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ON DERIVATIVE OF ENERGY FUNCTIONAL FOR ELASTIC BODIES WITH CRACKS AND UNILATERAL CONDITIONS

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Abstract. In this paper we consider elasticity equations in a domain having a cut (a crack) with unilateral boundary conditions considered at the crack faces. The boundary conditions provide a mutual nonpenetration between the crack faces, and the problem as a whole is nonlinear. Assuming that a general perturbation of the cut is given, we find the derivative of the energy functional with respect to the perturbation parameter. It is known that a calculation of the material derivative for similar problems has the difficulty of finding boundary conditions at the crack faces. We use a variational property of the solution, thus avoiding a direct calculation of the material derivative.

There are many results related to the differentiation of the potential energy functional with respect to variable domains (see, e.g., [9, 4, 5, 16, 18, 17, 3]). The general theory of calculating material and shape derivatives in linear and nonlinear boundary value problems is developed in [6].

Derivatives of energy functionals with respect to the crack length in classical linear elasticity can be found by different ways. It is well known that the classical approach to the crack problem is characterized by the equality-type boundary conditions considered at the crack faces [13, 4, 7, 14, 12, 15]. As for the analysis of solution dependence on the shape domain for a wide class of elastic problems, we refer the reader to [8].

In the works [1, 2] the appropriate technique of finding derivatives of the energy functional with respect to the crack length for unilateral boundary conditions is proposed,

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which can be used for a wide class of the unilateral problems. Qualitative properties of solutions (solution existence, solution regularity, dependence of solutions on parameters, etc.) in the crack problem for plates, shells, two- and three-dimensional bodies with unilateral conditions on the crack faces are analysed in [2] (see also [20, 19, 10, 11]).

1. Problem formulation. Let $D \subset R^3$ be a bounded domain with smooth boundary Γ, and $\Xi \subset D$ be a smooth two-dimensional surface. We assume that this surface can be extended up to the outer boundary Γ in such a way that D is divided into subdomains D_1 and D_2 with Lipschitz boundaries. Assume that this inner surface Ξ is described parametrically by the equations

$$x_i = x_i(y_1, y_2), \quad i = 1, 2, 3, \tag{1}$$

where (y_1, y_2) belong to the closure of an open bounded connected set $\omega \subset R^2$ having a smooth boundary γ. We suppose that the rank of the Jacobi matrix $\partial x_i / \partial y_j$ equals 2 at every point $(y_1, y_2) \in \omega \cup \gamma$, and that the map (1) is one-to-one. Let $\nu = (\nu_1, \nu_2, \nu_3)$ be a unit normal vector to Ξ, for example,

$$\nu = \frac{\dfrac{\partial x}{\partial y_1} \times \dfrac{\partial x}{\partial y_2}}{\left| \dfrac{\partial x}{\partial y_1} \times \dfrac{\partial x}{\partial y_2} \right|}.$$

Denote $\Omega = D \backslash \Xi$. In the domain Ω, we consider the following boundary value problem for finding a function $u = (u_1, u_2, u_3)$:

$$-\sigma_{ij,j} = f_i, \quad i = 1, 2, 3, \tag{2}$$

$$\sigma_{ij} = a_{ijkl}\varepsilon_{kl}, \quad i, j = 1, 2, 3, \tag{3}$$

$$u = 0 \quad \text{on } \Gamma, \tag{4}$$

$$[u]\nu \geq 0, \quad \sigma_\nu \leq 0, \quad [\sigma_\nu] = 0, \quad \sigma_\tau = 0, \quad \sigma_\nu [u]\nu = 0 \quad \text{on } \Xi. \tag{5}$$

Here $\varepsilon_{kl} = \varepsilon_{kl}(u) = \frac{1}{2}(u_{k,l} + u_{l,k})$ are strain tensor components, $u_{k,l} = \frac{\partial u_k}{\partial x_l}$; $\sigma_{ij} = \sigma_{ij}(u)$ denote the stress tensor components,

$$\{\sigma_{ij}\nu_j\}_{i=1}^3 = \sigma_\tau + \sigma_\nu \nu, \qquad \sigma_\nu = \sigma_{ij}\nu_j\nu_i.$$

The brackets $[v] = v^+ - v^-$ mean the jump of v across Ξ, where v^+, v^- stand for the values of v on Ξ^+, Ξ^-, respectively, and where Ξ^+, Ξ^- are defined for a given choice of positive and negative directions of ν on Ξ. Coefficients a_{ijkl} are assumed to be constant, satisfying the usual conditions of symmetry and positive definiteness, i.e.,

$$a_{ijkl} = a_{jikl} = a_{ijlk}, \quad a_{ijkl}\xi_{kl}\xi_{ij} \geq c|\xi|^2, \quad c > 0, \quad \xi_{ij} = \xi_{ji}.$$

The function $f = (f_1, f_2, f_3) \in C^1(R^3)$ is given.

The boundary value problem (2)–(5) describes an equilibrium state of an elastic body occupying the domain Ω in its nondeformable state, and the surface Ξ corresponds to a crack in a body. Conditions (5) provide the mutual nonpenetration between the crack faces without friction [8]. Considering the problem (2)–(5), we have in mind its variational formulation. Denote

$$K_0 = \{u = (u_1, u_2, u_3) \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma; [u]\nu \geq 0 \text{ on } \Xi\}.$$

Then (2)–(5) correspond to the following minimization problem:

$$\min_{u\in K_0}\left\{\frac{1}{2}\int_\Omega a_{ijkl}\varepsilon_{kl}(u)\varepsilon_{ij}(u)-\int_\Omega fu\right\}. \tag{6}$$

By the assumptions imposed on Ω, a_{ijkl}, f, the problem (6) has a unique solution u satisfying the following variational inequality:

$$u\in K_0: \quad \int_\Omega a_{ijkl}\varepsilon_{kl}(u)(\varepsilon_{ij}(\overline{u})-\varepsilon_{ij}(u))\geq\int_\Omega f(\overline{u}-u)\quad \forall\overline{u}\in K_0. \tag{7}$$

In the paper, we consider a general perturbation of the boundary value problem (2)–(5) and find the derivative of the energy functional with respect to the perturbation parameter. Note that the obtained result holds true for other boundary conditions. For example, we may assume that $\Gamma=\Gamma_1\cup\Gamma_2, \Gamma_1\cap\Gamma_2=\varnothing, \operatorname{meas}\Gamma_1>0, u=0$ on $\Gamma_1, \sigma_{ij}n_j=0$ on Γ_2. Here $n=(n_1,n_2,n_3)$ is the unit normal vector to Γ.

Let Ω_t be a family of domains such that, for each t, there exists a one-to-one mapping

$$y=\Phi_t(x),\quad x\in\Omega_t,\quad y\in\Omega, \tag{8}$$

with the positive Jacobian $|\frac{\partial\Phi_t}{\partial x}|>c>0, \Phi_t=(\Phi_t^1,\Phi_t^2,\Phi_t^3)$. We assume that $\Phi_0(x)=x, \Phi\in C^2(0,T;W^{2,\infty}_{\mathrm{loc}}(R^3))$. Let

$$x=x(t,y)=\Phi_t^{-1}(y) \tag{9}$$

be the mapping inverse to Φ_t. By fixing y in (8) and differentiating (8) with respect to t, we have

$$0=\frac{\partial\Phi_t}{\partial t}+\frac{\partial\Phi_t}{\partial x}\frac{dx(t)}{dt},$$

whence

$$\frac{dx(t)}{dt}=-\left(\frac{\partial\Phi_t}{\partial x}\right)^{-1}\frac{\partial\Phi_t}{\partial t}. \tag{10}$$

It is clear that (10) can be viewed as a system of ordinary differential equations; thus

$$\frac{dx(t)}{dt}=V(t,x(t)), \tag{11}$$

$$x(0)=y, \tag{12}$$

where

$$V(t,x(t))=-\left(\frac{\partial\Phi_t(x(t))}{\partial x}\right)^{-1}\frac{\partial\Phi_t(x(t))}{\partial t}, \tag{13}$$

and hence for the solution $x(t)$ of (11)–(12) we have $x(t)=x(t,y)$. Note that, by (9),

$$\frac{dx(t,y)}{dt}=\frac{\partial\Phi_t^{-1}(y)}{\partial t},$$

hence

$$V(t,x(t))=\frac{\partial\Phi_t^{-1}(y)}{\partial t},\quad x(t)=x(t,y).$$

Let $\Xi_t = \Phi_t^{-1}(\Xi)$ and $\Gamma_t = \Phi_t^{-1}(\Gamma)$. We can assume that Ξ_t has no self-intersections and consider the boundary value problem similar to (2)–(5) for the domain $\Omega_t = \Phi_t^{-1}(\Omega)$. Namely, in the domain Ω_t we want to find a function $u^t = (u_1^t, u_2^t, u_3^t)$ such that

$$-\sigma_{ij,j}^t = f_i, \quad i = 1,2,3, \tag{14}$$

$$\sigma_{ij}^t = a_{ijkl}\varepsilon_{kl}^t, \quad i,j = 1,2,3, \tag{15}$$

$$u^t = 0 \quad \text{on } \Gamma_t, \tag{16}$$

$$[u^t]\nu^t \geq 0, \quad \sigma_{\nu^t}^t \leq 0, \quad [\sigma_{\nu^t}^t] = 0, \quad \sigma_{\tau^t}^t = 0, \quad \Phi_{\nu^t}^t[u^t]\nu^t = 0 \quad \text{on } \Xi_t. \tag{17}$$

Here ν^t is a unit normal vector to $\Xi_t, \varepsilon_{kl}^t(u^t) = \frac{1}{2}(u_{k,l}^t + u_{l,k}^t)$. All the rest of the notation is similar to that of (2)–(5). In fact, the problem (14)–(17) is written in the variational form

$$u^t \in K_t : \quad \int_{\Omega_t} a_{ijkl}\varepsilon_{kl}(u^t)(\varepsilon_{ij}(\overline{u}^t) - \varepsilon_{ij}(u^t)) \geq \int_{\Omega_t} f(\overline{u}^t - u^t) \quad \forall \overline{u}^t \in K_t, \tag{18}$$

where

$$K_t = \{u = (u_1, u_2, u_3) \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma_t; [u]\nu^t \geq 0 \text{ on } \Xi_t\}.$$

We impose one more condition on the mapping Φ_t. Assume that the condition $v(y) \in K_0$ implies $v^t(x) \in K_t, v^t(x) = v(y), x \in \Omega_t, y \in \Omega, y = \Phi_t(x)$, and conversely, if $v(x) \in K_t$, then $v_t(y) \in K_0, v_t(y) = v(x), x = x(t,y)$. Note that this condition is not very restricting, and it holds in many cases [2].

Let u, u^t be the solutions of the problems (7), (18), respectively. Consider the energy functionals

$$J(\Omega) = \frac{1}{2}\int_\Omega a_{ijkl}u_{k,l}u_{i,j} - \int_\Omega fu,$$

$$J(\Omega_t) = \frac{1}{2}\int_{\Omega_t} a_{ijkl}u_{k,l}^t u_{i,j}^t - \int_{\Omega_t} fu^t.$$

Our purpose is to find the derivative of $J(\Omega_t)$ with respect to the parameter t, namely,

$$\left.\frac{dJ(\Omega_t)}{dt}\right|_{t=0} = \lim_{t\to 0}\frac{J(\Omega_t) - J(\Omega)}{t}. \tag{19}$$

2. Solution convergence. First of all we prove the convergence of u^t to u in a proper sense. Namely, let $u^t(x) = u_t(y), x \in \Omega_t, y \in \Omega, x = x(t,y)$. Denote by $\|\cdot\|_{1,\Omega}$ the norm in the space $H^1(\Omega)$.

LEMMA. The following estimate holds:

$$\|u_t - u\|_{1,\Omega} \leq ct,$$

where c is a constant independent of t.

Proof. The functions u, u^t satisfy the following variational inequalities:

$$u \in K_0 : \quad \int_\Omega a_{ijkl}u_{k,l}(\overline{u}_{i,j} - u_{i,j}) \geq \int_\Omega f(\overline{u} - u) \quad \forall \overline{u} \in K_0, \tag{20}$$

$$u^t \in K_t : \quad \int_{\Omega_t} a_{ijkl}u_{k,l}^t(\overline{u}_{i,j}^t - u_{i,j}^t) \geq \int_{\Omega_t} f(\overline{u}^t - u^t) \quad \forall \overline{u}^t \in K_t. \tag{21}$$

Denote $f_t(y) = f(x(t,y)), q_t(y) = |\frac{\partial \Phi_t^{-1}(y)}{\partial y}|$. We have

$$u^t_{k,l}(x) = u_{tk,p}(y)\Phi^p_{t,l}(x), \quad k,l = 1,2,3.$$

Consequently, the inequality (21) can be rewritten in the form

$$u_t \in K_0 : \quad \int_\Omega a_{ijkl} u_{tk,p} \Phi^p_{t,l}(\overline{u}_{ti,s}\Phi^s_{t,j} - u_{ti,s}\Phi^s_{t,j})q_t \geq \int_\Omega f_t(\overline{u}_t - u_t)q_t \quad \forall \overline{u}_t \in K_0. \tag{22}$$

It is important to note that $\overline{u}_t \in K_0$ due to the assumption imposed on Φ_t. Here we used the one-to-one mapping between K_t and K_0. In the inequality (22), we have $\Phi^p_{t,l} = \Phi^p_{t,l}(x(t,y))$. Denote by δ^p_l the Kronecker symbol. The following equalities hold:

$$\Phi^p_{t,l}(x(t,y)) = \Phi^p_{0,l}(x(t,y)) + \frac{\partial \Phi^p_{\xi,l}(x(t,y))}{\partial \xi} t, \quad \xi \in (0,t).$$

Since $\Phi^p_{0,l} = \delta^p_l$, $p,l = 1,2,3$, these equalities can be rewritten as follows:

$$\Phi^p_{t,l}(x(t,y)) = \delta^p_l + \Phi^{pl}_\xi(x(t,y))t, \tag{23}$$

where $\xi = \xi(t,p,l)$, and we have denoted $\frac{\partial \Phi^p_{\xi,l}}{\partial \xi}$ by Φ^{pl}_ξ,

$$\|\Phi^{pl}_\xi\|_{L^\infty(\Omega)} \leq c \quad \text{uniformly in } \xi \in (0,T). \tag{24}$$

Moreover, $q_t(y) = q_0(y) + \frac{\partial q_\xi(y)}{\partial \xi} t$, $\xi \in (0,t)$, and $q_0(y) = 1$. Denote $\frac{\partial q_\xi(y)}{\partial \xi}$ by $\overline{q}_\xi(y)$ which gives

$$q_t(y) = 1 + \overline{q}_\xi(y)t, \tag{25}$$

with the uniform in ξ estimate $\|\overline{q}_\xi\|_{L^\infty(\Omega)} \leq c$. By (23), the inequality (22) can be written in the form

$$\int_\Omega a_{ijkl} u_{tk,p}(\delta^p_l + t\Phi^{pl}_\xi)[\overline{u}_{ti,s}(\delta^s_j + t\Phi^{sj}_\xi) - u_{ti,s}(\delta^s_j + t\Phi^{sj}_\xi)]q_t \geq \int_\Omega f_t(\overline{u}_t - u_t)q_t. \tag{26}$$

Now substitute $\overline{u} = u_t, \overline{u} = u$ in (20), (26), respectively. By (25), this yields

$$\int_\Omega a_{ijkl} u_{k,l}(u_{ti,j} - u_{i,j}) \geq \int_\Omega f(u_t - u), \tag{27}$$

$$\begin{aligned}\int_\Omega a_{ijkl} u_{tk,l}(u_{i,j} - u_{ti,j})(1 + \overline{q}_\xi t) + t\int_\Omega a_{ijkl} u_{tk,p}\Phi^{pl}_\xi(u_{i,j} - u_{ti,j})q_t \\ + t\int_\Omega a_{ijkl} u_{tk,l}(u_{i,s}\Phi^{sj}_\xi - u_{ti,s}\Phi^{sj}_\xi)q_t \\ + t^2\int_\Omega a_{ijkl} u_{tk,p}\Phi^{pl}_\xi(u_{i,s}\Phi^{sj}_\xi - u_{ti,s}\Phi^{sj}_\xi)q_t \geq \int_\Omega f_t(u - u_t)q_t.\end{aligned} \tag{28}$$

Summing (27) and (28) we obtain

$$\begin{aligned}\int_\Omega a_{ijkl}(u_{k,l}-u_{tk,l})(u_{i,j}-u_{ti,j}) &\le t\int_\Omega \overline{q}_\xi a_{ijkl}u_{tk,l}(u_{i,j}-u_{ti,j})\\ &+t\int_\Omega [a_{ijkl}u_{tk,p}\Phi_\xi^{pl}(u_{i,j}-u_{ti,j})+a_{ijkl}u_{tk,l}(u_{i,s}\Phi_\xi^{sj}-u_{ti,s}\Phi_\xi^{sj})]q_t\\ &+t^2\int_\Omega a_{ijkl}u_{tk,p}\Phi_\xi^{pl}(u_{i,s}\Phi_\xi^{sj}-u_{ti,s}\Phi_\xi^{sj})q_t\\ &-\int_\Omega f(u_t-u)+\int_\Omega f_t(u_t-u)q_t. \qquad (29)\end{aligned}$$

Taking $\overline{u}_t=0$ in (26), we derive the uniform in $t\in(0,T)$ estimate

$$\|u_t\|_{1,\Omega}\le c.$$

Consequently, by (24), (25), from (29) it follows that

$$\int_\Omega a_{ijkl}(u_{i,j}-u_{ti,j})(u_{k,l}-u_{tk,l})\le ct^2+\int_\Omega |u-u_t|\,|f-f_t(1+\overline{q}_\xi t)|, \qquad (30)$$

where the constant c is independent of $t\in(0,T)$. Since

$$f_k(y)-f_{kt}(y)(1+\overline{q}_\xi t)=f_k(y)-\left[f_k(y)+\frac{\partial f_k}{\partial x_i}\frac{dx_i(\xi_1)}{d\xi_1}(1+\overline{q}_\xi t)t\right],$$
$$\xi_1\in(0,t)\quad k=1,2,3,$$

the inequality (30) implies

$$\|u-u_t\|_{1,\Omega}^2\le ct^2,$$

which completes the proof of the lemma. □

3. Main result. To find the derivative of the energy functional, we shall use the variational property of the solution. Introduce first some notation:

$$\Pi(\Omega_t,\varphi)=\frac{1}{2}\int_{\Omega_t} a_{ijkl}\varphi_{k,l}\varphi_{i,j}\,dx-\int_{\Omega_t} f\varphi\,dx,$$
$$\Pi_t(\Omega,\varphi)=\frac{1}{2}\int_\Omega a_{ijkl}\varphi_{k,p}\Phi_{t,l}^p\varphi_{i,s}\Phi_{t,j}^s q_t\,dy-\int_\Omega f_t\varphi q_t\,dy.$$

Since we have the one-to-one mapping between K_t and K_0, the following equality holds:

$$\min_{\varphi\in K_0}\Pi_t(\Omega;\varphi)=\min_{\varphi\in K_t}\Pi(\Omega_t;\varphi).$$

Note also that

$$J(\Omega)=\Pi(\Omega;u),\qquad J(\Omega_t)=\Pi(\Omega_t;u^t),$$

where u,u^t are the solutions of (7) and (18), respectively. Consequently, we conclude

$$\begin{aligned}\frac{J(\Omega_t)-J(\Omega)}{t}&=\frac{\Pi(\Omega_t;u^t)-\Pi(\Omega;u)}{t}\\ &=\frac{\Pi_t(\Omega;u_t)-\Pi(\Omega;u)}{t}\le\frac{\Pi_t(\Omega;u)-\Pi(\Omega;u)}{t}.\end{aligned}$$

This implies

$$\limsup_{t\to 0}\frac{J(\Omega_t)-J(\Omega)}{t}\le \limsup_{t\to 0}\frac{\Pi_t(\Omega;u)-\Pi(\Omega;u)}{t}. \tag{31}$$

On the other hand,

$$\limsup_{t\to 0}\frac{J(\Omega_t)-J(\Omega)}{t}=\frac{\Pi(\Omega_t;u^t)-\Pi(\Omega;u)}{t}\ge\frac{\Pi_t(\Omega;u_t)-\Pi(\Omega;u_t)}{t},$$

whence

$$\liminf_{t\to 0}\frac{J(\Omega_t)-J(\Omega)}{t}\ge \liminf_{t\to 0}\frac{\Pi_t(\Omega;u_t)-\Pi(\Omega;u_t)}{t}. \tag{32}$$

Now we aim to show that the right-hand sides of (31), (32) coincide, which implies an existence of the limit

$$\lim_{t\to 0}\frac{J(\Omega_t)-J(\Omega)}{t}.$$

Let us find the right-hand side of (31). It suffices to find the derivative

$$\frac{d}{dt}\Pi_t(\Omega;u)|_{t=0}=\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega a_{ijkl}u_{k,p}\Phi^p_{t,l}u_{i,s}\Phi^s_{t,j}q_t-\int_\Omega f_t u q_t\right\}\bigg|_{t=0}. \tag{33}$$

By denoting

$$\Lambda(y)=-V(0,y)=\frac{\partial\Phi_t(x(t,y))}{\partial t}\bigg|_{t=0}, \tag{34}$$

we have

$$\Phi^p_{t,l}(x(t,y))|_{t=0}=\delta^p_l,\qquad \frac{\partial\Phi^p_{t,l}(x(t,y))}{\partial t}\bigg|_{t=0}=\Lambda^p_{,l}(y), \tag{35}$$

and, moreover, as $t\to 0$,

$$\Phi^p_{t,l}(x(t,y))\to\delta^p_l,\qquad \frac{\partial\Phi^p_{t,l}(x(t,y))}{\partial t}\to\Lambda^p_{,l}(y)\quad\text{in } L^\infty(\Omega). \tag{36}$$

Indeed,

$$\begin{aligned}\Phi^p_{t,l}(x(t,y))&=\Phi^p_{t,l}(x(t,y))-\Phi^p_{0,l}(x(t,y))+\Phi^p_{0,l}(x(t,y))\\&=\frac{\partial\Phi^p_{\xi,l}(x(t,y))}{\partial\xi}t+\Phi^p_{0,l}(x(t,y)),\quad \xi\in(0,t).\end{aligned}$$

By (24), we have

$$\frac{\partial\Phi^p_{\xi,l}(x(t,y))}{\partial\xi}\text{ are bounded in } L^\infty(\Omega)\text{ uniformly in } \xi,t\in(0,T),$$

which together with the first equality of (35) implies the first convergence of (36). Similarly, since

$$\frac{\partial^2\Phi^p_{\xi,l}(x(t,y))}{\partial\xi^2}\text{ are bounded in } L^\infty(\Omega)\quad\text{for all } \xi,t\in(0,T)$$

and $\frac{\partial \Phi^p_{0,l}(x)}{\partial \xi}$ satisfy the Lipschitz property in x, from the equality

$$\begin{aligned}\frac{\partial \Phi^p_{t,l}(x(t,y))}{\partial t} &= \frac{\partial \Phi^p_{t,l}(x(t,y))}{\partial t} - \frac{\partial \Phi^p_{0,l}(x(t,y))}{\partial t} + \frac{\partial \Phi^p_{0,l}(x(t,y))}{\partial t} \\ &= \frac{\partial^2 \Phi^p_{\xi,l}(x(t,y))}{\partial \xi^2} t + \frac{\partial \Phi^p_{0,l}(x(t,y))}{\partial t}, \quad \xi \in (0,t),\end{aligned}$$

we conclude that the second convergence of (36) takes place.

It is well known that [21]

$$\frac{\partial q_t(y)}{\partial t} = q_t(y) \operatorname{div} V(t, x(t,y)), \tag{37}$$

and by (25), (34),

$$\left.\frac{\partial q_t(y)}{\partial t}\right|_{t=0} = -\operatorname{div} \Lambda(y).$$

We next obtain

$$\operatorname{div} V(t, x(t,y)) = \operatorname{div} V(0,y) + \frac{d}{d\xi} \operatorname{div} V(\xi, x(\xi,y)) t, \quad \xi \in (0,t),$$

$$\left\| \frac{d}{d\xi} \operatorname{div} V(\xi, x(\xi,y)) \right\|_{L^\infty(\Omega)} \le c \quad \text{uniformly in } \xi \in (0,T).$$

Hence, taking into account (25), as $t \to 0$,

$$q_t(y) \operatorname{div} V(t, x(t,y)) \to -\operatorname{div} \Lambda(y) \quad \text{in } L^\infty(\Omega).$$

By (37), this gives as $t \to 0$,

$$\frac{\partial q_t}{\partial t} \to -\operatorname{div} \Lambda \quad \text{in } L^\infty(\Omega). \tag{38}$$

Also note that (35) implies $\Phi^p_{t,ls}(x(t,y))|_{t=0} = 0$, $p, l, s = 1, 2, 3$, so that

$$\left.\frac{d\Phi^p_{t,l}(x(t,y))}{dt}\right|_{t=0} = \left.\frac{\partial \Phi^p_{t,l}(x(t,y))}{\partial t}\right|_{t=0}.$$

Hence, by (36), (38), we can calculate the right-hand side of (33), i.e., the right-hand side of (31):

$$\begin{aligned}\frac{d}{dt}\Pi_t(\Omega; u)|_{t=0} &= \frac{1}{2}\int_\Omega (a_{ijkl} u_{k,p} u_{i,s} \Lambda^p_{,l} \delta^s_j + a_{ijkl} u_{k,p} u_{i,s} \Lambda^s_{,j} \delta^p_l) \\ &\quad - \frac{1}{2}\int_\Omega a_{ijkl} u_{k,p} u_{i,s} \delta^p_l \delta^s_j (\operatorname{div} \Lambda) + \int_\Omega u_k (\nabla f_k \Lambda) + \int_\Omega f u \operatorname{div} \Lambda \\ &= \int_\Omega \left\{ \sigma_{kl} u_{k,p} \Lambda^p_{,l} - \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \operatorname{div} \Lambda \right\} + \int_\Omega u_k \operatorname{div}(f_k \Lambda).\end{aligned} \tag{39}$$

Now we find the right-hand side of (32). To this end we consider the term

$$\Delta_t = \frac{1}{t}\int_\Omega (u_{tk,p} u_{ti,s} \Phi^p_{t,l} \Phi^s_{t,j} q_t - u_{tk,l} u_{ti,j})$$

for fixed k, l, i, j. It is possible to write Δ_t in the form

$$\begin{aligned}\Delta_t = &\frac{1}{t}\int_\Omega (u_{tk,p}u_{ti,s}\Phi^p_{t,l}\Phi^s_{t,j} - u_{tk,p}\delta^p_l u_{ti,s}\Phi^s_{t,j})q_t \\ &+ \frac{1}{t}\int_\Omega (u_{tk,p}\delta^p_l u_{ti,s}\Phi^s_{t,j} - u_{tk,l}u_{ti,j}) \\ &+ \frac{1}{t}\int_\Omega (u_{tk,p}u_{ti,s}\Phi^p_{t,l}\Phi^s_{t,j} - u_{tk,p}\delta^p_l u_{ti,s}\Phi^s_{t,j}) \\ &+ \frac{1}{t}\int_\Omega (u_{tk,p}\delta^p_l u_{ti,s}\Phi^s_{t,j} - u_{tk,p}\delta^p_l u_{ti,s}\delta^s_j)q_t \\ &+ \frac{1}{t}\int_\Omega (u_{tk,p}\delta^p_l u_{ti,s}\delta^s_j - u_{tk,p}\Phi^p_{t,l}u_{ti,s}\delta^s_j)q_t \\ &+ \frac{1}{t}\int_\Omega (u_{tk,p}\Phi^p_{t,l}u_{ti,s}\delta^s_j q_t - u_{tk,p}\Phi^p_{t,l}u_{ti,s}\delta^s_j) \\ &+ \frac{1}{t}\int_\Omega (u_{tk,p}\Phi^p_{t,l}u_{ti,s}\delta^s_j - u_{tk,p}\Phi^p_{t,l}u_{ti,s}\Phi^s_{t,j}).\end{aligned} \tag{40}$$

Recall that, as $t \to 0$,

$$u_t \to u \quad \text{in } H^1(\Omega), \qquad q_t \to 1 \quad \text{in } L^\infty(\Omega). \tag{41}$$

Since

$$\frac{q_t(y) - q_0(y)}{t} = \frac{\partial q_\xi(y)}{\partial \xi}, \quad \xi \in (0, t),$$
$$\frac{\Phi^p_{t,l}(x(t,y)) - \delta^p_l}{t} = \frac{\partial \Phi^p_{\xi,l}(x(t,y))}{\partial \xi}, \quad \xi \in (0, t),$$

by the convergences (36), (38), we can find the limit of each part of the right-hand side of (40) as $t \to 0$, which implies

$$\begin{aligned}\lim_{t\to 0}\Delta_t &= \int_\Omega \{u_{k,p}\Lambda^p_{,l}u_{i,j} + u_{k,l}u_{i,s}\Lambda^s_{,j} + u_{i,j}u_{k,p}\Lambda^p_{,l} + u_{k,l}u_{i,s}\Lambda^s_{,j} \\ &\quad - u_{i,j}u_{k,p}\Lambda^p_{,l} - u_{k,l}u_{i,j}(\operatorname{div}\Lambda) - u_{k,l}u_{i,s}\Lambda^s_{,j}\} \\ &= \int_\Omega \{u_{i,j}u_{k,p}\Lambda^p_{,l} + u_{k,l}u_{i,s}\Lambda^s_{,j} - u_{k,l}u_{i,j}(\operatorname{div}\Lambda)\}.\end{aligned} \tag{42}$$

In addition to this, as $t \to 0$,

$$\frac{f_{kt} - f_k}{t} \to -\nabla f_k \cdot \Lambda \quad \text{in } L^\infty(\Omega), \quad k = 1, 2, 3;$$

hence

$$\begin{aligned}\lim_{t\to 0}\int_\Omega \frac{f_t q_t u_t - f u_t}{t} &= \lim_{t\to 0}\int_\Omega \frac{(f_t - f)q_t u_t}{t} + \lim_{t\to 0}\int_\Omega \frac{f u_t(q_t - 1)}{t} \\ &= -\int_\Omega u_k(\nabla f_k \Lambda) - \int_\Omega f u \operatorname{div}\Lambda.\end{aligned} \tag{43}$$

From (42), (43) it follows that

$$\begin{aligned}\lim_{t\to 0} &\frac{\Pi_t(\Omega; u_t) - \Pi(\Omega; u_t)}{t} \\ &= \int_\Omega \{a_{ijkl}u_{i,j}u_{k,p}\Lambda^p_{,l} - \frac{1}{2}a_{ijkl}u_{i,j}u_{k,l}(\operatorname{div}\Lambda)\} \\ &\quad + \int_\Omega u_k(\nabla f_k \Lambda) + \int_\Omega f u \operatorname{div}\Lambda \\ &= \int_\Omega \left\{\sigma_{kl}u_{k,p}\Lambda^p_{,l} - \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\operatorname{div}\Lambda\right\} + \int_\Omega u_k \operatorname{div}(f_k\Lambda).\end{aligned} \tag{44}$$

By (39), (44), we conclude that the right-hand sides of (31), (32) coincide and we obtain the following statement.

THEOREM. Let the hypotheses concerning Φ_t be fulfilled. Then the derivative of the energy functional is given by the formula

$$\left.\frac{dJ(\Omega_t)}{dt}\right|_{t=0} = \int_\Omega \left\{\sigma_{kl}u_{k,p}\Lambda^p_{,l} - \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\operatorname{div}\Lambda\right\} + \int_\Omega u_k \operatorname{div}(f_k\Lambda), \tag{45}$$

where the vector field Λ is defined by (34).

In conclusion, note that the formulae similar to (45) were obtained for isotropic two- and three-dimensional cracked bodies with conditions (5) at the crack faces provided that the perturbation Φ_t of the domain Ω_t describes the crack length change [1, 2].

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