 2\int_0^\infty \nu(\sigma)\langle \eta_\varepsilon(\sigma), \tilde{\lambda} \rangle_H\, d\sigma + k_\varepsilon(t)\end{aligned} \tag{4.12}$$

where

$$k_\varepsilon(t) = 2h_\varepsilon(t)\sup_{t \in I}\|\omega(t)\|_{\mathcal{M}}.$$

By (2.1), (2.3), (K5), and (H4)–(H5), we see at once that

$$\begin{aligned}&2\int_0^\infty \nu(\sigma)\langle \eta_\varepsilon(\sigma), \omega \rangle_H\, d\sigma - 2\int_0^\infty \nu(\sigma)\langle \eta_\varepsilon(\sigma), \tilde{\lambda} \rangle_H\, d\sigma \\ &\qquad \leq 4\int_0^\infty \nu(\sigma)\|\eta_\varepsilon(\sigma)\|^2_H\, d\sigma + 2a_0\|\omega\|^2_H + 2a_0\Lambda_0^2\|\chi\|^2_H.\end{aligned} \tag{4.13}$$

Concerning the second integral of the right-hand side of (4.12), since

$$\nabla\tilde{\lambda} = (\lambda'(\chi_1) - \lambda'(\chi_2))\nabla\chi_1 - \lambda'(\chi_2)\nabla\chi,$$

exploiting again (H4)–(H5) we have

$$\begin{aligned}\|\nabla(\lambda(\chi_1) - \lambda(\chi_2))\|^2_{H^3} &\leq 2\Lambda_1^2(\|\chi\nabla\chi_1\|^2_{H^3} + \|\nabla\chi\|^2_{H^3}) \\ &\leq 2\Lambda_1^2(\|\chi\|^2_{L^4(\Omega)}\|\nabla\chi_1\|^2_{(L^4(\Omega))^3} + \|\nabla\chi\|^2_{H^3}).\end{aligned}$$

Thus, by virtue of (2.3) and the continuous embedding $L^4(\Omega) \hookrightarrow V$, we conclude that

$$-2\int_0^\infty \nu(\sigma)\langle \eta_\varepsilon(\sigma), \tilde{\lambda} \rangle_V\, d\sigma \leq C_3(1 + \|\chi_1\|_W)\|\chi\|_V\|\eta_\varepsilon\|_{\mathcal{M}} \tag{4.14}$$

for some $C_3 > 0$. Introduce now, for any $t \in I$,

$$\begin{aligned}\Phi_\varepsilon^2(t) &= \|\omega(t)\|^2_H + \|\eta_\varepsilon^t\|^2_{\mathcal{M}}, \\ \Phi^2(t) &= \|\omega(t)\|^2_H + \|\eta^t\|^2_{\mathcal{M}}.\end{aligned}$$

Clearly, $\Phi_\varepsilon^2(t) \to \Phi^2(t)$ as $\varepsilon \to 0$ for every $t \in I$. Adding (4.11) and (4.12), with (4.13)–(4.14) in mind, and picking $C_4 = C_4(C_0) > 0$ large enough, we obtain the inequality

$$\frac{d}{dt}\Phi_\varepsilon^2(t) \leq C_4\Phi_\varepsilon^2(t) + C_4\|\chi(t)\|_H^2 + k_\varepsilon(t) + \rho(t)\|\chi(t)\|_V\Phi_\varepsilon(t) + 2\|f(t)\|_H\Phi_\varepsilon(t)$$

for a.e. $t \in I$, where we set

$$\rho(t) = C_4(1 + \|\chi_1(t)\|_W).$$

Notice that by (3.14)

$$\|\rho\|_{L^2(I)}^2 = C_5 = C_5(C_0) < \infty.$$

Lemma 2.3, (2.1), and the Hölder inequality then lead to

$$\Phi_\varepsilon^2(t) \leq 2e^{C_4(T-\tau)}\Phi_\varepsilon^2(\tau) + 2(C_4e^{C_4(T-\tau)} + C_5e^{2C_4(T-\tau)})\int_\tau^t \|\chi(y)\|_V^2\,dy$$
$$+ 2e^{C_4(T-\tau)}\int_\tau^T k_\varepsilon(y)\,dy + 8e^{2C_4(T-\tau)}\|f\|_{L^1(I,H)}^2 \quad \forall t \in I. \tag{4.15}$$

Letting $\varepsilon \to 0$, and applying the dominated convergence theorem, we conclude that

$$\Phi^2(t) \leq C_6\Phi^2(\tau) + C_6\int_\tau^t \|\chi(y)\|_V^2\,dy + C_6\|f\|_{L^1(I,H)}^2 \quad \forall t \in I \tag{4.16}$$

for some $C_6 = C_6(C_0,T) > 0$. If ω does not have the required space regularity, we can overcome the obstacle by introducing a regularization. Following [15, Appendix], let $J = -\Delta + \mathbb{1}$ denote the Riesz map from V onto V^*. For every $\alpha > 0$ consider the positive operator $A_\alpha = (\mathbb{1} + \alpha^2 J)^{-1}$. The relations

$$\|A_\alpha u\|_H \leq \|u\|_H \quad \text{and} \quad \|A_\alpha u - u\|_H \to 0 \quad \text{as } \alpha \to 0 \tag{4.17}$$

hold for every $u \in H$. Then we repeat the calculations leading to (4.11) and (4.12), except we now take $v = A_\alpha^2\omega$ in equation (4.1) and we multiply by $A_\alpha^2\eta_\varepsilon$ the regularized version of (4.3). With no substantial changes, and exploiting (4.17), we end up with (4.16), with $\{\omega,\eta\}$ replaced by $\{A_\alpha\omega, A_\alpha\eta\}$. Then, taking the limit as $\alpha \to 0$, we recover (4.16) for the original quantities.

To conclude the proof, let $M > C_6 + 1$, and set

$$\Psi^2(t) = \|\omega(t)\|_H^2 + M\|\chi(t)\|_H^2 + \|\eta^t\|_{\mathcal{M}}^2.$$

Addition of M-times (4.10) and (4.16) entails

$$\Psi^2(t) + \int_\tau^t \|\chi(y)\|_V^2\,dy \leq C_7\Psi^2(\tau) + C_7\int_\tau^t \Psi^2(y)\,dy + C_7\|f\|_{L^1(I,H)}^2$$

for every $t \in I$ and some $C_7 = C_7(C_0,T) > 0$. The Gronwall Lemma in the integral form then yields

$$\Psi^2(t) \leq C_7(\Psi^2(\tau) + \|f\|_{L^1(I,H)}^2)e^{C_7(T-\tau)}.$$

Finally, from (2.3) and (H4)–(H5),

$$\frac{1}{2}\|\vartheta\|_H^2 - 2\Lambda_0^2\|\chi\|_H^2 \leq \|\omega\|_H^2 \leq \frac{1}{2}\|\vartheta\|_H^2 + 2\Lambda_0^2\|\chi\|_H^2$$

and the desired inequality (3.15) follows at once taking $M > 2\Lambda_0^2$, which we may stipulate. In particular, when the two sets of data coincide, we get $\omega = \chi = \eta \equiv 0$ (and therefore $\vartheta_1 \equiv \vartheta_2$), and from (4.2) we deduce that $\xi \equiv 0$.

5. Proof of Theorem 3.3. Thanks to Theorem 3.2, we just need to prove the existence of a solution. We do that by means of a Faedo-Galerkin approximation scheme. Just for the sake of simplicity, we suppose $\Gamma_D \equiv \varnothing$ so that $V_0 \equiv V$ and $W_0 \equiv W$. Also, we let $\nu_0 = 1$. However, the proof can be easily adapted to the other cases. Assume for the moment that ϕ is convex, nonnegative, and continuously differentiable, with $\phi' = \beta$ Lipschitz continuous.

5.1. *Faedo-Galerkin approximation.* In order to prove the existence result we follow a Faedo-Galerkin method (cf. [14]). Let $\{v_j\}_{j=1}^{\infty}$ be a smooth orthonormal basis of H that is also orthogonal in V. For instance, take a complete set of normalized eigenfunctions for $-\Delta$ in V with Neumann boundary conditions, that is,

$$\begin{aligned} -\Delta v_j &= \alpha_j v_j \quad && \text{in } \Omega, \\ \partial_{\mathbf{n}} v_j &= 0 && \text{on } \partial\Omega, \end{aligned}$$

α_j being the eigenvalue corresponding to v_j. Next we select an orthonormal basis $\{\zeta_j\}_{j=1}^{\infty}$ of $\mathcal{M}$ that also belongs to $\mathcal{D}(\mathbb{R}^+, W)$. Here we assume for simplicity that ν and μ are strictly positive. If $s_\nu = \sup\{s : \nu(s) > 0\}$ and $s_\mu = \sup\{s : \mu(s) > 0\}$, we just replace $\mathcal{D}(\mathbb{R}^+, W)$ with $\mathcal{D}((0, s_m) \cup (s_m, s_M), W)$, where $s_m = \min\{s_\nu, s_\mu\}$ and $s_M = \max\{s_\nu, s_\mu\}$. This position is relevant in subsection 5.4, where the interested reader will have no difficulties in realizing the minor modification of the argument needed in that case.

Given an integer n, denote by P_n and Q_n the projections on the subspaces

$$V_n = \mathrm{Span}\{v_1, \dots, v_n\} \subset V \quad \text{and} \quad \mathcal{M}_n = \mathrm{Span}\{\zeta_1, \dots, \zeta_n\} \subset \mathcal{M}$$

respectively.

We are now ready to introduce the sequence of approximating problems.

Problem $\mathbf{P}_n$. Find $t_n \in (\tau, T]$ and $a_j, b_j, c_j \in W^{1,1}((\tau, t_n))$ $(j = 1, \dots, n)$, such that, setting

$$\vartheta_n(t) = \sum_{j=1}^{n} a_j(t) v_j, \quad \chi_n(t) = \sum_{j=1}^{n} b_j(t) v_j, \quad \eta_n^t(s) = \sum_{j=1}^{n} c_j(t) \zeta_j(s),$$

the triplet $(\vartheta_n, \chi_n, \eta_n)$ satisfies

$$\begin{aligned} &\vartheta_n \in W^{1,1}((\tau, t_n), V), \\ &\chi_n \in W^{1,1}((\tau, t_n), W), \\ &\eta_n \in W^{1,1}((\tau, t_n), L^2_\nu(\mathbb{R}^+, W) \cap L^2_\mu(\mathbb{R}^+, W)), \end{aligned}$$

and fulfills the system

$$\langle\langle \partial_t(\vartheta_n + \lambda(\chi_n)), v\rangle\rangle_V + \langle \vartheta_n, v\rangle_H + \int_0^\infty \nu(\sigma)\langle \eta_n(\sigma), v\rangle_H \, d\sigma + \int_0^\infty \mu(\sigma)\langle \nabla\eta_n(\sigma), \nabla v\rangle_{H^3} \, d\sigma = \langle f(t), v\rangle_H,$$

$$\langle\langle \partial_t \chi_n, v\rangle\rangle_V + \langle \nabla\chi_n, \nabla v\rangle_{H^3} + \langle \beta(\chi_n) - \gamma(\chi_n) - \lambda'(\chi_n)\vartheta_n, v\rangle_H = 0,$$
$$\langle \partial_t \eta_n, \zeta\rangle_{\mathcal{M}} + \langle \partial_s \eta_n, \zeta\rangle_{\mathcal{M}} = \langle \vartheta_n, \zeta\rangle_{\mathcal{M}}$$

for every $v \in V_n$ and $\zeta \in \mathcal{M}_n$, and almost everywhere in (τ, t_n), with initial conditions

$$\begin{aligned} \vartheta_n(\tau) &= \vartheta_{0n} = P_n\vartheta_0, \\ \chi_n(\tau) &= \chi_{0n} = P_n\chi_0 \quad \text{a.e. in } \Omega, \\ \eta_n^\tau &= \eta_{0n} = Q_n\eta_0 \quad \text{a.e. in } \Omega \times \mathbb{R}^+. \end{aligned}$$

Owing to standard ODE results, problem $\mathbf{P}_n$ admits a unique solution. Moreover, the a priori estimates proved in the next subsection imply that, in fact, $t_n = T$.

5.2. *A priori estimates.* In this subsection, c will denote a positive generic constant independent of n. Consider problem $\mathbf{P}_n$. Take $v = \vartheta_n$ in the equation for ϑ_n and $v = \partial_t\chi_n$ in the equation for χ_n. We get, respectively,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\vartheta_n\|_H^2 = -\|\vartheta_n\|_H^2 - \langle \lambda'(\chi_n)\partial_t\chi_n, \vartheta_n\rangle_H + \int_0^\infty \mu(\sigma)\langle \eta_n(\sigma), \vartheta_n\rangle_H \, d\sigma \\ - \int_0^\infty \nu(\sigma)\langle \eta_n(\sigma), \vartheta_n\rangle_H \, d\sigma - \int_0^\infty \mu(\sigma)\langle \eta_n(\sigma), \vartheta_n\rangle_V \, d\sigma + \langle f, \vartheta_n\rangle_H \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla\chi_n\|_{H^3}^2 = -\|\partial_t\chi_n\|_H^2 - \langle \beta(\chi_n), \partial_t\chi_n\rangle_H + \langle \gamma(\chi_n), \partial_t\chi_n\rangle_H \\ + \langle \lambda'(\chi_n)\vartheta_n, \partial_t\chi_n\rangle_H. \end{aligned} \tag{5.2}$$

Take now $\zeta = \eta_n$ in the equation for η_n, obtaining

$$\frac{1}{2}\frac{d}{dt}\|\eta_n\|_{\mathcal{M}}^2 = -\langle \partial_s\eta_n, \eta_n\rangle_{\mathcal{M}} + \int_0^\infty \nu(\sigma)\langle \eta_n(\sigma), \vartheta_n\rangle_H \, d\sigma + \int_0^\infty \mu(\sigma)\langle \eta_n(\sigma), \vartheta_n\rangle_V \, d\sigma. \tag{5.3}$$

Adding (5.1)–(5.3), by force of (2.1), (2.3), (H3), and taking (K5) into account, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\vartheta_n\|_H^2 + \|\nabla\chi_n\|_{H^3}^2 + \|\eta_n\|_{\mathcal{M}}^2 + 2\int_\Omega \phi(\chi_n)\right) \\ &\quad = -\|\vartheta_n\|_H^2 - \|\partial_t\chi_n\|_H^2 + \langle f, \vartheta_n\rangle_H + \langle \gamma(\chi_n), \partial_t\chi_n\rangle_H \\ &\qquad + \int_0^\infty \mu(\sigma)\langle \eta_n(\sigma), \vartheta_n\rangle_H \, d\sigma - \langle \partial_s\eta_n, \eta_n\rangle_{\mathcal{M}} \\ &\quad \le c(1 + \|\vartheta_n\|_H^2 + \|\chi_n\|_V^2 + \|\eta_n\|_{\mathcal{M}}^2) + \|f\|_H\|\vartheta_n\|_H - \frac{3}{4}\|\partial_t\chi_n\|_H^2. \end{aligned} \tag{5.4}$$

In the last inequality we used (K3) together with an integration by parts to handle the term $\langle \partial_s \eta_n, \eta_n \rangle_{\mathcal{M}}$. Indeed (see [21] for the details)

$$\langle \partial_s \eta_n, \eta_n \rangle_{\mathcal{M}} = -\frac{1}{2} \int_0^\infty \nu'(\sigma) \|\eta_n(\sigma)\|_H^2 \, d\sigma - \frac{1}{2} \int_0^\infty \mu'(\sigma) \|\eta_n(\sigma)\|_V^2 \, d\sigma \geq 0.$$

Denote for simplicity

$$\Phi^2(t) = 1 + \|\vartheta_n(t)\|_H^2 + \|\chi_n(t)\|_V^2 + \|\eta_n^t\|_{\mathcal{M}}^2 + 2 \int_\Omega \phi(\chi_n(t)) + \int_\tau^t \|\partial_t \chi_n(y)\|_H^2 \, dy.$$

Adding (5.4) and the straightforward inequality

$$\frac{1}{2} \frac{d}{dt} \|\chi_n\|_H^2 \leq \frac{1}{4} \|\partial_t \chi_n\|_H^2 + c\|\chi_n\|_H^2,$$

we find

$$\frac{d}{dt} \Phi^2(t) \leq c\Phi^2(t) + 2\|f(t)\|_H \Phi(t) \quad \text{for a.e. } t \in I,$$

and Lemma 2.3 yields

$$\Phi^2(t) \leq 2e^{c(T-\tau)} \Phi^2(\tau) + 4e^{2c(T-\tau)} \left(\int_\tau^T \|f(y)\|_H \, dy \right)^2 \quad \forall t \in I.$$

We deduce then the a priori estimates

$$\|\vartheta_n\|_{L^\infty(I,H)} \leq c, \tag{5.5}$$

$$\|\chi_n\|_{L^\infty(I,V) \cap H^1(I,H)} \leq c, \tag{5.6}$$

$$\|\eta_n\|_{L^\infty(I,\mathcal{M})} \leq c, \tag{5.7}$$

$$\|\phi(\chi_n)\|_{L^\infty(I,L^1(\Omega))} \leq c \tag{5.8}$$

for some positive constant c independent of n.

Finally, take $v = -\Delta \chi_n$ in the equation for χ_n to get

$$\begin{aligned}\frac{1}{2} \frac{d}{dt} \|\nabla \chi_n\|_{H^3}^2 = &-\|\Delta \chi_n\|_H^2 - \langle \beta'(\chi_n) \nabla \chi_n, \nabla \chi_n \rangle_{H^3} \\ &- \langle \gamma(\chi_n), \Delta \chi_n \rangle_H - \langle \lambda'(\chi_n) \vartheta_n, \Delta \chi_n \rangle_H.\end{aligned}$$

Using the fact that $\beta'(r) \geq 0$ for any $r \in \mathbb{R}$ and repeating the above arguments, we find

$$\|\chi_n\|_{L^2(I,W)} \leq c. \tag{5.9}$$

5.3. *Contracting estimate.* Here we prove that, due to (3.15), $\{(\vartheta_n, \chi_n, \eta_n)\}$ is a Cauchy sequence in a suitable Banach space. In fact, owing to (5.5)–(5.6) and (5.9), this sequence satisfies the bound (3.14). Then, (3.15) entails

$$\begin{aligned}\|\vartheta_n(t) - \vartheta_m(t)\|_H^2 + \|\chi_n(t) - \chi_m(t)\|_H^2 + \int_\tau^t \|\chi_n(y) - \chi_m(y)\|_V^2 \, dy + \|\eta_n^t - \eta_m^t\|_{\mathcal{M}}^2 \\ \leq C(\|\vartheta_{0n} - \vartheta_{0m}\|_H^2 + \|\chi_{0n} - \chi_{0m}\|_H^2 + \|\eta_{0n} - \eta_{0m}\|_{\mathcal{M}}^2)\end{aligned} \tag{5.10}$$

for any $t \in I$.

5.4. *Passage to limit.* On account of the above estimates (5.5)–(5.7), (5.9)–(5.10), we have that, up to subsequences,

$$\begin{aligned} &\vartheta_n \to \vartheta \quad \text{in } C^0(I,H),\\ &\chi_n \to \chi \quad \text{weak* in } L^\infty(I,V),\\ &\chi_n \to \chi \quad \text{weak in } H^1(I,H)\cap L^2(I,W),\\ &\eta_n \to \eta \quad \text{in } C^0(I,\mathcal{M}). \end{aligned}$$

Using classical compactness arguments or, alternatively, estimate (3.15), we also infer that

$$\chi_n \to \chi \quad \text{in } C^0(I,H)\cap L^2(I,V). \tag{5.11}$$

It is then readily seen (cf. [14]) that Eq. (3.9) is satisfied by ϑ and χ as well, and that $\beta(\chi)\in L^2(I,H)$.

Concerning Eq. (3.10), we need a little bit of extra work. Fix an integer $m>0$ and choose $\zeta\in\mathcal{D}(I,\mathcal{D}(\mathbb{R}^+,W))$ of the form

$$\zeta^t(s)=\sum_{j=1}^m \tilde{c}_j(t)\zeta_j(s)$$

with $\tilde{c}_j\in\mathcal{D}(I,\mathbb{R})$. Then for every $n\geq m$ we have the equality

$$\langle\partial_t\eta_n,\zeta\rangle_{\mathcal{M}}+\langle\partial_s\eta_n,\zeta\rangle_{\mathcal{M}}=\langle\vartheta_n,\zeta\rangle_{\mathcal{M}} \quad \text{for a.e. } t\in I.$$

Note first that the above convergence of $\{\eta_n\}$ implies convergence in $\mathcal{D}'(I,\mathcal{D}'(\mathbb{R}^+,W^*))$ and this yields

$$\langle\langle\partial_t\eta+\partial_s\eta,\zeta\rangle\rangle_{\mathcal{D}(I,\mathcal{D}(\mathbb{R}^+,W))}=\langle\langle\vartheta,\zeta\rangle\rangle_{\mathcal{D}(I,\mathcal{D}(\mathbb{R}^+,W))}.$$

Applying a density argument, we conclude that the equality $\partial_t\eta+\partial_s\eta=\vartheta$ holds in H, for almost any $t\in I$ and $s\in\mathbb{R}^+$. Therefore, (3.6) holds. Assume now for simplicity that $\nu\equiv 0$ (the argument being exactly the same in the general case). Integrations by parts and the above convergences yield

$$\begin{aligned} \int_\tau^T\langle\partial_t\eta_n,\zeta\rangle_{\mathcal{M}} &= -\int_\tau^T\langle\eta_n,\partial_t\zeta\rangle_{\mathcal{M}}\\ &\to -\int_\tau^T\langle\eta,\partial_t\zeta\rangle_{\mathcal{M}}\\ &= -\int_\tau^T\int_0^\infty\mu\langle\eta,\partial_t\zeta\rangle_H+\int_\tau^T\int_0^\infty\nu\langle\eta,\Delta\partial_t\zeta\rangle_H \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} \int_\tau^T\langle\vartheta_n,\zeta\rangle_{\mathcal{M}} &= \int_\tau^T\int_0^\infty\mu\langle\vartheta_n,\zeta\rangle_H-\int_\tau^T\int_0^\infty\mu\langle\vartheta_n,\Delta\zeta\rangle_H\\ &\to \int_\tau^T\int_0^\infty\mu\langle\vartheta,\zeta\rangle_H-\int_\tau^T\int_0^\infty\mu\langle\vartheta,\Delta\zeta\rangle_H. \end{aligned} \tag{5.13}$$

Further, observe that for every fixed $t_0\in I$,

$$\zeta^{t_0}\in H^1_\mu(\mathbb{R}^+,W)\cap L^2_{(\mu')^2/\mu}(\mathbb{R}^+,W).$$

Thus, integration by parts and the above convergences entail

$$\begin{aligned}
&\int_\tau^T \langle \partial_s, \eta_n, \zeta\rangle_{\mathcal{M}} \\
&\quad = -\int_\tau^T \int_0^\infty \mu'\langle \eta_n, \zeta\rangle_H + \mu\langle \eta_n, \partial_s\zeta\rangle_H + \int_\tau^T \int_0^\infty \mu'\langle \eta_n, \Delta\zeta\rangle_H + \mu\langle \eta_n, \Delta\partial_s\zeta\rangle_H \\
&\quad \to -\int_\tau^T \int_0^\infty \mu'\langle \eta, \zeta\rangle_H + \mu\langle \eta, \partial_s\zeta\rangle_H + \int_\tau^T \int_0^\infty \mu'\langle \eta, \Delta\zeta\rangle_H + \mu\langle \eta, \Delta\partial_s\zeta\rangle_H. \quad (5.14)
\end{aligned}$$

Exploiting the distributional inequality

$$\int_\tau^T \int_0^\infty \mu\langle \eta, \Delta\partial_t\zeta\rangle_H + \mu'\langle \eta, \Delta\zeta\rangle_H + \mu\langle \eta, \Delta\partial_s\zeta\rangle_H = -\int_\tau^T \int_0^\infty \mu\langle \partial_t\eta + \partial_s\eta, \Delta\zeta\rangle_H$$

and collecting (5.12)–(5.14), we conclude that

$$-\int_\tau^T \int_0^\infty \mu\langle \partial_t\eta + \partial_s\eta, \Delta\zeta\rangle_H = -\int_\tau^T \int_0^\infty \mu\langle \vartheta, \Delta\zeta\rangle_H$$

and analogously for the other term. Letting $m \to \infty$, and using a density argument, we get (3.10). Besides, it is straightforward to derive the equation

$$\begin{aligned}
\langle\langle \partial_t\vartheta, v\rangle\rangle_{\mathcal{D}(I,V)} + \int_\tau^T \langle \lambda'(\chi)\chi_t + \vartheta, v\rangle_H + \int_\tau^T \int_0^\infty \nu\langle \eta, v\rangle_H& \\
+ \int_\tau^T \int_0^\infty \mu\langle \eta, v\rangle_H + \int_\tau^T \int_0^\infty \mu\langle \nabla\eta, \nabla v\rangle_{H^3}& \\
= \int_\tau^T \langle f, v\rangle_H \quad \forall v \in \mathcal{D}(I, V)&,
\end{aligned}$$

and, by comparison, we deduce that

$$\partial_t\vartheta \in L^\infty(I, V^*) + L^1(I, H).$$

Therefore, the above equation can be set in the form (3.8). Applying Fatou's lemma to (5.8), we get that $\phi(\chi) \in L^\infty(I, L^1(\Omega))$. Of course, properties (3.1)–(3.6) and the initial conditions (3.11)–(3.13) easily follow from the above convergences and equations (3.8)–(3.10). Note that, due to the uniqueness, the whole sequence $\{(\vartheta_n, \chi_n, \eta_n)\}$ converges to the solution we found.

5.5. *Approximating β.* Assume now that ϕ and β satisfy (H1)–(H2). Arguing as in [14], for any $\varepsilon > 0$ we approximate ϕ and β with ϕ_ε and β_ε (cf., e.g., [7], Proposition 2.6 and Proposition 2.11), where β_ε is the Yosida approximation of β (and therefore Lipschitz continuous), and ϕ_ε is a nonnegative, convex, and continuously differentiable function such that

$$\beta_\varepsilon = \phi'_\varepsilon = \partial\phi_\varepsilon$$

and

$$0 \le \phi_\varepsilon(y) \uparrow \phi(y) \quad \text{as } \varepsilon \to 0. \quad (5.15)$$

We know that for every $\varepsilon > 0$ there is a solution $(\vartheta_\varepsilon, \chi_\varepsilon, \eta_\varepsilon)$ of problem $\mathbf{P}$ according to Definition 3.1 with β_ε in place of β. Nonetheless, estimate (5.5)–(5.8) are independent of ε. Thus, as in the former case, we can take weak and weak* limits as $\varepsilon \to 0$. The argument is treated in detail in [14], to which the interested reader is referred. We just recall that the main point is to recover a uniform bound for $\beta_\varepsilon(\chi_\varepsilon)$ in $L^2(\Omega \times I)$. This can be done via a comparison argument applied to the approximation of (3.9). Besides, let us mention how to prove that, calling χ the limit of χ_ε, (5.8) holds for χ as well. Indeed, from (5.11) we get that for every fixed $t \in I$, $\chi_\varepsilon \to \chi$ a.e. in Ω (up to a subnet). Setting $\varepsilon_0 > 0$, exploiting the continuity of ϕ_{ε_0}, Fatou's lemma, and (5.15),

$$\int_\Omega \phi_{\varepsilon_0}(\chi)\,dx \leq \liminf_{\varepsilon\to 0} \int_\Omega \phi_{\varepsilon_0}(\chi_\varepsilon)\,dx \leq \liminf_{\varepsilon\to 0} \int_\Omega \phi_\varepsilon(\chi_\varepsilon)\,dx \leq c \tag{5.16}$$

by force of (5.8). Consequently, an application of the monotone convergence theorem to (5.16) yields $\phi(\chi) \in L^\infty(I, L^1(\Omega))$. Finally, on account of (3.3)–(3.4) and using Lemma 3.3 in [7], we can deduce $\phi(\chi) \in W^{1,1}(I, L^1(\Omega))$.

6. Proof of Theorem 3.5.

According to the hypothesis, set

$$F = \sup_{f\in\mathcal{F}} \|f\|_{\mathcal{T}}^2 \tag{6.1}$$

and let $z_0 = (\vartheta_0, \chi_0, \eta_0) \in B_\phi(0, R)$, for some $R > 0$.

Let us examine first the case $\nu_0 = 1$, corresponding to assumption (1.1). We perform some a priori estimates, which clearly hold in a Faedo-Galerkin scheme. Thus, we can proceed formally. Namely, add (5.1)–(5.2), with (ϑ, χ, η) in place of $(\vartheta_n, \chi_n, \eta_n)$, to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\vartheta\|_H^2 + \|\nabla\chi\|_{H^3}^2 + 2\int_\Omega \phi(\chi)\,dx - 2\int_\Omega \hat\gamma(\chi)\,dx\right)\\ &= -\|\vartheta\|_H^2 - \|\partial_t\chi\|_H^2 - \int_0^\infty \nu(\sigma)\langle\eta(\sigma),\vartheta\rangle_H\,d\sigma - \int_0^\infty \mu(\sigma)\langle\nabla\eta(\sigma),\nabla\vartheta\rangle_{H^3}\,d\sigma + \langle f,\vartheta\rangle_H.\end{aligned} \tag{6.2}$$

Then multiply Eq. (3.9) by $\kappa\chi$, for $\kappa > 0$, and integrate over Ω, so obtaining

$$\frac{1}{2}\frac{d}{dt}\kappa\|\chi\|_H^2 = -\kappa\|\nabla\chi\|_{H^3}^2 - \kappa\langle\xi,\chi\rangle_H + \kappa\langle\gamma(\chi),\chi\rangle_H + \kappa\langle\lambda'(\chi)\vartheta,\chi\rangle_H \tag{6.3}$$

where $\xi \in \beta(\chi)$. Besides, define

$$\rho_{\mathcal{M}}(\eta) = \int_0^\infty \nu(\sigma)\|\eta(\sigma)\|_H^2\,d\sigma + \int_0^\infty \mu(\sigma)\|\nabla\eta(\sigma)\|_{H^3}^2\,d\sigma$$

and consider Eq. (3.10) in the strong sense, i.e., almost everywhere in $I \times \Omega \times \mathbb{R}^+$. Multiply that equation by $\nu\eta$, then take the gradient of both sides and multiply by $\mu\nabla\eta$, add the resulting equations and integrate on $\Omega \times \mathbb{R}^+$. This procedure yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\rho_{\mathcal{M}}(\eta) = &-\frac{1}{2}\int_0^\infty \nu(\sigma)\frac{d}{d\sigma}\|\eta(\sigma)\|_H^2\,d\sigma - \frac{1}{2}\int_0^\infty \mu(\sigma)\frac{d}{d\sigma}\|\nabla\eta(\sigma)\|_{H^3}^2\,d\sigma\\ &+\int_0^\infty \nu(\sigma)\langle\vartheta,\eta(\sigma)\rangle_H\,d\sigma + \int_0^\infty \mu(\sigma)\langle\nabla\vartheta,\nabla\eta(\sigma)\rangle_{H^3}\,d\sigma.\end{aligned} \tag{6.4}$$

Using (K4) and performing an integration by parts, we have that

$$\int_0^\infty \nu(\sigma)\frac{d}{d\sigma}\|\eta(\sigma)\|_H^2\,d\sigma + \int_0^\infty \mu(\sigma)\frac{d}{d\sigma}\|\nabla\eta(\sigma)\|_{H^3}^2\,d\sigma \geq \delta\rho_{\mathcal{M}}(\eta). \tag{6.5}$$

From (2.3) and (H4),

$$\kappa\langle\lambda'(\chi)\vartheta,\chi\rangle_H \le \frac{\kappa\Lambda_0^2}{2m_1}\|\vartheta\|_H^2 + \frac{\kappa m_1}{2}\|\chi\|_H^2. \tag{6.6}$$

Choose then $\kappa = m_1/\Lambda_0^2$ (if $\Lambda_0 = 0$, then κ can be chosen equal to 1). Adding (6.2)–(6.4) and taking (H11) and (6.5)–(6.6) into account, we infer that

$$\begin{aligned}
&\frac{d}{dt}\left(\|\vartheta\|_H^2 + \kappa\|\chi\|_H^2 + \|\nabla\chi\|_{H^3}^2 + \rho_{\mathcal{M}}(\eta) + 2\int_\Omega \phi(\chi) - 2\int_\Omega \hat{\gamma}(\chi)\right)\\
&\quad\le -\left(\|\vartheta\|_H^2 + \kappa m_1\|\chi\|_H^2 + 2\kappa\|\nabla\chi\|_{H^3}^2 + \delta\rho_{\mathcal{M}}(\eta) + 2\bar{\varepsilon}\kappa\int_\Omega \phi(\chi) - 2\bar{\varepsilon}\kappa\int_\Omega \hat{\gamma}(\chi)\right)\\
&\quad\quad + 2\kappa|\Omega|m_2 + 2\|f\|_H\|\vartheta\|_H
\end{aligned} \tag{6.7}$$

for every $\bar{\varepsilon} \le \varepsilon_0$. By force of (H12), set, for any $t \ge \tau$,

$$\begin{aligned}
\Phi^2(t) = {}& \|\vartheta(t)\|_H^2 + \kappa\|\chi(t)\|_H^2 + \|\nabla\chi(t)\|_{H^3}^2 + \rho_{\mathcal{M}}(\eta^t)\\
&+ 2\int_\Omega \phi(\chi(t)) - 2\int_\Omega \hat{\gamma}(\chi(t)) + 2|\Omega|L
\end{aligned}$$

and let

$$\varepsilon = \min\{\kappa\varepsilon_0, 1, m_1, 2\kappa, \delta\}$$

and choose

$$\bar{\varepsilon} = \frac{\varepsilon}{2\kappa}.$$

Denoting $L_0 = 2\varepsilon|\Omega|L + 2\kappa|\Omega|m_2$, m_2 being fixed by the choice of $\bar{\varepsilon}$, we get

$$\frac{d}{dt}\Phi^2(t) + \varepsilon\Phi^2(t) \le L_0 + 2\|f(t)\|_H\Phi(t) \tag{6.8}$$

for almost any $t > \tau$. Exploiting Lemma 2.4, and recalling (6.1), inequality (6.8) entails

$$\Phi^2(t) \le 2\Phi^2(\tau)e^{-\varepsilon(t-\tau)} + K_1 \tag{6.9}$$

for any $t \ge \tau$, with

$$K_1 = \frac{2L_0}{\varepsilon} + \frac{4e^{\varepsilon}F}{(1-e^{-\varepsilon/2})^2}.$$

Notice that, in view of (3.17), since $z_0 = (\vartheta_0, \chi_0, \eta_0) \in B_\phi(0,R)$,

$$\Phi^2(\tau) \le C_1(R) = 2R + \max\{1,\kappa,2\Gamma\}R^2 + 2|\Omega|L + |\Omega|\frac{|\gamma(0)|}{\Gamma}. \tag{6.10}$$

Denote

$$\begin{aligned}
\Psi^2(t) = {}& \|\vartheta(t)\|_H^2 + \kappa\|\chi(t)\|_H^2 + \|\nabla\chi(t)\|_{H^3}^2 + \|\eta^t\|_{\mathcal{M}}^2\\
&+ 2\int_\Omega \phi(\chi(t)) - 2\int_\Omega \hat{\gamma}(\chi(t)) + 2|\Omega|L
\end{aligned}$$

for any $t \geq \tau$, and repeat the same arguments leading to (6.8), the only difference being that we have to take the inner product in $\mathcal{M}$ of the strong version of Eq. (3.10) with η in place of (6.4) (see also Sec. 4). The result is the following inequality, similar to (6.8):

$$\frac{d}{dt}\Psi^2 + \varepsilon\Psi^2 \leq L_0 + 2\|f\|_H\Psi + 2\int_0^\infty \mu(\sigma)\langle\eta(\sigma), \vartheta\rangle_H \, d\sigma \tag{6.11}$$

for almost any $t \geq \tau$. On the other hand, recalling (K5) and (2.3), we have

$$2\int_0^\infty \mu(\sigma)\langle\eta(\sigma), \vartheta\rangle_H \, d\sigma \leq \frac{\varepsilon}{2}\|\eta\|^2_{\mathcal{M}} + \frac{2a_0}{\varepsilon}\|\vartheta\|^2_H \leq \frac{\varepsilon}{2}\Psi^2 + \frac{2a_0}{\varepsilon}\Phi^2. \tag{6.12}$$

Thus (6.9)–(6.12) entail

$$\frac{d}{dt}\Psi^2(t) + \frac{\varepsilon}{2}\Psi^2(t) \leq K_2 + 2\|f(t)\|_H\Psi(t) + C_2(R)e^{-\varepsilon(t-\tau)} \tag{6.13}$$

for almost any $t \geq \tau$, where we set

$$K_2 = L_0 + \frac{2a_0K_1}{\varepsilon} \quad \text{and} \quad C_2(R) = \frac{4a_0C_1(R)}{\varepsilon}.$$

Lemma 2.4 applied to (6.13) gives

$$\begin{aligned} \Psi^2(t) &\leq 2\Psi^2(\tau)e^{-(\varepsilon/2)(t-\tau)} + K_3 + 2C_2(R)\int_\tau^t e^{-\varepsilon(y-\tau)}e^{-(\varepsilon/2)(t-y)} \, dy \\ &\leq 2\left(\Psi^2(\tau) + \frac{2C_2(R)}{\varepsilon}\right)e^{-(\varepsilon/2)(t-\tau)} + K_3 \end{aligned} \tag{6.14}$$

for any $t \geq \tau$, with

$$K_3 = \frac{4K_2}{\varepsilon} + \frac{4e^{(\varepsilon/2)}F}{(1-e^{-\varepsilon/4})^2}.$$

Notice that (6.10) holds for Ψ as well; hence, setting

$$C_3(R) = 2\left(C_1(R) + \frac{2C_2(R)}{\varepsilon}\right)$$

and $M = \max\{1, 1/\kappa\}$, we rewrite (6.14) as

$$\|U_f(t,\tau)z_0\|^2_{\mathcal{H}} \leq M\Psi^2(t) \leq MC_3(R)e^{-(\varepsilon/2)(t-\tau)} + MK_3.$$

Choosing $R_0 = 2MK_3$, (3.18) is satisfied for

$$t^* = \max\left\{0, \frac{2}{\varepsilon}\log\frac{C_3(R)}{K_3}\right\}.$$

Finally, addition of (6.2) and (6.4), and integration from τ to t, lead to

$$2\int_\tau^t \|\partial_t\chi(y)\|^2_H \, dy \leq \Phi^2(\tau) + 2\int_\tau^t \|f(y)\|_H\Phi(y) \, dy.$$

Hence, if $\mathcal{F}$ is a bounded subset of $L^1(\mathbb{R}, H)$, then (3.19) follows at once from (6.10).

Consider now the case $\nu_0 = -1$, corresponding to assumption (1.2), together with $|\Gamma_D| > 0$ and (K6). In light of Remark 3.6, we work in $\widetilde{\mathcal{M}}$ rather than in $\mathcal{M}$. Arguing as in the former case (cf. (6.2)–(6.6), and notice that we no longer have the cancellation of the term $\int \nu\langle \eta, \vartheta\rangle_H$), we get

$$\begin{aligned}
&\frac{d}{dt}\left(\|\vartheta\|_H^2 + \kappa\|\chi\|_H^2 + \|\nabla\chi\|_{H^3}^2 + \|\eta\|_{\widetilde{\mathcal{M}}}^2 + 2\int_\Omega \phi(\chi) - 2\int_\Omega \hat{\gamma}(\chi)\right)\\
&\quad\le -\left((2-l)\|\vartheta\|_H^2 + \kappa m_1\|\chi\|_H^2 + 2\kappa\|\nabla\chi\|_{H^3}^2 + \delta\|\eta\|_{\widetilde{\mathcal{M}}}^2 + 2\bar{\varepsilon}\kappa\int_\Omega \phi(\chi) - 2\bar{\varepsilon}\kappa\int_\Omega \hat{\gamma}(\chi)\right)\\
&\qquad + 2\kappa|\Omega|m_2 + 2\|f\|_H\|\vartheta\|_H + 2\int_0^\infty \nu(\sigma)\|\eta(\sigma)\|_H\|\vartheta\|_H\,d\sigma.
\end{aligned} \tag{6.15}$$

Here we made the finer choice $\kappa = m_1(2-l)/\Lambda_0^2$ in (6.6), having set

$$l = 1 - \frac{2\delta_0}{\delta_0 + \delta}.$$

Observe now that, due to (2.2)–(2.3) and (K6),

$$\begin{aligned}
2\int_0^\infty \nu(\sigma)\|\eta(\sigma)\|_H\|\vartheta\|_H\,d\sigma &\le (2-2l)\|\vartheta\|_H^2 + \frac{a_0}{2-2l}\int_0^\infty \nu(\sigma)\|\eta(\sigma)\|_H^2\,d\sigma\\
&\le (2-2l)\|\vartheta\|_H^2 + \frac{\delta_0}{1-l}\|\vartheta\|_{\widetilde{\mathcal{M}}}^2\\
&= (2-2l)\|\vartheta\|_H^2 + \frac{\delta+\delta_0}{2}\|\eta\|_{\widetilde{\mathcal{M}}}^2.
\end{aligned} \tag{6.16}$$

Thus, collecting (6.15)–(6.16) we conclude that

$$\begin{aligned}
&\frac{d}{dt}\left(\|\vartheta\|_H^2 + \kappa\|\chi\|_H^2 + \|\nabla\chi\|_{H^3}^2 + \|\eta\|_{\widetilde{\mathcal{M}}}^2 + 2\int_\Omega \phi(\chi) - 2\int_\Omega \hat{\gamma}(\chi)\right)\\
&\quad\le -\left(l\|\vartheta\|_H^2 + \kappa m_1\|\chi\|_H^2 + 2\kappa\|\nabla\chi\|_{H^3}^2 + \frac{\delta-\delta_0}{2}\|\eta\|_{\widetilde{\mathcal{M}}}^2 + 2\bar{\varepsilon}\kappa\int_\Omega \phi(\chi) - 2\bar{\varepsilon}\kappa\int_\Omega \hat{\gamma}(\chi)\right)\\
&\qquad + 2\kappa|\Omega|m_2 + 2\|f\|_H\|\vartheta\|_H + 2\int_0^\infty \nu(\sigma)\|\eta(\sigma)\|_H\|\vartheta\|_H\,d\sigma.
\end{aligned}$$

Choosing now ε properly (mimicking the former case), we find a differential inequality like (6.8), with the difference that in this case Φ is equivalent to the norm of (ϑ, χ, η) in $\mathcal{H}$. Hence the proof is carried out by Lemma 2.4, and by an estimate similar to (6.10).

7. Proof of Proposition 3.8. Taking the notation of Sec. 4 into account, let us consider Eq. (4.2) and multiply both sides by $-\Delta\chi$. Integrating the resulting equation over $\Omega \times (\tau, t)$, it follows that

$$\begin{aligned}
&\frac{1}{2}\|\nabla\chi(t)\|^2_{H^3} + \int_\tau^t \|\Delta\chi(y)\|^2_H \, dy \\
&\quad = \frac{1}{2}\|\nabla\chi_0\|^2_{H^3} + \int_\tau^t \langle (\chi_1(y))^3 - (\chi_2(y))^3, \Delta\chi(y)\rangle_H \, dy \\
&\qquad - \int_\tau^t \langle \gamma(\chi_1(y)) - \gamma(\chi_2(y)), \Delta\chi(y)\rangle_H \, dy \\
&\qquad - \int_\tau^t \langle \lambda_0 \vartheta(y), \Delta\chi(y)\rangle_H \, dy \\
&\quad = \frac{1}{2}\|\nabla\chi_0\|^2_{H^3} + \sum_{j=1}^3 I_j(t).
\end{aligned} \tag{7.1}$$

Recalling (H3), thanks to Lemma 2.1, we easily obtain

$$I_2(t) + I_3(t) \leq c\left(\int_\tau^t \|\chi(y)\|^2_H + \|\vartheta(y)\|^2_H \, dy\right) + \frac{1}{4}\int_\tau^t \|\Delta\chi(y)\|^2_H \, dy. \tag{7.2}$$

On the other hand, still using Lemma 2.1, we get

$$I_1(t) \leq \int_\tau^t \|(\chi_1(y))^3 - (\chi_2(y))^3\|^2_H \, dy + \frac{1}{4}\int_\tau^t \|\Delta\chi(y)\|^2_H \, dy. \tag{7.3}$$

Observe now that, owing to the Hölder inequality and Lemma 2.2,

$$\begin{aligned}
&\int_\tau^t \|(\chi_1(y))^3 - (\chi_2(y))^3\|^2_H \, dy \\
&\quad \leq c\int_\tau^t \int_\Omega (|\chi_1|^4 + |\chi_2|^4 + 1)|\chi|^2 \\
&\quad \leq c\int_\tau^t (\|\chi_1(y)\|^4_{L^6(\Omega)} + \|\chi_2(y)\|^4_{L^6(\Omega)} + 1)\|\chi(y)\|^2_{L^6(\Omega)} \, dy \\
&\quad \leq c\int_\tau^t (\|\chi_1(y)\|^4_V + \|\chi_2(y)\|^4_V + 1)\|\chi(y)\|^2_V \, dy
\end{aligned} \tag{7.4}$$

where c indicates a positive constant that may vary from line to line. Then, on account of bound (3.18), we infer that

$$\int_\tau^t \|(\chi_1(y))^3 - (\chi_2(y))^3\|^2_H \, dy \leq c\int_\tau^t \|\chi(y)\|^2_V \, dy. \tag{7.5}$$

Combining (7.2)–(7.3), (7.5) with (7.1), we deduce that

$$\begin{aligned}
&\frac{1}{2}\|\nabla\chi(t)\|^2_{H^3} + \frac{1}{2}\int_\tau^t \|\Delta\chi(y)\|^2_H \, dy \\
&\qquad \leq \frac{1}{2}\|\nabla\chi_0\|^2_{H^3} + c\int_\tau^t (\|\chi(y)\|^2_V + \|\vartheta(y)\|^2_H) \, dy.
\end{aligned} \tag{7.6}$$

Finally, thanks to (3.15), from (7.6) we derive (3.21).

8. Proof of Theorem 3.9. Consider again the Faedo-Galerkin approximating scheme used in Sec. 5 and observe that a priori estimates (5.5)–(5.8) hold in this case as well. Of course, (H4) and (H16) do not allow us to recover (5.9). Anyway, following the proof of Lemma 3.2 in [11], we can obtain the bounds

$$\|\beta(\chi_n)\|_{L^2(I,L^{3/2}(\Omega))} \le c, \tag{8.1}$$

$$\|\chi_n\|_{L^2(I,W^{2,3/2}(\Omega))} \le c \tag{8.2}$$

for some $c > 0$ independent of n. Estimates (5.5)–(5.7) and (8.1)–(8.2) lead to the convergences

$$\vartheta_n(t) \to \vartheta \quad \text{weak* in } L^\infty(I,H), \tag{8.3}$$

$$\chi_n \to \chi \quad \text{weak* in } L^\infty(I,V),\ \text{weak in } H^1(I,H)\cap L^2(I,W^{2,3/2}(\Omega)), \tag{8.4}$$

$$\beta(\chi_n) \to \xi \quad \text{weak in } L^2(I,L^{3/2}(\Omega)), \tag{8.5}$$

$$\eta_n \to \eta \quad \text{weak* in } L^\infty(I,\mathcal{M}), \tag{8.6}$$

and we can still deduce the strong convergence (5.11) which implies $\xi = \beta(\chi)$. In addition, (H4), (H16), (5.11), and (8.3)–(8.4) entail

$$\lambda'(\chi_n)\vartheta_n \to \lambda'(\chi)\vartheta \qquad \text{weak in } L^2(I,L^{3/2}(\Omega)), \tag{8.7}$$

$$\lambda'(\chi_n)\partial_t\chi_n \to \lambda'(\chi)\partial_t\chi \quad \text{weak in } L^1(I,L^{3/2}(\Omega)), \tag{8.8}$$

and using Fatou's lemma and (5.8), we infer (3.27).

Summing up, by (5.8), (5.11), and (8.3)–(8.8), we can easily show that (ϑ,χ,η) fulfills (3.2), (3.7)–(3.12), and (3.22)–(3.27), whenever β is Lipschitz continuous. If β satisfies (H1)–(H2) only, then we can argue as in subsection 5.5, taking into account that β induces a maximal monotone operator from $L^2(I,L^3(\Omega))$ to $L^2(I,L^{3/2}(\Omega))$ (cf. proof of Theorem 2.3 in [11]).

Regarding uniqueness, observe that the pair (χ,ξ) solves the Cauchy-Neumann problem (3.7), (3.9), (3.12), (3.24). If $\chi_0 \in L^\infty(\Omega)$, taking advantage of Lemma 3.1 and Lemma 3.3 in [12], we deduce that (χ,ξ) enjoys (3.4) and (3.28). Then, thanks to the boundedness of χ, to get uniqueness we can just exploit the same argument used in Sec. 4 to deduce (3.15) without using the boundedness of λ' (cf. (H5)). Also, we can take advantage of the contracting estimate (5.10) to prove that (3.1), (3.5), and (3.13) are satisfied.

9. Proof of Theorem 3.10. Also in this proof we suppose $\Gamma_D \equiv \varnothing$, just for the sake of simplicity. Theorems 3.2 and 3.3 allow us to construct a family $\{(\vartheta_\varepsilon,\chi_\varepsilon,\xi_\varepsilon,\eta_\varepsilon)\}$ of solutions to problem **P** from which we can extract a subsequence converging to the solution we are looking for. Suppose for the moment that β is a Lipschitz continuous function and take $\{\chi_{0\varepsilon}\}$ such that

$$\chi_{0\varepsilon} \in V \quad \forall\varepsilon > 0, \tag{9.1}$$

$$\int_\Omega \phi(\chi_{0\varepsilon}) \le \int_\Omega \phi(\chi_0) \quad \forall\varepsilon > 0, \tag{9.2}$$

$$\chi_{0\varepsilon} \to \chi_0 \quad \text{in } H. \tag{9.3}$$

For the existence of a sequence of this sort, the reader is referred to Sec. 3 of [15].

Thanks to our assumptions and (9.1)–(9.2), Theorem 3.3 applies and we obtain a unique quadruplet $(\vartheta_\varepsilon, \chi_\varepsilon, \xi_\varepsilon, \eta_\varepsilon)$ that solves $\mathbf{P}$ according to Definition 3.1 with χ_0 substituted by $\chi_{0\varepsilon}$.

Set now $\omega_\varepsilon = \vartheta_\varepsilon + \lambda(\chi_\varepsilon)$ and, recalling (4.1)–(4.3), observe that

$$\langle\langle \partial_t \omega_\varepsilon, v\rangle\rangle_V + \langle \omega_\varepsilon, v\rangle_H = -\int_0^\infty \nu_0 \nu(\sigma)\langle \eta_\varepsilon(\sigma), v\rangle_H \, d\sigma$$

$$-\int_0^\infty \mu(\sigma)\langle \nabla\eta_\varepsilon(\sigma), \nabla v\rangle_{H^3} \, d\sigma + \langle \lambda(\chi_\varepsilon), v\rangle_H + \langle f, v\rangle_H \quad \forall v \in V, \text{ a.e. in } I, \tag{9.4}$$

$$\partial_t \chi_\varepsilon - \Delta\chi_\varepsilon + \xi_\varepsilon = \gamma(\chi_\varepsilon) + \lambda'(\chi_\varepsilon)(\omega_\varepsilon - \lambda(\chi_\varepsilon)) \quad \text{a.e. in } \Omega \times I, \tag{9.5}$$

$$\int_0^\infty \nu(\sigma)\langle \partial_t\eta_\varepsilon(\sigma) + \partial_s\eta_\varepsilon(\sigma), \zeta(\sigma)\rangle_H \, d\sigma - \int_0^\infty \mu(\sigma)\langle \partial_t\eta_\varepsilon(\sigma) + \partial_s\eta_\varepsilon(\sigma), \Delta\zeta(\sigma)\rangle_H \, d\sigma$$

$$= \int_0^\infty \nu(\sigma)\langle \omega_\varepsilon - \lambda(\chi_\varepsilon), \zeta(\sigma)\rangle_H \, d\sigma - \int_0^\infty \mu(\sigma)\langle \omega_\varepsilon - \lambda(\chi_\varepsilon), \Delta\zeta(\sigma)\rangle_H \, d\sigma$$

$$\forall \zeta \in L^2_\nu(\mathbb{R}^+, H) \cap L^2_\mu(\mathbb{R}^+, W), \text{ a.e. in } I. \tag{9.6}$$

Let us multiply Eq. (9.5) by χ_ε and integrate over Ω. This yields

$$\frac{1}{2}\frac{d}{dt}\|\chi_\varepsilon\|_H^2 = -\|\nabla\chi_\varepsilon\|_{H^3}^2 - \langle \xi_\varepsilon, \chi_\varepsilon\rangle_H + \langle \gamma(\chi_\varepsilon), \chi_\varepsilon\rangle_H + \langle \lambda'(\chi_\varepsilon)\omega_\varepsilon, \chi_\varepsilon\rangle_H. \tag{9.7}$$

Using (H2)–(H3) we get at once that

$$-\langle \xi_\varepsilon, \chi_\varepsilon\rangle_H \le 0 \tag{9.8}$$

and

$$\langle \gamma(\chi_\varepsilon), \chi_\varepsilon\rangle_H \le c(1 + \|\chi_\varepsilon\|_H^2) \tag{9.10}$$

whereas (H4)–(H5) and (2.3) imply

$$\langle \lambda'(\chi_\varepsilon)\omega_\varepsilon, \chi_\varepsilon\rangle_H \le c(\|\omega_\varepsilon\|_H^2 + \|\chi_\varepsilon\|_H^2). \tag{9.11}$$

Here and in the sequel of the proof, c stands for a generic positive constant independent of ε.

Thus, owing to (9.8)–(9.11), equality (9.7) gives

$$\frac{d}{dt}\|\chi_\varepsilon\|_H^2 + \|\nabla\chi_\varepsilon\|_{H^3}^2 \le c(\|\omega_\varepsilon\|_H^2 + \|\chi_\varepsilon\|_H^2) \tag{9.12}$$

and integrating (9.12) with respect to time from τ to $t \le T$, we get

$$\|\chi_\varepsilon(t)\|_H^2 + \int_\tau^t \|\chi_\varepsilon(y)\|_V^2 \, dy \le \|\chi_{0\varepsilon}\|_H^2 + c\left(\int_\tau^t \|\omega_\varepsilon(y)\|_H^2 \, dy + \int_\tau^t \|\chi_\varepsilon(y)\|_H^2 \, dy\right). \tag{9.13}$$

Regarding the remaining equations (9.4) and (9.6), we proceed formally along the lines of Sec. 4. Of course, the whole procedure can be made rigorous by using a regularization argument like the one used in Sec. 4. Taking ω_ε in (9.4) and arguing as in Sec. 4, we obtain (see (4.11))

$$\begin{aligned}\frac{d}{dt}\|\omega_\varepsilon\|_H^2 &\le (k_0 + a_0 - 1)\|\omega_\varepsilon\|_H^2 + \Lambda_0^2\|\chi_\varepsilon\|_H^2 + \|\eta_\varepsilon\|_{\mathcal{M}}^2 \\ &\quad - 2\int_0^\infty \mu(\sigma)\langle\eta_\varepsilon(\sigma), \omega_\varepsilon\rangle_V \, d\sigma + 2\|f\|_H\|\omega_\varepsilon\|_H.\end{aligned} \tag{9.14}$$

Then, consider Eq. (9.6) in the stronger form

$$\langle \partial_t\eta_\varepsilon + \partial_s\eta_\varepsilon, \zeta\rangle_{\mathcal{M}} = \langle \omega_\varepsilon - \lambda(\chi_\varepsilon), \zeta\rangle_{\mathcal{M}} \quad \forall \zeta \in \mathcal{M}, \text{ a.e. in } I$$

and let $\zeta = \eta_\varepsilon$. We deduce the identity (cf. (5.3))

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\eta_\varepsilon\|_{\mathcal{M}}^2 &= -\langle\partial_s\eta_\varepsilon, \eta_\varepsilon\rangle_{\mathcal{M}} \\ &\quad + \int_0^\infty \nu(\sigma)\langle\eta_\varepsilon(\sigma), \omega_\varepsilon\rangle_H \, d\sigma + \int_0^\infty \mu(\sigma)\langle\eta_\varepsilon(\sigma), \omega_\varepsilon\rangle_V \, d\sigma \\ &\quad - \int_0^\infty \nu(\sigma)\langle\eta_\varepsilon(\sigma), \lambda(\chi_\varepsilon)\rangle_H \, d\sigma - \int_0^\infty \mu(\sigma)\langle\eta_\varepsilon(\sigma), \lambda(\chi_\varepsilon)\rangle_V \, d\sigma.\end{aligned} \tag{9.15}$$

Adding (9.14) and (9.15) together, since $\langle\partial_s\eta_\varepsilon, \eta_\varepsilon\rangle_{\mathcal{M}} \ge 0$, we infer the inequality

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\omega_\varepsilon\|_H^2 + \|\eta_\varepsilon\|_{\mathcal{M}}^2) \\ &\quad \le -\|\omega_\varepsilon\|_H^2 - \int_0^\infty \nu(\sigma)\langle\eta_\varepsilon(\sigma), \omega_\varepsilon\rangle_H \, d\sigma \\ &\qquad - \int_0^\infty \mu(\sigma)\langle\nabla\eta_\varepsilon(\sigma), \nabla\omega_\varepsilon\rangle_{H^3} \, d\sigma - \langle\lambda(\chi_\varepsilon), \omega_\varepsilon\rangle_H \\ &\qquad + \int_0^\infty \nu(\sigma)\langle\eta_\varepsilon(\sigma), \omega_\varepsilon\rangle_H \, d\sigma + \int_0^\infty \mu(\sigma)\langle\eta_\varepsilon(\sigma), \omega_\varepsilon\rangle_V \, d\sigma \\ &\qquad - \int_0^\infty \nu(\sigma)\langle\eta_\varepsilon(\sigma), \lambda(\chi_\varepsilon)\rangle_H \, d\sigma - \int_0^\infty \mu(\sigma)\langle\eta_\varepsilon(\sigma), \lambda(\chi_\varepsilon)\rangle_V \, d\sigma.\end{aligned} \tag{9.16}$$

Set now, for any $t \in I$,

$$\Psi_\varepsilon^2(t) = \|\omega_\varepsilon(t)\|_H^2 + \|\eta_\varepsilon^t\|_{\mathcal{M}}^2$$

and observe that from inequality (9.16), by force of (H5), we deduce that

$$\frac{d}{dt}\Psi_\varepsilon^2(t) \le c(\Psi_\varepsilon^2(t) + \|\chi_\varepsilon(t)\|_V^2) \quad \text{for a.e. } t \in I.$$

Hence, Lemma 2.3 entails

$$\Psi_\varepsilon^2(t) \le 2e^{c(T-\tau)}(\|\omega_{0\varepsilon}\|_H^2 + \|\eta_0\|_{\mathcal{M}}^2) + 2ce^{c(t-\tau)}\int_\tau^t \|\chi_\varepsilon(y)\|_V^2 \, dy \quad \forall t \in I \tag{9.17}$$

where $\omega_{0\varepsilon} = \vartheta_0 + \lambda(\chi_{0\varepsilon})$. Combining (9.13) with (9.17), recalling (H5) and (9.3), and using Lemma 2.3, we easily get the bound

$$\|\vartheta_\varepsilon\|_{L^\infty(I,H)} + \|\eta_\varepsilon\|_{L^\infty(I,\mathcal{M})} + \|\chi_\varepsilon\|_{L^\infty(I,H)\cap L^2(I,V)} \le c. \tag{9.18}$$

Thus, thanks to (9.18), there exists a subsequence $\{\varepsilon_n\}$ converging to 0 such that

$$\vartheta_{\varepsilon_n} \to \vartheta \quad \text{weak* in } L^\infty(I,H), \tag{9.19}$$

$$\chi_{\varepsilon_n} \to \chi \quad \text{weak* in } L^\infty(I,H),\ \text{weak in } L^2(I,V), \tag{9.20}$$

$$\eta_{\varepsilon_n} \to \eta \quad \text{weak* in } L^\infty(I,\mathcal{M}). \tag{9.21}$$

To conclude, we need a uniform bound for $\|\xi_\varepsilon\|_{L^2(I,H)}$, which can be achieved by arguing as in Sec. 3 of [15]. More precisely, we observe that

$$\partial_t\chi_\varepsilon - \Delta\chi_\varepsilon + \xi_\varepsilon = \gamma(\chi_\varepsilon) + \lambda'(\chi_\varepsilon)\vartheta_\varepsilon \quad \text{a.e. in } \Omega\times I. \tag{9.22}$$

Then we multiply both members of (9.5) by ξ_ε and we integrate over $\Omega\times[\tau,t]$ with $t\in I$. Taking advantage of (H1)–(H5), (3.7), and (9.2), we obtain (see inequality (3.10) in [15])

$$\int_\Omega \phi(\chi_\varepsilon(t)) + \frac{1}{2}\int_\tau^t \|\xi_\varepsilon(y)\|_H^2\,dy \le c(\|\vartheta_\varepsilon\|^2_{L^2(I,H)} + \|\chi_\varepsilon\|^2_{L^2(I,H)}) + \int_\Omega \phi(\chi_{0\varepsilon}) \tag{9.23}$$

for any $t\in I$. Thus, owing to (9.2) and (9.18), from (9.23) we deduce the bound

$$\int_\Omega \phi(\chi_\varepsilon(t)) + \frac{1}{2}\int_\tau^t \|\xi_\varepsilon(y)\|_H^2\,dy \le c. \tag{9.24}$$

Consequently, we have, up to a subsequence still named $\{\varepsilon_n\}$,

$$\xi_{\varepsilon_n} \to \xi \quad \text{weak in } L^2(I,H), \tag{9.25}$$

and χ enjoys (3.27) thanks to Fatou's lemma. Also, (3.7) easily follows.

Equation (9.22) implies the variational identity

$$\langle\langle\partial_t\chi_{\varepsilon_n}, v\rangle\rangle_V + \langle\nabla\chi_{\varepsilon_n}, \nabla v\rangle_{H^3} + \langle\xi_{\varepsilon_n}, v\rangle_H = \langle\gamma(\chi_{\varepsilon_n}), v\rangle_H + \langle\lambda'(\chi_{\varepsilon_n})\vartheta_{\varepsilon_n}, v\rangle_H \tag{9.26}$$

for any $v\in V$ and almost everywhere in I. Then, by comparison in (9.26), we recover

$$\|\partial_t\chi_{\varepsilon_n}\|_{L^2(I,V^*)} \le c. \tag{9.27}$$

Hence, we have

$$\partial_t\chi_{\varepsilon_n} \to \partial_t\chi \quad \text{weak in } L^2(I,V^*) \tag{9.28}$$

and, owing to (9.20) and (9.28), a well-known compactness result implies

$$\chi_{\varepsilon_n} \to \chi \quad \text{in } C^0(I,H). \tag{9.29}$$

We now have all the ingredients, namely (9.19)–(9.21), (9.25), (9.28)–(9.29), to show that $(\vartheta,\chi,\eta,\xi)$ enjoys the properties stated in Theorem 3.10. In particular, to get (3.34) we simply integrate its approximate version with respect to time over $[\tau,t]$, $t\in(\tau,T]$. With the help of (H4)–(H5), (9.19)–(9.21), we can pass to the limit in the integrated equation. Property (3.30) can be deduced by comparison in the limit equation, thanks to (H4)–(H5) and (3.33). Then, the limit equation can be differentiated with respect to time and this gives (3.34). If β satisfies (H1)–(H2) only, then we can reproduce the argument sketched in subsection 5.5.

Whenever λ satisfies (H15), a careful analysis of the proof of (3.15) (see Sec. 4) shows that the bound C_0 is no longer needed. Hence, as a consequence of (3.15), the approximating family $\{\omega_\varepsilon,\chi_\varepsilon,\eta_\varepsilon\}$ satisfies a uniform contraction estimate. This allows us to conclude that the limit solution enjoys (3.1), (3.5)–(3.6), (3.11), and (3.13).

Appendix: The model equation. We consider a homogeneous rigid heat conductor belonging to the class of *simple materials* which occupies a bounded domain Ω (see, e.g., [18]). As a consequence, at any point $x \in \Omega$, the evolution is described by a causal input-output system whose input-space is independent of the nature of the material. Because of rigidity, each input process is given by the pair of functions $(\partial_t\theta, \nabla\theta)$ defined on $I = [\tau, T]$, where $\theta > 0$ is the *absolute temperature*. The state-space at each point x of such a system must reflect the features of the material itself.

Here we consider a rigid heat conductor that undergoes some solid-liquid transition at low temperature. Thus, we are forced to assume that the internal energy and heat flux depend on temperature through hereditary constitutive equations, in order to account for the second-sound effect literature (see, e.g., [26, 27] and references therein). In addition, we assume that the transition process is macroscopically described by a non-conserved *phase variable* χ which plays the role of an internal variable for the material. Accordingly, the state of the system at time t is represented by the vector $(\chi(t), \nabla\chi(t), \theta(t), \theta^t)$, where θ^t is a causal function, called the *past history of the temperature up to* t, defined by

$$\theta^t(s) = \theta(t-s) \quad s \geq 0.$$

Note that, unlike other models considered in the literature (see, e.g., [16]), we neglect memory effects in the phase variable. As we shall see, this choice is compatible with thermodynamics.

When a heat source f is given, the evolution of the temperature in a rigid body is governed by the *energy balance equation*

$$\partial_t e + \operatorname{div} \mathbf{q} = f$$

jointly with proper constitutive equations for the *internal energy* e, which is a state function, and the *heat flux vector* $\mathbf{q}$. Here, paralleling the procedure followed in [13], we assume that the internal energy is the sum of a function of $(\chi, \nabla\chi)$ and a function of (θ, θ^t), namely,

$$e(\chi, \nabla\chi, \theta, \theta^t) = G(\chi, \nabla\chi) + G_1(\theta, \theta^t) \tag{A.1}$$

where G and G_1 are suitable smooth functions. We also assume that the heat flux vector $\mathbf{q}$ is independent of the phase-field.

According to well-established theories of heat flow with memory (see [10, 23]), $\mathbf{q}$ depends on the history of the temperature gradient, that is,

$$\mathbf{q} = \mathbf{Q}(\theta, \theta^t, \nabla\theta^t). \tag{A.2}$$

Furthermore, a constitutive equation that describes the *phase kinetics* is needed. To take into account phase diffusion and relaxation, the evolution equation is required to involve time and space derivatives of χ, as in standard phase-field models. Such models are mainly due to Cahn and coworkers [2, 9] and are physically based on the assumption that $\chi = 0$ or $\chi = 1$ in most of the conductor, and the two phases are separated by a thin diffusive interface. In particular, the expressions of the internal and free energy densities contain a term $\varepsilon|\nabla\chi|^2$, $\varepsilon > 0$, representing the *interfacial energy* contribution. In view of this consideration, (A.1) reads

$$e = G_0(\chi) + G_1(\theta, \theta^t) + \varepsilon|\nabla\chi|^2 \tag{A.3}$$

for some smooth function G_0. Following standard variational procedures (see, e.g., [34]), the following rate-type constitutive equation for $\partial_t\chi$ is obtained:

$$\partial_t\chi = \Sigma(\chi, \theta, \theta^t) + \varepsilon\Delta\chi \tag{A.4}$$

where Σ is a function to be chosen properly (see below).

This description differs from the classical Stefan problem with phase relaxation, where the interfacial energy is neglected, and the indicator function $I(\chi)$ is present in the expression of the free energy, to make χ assume values inside the interval $[0,1]$ only.

Constitutive functions $G_0, G_1, \mathbf{Q}$, and Σ cannot be arbitrarily chosen; indeed, they have to satisfy the *Second Principle of Thermodynamics*. We recall that, after introducing the *Helmholtz free energy density* ψ and the *entropy density* h, the Second Principle is stated in a local form by the *Clausius-Duhem inequality*

$$\partial_t\psi + h\partial_t\theta + \frac{\mathbf{q}\cdot\nabla\theta}{\theta} \leq 0 \tag{A.5}$$

where ψ and h are state functions, related to the internal energy e by the standard relation

$$e = \psi + \theta h. \tag{A.6}$$

It is worth noting that deformations are negligible because of rigidity, and the power of internal stresses does not appear in (A.5).

In order to check thermodynamic compatibility, by means of (A.5)–(A.6), we follow a local procedure. Due to the nonlocal character of the interfacial energy, we are forced to consider the limit case $\varepsilon \to 0$. In particular, we must replace ψ, h, and e by their limits as $\varepsilon \to 0$, namely, ψ_0, h_0, and e_0. Therefore, from (A.3) we obtain

$$e_0(\chi, \theta, \theta^t) = G_0(\chi) + G_1(\theta, \theta^t). \tag{A.7}$$

In spite of this, all compatibility results still hold, even if a nonlocal interfacial energy term is added to e_0 a posteriori. Since the free energy ψ_0 depends on time through the state variables $\chi(t), \theta(t)$, and θ^t, we get

$$\partial_t\psi_0(t) = \frac{\partial\psi_0}{\partial\theta}(\chi, \theta, \theta^t)\partial_t\theta + \frac{\partial\psi_0}{\partial\chi}(\chi, \theta, \theta^t)\partial_t\chi + \delta\psi_0(\chi, \theta, \theta^t|\partial_t\theta^t)$$

where $\delta\psi_0(\chi, \theta, \theta^t|\partial_t\theta^t)$ is the Fréchet differential of ψ_0 with respect to θ^t. This approach is closely related to the one followed in [23], where the functional depends on the summed past history rather than temperature, as in the present case. Using this fact, (A.5) becomes

$$\left(\frac{\partial\psi_0}{\partial\theta} + h_0\right)\partial_t\theta + \frac{\partial\psi_0}{\partial\chi}\Sigma + \delta\psi_0 + \frac{\nabla\theta\cdot\mathbf{Q}}{\theta} \leq 0. \tag{A.8}$$

From the arbitrariness of the heat supply f, we can choose the values of the input process $(\partial_t\theta, \nabla\theta)$ independently of the state variables, so that inequality (A.8) implies the following relations:

$$h_0 = -\frac{\partial \psi_0}{\partial \theta}, \tag{A.9}$$

$$\delta \psi_0 + \frac{\partial \psi_0}{\partial \chi} \Sigma \leq 0, \tag{A.10}$$

$$\frac{\nabla \theta \cdot \mathbf{Q}}{\theta} \leq 0, \tag{A.11}$$

which are thermodynamic restrictions on the constitutive equations (A.2)–(A.4). In view of (A.6) and (A.7), equality (A.9) can be used to find a general expression of the free energy ψ_0. Indeed, it can be obtained as the solution of the differential equation

$$\psi_0 - \theta \frac{\partial \psi_0}{\partial \theta} = G_0(\chi) + G_1(\theta, \theta^t).$$

Letting θ_c denote the *critical temperature* at which transition occurs, a straightforward calculation leads to

$$\psi_0(\chi, \theta, \theta^t) = -\theta \Lambda(\theta, \theta^t) + \theta B(\chi) - (\theta - \theta_c)\lambda(\chi), \tag{A.12}$$

where Λ and λ are two suitable smooth functions such that

$$G_0(\chi) = \theta_c \lambda(\chi) \quad \text{and} \quad G_1(\theta, \theta^t) = \theta^2 \frac{\partial \Lambda}{\partial \theta}(\theta, \theta^t),$$

and B is an arbitrary function whose properties will be discussed below. In order to satisfy (A.10), we assume for simplicity

$$\delta \psi_0 \leq 0, \tag{A.13}$$

$$\frac{\partial \psi_0}{\partial \chi} \Sigma \leq 0. \tag{A.14}$$

It is easy to check that the following choice of Σ satisfies (A.14):

$$\Sigma(\chi, \theta, \theta^t) = -\frac{1}{m\theta} \frac{\partial \psi_0}{\partial \chi}(\chi, \theta, \theta^t) \tag{A.15}$$

where $m > 0$ is the reciprocal of the *mobility constant*. In connection with (A.12) and (A.15), relation (A.4) becomes

$$m \partial_t \chi + B'(\chi) = \left(1 - \frac{\theta_c}{\theta}\right) \lambda'(\chi) + m_0 \Delta \chi \tag{A.16}$$

with $m_0 = m\varepsilon$. Such an equation is a thermodynamically compatible constitutive relation governing the evolution of the phase variable χ and represents a generalization to hereditary conductors of a phase-field model proposed in recent years by Penrose and Fife [31], on the basis of thermodynamical arguments.

Since we are mainly interested in phase transition phenomena involving temperatures close to the critical value θ_c, we restrict our attention to small variations of the absolute temperature around θ_c, and small temperature gradients. As a consequence, linearizing with respect to the temperature variable, the local state variables of the material can be represented by $(\chi, \vartheta, \vartheta^t)$, where

$$\vartheta(x, t) = \frac{\theta(x, t) - \theta_c}{\theta_c}$$

is the *temperature variation field*, and ϑ^t represents its past history up to t. Accordingly, we can reasonably suppose that the temperature dependent part of the internal energy G_1 and the heat flux $\mathbf{Q}$ depend linearly on (ϑ, ϑ^t) and $\nabla\vartheta^t$, respectively. In particular, they are assumed here to obey the linear hereditary laws arising from the linearized theory of Gurtin and Pipkin [23]. Therefore, we have

$$e(x,t) = e_c + c_v\theta_c\vartheta(x,t) + \int_0^\infty a(s)\vartheta^t(x,s)\,ds + \theta_c\lambda(\chi(x,t)), \tag{A.17}$$

$$\mathbf{q}(x,t) = -\int_0^\infty k(s)\nabla\vartheta^t(x,s)\,ds. \tag{A.18}$$

The usual properties of the internal energy compel the specific heat c_v to be positive. Moreover, as shown in [19], thermodynamic restrictions can be expressed in terms of the Fourier transform of the memory kernels a and k. If they are both assumed to be summable, as usual, these conditions are satisfied if and only if

$$k_c(\omega) > 0 \quad \text{and} \quad \omega a_s(\omega) > 0 \quad \forall\omega \neq 0 \tag{A.19}$$

where the subscripts c and s stand for the cosine and sine half-range Fourier transforms, respectively.

In particular, (A.19) is satisfied if a and k are positive and monotone decreasing. Recalling the Introduction, suppose now that a is positive, bounded, and monotonically increasing. In this case, the same thermodynamic argument yields

$$k_c(\omega) > 0 \quad \text{and} \quad a(0) + a_c'(\omega) > 0 \quad \forall\omega \in \mathbb{R}. \tag{A.20}$$

Indeed, by virtue of well-known results in the theory of Fourier transforms, when a is summable, as k, then

$$a(0) + a_c'(\omega) = \omega a_s(\omega).$$

Now, it is easy to check that conditions (A.20) are satisfied when (K1)–(K3) hold. In particular, accounting for exponential kernels, namely

$$k(s) = k_0\exp(-\delta s) \quad \text{and} \quad a(s) = a_0 + \alpha(1-\exp(-\delta s)) \quad \forall s \geq 0,$$

we have

$$k_c(\omega) = \frac{k_0\delta}{\omega^2+\delta^2} \quad \text{and} \quad a(0) + a_c'(\omega) = a_0 + \frac{\alpha\delta^2}{\omega^2+\delta^2}$$

and (A.20) follows for all real ω and for arbitrary positive constants k_0, a_0, α, and δ.

Finally, applying the linearization scheme to (A.16), the resulting constitutive equation for the phase-field kinetics reads

$$m\partial_t\chi - m_0\Delta\chi + B'(\chi) = \vartheta\lambda'(\chi). \tag{A.21}$$

A similar situation has been investigated by Caginalp [8] within the framework of the standard phase-field model. It is convenient to assume a quite general expression for B, namely,

$$B(\chi) = \varphi(\chi) - \hat\gamma(\chi)$$

where ϕ is a proper convex and lower semicontinuous function with $\phi(0) = 0$ (e.g., the indicator function of $[0,1]$) and $\hat\gamma$ is a smooth function with quadratic growth at most.

This choice of B may allow us, for instance, to avoid χ taking values outside the unit interval $[0,1]$. Setting $\gamma = \hat{\gamma}'$ and $\beta = \partial\phi$, Eq. (A.19) becomes the differential inclusion

$$m\partial_t \chi - m_0 \Delta \chi + \beta(\chi) \ni \gamma(\chi) + \vartheta \lambda'(\chi). \tag{A.22}$$

Acknowledgments. The authors thank Pierluigi Colli and Fabio Luterotti for helpful discussions and comments.

This work has been partially supported by the Italian MURST Research Projects "Modelli Matematici per la Scienza dei Materiali", "Equazioni Differenziali: Metodi Analitici, Geometrici e Funzionali, e Applicazioni", and "Problemi e Metodi nella Teoria delle Equazioni Iperboliche".

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