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EXISTENCE OF WEAK SOLUTIONS TO THE ELASTIC STRING EQUATIONS IN THREE DIMENSIONS

By

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Abstract. Many applied problems resulting in hyperbolic conservation laws are nonstrictly hyperbolic. As of yet, there is no comprehensive theory to describe the solutions of these systems. We examine the equations modeling an elastic string of infinite length in three-dimensional space, restricted to possess non-simple eigenvalues of constant multiplicity. We show that there exists a weak solution of the nonstrictly hyperbolic conservation law when the total variation of the initial data is sufficiently small. The proof technique is similar to Glimm's classical existence for hyperbolic conservation laws, but necessarily departs from Glimm's proof by not requiring strict hyperbolicity.

1. Introduction. Consider an elastic string of infinite length that is free to move in three-dimensional space (u, v, w) (see [1], [6]). We will describe the string by using a reference configuration when the string is under uniform tension $T_0 > 0$ and constant density ρ_0. Let x be the arclength when the string is under uniform tension. Let $u(x,t), v(x,t)$, and $w(x,t)$ represent the components in the directions $\vec{i}, \vec{j}$, and $\vec{k}$ of the point on the string that is at a distance x from the origin in the reference state. We assume that the only force acting on the string is tension, with magnitude T, which acts in a direction tangent to the string. The *strain*, ε, is defined to be

$$\varepsilon = \sqrt{u_x^2 + v_x^2 + w_x^2} - 1.$$

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We will assume a stress-strain relation of the form $T = T(\varepsilon)$. The equations of motion of the string can be written in the form (see [2]):

$$\rho_0 u_{tt} = \left(\frac{T}{1+\varepsilon} u_x\right)_x,$$
$$\rho_0 v_{tt} = \left(\frac{T}{1+\varepsilon} v_x\right)_x,$$
$$\rho_0 w_{tt} = \left(\frac{T}{1+\varepsilon} w_x\right)_x.$$

This is an idealized system of equations, in which the string is taken to be infinitely thin. For simplicity, we will let $T_0 = 1$ and $\rho_0 = 1$. To simplify the equations further, we let $r = \varepsilon + 1$ and $r\phi = r\phi(r) = T$.

Rewriting the equations as a first-order system, where we take

$$U = (u_x, u_t, v_x, v_t, w_x, w_t),$$

we obtain

$$U_t + F(U)_x = 0, \tag{1}$$

a system of six conservation laws with flux function

$$F(U) = (-u_t, -\phi u_x, -v_t, -\phi v_x, -w_t, -\phi w_x).$$

The eigenvalues for this system are $\lambda_1 = -\sqrt{T_\varepsilon}, \lambda_2 = \lambda_3 = -\sqrt{\phi}, \lambda_4 = \lambda_5 = \sqrt{\phi}, \lambda_6 = \sqrt{T_\varepsilon}$. Hence the system is hyperbolic if T is nonnegative, differentiable, and monotone increasing. A brief introduction to hyperbolic conservation laws is given in the appendix, which contains the standard definitions used within this paper. The reader can find excellent introductions to hyperbolic conservation laws in a number of references, including [8], [9], [10], and [12].

Assume that ϕ is both nonnegative and monotone increasing and that T is continuously differentiable. The eigenvalues $\pm\sqrt{\phi}$ each have multiplicity two, and, hence, the system is nonstrictly hyperbolic. Thus, these characteristic families must necessarily be linearly degenerate (see [4]), and so the waves associated with these fields are contact discontinuities. This agrees with statements made by Cristescu in [2] that the speeds $\pm\sqrt{\phi}$ are characteristic of the propagation of transverse waves, or of changes in the shape of the string without changes in tension. Since $T_\varepsilon = \phi + r\phi'$, it is clear that in the neighborhood of a state where $r\phi' = 0$, the eigenvalues are no longer of constant multiplicity. In this paper, we limit ourselves to the study of eigenvalues of constant multiplicity and for this reason we assume that ϕ is strictly monotone increasing. We will later require the assumption that ϕ is convex, implying T_ε is strictly increasing. In reference [7] of Keyfitz and Kranzer, which describes a string in the plane, the characteristics have varying multiplicity; that is to say, the eigenvalues coalesce on a subset of phase space where $r\phi' = 0$. However, in an open region of phase space, the model for the string in the plane does not exhibit eigenvalues of constant multiplicity greater than one.

We shall now fix notation which will be used throughout the paper.

(i) For $V \in \mathfrak{R}^m$, let $\|V\|$ denote the Euclidean norm of V.

(ii) For $V(x) : \mathfrak{R} \to \mathfrak{R}^m$, let $\|V\|_{TV}$ be the total varation of V.

(iii) Let $U|_S$ denote the restriction of U to the set S.

2. The Riemann problem. Before discussing the Glimm scheme and wave interactions, let us first analyze the Rankine-Hugoniot relation. Here, and for the remainder of the paper, we use the notation $U^o \equiv P^o U = (U_1, U_3, U_5)$, where P^o is the projection onto the odd components of the vector U. In addition, let $U^e \equiv (U_2, U_4, U_6)$. Let us determine which states can be connected to a state U_L by a backward or a forward rarefaction, a backward or a forward shock, and by a backward or a forward linear wave. Let $\mathbf{r_i}$ be an eigenvector associated with the characteristic value λ_i.

For the nonlinear waves associated with the first and the last characteristic families, $\lambda_i = \pm\sqrt{T_\varepsilon}$ and has associated eigenvector $\mathbf{r_i}$ such that $\mathbf{r}_i^o = \frac{1}{\|U^o\|}U^o$ and $\mathbf{r}_i^e = -\lambda_i \mathbf{r}_i^o$. If U_R can be connected to U_L by a nonlinear wave, then it can be determined that

$$U_R^o = \frac{r_R}{r_L} U_L^o \quad \text{and} \quad U_R^e = U_L^e \pm \frac{A}{r_L} U_L^o, \tag{2}$$

where the last term is added in the case of a backward wave and subtracted in the case of a forward wave. Here,

$$A = \begin{cases} \sqrt{\dfrac{\phi_R r_R - \phi_L r_L}{r_R - r_L}}(r_R - r_L) & \text{for shocks,} \\ \displaystyle\int_L^R \sqrt{T_\varepsilon}\, dr & \text{for rarefaction waves,} \end{cases}$$

and $r_L = \|U_L^o\|$ and $r_R = \|U_R^o\|$.

For the linear waves, the two-dimensional eigenspace R_i associated with the characteristic field $\lambda_i = \pm\sqrt{\phi}$ satisfies $P^o R_i = \{U^o\}^\perp$ and $\mathbf{r}_i^e = \pm\sqrt{\phi}\mathbf{r}_i^o$ for $\mathbf{r}_i \in R_i$. If U_R is connected to U_L by a contact discontinuity, then for $i = 2, \ldots, 5$,

$$\lambda_i(U_R) \equiv \pm\sqrt{\phi_R} = \pm\sqrt{\phi_L} = \lambda_i(U_L).$$

This, along with the assumption that ϕ is strictly monotone, implies $r_L = r_R$. In addition, it can be determined from the Rankine-Hugoniot relation that

$$U_R^e = U_L^e \pm \sqrt{\phi_L}(U_R^0 - U_L^0), \tag{3}$$

where the speed s of the contact discontinuity is also $\pm\sqrt{\phi_L}$. Here, the sign in the last term is the plus sign in the case of a backward contact and the minus sign in the case of a forward contact.

In order to show existence and uniqueness of the solution to the Riemann problem, we show that there exists a unique parameterized curve connecting states U_L and U_R whenever U_R is within a small enough neighborhood of U_L. First, let us find the one-parameter family of states that can be connected to U_L by a rarefaction wave. A general theorem and the proof are given in Smoller [12], Lemma 17.8. Notably, the proof is independent of the existence of multiple eigenvalues in the other families.

THEOREM 2.1. There is a one-parameter family of states $U(\epsilon_i)$, which can be connected to U_L on the left by a rarefaction wave, so that $U(0) = U_L$ and $\frac{dU}{d\epsilon_i}|_{\epsilon_i=0} = \mathbf{r}_i(U_L)$ for $i = 1$ (a backward rarefaction wave) and for $i = 6$ (a forward rarefaction wave).

Proof. One can prove the result by choosing $\epsilon_i = \|U^o\| - r_L$ in Eq. (2). □

The other type of nonlinear wave that we need to analyze is the shock wave. An analysis of the one-parameter family of states that can be connected to a left state by a shock wave follows. The proof given here is specific to the particular problem that we are solving, the equations modeling the motion of an elastic string. In contrast to the general proof of existence of such a parameterized curve in Smoller [12] (pp. 328–330), this proof does not (and must not) require distinct or simple eigenvalues. In addition, the proof given here is not restricted to a small neighborhood of U_L.

THEOREM 2.2. There exists a one-parameter family of states that can be connected to U_L by a shock wave so that $U(0) = U_L$ and $\frac{dU}{d\epsilon_i}|_{\epsilon_i=0} = \mathbf{r}_i(U_L)$ for $i = 1$ (a backward shock) and for $i = 6$ (a forward shock.)

Proof. It follows from the Rankine-Hugoniot relation that, if U is connected to U_L on the left by a shock wave,

$$U^o = \left(\frac{s^2 - \phi_L}{s^2 - \phi(r)}\right) U_L^o, \tag{4}$$

and

$$U^e = U_L^e - s(U^o - U_L^o), \tag{5}$$

where $r = \|U^o\|$ and s, the speed of the shock wave, satisfies

$$s^2 = \frac{\phi(r)r - \phi_L r_L}{r - r_L}.$$

Moreover, the Lax entropy condition, along with the assumption that ϕ is monotone increasing, gives $\frac{s^2-\phi_L}{s^2-\phi(r)} > 0$ in the case of a shock. This, along with Eq. (4), shows that U^o and U_L^o are on the same radial line, and we can write U as a function of r:

$$U^o(r) = U_L^o + \left(\frac{r - r_L}{r_L}\right) U_L^o,$$

and

$$U^e(r) = U_L^e - s\left(\frac{r - r_L}{r_L}\right) U_L^o.$$

The proof is completed by letting $\epsilon_i = r - r_L$. □

In the previous two theorems, it was shown that there exist two one-parameter families of states that can be connected to U_L by nonlinear waves. The case of the linear waves must be handled a little differently. For each of the linear families, there will be a two-parameter family of curves that can be connected to U_L by a linear wave, i.e., a contact discontinuity. This naturally follows since each of the linear families possesses a two-dimensional eigenspace. In the proof, an *artificial* intermediate state will be introduced so that the two-parameter family can be thought of as a composition of two one-parameter families. Unlike the previous theorem, this theorem holds only locally, and the solutions exist only in a small neighborhood of U_L.

THEOREM 2.3. Assume that U_L is such that $U_L^o \neq 0$. There exist two-parameter families of states $U(\epsilon_i, \epsilon_{i+1})$, for $i = 2$ and 4, which can be connected to U_L on the left with an intermediate state U_I, such that $U_I = U(\epsilon_i, 0)$ and $U(0,0) = U_L$. Moreover, there exists a basis $\{\mathbf{r}_i, \mathbf{r}_{i+1}\}$ of the eigenspace associated with λ_i so that $\frac{dU_I}{d\epsilon_i}|_{(0,0)} = \mathbf{r}_i(U_L)$ and, for ϵ_i fixed, $\frac{dU}{d\epsilon_{i+1}}|_{(\epsilon_i,0)} = \mathbf{r}_{i+1}(U_I)$.

Proof. Since $r_L = \|U_L^o\| > 0$, there exists some component of U_L^o that is nonzero. Without loss of generality, we can assume that $(U_L^o)_1$ or $(U_L^o)_2$ is nonzero (otherwise implement a change of coordinates). Writing U_L^o in polar representation, let

$$U_L^o = \|U_L^o\|(\cos\theta_L \sin\psi_L, \sin\theta_L \sin\psi_L, \cos\psi_L),$$

with $\psi_L \in (0, \pi)$. Recall that the projection of the eigenspace of λ_i onto its odd components is the space $\{U^o\}^\perp$. Now writing U^o in polar representation, let

$$U^o = \|U^o\|(\cos\theta \sin\psi, \sin\theta \sin\psi, \cos\psi).$$

Now we have $\{U^o\}^\perp = \operatorname{span}\{\mathbf{r}_i^o, \mathbf{r}_{i+1}^o\}$, where we define the curvilinear coordinate system

$$\mathbf{r}_i^o = \|U_L^o\| \sin\psi_L(-\sin\theta, \cos\theta, 0)$$

and

$$\mathbf{r}_{i+1}^o = \|U_L^o\|(\cos\theta\cos\psi, \sin\theta\cos\psi, -\sin\psi).$$

For θ satisfying $|\theta - \theta_L| < \pi$, define $V(\theta)$ as the solution of the differential equation

$$\frac{dV}{d\theta} = \mathbf{r}_i^o(\theta) \quad \text{with } V(\theta_L) = U_L^o.$$

For $\epsilon_i \in (-\pi, \pi)$, define

$$U_I^o(\epsilon_i) \equiv V(\epsilon_i + \theta_L) \tag{6}$$

and

$$U_I^e(\epsilon_i) \equiv U_L^e \pm \sqrt{\phi_L}(U_I^o(\epsilon_i) - U_L^o). \tag{7}$$

Since λ_i is an eigenvalue associated with a linearly degenerate field, it follows that the derivative $\frac{d}{d\epsilon_i}(\lambda_i(U_I(\epsilon_i)) = 0$, implying $\lambda_i(U_I(\epsilon_i))$ is constant and equals $\lambda_i(U_I(0)) = \lambda_i(U_L)$. Thus, $U_I(\epsilon_i)$ is connected to U_L on the left by a backward or forward contact discontinuity when the last term of (7) is added or subtracted, respectively.

Now fixing θ, we define $W(\theta, \psi)$ to be the solution to the differential equation

$$\frac{dW(\theta,\psi)}{d\psi} = \mathbf{r}_{i+1}^o(\theta, \psi) \quad \text{with } W(\theta, \psi_L) = U_I^o(\theta).$$

For $\epsilon_{i+1} \in (-\psi_L, \pi - \psi_L)$, let

$$U^o(\epsilon_i, \epsilon_{i+1}) \equiv W(\epsilon_i + \theta_L, \epsilon_{i+1} + \psi_L) \tag{8}$$

and

$$U^e(\epsilon_i, \epsilon_{i+1}) \equiv U_I^e \pm \sqrt{\phi_I}(U^o(\epsilon_i, \epsilon_{i+1}) - U_I^o(\epsilon_i)), \tag{9}$$

where $\phi_I = \phi(\|U_I^o\|)$. Since λ_{i+1} is an eigenvalue associated with a linearly degenerate field, it follows by the construction of $U(\epsilon_i, \epsilon_{i+1})$ that $\lambda_{i+1}(U(\epsilon_i, \epsilon_{i+1}))$ is constant and must equal $\lambda_{i+1}(U_I(\epsilon_i))$. Thus, $U(\epsilon_i, \epsilon_{i+1})$ is connected to U_I on the left by a contact discontinuity. Using the fact that $\lambda_{i+1}(U_I) = \lambda_i(U_L)$,

$$U^e(\epsilon_i, \epsilon_{i+1}) \equiv U_L^e \pm \sqrt{\phi_L}(U^o(\epsilon_i, \epsilon_{i+1}) - U_L^o), \tag{10}$$

and we see that $U(\epsilon_i, \epsilon_{i+1})$ is connected to U_L on the left by a contact discontinuity with speed $\pm\sqrt{\phi_L}$. This completes the construction of a two-parameter family $U(\epsilon_i, \epsilon_{i+1})$ that can be connected to U_L on the left by a contact discontinuity with artificial intermediate state U_I. □

In the previous three theorems, we characterized the parameterized curves of those states that can be connected to U_L on the left by a rarefaction, shock, or linear wave. Now that we have established that these curves exist and have derived their Taylor expansions up to the first order at $\epsilon_i = 0$, we will use the inverse function theorem to show that the composition of these curves is an invertible map. In this manner, we will be able to show existence and uniqueness for the Riemann problem given any two states U_L and U_R sufficiently close. The following theorem and proof is the classical one as given in Smoller [12]. What actually distinguishes this result from the classical one is the introduction of two-parameter families to represent two-dimensional surfaces of states connected by a contact discontinuity. The general theorem, as given in [12], requires that the conservation law be strictly hyperbolic which is not the case here.

THEOREM 2.4. For $\|U_L^o\| > 0$, we can find a constant $\delta > 0$, depending only on the nonlinear flux function F, such that there exists a unique solution to the Riemann problem whenever $\|U_L - U_R\| < \delta$.

Proof. Define $T_{\epsilon_i}(U_L)$ to be the parameterized curves described in the previous theorems 2.1, 2.2, and 2.3. Here, for $i = 2, 4$, $T_{\epsilon_i} = U_I(\epsilon_i)$ and $T_{\epsilon_{i+1}} = U(0, \epsilon_{i+1})$. Now

$$T_{\epsilon_i}(U_L) = U_L + \epsilon_i \mathbf{r}_i(U_L) + \mathcal{O}(\epsilon_i^2). \tag{11}$$

Let $\mathcal{B}$ be a neighborhood of U_L in which all the $T_{\epsilon_i}(U_L)$ are defined. Now define the function $\mathcal{F} : \mathcal{B} \to \Re^6$ by

$$\mathcal{F}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6) = T_{\epsilon_6} T_{\epsilon_5} \cdots T_{\epsilon_2} T_{\epsilon_1}(U_L) - U_L.$$

Using the Taylor expansion in (11), it can be shown that

$$d\mathcal{F}(0) = (\mathbf{r}_1(U_L), \mathbf{r}_2(U_L), \mathbf{r}_3(U_L), \mathbf{r}_4(U_L), \mathbf{r}_5(U_L), \mathbf{r}_6(U_L)),$$

which is nonsingular since the $\{\mathbf{r}_i\}$ form a linearly independent set at U_L. Thus, using the Inverse Function Theorem, there exists a neighborhood of $\epsilon = 0$ in which $\mathcal{F}$ is invertible and the theorem holds. □

This theorem shows that the Riemann problem is uniquely solvable in a sufficiently small neighborhood of U_L. Using solutions of the Riemann problem, we will be able to adapt the Glimm scheme to show existence of weak solutions to the multiply characteristic elastic string problem. In the next section, we give a brief description of the Glimm scheme.

3. Glimm's scheme. Consider the Cauchy problem for a system of conservation laws

$$v_t + f(v)_x = 0,$$

with

$$v(x, 0) = v_0(x).$$

Given some sufficiently small value of time, say t', an approximation to $v(x,t)$ at time t' is given by the solution to the conservation law system

$$v_t + f(v)_x = 0,$$

with

$$v(x,0) = \tilde{v}_0(x),$$

where $\tilde{v}_0$ is a piecewise constant approximation to the initial data v_0. An approximate solution to the problem is constructed by solving a Riemann problem at every discontinuity in $\tilde{v}_0$. This process forms the fundamental basis for each time step of Glimm's scheme. Let $l = \Delta x$ be a discretization of space and $k = \Delta t$ be a discretization of time. Glimm's scheme generates an approximate solution at (x,t), say $u_h(x,t)$, where the mesh size $h = (l,k)$. Letting $\theta \equiv \{\theta_i\}$ be a sequence of random numbers from a uniform distribution on $[-1,1]$, an algorithmic description of Glimm's scheme is given below. A more detailed explanation can be found in the paper by Glimm (see [5]).

Glimm's Scheme

(i) Initialize $u_h(x,0)$ to be piecewise constant: For x in the interval $[(2m-1)l, (2m+1)l]$, set $u_h(x,0) = u_0((2m+\theta_0)l)$.

(ii) For $n = 1,2,\ldots$ define $u_h(x,t)$ in the nth time interval: When $m+n$ is even, solve the following Riemann problem:

$$v_t + f(v)_x = 0,$$
$$v(x,(n-1)k) = \begin{cases} u_h((m-1)l,(n-1)k) & \text{for } x < ml, \\ u_h((m+1)l,(n-1)k) & \text{for } x > ml. \end{cases}$$

Set $u_h(x,t) \equiv v(x,t)$ when $x \in [(m-1)l,(m+1)l]$ and $t \in ((n-1)l, nk)$.

(iii) Choose piecewise constant data for the next time step. Choose $u_h(x,nk)$ to have constant value $u_h(x,nk) = v((m+\theta_n)l,nk)$ on the interval $[(m-l)l,(m+1)l]$, when $m+n$ is even.

At any time $t = nk$, Glimm's scheme generates an approximate solution $u_h(x,nk)$, which is constructed to be constant on every interval (or cell) $[(m-1)l,(m+1)l]$, where $m+n$ is even. Between time steps, the approximate solution, $u_h(x,t)$, is the exact solution to Riemann problems solved at the cell interfaces of the previous time step. In addition, since it is necessary to avoid wave interactions between Riemann problems, the ratio k/l is chosen to be smaller than the reciprocal of the largest possible wave speed. This condition is commonly known as the *Courant-Friedrichs-Lewy condition* or *CFL condition* (see for instance [8] and [10] among numerous other references).

Consider the set $\{a_{m,n}\}$ consisting of the randomly sampled points $a_{m,n} \equiv ((m+\theta_n)l, nk)$ with $m+n$ even. The set $\{a_{m,n}\}$ contains the points that are sampled to determine the piecewise constant approximation u_h at each time step n. Note that we can consider $\{a_{m,n}\}$ as forming a discrete mesh. Since it will later be of value to estimate wave interactions across this mesh, we now introduce the following definition.

DEFINITION 3.1. A mesh curve is a piecewise linear curve consisting of a set of mesh points and the line segments joining them, where, if $a_{m,n}$ belongs to the curve, then either $a_{m+1,n-1}$ or $a_{m+1,n+1}$ belongs to the curve but not both.

We introduce a partial order "$\succ$" on mesh curves, as follows. We say the mesh curve $I_1 \succ I_2$ if every point of I_1 is a point of I_2 or lies above the mesh curve I_2. A mesh curve I is called an immediate successor to J if $I \succ J$ and every mesh point of I except one is on J. In addition, we let O be the unique mesh curve that passes through the mesh points on $t = 0$ and $t = \Delta t$.

4. Interaction estimates and bounds on growth. To show that the Glimm scheme produces a convergent subsequence, we use the classical technique developed for strictly hyperbolic conservation laws (see [5]). The proof is broken down into two components: first we show that the Glimm scheme produces a convergent subsequence of approximate solutions and second, we demonstrate that this subsequence converges to a weak solution. This will rely upon the application of Helly's theorem. Loosely stated, Helly's theorem guarantees that if a set of functions is both uniformly bounded and has uniformly bounded total variation, then it is relatively compact and, hence, contains a convergent subsequence (see [3]). To satisfy the hypothesis of Helly's theorem, we must show that the approximate solutions generated by the Glimm scheme, $\{U_h\}$, are uniformly bounded in L^∞ as well as uniformly bounded in total variation. In this section, we will prove that the total variation is equivalent to a metric measuring the wave strengths, and that this equivalent metric stays bounded. Once these bounds are shown, we have that the Glimm scheme is defined for all time $t \in \Re^+$.

Consider data U_L, U_R, and U_M. We assume that $\|U_L^o\| \geq 0$, and that U_L, U_M, U_R are sufficiently close together that all Riemann problems that arise in the Glimm scheme are uniquely solvable. Here we will use the parameters from the curves T_{ϵ_i} to describe the solutions to the following Riemann problems:

$$(U_L, U_R) = [(U_0, U_1, \ldots, U_6)/(\epsilon_1, \epsilon_2, \ldots, \epsilon_6)], \tag{12}$$

$$(U_L, U_M) = [(U_0', U_1', \ldots, U_6')/(\gamma_1, \gamma_2, \ldots, \gamma_6)], \tag{13}$$

$$(U_M, U_R) = [(U_0'', U_1'', \ldots, U_6'')]/(\delta_1, \delta_2, \ldots, \delta_6). \tag{14}$$

In an abuse of notation, we will refer to ϵ_i as the i-wave connecting the states U_{i-1} and U_i. Recall that the parameters associated with the nonlinear families ($i = 1, 6$) are $\epsilon_i = \|U_i^o\| - \|U_{i-1}^o\|$. Regarding the linear families, it is important to note that the states U_2 and U_4 are both artificial states created in the parameterization of states connected by a contact discontinuity. In Theorem 2.3, U_2 and U_4 correspond to the artificial state U_I. Thus, for the Riemann problem (U_L, U_R), the state U_1 is connected directly to U_3 by a backward contact, and then U_3 is connected directly to U_5 by a forward contact.

Without loss of generality, we can also assume that one of the first two components of U_L^o, either$(U_L^o)_1$ or $(U_L^o)_2$ is nonzero. In a sufficiently small neighborhood of U_L, using the two-parameter curves of states connected by a contact discontinuity, we observe that U_3^o is obtained from U_1^o by rotating by ϵ_2 in the azimuth (i.e., counter-clockwise in the horizontal direction for ϵ_2 positive) and by changing the vertical elevation by angle ϵ_3.

Likewise, U_5^o can be obtained from U_3^o by rotating by ϵ_4 in the azimuth and then by changing the vertical elevation by angle ϵ_5.

We now develop an interaction estimate that bounds the strength of the ϵ_i wave of the Riemann problem (U_L, U_R) by the interactions of waves γ_i and δ_i of the Riemann problems (U_L, U_M) and (U_M, U_R).

THEOREM 4.1. Given U_L, U_M, and U_R as in (12), (13), and (14), then

$$\epsilon_i = \gamma_i + \delta_i + \mathcal{O}(|\gamma|\,|\delta|). \tag{15}$$

Proof. See Smoller [12]. Although the proof is for strictly hyperbolic conservation laws with distinct eigenvalues, the proof of our theorem can be made similar by the introduction of the artificial intermediate states U_2 and U_4. □

The preceding result is not strong enough to guarantee that all wave interactions are bounded. We can improve upon the estimate of Theorem 4.1 by showing that the difference between the sum of the interacting waves γ and δ and the resulting wave ϵ is actually of the order of quadratic terms involving *only* the approaching waves. We will need to be a little careful in our definition of approaching waves since the elastic string problem we are solving possesses multiple eigenvalues. We say that the i-wave γ_i approaches the j-wave δ_j if either

(i) $\lambda_i > \lambda_j$,

or

(ii) $i = j$ and one of the waves is a shock wave.

Note that for approaching waves of type (ii), one of the waves is a shock, which implies that $i = 1$ or 6. It is the first case that distinguishes our definition of approaching waves from the classical one (see [5]). Notice that if $i = j + 1$ and $i = 3$ or 5, then the wave γ_i travels with the same speed as the wave δ_j and these waves *do not* interact. The method of proof is similar to the one given in Smoller [12], but with the variation on the types of approaching waves.

THEOREM 4.2. Letting U_L, U_M, and U_R be defined as in (12), (13), and (14), then

$$\epsilon_i = \gamma_i + \delta_i + \mathcal{O}(1)D(\gamma, \delta) \quad \text{as } |\gamma| + |\delta| \to 0, \tag{16}$$

where $D(\gamma, \delta) = \sum_{(i,j)\in\mathcal{A}} |\gamma_i|\,|\delta_j|$ and $\mathcal{A} = \{(i,j) :$ the γ_i wave approaches the δ_j wave$\}$.

Proof. The proof proceeds by induction. First consider the case when $D = 0$. Suppose that $\gamma = (\gamma_1, \dots, \gamma_k, 0, \dots, 0)$. Then the interactions are of the following form depending on k:

(i) Case 1) $k = 2$ or 4. Since $D = 0$, this implies that the wave $\delta = (0, \dots, 0, \delta_k, \dots, \delta_6)$. The state $U'_k = U_M$ is connected to U'_{k-1} by the contact γ_k, and thus U_M^o can be obtained by rotating the azimuth of $(U'_{k-1})^o$ by angle γ_k. In addition, the state $U''_{k-1} = U_M$ is connected to U''_k by the contact δ_k, and so $(U''_k)^o$ can be obtained by rotating the azimuth of $(U''_{k-1})^o$ by δ_k. Thus, U''_k is connected to U'_{k-1} by a contact ϵ_k. Moreover, $(U''_k)^o$ can be obtained by rotating $(U'_{k-1})^o$ by $\epsilon_k = \gamma_k + \delta_k$. In addition, $\epsilon_i = \gamma_i$ for all $i < k$ and $\epsilon_i = \delta_i$ for all $i > k$. So Eq. (16) is satisfied in this case.

(ii) Case 2) $k = 3$ or 5. Since $D = 0$, we determine that $\delta = (0, \dots, 0, \delta_{k-1}, \dots, \delta_6)$. Note that $U'_k = U_M$ is connected to U'_{k-2} by a contact discontinuity with artificial intermediate state U'_{k-1}. The state U^o_M can be obtained by rotating the azimuth of $(U'_{k-2})^o$ by γ_{k-1} and then by changing the vertical elevation by γ_k. In addition, U''_k is connected to $U_M = U''_{k-2}$ by a contact discontinuity with artificial intermediate state U''_{k-1}. This implies that $(U''_k)^o$ can be obtained from U^o_M by rotating the azimuth by δ_{k-1} and then by changing the elevation by δ_k. Thus, U''_k can be connected to U'_{k-2} by a contact discontinuity. Moreover, since $(U''_k)^o$ can be obtained by rotating the azimuth of $(U'_{k-2})^o$ by $\gamma_{k-1}+\delta_{k-1}$ and then by changing the vertical elevation by $\gamma_k+\delta_k$, it follows that $\epsilon_{k-1} = \gamma_{k-1}+\delta_{k-1}$ and $\epsilon_k = \gamma_k + \delta_k$. In addition, $\epsilon_i = \gamma_i$ for all $i < k-1$ and $\epsilon_i = \delta_i$ for all $i > k$. Thus, Eq. (16) is satisfied in the second case.

(iii) Case 3) $k = 1$ or 6. There are two possibilities for this case: either γ_k is a shock or it is a rarefaction wave. If γ_k is a rarefaction, then, since $D = 0$, this implies that δ_k cannot be a shock. So in this case the two rarefaction waves combine, and we have $\epsilon_i = \gamma_i$ for all $i < k$, $\epsilon_i = \delta_i$ for all $i > k$, and $\epsilon_k = \gamma_k + \delta_k$. In the final case, if $k = 6$ and γ_6 is a shock wave, then $\delta = 0$ since $D = 0$, and the theorem is trivially true ($\epsilon_i = \gamma_i$). Otherwise, if $k = 1$ and γ_1 is a shock wave, then $\delta = (0, \delta_2, \ldots, \delta_6)$. In this case, there is no wave interaction, and the two waves simply add, so that $\epsilon_i = \gamma_i$ for $i \le k$ and $\epsilon_i = \delta_i$ if $i > k$.

Thus we have shown that if $D = 0$ the theorem is true. The remainder of the proof proceeds by induction on the δ-wave and is similar to the proof in Smoller ([12], Theorem 19.2). □

We utilize the preceding interaction estimate to define a functional L that bounds the sequence $\{U_h\}$. Once this is completed, we show that the metric L is equivalent to the total variation. Define the following functionals:

$$L(J) = \sum \{|\epsilon| : \epsilon \text{ crosses } J\},$$

$$Q(J) = \sum \{|\alpha|\,|\beta| : \alpha \text{ and } \beta \text{ cross } J \text{ and } \alpha \text{ approaches } \beta\}.$$

THEOREM 4.3. Let I and J be mesh curves with $J \succ I$. If $L(I)$ is sufficiently small, then $Q(I) \ge Q(J)$, and there exists a constant k independent of J, such that $L(J) + kQ(J) \le L(I) + kQ(I)$.

Proof. The proof is identical to the one given in Smoller [12], and differs only with regard to our different definition of approaching waves. □

Since any mesh curve J is a successor of the unique mesh curve O, $J \succ O$, and we obtain

$$L(J) + kQ(J) \le L(O) + kQ(O).$$

Using the fact that $Q(J) \le (L(J))^2$ for any J-curve, this inequality becomes

$$L(J) + kQ(J) \le L(O) + k(L(O))^2.$$

Thus, if $L(O) < 1/k$, then

$$L(J) + kQ(J) \le 2L(O). \tag{17}$$

We have now shown that for any mesh curve J, $L(J) \le 2L(O)$. We later show that the assumption that $L(O)$ is small is equivalent to assuming that the total variation of the initial data is small.

Now that we have shown that the constructed metric L is bounded, we need to show that it is equivalent to the total variation norm. This will give the desired result that the total variation of the approximate solutions is uniformly bounded. Rather than working with the parameters ϵ_i to measure wave strengths, we will now choose the parameters as in [11]. Until now it was convenient to think of the Riemann solution as being determined by the six parameters ϵ_i, $i = 1, \dots, 6$. In particular, states connected by a contact discontinuity required two parameters. We now proceed to define the parameters of the Riemann solution as follows: letting U_L and U_R be two states that are sufficiently close, rewrite the solution to the Riemann problem as

$$(U_L, U_R) = [(U_L = U_0, U_1, U_2, U_3, U_4 = U_R)/(\tilde{\epsilon}_1, \tilde{\epsilon}_2, \tilde{\epsilon}_3, \tilde{\epsilon}_4)].$$

We now define the wave strengths $\tilde{\epsilon}_i$. Define the backward nonlinear wave as before, so that the states U_0 and U_1, which are connected by a shock or rarefaction, have associated wave strength $\tilde{\epsilon}_1 = \|U_1^o\| - \|U_0^o\|$. Likewise, states U_3 and U_4, which are connected by a shock or rarefaction, have associated wave strength $\tilde{\epsilon}_4 = \|U_4^o\| - \|U_3^o\|$. It is the parameters for the linear families that will be chosen differently. The states U_1 and U_2 are connected by a contact discontinuity, and hence $\|U_1^o\| = \|U_2^o\|$. We will define the wave strength to be the angle between the odd components of the two states so that $\tilde{\epsilon}_2 = \arccos(U_1^o \cdot U_2^o/\|U_1^o\|^2)$. Similarly, states U_3 and U_2 are connected by a contact discontinuity which implies $\|U_3^0\| = \|U_2^0\|$. Define the associated wave strength to be $\tilde{\epsilon}_3 = \arccos(U_3^o \cdot U_2^o/\|U_3^o\|^2)$. Recall that in using the six-parameter family, we used two parameters to describe a linear wave: one parameter corresponding to a change in azimuth of the odd components and the other corresponding to a vertical, or longitudinal change of the odd components. This was necessary to obtain the interaction estimate in Theorem 4.1. It is simpler, however, to deal with the four parameters to show the equivalence relation.

Define the functional $\mathcal{L}$ so that

$$\mathcal{L}(J) = \sum \{|\tilde{\epsilon}| : \tilde{\epsilon} \text{ crosses } J\}.$$

We will now show that the two metrics $\mathcal{L}$ and L are equivalent.

LEMMA 4.1. There exist positive constants A_1 and A_2, both independent of the mesh size and U_h, so that for any mesh curve J,

$$A_1 L(J) \leq \mathcal{L}(J) \leq A_2 L(J).$$

Proof. Consider the following Riemann problem where the states U_L and U_R are sufficiently close to give the unique solution to the Riemann problem described as

$$(U_L, U_R) = [(U_L = U_0, U_1, U_2, \dots, U_5, U_6 = U_R)/(\epsilon_1, \dots, \epsilon_6)],$$

or

$$(U_L, U_R) = [(U_L = u_0, u_1, u_2, u_3, u_4 = U_R)/(\tilde{\epsilon}_1, \dots, \tilde{\epsilon}_4)].$$

Recall that the states U_2 and U_4 were artificially introduced in the construction of the two-parameter family characterizing the states connected by a contact discontinuity. By uniqueness of the solution to the Riemann problem, $U_1 = u_1, U_3 = u_2$, and $U_5 = u_3$. It clearly follows that for the backward nonlinear waves,

$$\epsilon_1 = \|U_1^o\| - \|U_0^o\| = \|u_1^o\| - \|u_0^o\| = \tilde{\epsilon}_1. \tag{18}$$

Similarly, for the forward nonlinear waves,

$$\epsilon_6 = \|U_6^o\| - \|U_5^o\| = \|u_4^o\| - \|u_3^o\| = \tilde{\epsilon}_4. \tag{19}$$

Letting $\angle(u, v) \in [0, \pi)$ denote the angle between u and v, then we have $\angle(U_i^o, U_{i+1}^o) = |\epsilon_i|$ and $\angle(U_{i+1}^o, U_{i+2}^o) = |\epsilon_{i+1}|$ for $i = 2, 4$. Clearly,

$$\tilde{\epsilon}_2 = \angle(u_1^o, u_2^o) = \angle(U_1^o, U_3^o) \le |\epsilon_2| + |\epsilon_3|. \tag{20}$$

Similarly, we obtain

$$\tilde{\epsilon}_3 \le |\epsilon_4| + |\epsilon_5|. \tag{21}$$

Now, we will show that there exists a constant so that

$$\tilde{\epsilon}_2 > A_1(|\epsilon_2| + |\epsilon_3|).$$

Since U_1 and U_3 are connected by a linear wave, $\|U_1^o\| = r = \|U_3^o\|$. Let

$$U_1^o = r(\cos\theta_1 \sin\Psi_1, \sin\theta_1 \sin\Psi_1, \cos\Psi_1)$$

and

$$U_3^o = r(\cos\theta_3 \sin\Psi_3, \sin\theta_3 \sin\Psi_3, \cos\Psi_3).$$

We can then write

$$\tilde{\epsilon}_2 > \frac{1}{r}\|U_1^o - U_3^o\| = 2\left(\sin^2\left(\frac{\Psi_3 - \Psi_1}{2}\right) + \sin(\Psi_1)\sin(\Psi_3)\sin^2\left(\frac{\theta_3 - \theta_1}{2}\right)\right)^{1/2}.$$

Using the simple fact that $|\sin(\alpha/2)| \ge |\alpha|/\pi$ for $\alpha \in [-\pi, \pi]$, we obtain

$$\tilde{\epsilon}_2 > \frac{2}{\pi}|\epsilon_3|.$$

Since $|\epsilon_2| \in [0, \pi)$, we now have

$$|\epsilon_2| = \angle(U_1^o, U_2^o) \le \angle(U_1^o, U_3^o) + \angle(U_2^o, U_3^o) = |\tilde{\epsilon}_2| + |\epsilon_3|,$$

which implies that

$$|\epsilon_3| + |\epsilon_2| \le 2|\epsilon_3| + |\tilde{\epsilon}_2| \le (\pi + 1)|\tilde{\epsilon}_2|.$$

Thus, we have shown for $A_1 = \frac{1}{\pi+1}$,

$$A_1(|\epsilon_3| + |\epsilon_2|) \le |\tilde{\epsilon}_2|, \tag{22}$$

and similarly, one can show

$$A_1(|\epsilon_4| + |\epsilon_5|) \le |\tilde{\epsilon}_3|. \tag{23}$$

Thus, Eqs. (18)–(23) imply that

$$\sum\{A_1|\epsilon_i| : \epsilon_i \text{ crosses } J\} \le \sum\{|\tilde{\epsilon}_i| : \tilde{\epsilon}_i \text{ crosses } J\} \le \sum\{|\epsilon_i| : \epsilon_i \text{ crosses } J\}.$$

Taking $A_2 = 1$, this gives the final result. □

Since the two metrics $L(J)$ and $\mathcal{L}(J)$ are equivalent, and $L(J) < 2L(O)$ (by Eq. (17)), it follows that

$$\mathcal{L}(J) < \frac{2}{A_1}\mathcal{L}(O). \tag{24}$$

We now show that $\mathcal{L}(O)$ is bounded by a constant independent of the mesh size, but depending only on the bounds on the initial data. Once we establish this fact, by (24), we will have shown that metric $\mathcal{L}(J)$ is also bounded by a constant independent of the mesh size.

LEMMA 4.2. Suppose there exist constants c_1 and c_2 so that the initial data U_0 satisfies
(i) $c_1 \leq \|U_0^o\| \leq c_2$,
(ii) $\|U_0^e\|_{TV} \leq \int_{\frac{c_1}{2}}^{c_1} \sqrt{\phi}$,
and
(iii) U_0^o is contained in some octant of 3-space.
Then there exists a positive constant κ, independent of the mesh size, so that

$$\mathcal{L}(O) \leq \kappa \|U_0\|_{TV}.$$

Proof. Let $\overline{U}_0$ be the piecewise approximation to the initial data U_0. Recall that $\mathcal{L}(O)$ is the sum of the magnitude of the $\tilde{\epsilon}_i$ waves that cross the mesh curve O. These same waves result from the Riemann problems induced at the discontinuities in $\overline{U}_0$. Consider a single discontinuity in $\overline{U}_0$ with the two neighboring states U_L and U_R, to the left and right respectively. It is sufficient to show that

$$\sum_{i=1}^{4} |\tilde{\epsilon}_i| \leq \kappa \|U_L - U_R\|, \tag{25}$$

where the solution to the Riemann problem with left and right state U_L and U_R is denoted by

$$(U_L, U_R) = [(U_L = u_0, u_1, u_2, u_3, u_4 = U_R)/(\tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_4)].$$

Hence, we require that

$$|r_2 - r_L| + \theta_{L2} + \theta_{2R} + |r_R - r_2| \leq \kappa \|U_L - U_R\|, \tag{26}$$

where $r_L = \|U_L^o\|, r_2 = \|u_2^o\|, r_R = \|U_R^o\|$, θ_{L2} is the angle from U_L^o to u_2^o, and θ_{2R} is the angle from u_2^o to U_R^o. Using the assumption that ϕ is strictly monotone and convex, we have that T_ε is monotone increasing and that the Riemann problem can be characterized by r_L, r_R, and r_2. We examine the following cases:

(i) Case 1: $r_L \leq r_2 \leq r_R$. In this case, the left-hand side of (26) can be written as

$$\begin{aligned}\sum_{i=1}^{4} |\tilde{\epsilon}_i| &\leq r_R - r_L + \pi\left(\sin\frac{\theta_{L2}}{2} + \sin\frac{\theta_{2R}}{2}\right) \\ &= r_R - r_L + \frac{\pi}{2r_2}\left(\left\|u_2^o - \frac{r_2}{r_L}U_L^o\right\| + \left\|u_2^o - \frac{r_2}{r_R}U_R^o\right\|\right).\end{aligned} \tag{27}$$

In the event that $r_L \leq r_2 \leq r_R$, it can be observed that

$$2\sqrt{\phi_2}u_2^o = (U_R^e - U_L^e) + \frac{1}{r_R}\left(r_2\sqrt{\phi_2} + \int_{r_2}^{r_R}\sqrt{T_\varepsilon}\right)U_R^o + \frac{1}{r_L}(r_2\sqrt{\phi_2} - |s|(r_2 - r_L))U_L^o,$$

where s is the speed of the backward shock and $\phi_2 = \phi(r_2)$. Now using the Lax entropy condition, we obtain

$$\begin{aligned}\left\|u_2^o - \frac{r_2}{r_L}U_L^o\right\| &\leq \frac{r_2}{2}\left\|\frac{1}{r_R}U_R^o - \frac{1}{r_L}U_L^o\right\| \\ &\quad + \frac{1}{2\sqrt{\phi_2}}\left(\int_{r_2}^{r_R}\sqrt{T_\varepsilon} + \sqrt{T_\varepsilon|_{r=r_2}}(r_2 - r_L) + \|U_R^e - U_L^e\|\right).\end{aligned}$$

Since T_ε is monotone increasing,

$$\begin{aligned}\left\|u_2^o - \frac{r_2}{r_L}U_L^o\right\| &\leq \frac{r_2}{2}\left\|\frac{1}{r_R}U_R^o - \frac{1}{r_L}U_L^o\right\| \\ &\quad + \frac{1}{2\sqrt{\phi_2}}(\sqrt{T_\varepsilon|_{r=r_R}}(r_R - r_L) + \|U_R^e - U_L^e\|).\end{aligned} \tag{28}$$

Similarly, the same bounds can be shown for $\|u_2^o - \frac{r_2}{r_R}U_R^o\|$. Now using the fact that $r_R, r_2, r_L \in [c_1, c_2]$, along with (28), Eq. (27) gives

$$\begin{aligned}\sum_{i=1}^{4}|\tilde{\epsilon}_i| &\leq \left(1 + \frac{\pi\sqrt{T_\varepsilon|_{r=c_2}}}{2c_1\sqrt{\phi(c_1)}}\right)(r_R - r_L) \\ &\quad + \frac{\pi}{2}\left\|\frac{1}{r_R}U_R^o - \frac{1}{r_L}U_L^o\right\| + \frac{\pi}{2c_1\sqrt{\phi(c_1)}}\|U_R^e - U_L^e\|.\end{aligned}$$

Result (26) follows.

(ii) Case 2: $r_R \leq r_2 \leq r_L$. In this event we obtain

$$2\sqrt{\phi_2}u_2^o = (U_R^e - U_L^e) + \frac{1}{r_L}\left(\sqrt{\phi_2} + \int_{r_2}^{r_L}\sqrt{T_\varepsilon}\right)U_L^o + \frac{1}{r_R}(\sqrt{\phi_2} - |s|(r_2 - r_R))U_R^o,$$

where s is the speed of the forward shock and $\phi_2 = \phi(r_2)$. The proof is identical to Case 1.

(iii) Case 3: $r_2 > r_L, r_R$. Here, the left-hand side of Eq. (26) becomes

$$\begin{aligned}\sum_{i=1}^{4}|\tilde{\epsilon}_i| &\leq 2r_2 - r_R - r_L + \pi\left(\sin\frac{\theta_{L2}}{2} + \sin\frac{\theta_{2R}}{2}\right) \\ &= 2r_2 - r_L - r_R + \frac{\pi}{2r_2}\left(\left\|u_2^o - \frac{r_2}{r_L}U_L^o\right\| + \left\|u_2^o - \frac{r_2}{r_R}U_R^o\right\|\right).\end{aligned} \tag{29}$$

Using the assumption that ϕ is convex and strictly increasing, we observe that the first and last waves are shocks and derive

$$\begin{aligned}2\sqrt{\phi(r_2)}u_2^o &= (U_R^e - U_L^e) + \frac{1}{r_L}(r_2\sqrt{\phi_2} - |s_L|(r_2 - r_L))U_L^o \\ &\quad + \frac{1}{r_R}(r_2\sqrt{\phi_2} - |s_R|(r_2 - r_R))U_R^o,\end{aligned} \tag{30}$$

where s_L is the speed of the backward shock, s_R is the speed of the forward shock, and $\phi_2 = \phi(r_2)$. Using the Lax entropy condition, Eq. (30) can be used to derive

$$\left\| u_2^o - \frac{r_2}{r_L} U_L \right\| \leq \frac{1}{2\sqrt{\phi_2}} (\|U_R^e - U_L^e\| + \sqrt{T_\varepsilon|_{r=r_2}} (2r_2 - r_L - r_R)) + \left(r_2 + \frac{1}{2\sqrt{\phi_2}} \sqrt{T_\varepsilon|_{r=r_2}} (r_2 - r_L) \right) \sin \frac{\theta_{LR}}{2},$$

where θ_{LR} is the angle from U_L^o to U_R^o. Using the Lax entropy condition along with the assumption that ϕ is increasing, Eq. (30) can also be used to obtain

$$\begin{aligned} r_2 &\leq \frac{1}{2\sqrt{\phi(c_1)}} \|U_L^e - U_R^e\| + \frac{r_L + r_R}{2} \\ &\leq \frac{1}{2\sqrt{\phi(c_1)}} \left(\int_{c_1/2}^{c_1} \sqrt{\phi} \right) + c_2 \equiv c_3. \end{aligned} \tag{31}$$

Since $r_L \geq c_1$ and ϕ is both convex and strictly increasing, we have

$$\left\| u_2^o - \frac{r_2}{r_L} U_L \right\| \leq \frac{1}{2\sqrt{\phi(c_1)}} (\|U_R^e - U_L^e\| + \sqrt{T_\varepsilon|_{r=c_3}} (2r_2 - r_L - r_R)) + \left(c_3 + \frac{1}{2\sqrt{\phi(c_1)}} \sqrt{T_\varepsilon|_{r=c_3}} (c_3 - c_1) \right) \sin \frac{\theta_{LR}}{2}.$$

Similarly, we obtain

$$\left\| u_2^o - \frac{r_2}{r_R} U_R \right\| \leq \frac{1}{2\sqrt{\phi(c_1)}} (\|U_R^e - U_L^e\| + \sqrt{T_\varepsilon|_{r=c_3}} (2r_2 - r_L - r_R)) + \left(c_3 + \frac{1}{2\sqrt{\phi(c_1)}} \sqrt{T_\varepsilon|_{r=c_3}} (c_3 - c_1) \right) \sin \frac{\theta_{LR}}{2}.$$

Hence, there exist positive constants $\kappa_1, \kappa_2, \kappa_3$ so that Eq. (29) can be bounded as follows:

$$\sum_{i=1}^{4} |\tilde{\epsilon}_i| \leq \kappa_1 (2r_2 - r_L - r_R) + \kappa_2 \|U_R^e - U_L^e\| + \kappa_3 \sin \frac{\theta_{LR}}{2}. \tag{32}$$

Now substituting the first part of inequality (31), we obtain

$$\begin{aligned} \sum_{i=1}^{4} |\tilde{\epsilon}_i| &\leq \left(\frac{1}{\sqrt{\phi(c_1)}} \kappa_1 + \kappa_2 \right) \|U_R^e - U_L^e\| + \kappa_3 \sin \frac{\theta_{LR}}{2} \\ &\leq \left(\frac{1}{\sqrt{\phi(c_1)}} \kappa_1 + \kappa_2 \right) \|U_R^e - U_L^e\| + \frac{\kappa_3}{2c_1} \|U_R^o - U_L^o\|. \end{aligned}$$

The result (26) follows.

(iv) Case 4: $r_2 < r_L, r_R$. Here, the left-hand side of Eq. (26) becomes

$$\sum_{i=1}^{4} |\tilde{\epsilon}_i| \leq 2r_2 - r_L - r_R + \frac{\pi}{2r_2} \left(\left\| u_2^o - \frac{r_2}{r_L} U_L^o \right\| + \left\| u_2^o - \frac{r_2}{r_R} U_R^o \right\| \right). \tag{33}$$

Using the assumption that ϕ is convex and strictly increasing, we observe that the first and last waves are rarefaction waves so that

$$2\sqrt{\phi_2}u_2^o = (U_R^e - U_L^e) + \frac{1}{r_L}\left(r_2\sqrt{\phi_2} + \int_{r_2}^{r_L}\sqrt{T_\varepsilon}\right)U_L^o + \frac{1}{r_R}\left(r_2\sqrt{\phi_2}\int_{r_2}^{r_R}\sqrt{T_\varepsilon}\right)U_R^o, \tag{34}$$

where $\phi_2 = \phi(r_2)$. Since the assumptions on ϕ imply that T_ε is monotone increasing, Eq. (34) implies that

$$\begin{aligned}\left\|u_2^o - \frac{r_2}{r_L}U_L^o\right\| \le \frac{r_2}{2}&\left\|\frac{1}{r_R}U_R^o - \frac{1}{r_L}U_L^o\right\| \\ &+ \frac{1}{2\sqrt{\phi_2}}(\|U_R^e - U_L^e\| + \sqrt{T_\varepsilon|_{r=c_2}}(r_L + r_R - 2r_2)).\end{aligned}$$

By symmetry, the same bound holds for $\|u_2^o - \frac{r_2}{r_R}U_R^o\|$. Using these bounds in Eq. (33), we have

$$\begin{aligned}\sum_{i=1}^{4}|\tilde{\epsilon}_i| \le &\left(1 + \frac{\sqrt{T_\varepsilon|_{r=c_2}}\pi}{2r_2\sqrt{\phi_2}}\right)(r_L + r_R - 2r_2) \\ &+ \frac{\pi}{2}\left\|\frac{1}{r_R}U_R^o - \frac{1}{r_L}U_L^o\right\| + \frac{\pi}{2r_2\sqrt{\phi_2}}\|U_R^e - U_L^e\|.\end{aligned} \tag{35}$$

Equation (34), with the assumptions that $\theta_{LR} \le \frac{\pi}{2}$ and ϕ is convex, can be used to derive

$$r_L + r_R - 2r_2 \le 2\sqrt{r_L r_R}\sin\frac{\theta_{LR}}{2} + \frac{1}{\sqrt{\phi_2}}\|U_R^e - U_L^e\|. \tag{36}$$

Substituting into Eq. (35) and using the assumption that $r_L, r_R \in [c_1, c_2]$, we obtain

$$\begin{aligned}\sum_{i=1}^{4}|\tilde{\epsilon}_i| \le &\frac{1}{\sqrt{\phi_2}}\left(1 + \frac{\pi\sqrt{T_\varepsilon|_{r=c_2}}}{2r_2\sqrt{\phi_2}}\right)\|U_R^e - U_L^e\| \\ &+ 2c_2\left(1 + \frac{\sqrt{T_\varepsilon|_{r=c_2}}\pi}{2r_2\sqrt{\phi_2}}\right)\sin\frac{\theta_{LR}}{2} \\ &+ \frac{\pi}{2}\left\|\frac{1}{r_R}U_R^o - \frac{1}{r_L}U_L^o\right\| + \frac{\pi}{2r_2\sqrt{\phi(r_2)}}\|U_R^e - U_L^e\|.\end{aligned} \tag{37}$$

The result is completed by showing that r_2 is bounded below by a positive constant. Suppose $r_2 < \frac{c_1}{2}$. Using Eq. (34) along with the assumption that $\theta_{LR} \le \frac{\pi}{2}$, we obtain

$$2\sqrt{\phi_2}r_2 \ge \sqrt{2}\left(r_2\sqrt{\phi_2} + \int_{c_1/2}^{c_1}\sqrt{T_\varepsilon}\right) - \|U_R^e - U_L^e\|. \tag{38}$$

Using the assumption that ϕ is strictly convex along with assumption (iii) gives

$$(2 - \sqrt{2})\sqrt{\phi_2}r_2 \ge \int_{c_1/2}^{c_1}\sqrt{T_\varepsilon} - \int_{c_1/2}^{c_1}\sqrt{\phi} \equiv \kappa_0 > 0.$$

Since $r\sqrt{\phi}$ is a strictly monotone function of r, this implies there must be a unique value of $r > 0$, say $r = \tilde{c}_0$, satisfying

$$(2 - \sqrt{2})\sqrt{\phi(\tilde{c}_0)}\tilde{c}_0 = \kappa_0.$$

Hence, we have shown that $r_2 \geq c_0$ where $c_0 = \min\{\frac{c_1}{2}, \tilde{c}_0\}$. Using Eq. (37) and the fact that ϕ is monotone, $r_2 \in [c_0, c_2]$ implies that there exist positive constants so that

$$\sum_{i=1}^{4} |\tilde{\epsilon}_i| \leq \kappa_1 \|U_R^e - U_L^e\| + \kappa_2 \sin\frac{\theta_{LR}}{2} + \kappa_2 \left\|\frac{1}{r_R}U_R^o - \frac{1}{r_L}U_L^o\right\|.$$

We have now shown that the lemma holds for all possible cases of r_2. □

We employ this result in the following section to show that the solutions constructed from the Glimm scheme are bounded.

5. Bounds on the approximate solution $\{U_h\}$. In order to show that the total variation of the approximate solutions is bounded, we first restrict ourselves to bounding the variation only in the odd components. Before attempting this, however, we first show that the odd components of the approximate solutions are bounded.

COROLLARY 5.1. Assume that the total variation of the initial data U_0 is sufficiently small. If there exist positive constants c_1 and c_2 so that the initial data satisfies

$$0 < c_1 \leq \|U_0^o\| \leq c_2, \tag{39}$$

then there exist positive constants C_1 and C_2 such that for any mesh curve J,

$$C_1 \leq \|U_h^o|_J\| \leq C_2.$$

Proof. We begin the proof by showing that the oscillation in the odd components is bounded by the functional $\mathcal{L}$.

CLAIM 5.1. $\mathcal{L}(J) \geq \sup \|U^o|_J\| - \inf \|U^o|_J\|$.

Proof. Let $\tilde{\epsilon}_i$ be a wave that crosses J. Let u_l be the state to the left of the wave and u_r be the state to the right of the wave. Since either $\|u_l^o\| = \|u_r^o\|$ or $\tilde{\epsilon}_i = \|u_r^o\| - \|u_l^o\|$ in the case of a linear or nonlinear wave, respectively,

$$\begin{aligned}\sum\{|\tilde{\epsilon}_i| : \tilde{\epsilon}_i \text{ crosses } J\} &\geq \sum\{|\,\|u_r^o\| - \|u_l^o\|\,| : \text{ state } u_l \text{ is connected to } u_r \text{ on } J\} \\ &\geq \sup \|U^o|_J\| - \inf \|U^o|_J\|.\end{aligned}$$

This concludes the proof of the claim. □

Now we use the claim to show that $\|U^o|_J\|$ is bounded. Let J_o be a single segment of the mesh curve J. Then there is another mesh curve I containing the segment J_o and a segment O' from the mesh curve O. Applying the claim, we now have

$$\mathcal{L}(I) \geq \sup \|U^o|_I\| - \inf \|U^o|_I\| \geq |\,\|U^o|_{J_o}\| - \|U^o|_{O'}\|\,|. \tag{40}$$

Since there exists a point on O' where $t = 0$, assumption (39) gives

$$c_1 - \mathcal{L}(I) \leq \|U^o|_{J_o}\| \leq c_2 + \mathcal{L}(I).$$

Using the result $\mathcal{L}(I) \leq \frac{2}{A_1}\mathcal{L}(O)$, we obtain

$$c_1 - \frac{2}{A_1}\mathcal{L}(O) \leq \|U^o|_{J_o}\| \leq c_2 + \frac{2}{A_1}\mathcal{L}(O).$$

Since the choice of the segment J_o was arbitrary, the proof is completed by taking

$$\frac{2}{A_1}\mathcal{L}(O) < c_1$$

by choosing $\|U_0\|_{TV}$ small enough, as indicated in Lemma 4.2. $\square$

By the preceding corollary, restricting U_h to any mesh curve, we have a uniform bound on $\|U_h^o\|$. This result will be needed to show that the norm of total variation in the *odd* components of U_h is uniformly bounded.

LEMMA 5.1. There exist positive constants B_1 and B_2 so that, restricted to any mesh curve J,

$$B_1\mathcal{L}(J) \leq \|U_h^o|_J\|_{TV} \leq B_2\mathcal{L}(J).$$

Proof. Let J be any mesh curve. Since U_h (restricted to the mesh curve J) changes values only where some $\tilde{\epsilon}$ wave crosses the J curve, the total variation of U_h^o will increase along J only at those places where a wave crosses J. Suppose that the wave $\tilde{\epsilon}_i$ crosses J with u_l and u_r the states to the left and right of the wave, respectively. Thus, by showing

$$B_1|\tilde{\epsilon}_i| \leq \|u_r^o - u_l^o\| \leq B_2|\tilde{\epsilon}_i|,$$

we complete the proof. We now consider the only two possible cases: (i) $\tilde{\epsilon}_i$ is a nonlinear wave or (ii) $\tilde{\epsilon}_i$ is a linear wave.

(i) If $\tilde{\epsilon}_i$ is a nonlinear wave, then it is a shock or a rarefaction wave. By the structure of solutions to the Riemann problem, this implies that u_l^o and u_r^o lie on the same radial line. Thus,

$$\|u_l^o - u_r^o\| = |\,\|u_l^o\| - \|u_r^o\|\,| = |\tilde{\epsilon}_i|.$$

(ii) If $\tilde{\epsilon}_i$ is a linear wave, then it is a contact discontinuity. This implies that $\|u_l^o\| = \|u_r^o\|$, from which we obtain

$$\|u_l^o - u_r^o\|^2 = 4\|u_l^o\|\,\|u_r^o\|\sin^2(\theta/2),$$

where θ is the angle between u_l^o and u_r^o. We now use Corollary 5.1 and the fact that $\theta/\pi \leq \sin(\theta/2) \leq \theta/2$ for $\theta \in (0,\pi)$ to obtain inequality

$$\frac{2C_1}{\pi}\theta \leq \|u_l^o - u_r^o\| \leq C_2\theta.$$

Since $\tilde{\epsilon}_i = \theta$, we now have

$$\frac{2C_1}{\pi}|\tilde{\epsilon}_i| \leq \|u_l^o - u_r^o\| \leq C_2|\tilde{\epsilon}_i|.$$

Choosing

$$B_1 \equiv \min\left\{1, \frac{2C_1}{\pi}\right\}$$

and

$$B_2 \equiv \max\{1, C_2\},$$

the proof is complete. $\square$

Since it has been shown that $\mathcal{L}(J) \leq \frac{2}{A_1}\mathcal{L}(O)$ (Eq. (24)), by the equivalence of the metrics, we have

$$\|U_h^o|_J\|_{TV} \leq B_2\mathcal{L}(J) \leq \frac{2B_2}{A_1}\mathcal{L}(O) \leq \mathit{Const}\,\|U_0^o\|_{TV}, \tag{41}$$

where the constant is independent of the mesh size. Thus we have shown that the total variation of the odd components is bounded. We now show that the total variation of the even components is equivalent to the total variation of the odd components.

LEMMA 5.2. There exist positive constants $\widetilde{B}_1$ and $\widetilde{B}_2$ so that, restricted to any mesh curve J,

$$\widetilde{B}_1 \|U_h^o|_J\|_{TV} \le \|U_h^e|_J\|_{TV} \le \widetilde{B}_2 \|U_h^o|_J\|_{TV}.$$

Proof. Consider an arbitrary mesh curve J and suppose that the wave $\tilde{\epsilon}_i$ crosses J with u_l and u_r the two states on J to the left and the right of the wave $\tilde{\epsilon}_i$, respectively. Recalling the possible solutions to the Riemann problem, (2) and (3),

$$u_r^e = \begin{cases} u_l^e \pm \sqrt{\phi_l}(u_r^o - u_l^o) & \text{for linear waves,} \\ u_l^e \pm \frac{A}{r_l} u_l^o & \text{for nonlinear waves,} \end{cases}$$

where $r_l = \|u_l^o\|, r_r = \|u_r^o\|$, and

$$A = \begin{cases} \sqrt{\dfrac{\phi_r r_r - \phi_l r_l}{r_r - r_l}}(r_r - r_l) & \text{for shocks,} \\ \displaystyle\int_l^r \sqrt{T_\varepsilon}\, dr & \text{for rarefaction waves.} \end{cases}$$

To show that the result holds in each case, we show that there exist constants independent of u_l and u_r such that

$$\widetilde{B}_1 \|u_r^o - u_l^o\| \le \|u_r^e - u_l^e\| \le \widetilde{B}_2 \|u_r^o - u_l^o\|.$$

We now consider the two possible waves connecting the states: (i) u_l and u_r are connected by a linear wave, or (ii) u_l and u_r are joined by a nonlinear wave.

(i) Suppose states u_l and u_r are connected by a linear wave. In this event, we obtain

$$\|u_r^e - u_l^e\| = \sqrt{\phi_l}\|u_r^o - u_l^o\|.$$

Using Corollary 5.1 and the assumption that ϕ is a continuous function of $r = \|U^o\|$, there exist positive constants $\tilde{b}_1$ and $\tilde{b}_2$ so that

$$\tilde{b}_1 \|u_r^o - u_l^o\| \le \|u_r^e - u_l^e\| \le \tilde{b}_2 \|u_r^o - u_l^o\|.$$

(ii) Suppose states u_l and u_r are connected by a nonlinear wave. Let us first consider the case that the two states are connected by a shock wave. Since for nonlinear waves, $u_r^o = \frac{r_r}{r_l} u_l^o$, we obtain

$$\|u_r^e - u_l^e\| = \sqrt{\frac{\phi_r r_r - \phi_l r_l}{r_r - r_l}} \|u_r^o - u_l^o\|.$$

Applying the mean value theorem, we obtain

$$\|u_r^e - u_l^e\| = \sqrt{(\phi r)'|_\xi} \|u_r^o - u_l^o\|,$$

for ξ between r_l and r_r. Using Corollary 5.1 and the assumption that ϕ' is continuous, there exist positive constants b_1 and b_2 so that

$$b_1 \le \sqrt{(\phi r)'} \le b_2.$$

From this we obtain the desired inequality

$$b_1\|u_r^o - u_l^o\| \le \|u_r^e - u_l^e\| \le b_2\|u_r^o - u_l^o\|.$$

Let us now consider the other possible nonlinear wave, the rarefaction wave. In this case, two states u_r and u_l satisfy

$$\|u_r^e - u_l^e\| = \left|\int_l^r \sqrt{T_\varepsilon}\right| \left\|\frac{1}{r_l}u_l^o\right\|. \tag{42}$$

Since $T_\varepsilon = (\phi r)'$, Eq. (42) yields

$$b_1|r_r - r_l| \left\|\frac{1}{r_l}u_l^o\right\| \le \|u_r^e - u_l^e\| \le b_2|r_r - r_l| \left\|\frac{1}{r_l}u_l^o\right\|. \tag{43}$$

Since $u_r^o = \frac{r_r}{r_l}u_l^o$, we now have

$$b_1\|u_r^o - u_l^o\| \le \|u_r^e - u_l^e\| \le b_2\|u_r^o - u_l^o\|.$$

Thus, we have shown that the inequality holds in the case of a nonlinear wave, with the same constants for a rarefaction or a shock wave.

Taking $\widetilde{B}_1 = \min\{\tilde{b}_1, b_1\}$ and $\widetilde{B}_2 = \max\{\tilde{b}_2, b_2\}$, the proof is complete since $U_h|_J$ is piecewise constant except at the places where some $\tilde{\epsilon}_i$ crosses J. □

We have shown that there exists a constant, independent of the mesh curve J and the grid size, so that $\|U_h^o|_J\|_{TV} <$ *Const* $\|U_0^o\|_{TV}$. Thus, by Lemma 5.2, it follows that

$$\|U_h^e|_J\|_{TV} \le \widetilde{B}_2\|U_h^o|_J\|_{TV} \le \textit{Const}\ \widetilde{B}_2\|U_0^o\|_{TV}. \tag{44}$$

Thus, the total variation in the even components of the constructed solution remains bounded. Because the total variation is bounded in both the even and the odd components, the total variation of the constructed solutions stays bounded.

COROLLARY 5.2. If $\|U_0^o\|_{TV}$ is sufficiently small, there exist positive constants D_1 and D_2, independent of the grid size h, so that $D_1 \le \|U_h(x,t)\| \le D_2$.

Proof. Let $t \in [n\Delta t, (n+1)\Delta t]$ and let J be the unique mesh curve from $n\Delta t$ to $(n+1)\Delta t$. Since U_h is piecewise constant at every timestep $n\Delta t$ with constant values determined at the mesh points $a_{m,n}$, it follows that

$$\|U_h(x,t)\|_{TV} \le \|U_h|_J\|_{TV} \le \textit{Const}\ \|U_0^o\|_{TV}.$$

Given that the total variation of the initial data is bounded, there exists a constant $\overline{U}$ so that $\lim_{x\to-\infty} U_0 = \overline{U}$. This implies that for any time t, $\lim_{x\to-\infty} U_h(x,t) = \overline{U}$. Thus, we obtain the result

$$\|\overline{U}\| - \textit{Const}\ \|U_0^o\|_{TV} \le \|U_h(x,t)\| \le \|\overline{U}\| + \textit{Const}\ \|U_0^o\|_{TV}. \quad \square$$

We have shown that Glimm's scheme produces a sequence $\{U_h\}$ that is uniformly bounded and has uniformly bounded total variation. Using Helly's theorem, this is sufficient to show that, for a fixed time t, $\{U_h(\cdot,t)\}$ has a convergent subsequence. In order to prove that $\{U_h\}$ has a convergent subsequence, we require the following lemma.

LEMMA 5.3. Assume that $\Delta x/\Delta t$ satisfies the CFL condition and that, in addition, $\Delta x/\Delta t < \lambda_m$. There exists a positive constant C, independent of h such that

$$\int_{-\infty}^{\infty} |u_h(x,t) - u_h(x,t')|\, dx \le C|t - t'|.$$

The lemma and its proof are from Smoller, Corollary 19.8 in [12], and follow from the fact that the total variation is bounded by the total variation of the initial data.

6. Existence of weak solutions. Using the previous results of the previous section, which state that approximations generated by the Glimm scheme are bounded and that they are uniformly bounded in total variation, we show that the Glimm scheme has a convergent subsequence. The following classical theorem and proof showing that Glimm's scheme converges are due to Glimm [5].

Consider the net (a generalized sequence) $\{u_h\}$. Define the indexing set so that $h = (\Delta x, \Delta t)$ and $(\Delta x)/(\Delta t)$ satisfies the CFL condition. The partial order "$\succ$" on the index set is defined: if $i = (\Delta x_i, \Delta t_i)$ and $j = (\Delta x_j, \Delta t_j)$, then $j \succ i$ when $\Delta x_j \le \Delta x_i$ and $\Delta t_j \le \Delta t_i$.

THEOREM 6.1. If the net $\{u_h(x,t)\}$ satisfies

(i) $\sup_x \|u_h(x,t)\| \le C$,

(ii) $\|u_h(\cdot,t)\|_{TV} \le M_1$,

(iii) $\int_{\mathfrak{R}} |u_h(x,t) - u_h(x,s)|\, dx \le M_2|t - s|$,

then u_h has a subnet that converges in $L^1_{\text{loc}}(\mathfrak{R} \times \mathfrak{R}^+)$.

In the proof, hypotheses (i) and (ii) imply that Helly's theorem holds for any rational time t_i. Using a diagonalization process, we can show the existence of a subnet that converges for all rational time. Since the rationals are dense in the reals, the last hypothesis shows that the subnet converges for all real time.

Hence, Corollary 5.2, results (41) and (44), along with Lemma 5.3, give the convergence of the Glimm scheme. Now, we show that the limit u is a weak solution. For this, we must show that

$$\iint (u\phi_t + f(u)\phi_x)\, dx\, dt + \int u_0 \phi(x,0)\, dx = 0, \tag{45}$$

for any test function $\phi \in C_0^\infty$. Assume that the support of ϕ is contained in $I \times [0,T]$. Then by the Lebesgue Dominated Convergence Theorem, we have

$$\iint u_h \phi_t \to \iint u\phi_t.$$

Similarly, this implies that the limit u is a weak solution if

$$\iint (u_h\phi_t + f(u_h)\phi_x)\, dx\, dt + \int u_0\phi(x,0)\, dx \to 0.$$

We now make the dependence on the random sequence $\{\theta_i\}$ explicit. Define

$$J(\theta, h, \phi) \equiv \iint (u_h^\theta \phi_t + f(u_h^\theta)\phi_x)\, dx\, dt + \int u_0\phi(x,0)\, dx.$$

Once convergence of J to 0 is established for the sequence $\{u_h^\theta\}$ obtained by Glimm's scheme, then the function $u(x,t)$ for which $u_h \to u$ will be a weak solution of the Cauchy problem for the system of conservation laws (1).

Let $\mathcal{N}$ denote the set of natural numbers. Consider the product space $\Omega = (-1,1)^{\mathcal{N}}$, where $\{\theta_i\} \in (-1,1)^{\mathcal{N}}$. We endow the space Ω of sequences $A = \{a_i\} \in (-1,1)^{\mathcal{N}}$ with probability measure $d\nu(A)$. This is accomplished by normalizing; so $d\nu(a_i)$ equals half of the Lebesgue measure. Then the key result stated and proved by Glimm [5] is:

LEMMA 6.1. $\int_\Omega |J(A,h,\phi)|^2\, d\nu(A) \to 0$ as the mesh size goes to zero.

The proof is now complete, and we have shown that the approximations generated by the Glimm scheme converge to a weak solution for the Cauchy problem associated with Eq. (1) given that the total variation of the initial data is sufficiently small and bounded away from zero.

7. Conclusions. In summary, we have shown that there exists a weak solution to the Cauchy problem associated with the equations modeling an elastic string in three-dimensional space (1). Following the Glimm construction of approximate solutions, we showed that the approximate solutions satisfy the hypotheses of Helly's Theorem. As in Glimm's seminal paper, this was accomplished by showing that the constructed solutions were bounded in an equivalent functional. It is in the choice of the functional that the method used in this paper diverges from the literature. This was a necessary modification, since the standard choice of an equivalent functional (as in Glimm [5]) is not appropriate for conservation laws possessing multiple characteristics as is the case with problem (1). With our chosen functional, we show that wave interactions across an arbitrary mesh curve are proportional to the measure of wave interactions across the initial mesh curve. After establishing that the total variation norm is equivalent to the functional, the remainder of the existence proof, that the limit of the subsequence is a weak solution, was identical to Glimm's proof of existence for strictly hyperbolic conservation laws.

8. Appendix A. Let $x \in \Re, u(x,t) \in \Re^n$, and $f(u) \in \Re^n$. We say that the conservation law given by

$$u_t + f(u)_x = 0 \tag{46}$$

is *hyperbolic* if the Jacobian of $f(u)$ is diagonalizable. If the n eigenvalues of the Jacobian of f are real and distinct, we say that (46) is *strictly hyperbolic*. If the eigenvalues are real but not everywhere distinct, we say that (46) is *nonstrictly hyperbolic*.

Consider the conservation law system (46) where λ_i is an eigenvalue of the Jacobian of f with r_i an associated eigenvector. The Riemann problem for the conservation law (46) is defined to be the differential equation (46) along with the initial data:

$$u(x,0) = u = \begin{cases} u_L & \text{for } x < 0, \\ u_R & \text{for } x > 0. \end{cases}$$

DEFINITION 8.1. The ith characteristic family is said to be *genuinely nonlinear* if $\nabla\lambda_i \cdot r_i \neq 0$.

There are two (classical) types of nonlinear waves: centered rarefaction waves and shock waves. Rarefaction waves are continuous solutions to the differential equation, while shock waves are discontinuous solutions. We first discuss the rarefaction waves.

Since a centered rarefaction wave is a continuous self-similar solution, we perform the following change of variables $\xi = x/t$. With the change of variables, Eq. (46) becomes

$$(df(u) - \xi I)u_\xi = 0,$$

which implies that ξ is an eigenvalue of df and u_ξ is the corresponding eigenvector. Thus, $\xi = \lambda_i(u(\xi))$ and $u_\xi = r_i(u(\xi))$ for some i. We say that u_L is connected to u_R by a k-rarefaction wave if

$$\lambda_k(u_R) > \lambda_k(u_L),$$

and

$$\lambda_k(u(\xi)) = \xi \quad \text{for } \lambda_k(u_L) \leq \xi \leq \lambda_k(u_R).$$

Now consider a shock wave solving the Riemann problem. A k-shock is a discontinuous solution that satisfies the *Rankine-Hugoniot* relation:

$$s(u_R - u_L) = (f(u_R) - f(u_L)),$$

where s is the speed at which the shock travels. In addition, the k-shock must satisfy the *Lax entropy condition*,

$$\lambda_{k-1}(u_L) < s < \lambda_k(u_L) \quad \text{and} \quad \lambda_k(u_R) < s < \lambda_{k+1}(u_R).$$

In this case, the solution to the Riemann problem is

$$u(x,t) = \begin{cases} u_L & \text{for } x < st, \\ u_R & \text{for } x > st. \end{cases}$$

Having defined the possible nonlinear waves, we now define a linear wave.

DEFINITION 8.2. The ith characteristic family is said to be *linearly degenerate* if $\nabla\lambda_i \cdot r_i \equiv 0$.

Considering the Riemann problem, we say that the two states u_L and u_R are connected by a k-contact discontinuity if $\lambda_k(u_L) = \lambda_k(u_R)$. Here, the solution is given by

$$u(x,t) = \begin{cases} u_L & \text{for } x < st, \\ u_R & \text{for } x > st, \end{cases}$$

where the propagation speed is $s = \lambda_k(u_L)$.

For a fixed state u_L, let S_k be the curve in state space of all the states that can be connected to u_L on the left by a k-shock, let R_k be the curve containing all the states that can be connected to u_L on the left by a k-rarefaction wave, and let C_k be the subset containing all the states that can be connected to u_L by a k-contact discontinuity. The locus at state u_L is defined to be the union of all these curves R_i, S_i, and C_i for all i. Using the wave loci, we are able to determine the solution to the Riemann problem in a neighborhood of u_L.

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