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ASYMPTOTIC BEHAVIOUR FOR A PARTIALLY DIFFUSIVE RELAXATION SYSTEM

By

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Abstract. The time asymptotics of nonnegative and integrable solutions to a partially diffusive relaxation system is investigated. Under suitable assumptions on the relaxation term, the convergence to a self-similar source type solution, either of the heat equation or of the viscous Burgers equation, is proved. The proof relies on optimal decay rates and classical scaling arguments.

1. Introduction. We study the large time behaviour of nonnegative and integrable solutions to the system

$$u_t + u_x - u_{xx} = k(v - u^q) \text{ in } (0, +\infty) \times \mathbb{R}, \tag{1}$$

$$v_t = k(u^q - v) \text{ in } (0, +\infty) \times \mathbb{R}, \tag{2}$$

with initial data

$$(u, v)(0) = (u_0, v_0) \text{ in } \mathbb{R},$$

where k and q are positive real numbers and u_0, v_0 are nonnegative, bounded and integrable functions on $\mathbb{R}$. A first step towards the understanding of the large time behaviour of integrable solutions to (1)–(2) is performed in [GvDD], where a generalized version of (1)–(2) is considered as a model for the one-dimensional transport of a one-species contaminant through a porous medium accounting for adsorption reactions. Assuming the reaction to be fast, that is, $k \to +\infty$, the system (1)–(2) formally reduces to

$$(u + u^q)_t + u_x - u_{xx} = 0 \text{ in } (0, +\infty) \times \mathbb{R}. \tag{3}$$

Let us also mention that the fast reaction limit $k \to \infty$ for the system without the diffusive term u_{xx} has been considered in [KT] and [TW] for one space dimension, and in [KaTz] for general space dimension.

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Asymptotic expansions of solutions to (3) as $t \to \infty$ are constructed in [GvDD] according to the different values of q, while analytic results are provided in [EZ, EVZ, LS, Re]. Summarizing the results obtained in the above mentioned papers, five cases are to be distinguished: if $q \in (0,1)$, $u^q(t)$ converges in $L^1(\mathbb{R})$ to the self-similar source-type solution to the scalar conservation law $z_t - (z^q)_x = 0$ with total mass $M := |u_0 + u_0^q|_{L^1}$, while (3) is a linear equation for $q = 1$ whose large time behaviour follows at once from that of the linear heat equation. Introducing $U(t,x) = u(t, x+t)$, $U(t)$ converges to $L^1(\mathbb{R})$ to the self-similar source-type solution to $z_t - (z^q)_x = 0$, $z_t - (z^2)_x - z_{xx} = 0$ and $z_t - z_{xx} = 0$ with total mass M when $q \in (1,2)$, $q = 2$, and $q > 2$, respectively. We refer to the above mentioned papers for a more precise description of the large time behaviour of solutions to (3).

A possible expectation is that the study of the large time behaviour of solutions to (3) gives some clues towards the description of the large time behaviour of solutions to (1)–(2) for finite values of k, at least for some values of the parameter q. This is actually what we aim to prove in this paper when q ranges in $[2, +\infty)$. In view of the results already known for (3) when $q \in [2, +\infty)$, the large time behaviour of the first component u of the solutions to (1)–(2) is expected to be ruled by the diffusion term if $q > 2$, while it is given by a balance between diffusion and convection for $q = 2$. More precisely, our result reads as follows:

THEOREM 1. Assume that $q \in [2, +\infty)$ and consider two measurable functions u_0 and v_0 satisfying

$$(u_0, v_0) \in L^1(\mathbb{R}; \mathbb{R}^2) \cap L^\infty(\mathbb{R}; \mathbb{R}^2), \;\; u_0, v_0 \geq 0 \text{ a.e. in } \mathbb{R}. \tag{4}$$

Let (u, v) be the solution to (1)–(2) with initial data (u_0, v_0). Introducing $U(t,x) = u(t, x+t)$ for $(t,x) \in (0, +\infty) \times \mathbb{R}$ and $M = |u_0 + v_0|_{L^1}$, there holds

$$\lim_{t \to +\infty} t^{(1-1/p)/2} |U(t) - S_M(t)|_{L^p} = 0 \tag{5}$$

for $p \in [1, \infty)$, where

a: S_M is the unique nonnegative solution to

$$\begin{aligned} S_{M,t} - S_{M,xx} &= 0 \text{ in } (0, +\infty) \times \mathbb{R}, \\ S_M(0) &= M\delta, \end{aligned} \tag{6}$$

if $q > 2$, and

b: S_M is the unique nonnegative solution to

$$\begin{aligned} S_{M,t} - (S_M^2)_x - S_{M,xx} &= 0 \text{ in } (0, +\infty) \times \mathbb{R}, \\ S_M(0) &= M\delta, \end{aligned} \tag{7}$$

if $q = 2$.

Here δ denotes the Dirac mass centered at $x = 0$.

The description of the long time behaviour of the solutions to (1)–(2) is an open problem when $q < 2$. According to the formal analysis in [GvDD], the diffusive effects are expected to vanish as $t \to \infty$ and our approach does not allow to handle this case.

REMARK 1. For the sake of completeness, we briefly recall that for every fixed $M > 0$, the functions S_M defined by (6) and (7) are given by

$$S_M(t,x) = t^{-1/2} f_M(xt^{-1/2}), \quad \int_{-\infty}^{+\infty} f_M(z)\,dz = M$$

with

$$f_M(z) = C_M e^{-|z|^2/4} \text{ for (6)}$$

and

$$f_M(z) = e^{-|z|^2/4} \left\{ C_M + \int_{-\infty}^{z} e^{-|s|^2/4}\,ds \right\}^{-1} \text{ for (7)},$$

where C_M is a positive constant such that the total mass of f_M is M.

Before proceeding with the proof of Theorem 1, let us briefly describe the main steps. First we may take $k = 1$ without loss of generality. Following the analysis of [GvDD] we first perform a temporal shift on the space variable by putting $(U,V)(t,x) = (u,v)(t,x+t)$ for $(t,x) \in (0,+\infty) \times \mathbb{R}$. Then (U,V) solves

$$U_t - U_{xx} = V - U^q, \quad V_t - V_x = U^q - V. \tag{8}$$

As U is expected to become invariant with respect to a suitable scaling of the time and space variables for large times, we shall employ a scaling method to prove Theorem 1. Scaling arguments have been widely used in the study of the large time behaviour of parabolic and hyperbolic equations (see, e.g., [Va, LP] and the references therein). More precisely, in view of the expected result, we introduce

$$U_\lambda(t,x) = \lambda U(\lambda^2 t, \lambda x) \text{ and } V_\lambda(t,x) = \lambda^q V(\lambda^2 t, \lambda x)$$

for $(t,x) \in (0,+\infty) \times \mathbb{R}$ and $\lambda \geq 1$. Notice that the functions S_M defined above are invariant under the scaling performed on U. The pair (U_λ, V_λ) then satisfies the following system:

$$V_\lambda - U_\lambda^q = \lambda^{-1} V_{\lambda,x} - \lambda^{-2} V_{\lambda,t},$$
$$U_{\lambda,t} - U_{\lambda,xx} = \lambda^{2-q} V_{\lambda,x} - \lambda^{1-q} V_{\lambda,t}.$$

We now formally pass to the limit as $\lambda \to +\infty$ in the previous system and conclude that a possible limit (U_∞, V_∞) of (U_λ, V_λ) should satisfy $V_\infty = U_\infty^q$, the function U_∞ being a solution to (6) if $q > 2$ or to (7) if $q = 2$. In addition, the choice of the scaling performed on (U,V) warrants that $U_\infty(0) = (|u_0 + v_0|_{L^1})\delta$, which allows to uniquely determine U_∞. A rigorous justification of this formal limiting procedure is the purpose of this paper and requires the boundedness of (U_λ, V_λ) in suitable norms, or equivalently, optimal temporal decay rates for various norms of (U,V). After collecting some preliminary results in Sec. 2, Sec. 3 is devoted to the proof of temporal decay estimates for (U,V). The proof of Theorem 1 is performed in the last section.

REMARK 2. System (1)–(2) without the diffusive term, i.e.,

$$u_t + u_x = k(v - u^q) \text{ in } (0,+\infty) \times \mathbb{R}, \tag{9}$$

$$v_t = k(u^q - v) \text{ in } (0,+\infty) \times \mathbb{R}, \tag{10}$$

is a particular case of hyperbolic systems with relaxation considered by T.-P. Liu [L] (see [Na] for a recent review on the subject). As this author pointed out, under the strict subcharacteristic condition, the large time behaviour of these systems is described by a nonlinear diffusion equation. This has been proved subsequently for some particular cases (cf. [C, LN] and the references therein).

Let us notice here that the strict subcharacteristic condition is not fulfilled by (9)–(10). Actually, the same formal computation as in [L, Sec. 2] indicates that $W = (u+v)(t, x+t)$, behaves, as $t \to \infty$, like the solutions to

$$w_t - (w^q)_x - (w^q)_{xx} = 0,$$

and therefore, by the results proved in [LS], is asymptotic to integrable self-similar solutions to the hyperbolic equation

$$w_t - (w^q)_x = 0.$$

This also indicates that the large time behaviour of the system (1)–(2) and its inviscid counterpart (9)–(10) are of a different nature.

2. Well-posedness and basic properties. In this section we assume that $q > 1$ and denote by $\widetilde{G}(t)$ the linear semigroup in $L^1(\mathbb{R})$ generated by the operator $(\partial_{xx} - \partial_x)$. We first recall the well-posedness of (1)–(2) for initial data satisfying (4).

PROPOSITION 2. Let (u_0, v_0) be two measurable functions satisfying (4). There is a unique couple of nonnegative functions

$$(u, v) \in \mathcal{C}([0, +\infty); L^1(\mathbb{R}; \mathbb{R}^2)) \cap L^\infty((0, +\infty) \times \mathbb{R}; \mathbb{R}^2) \tag{11}$$

such that (u, v) satisfies

$$u(t) = \widetilde{G}(t)u_0 + \int_0^t \widetilde{G}(t-s)(v - u^q)(s)\, ds, \tag{12}$$

$$v(t) = e^{-t}v_0 + \int_0^t e^{s-t}u^q(s)\, ds, \tag{13}$$

for $t \in [0, +\infty)$. In addition,

$$\int (u(t,x) + v(t,x))\, dx = \int (u_0(x) + v_0(x))\, dx, t \in [0, +\infty). \tag{14}$$

For further use we introduce the notation $(u(t), v(t)) = S_t(u_0, v_0)$ for $t \geq 0$.

REMARK 3. It actually follows from (11) and (14) that the right-hand side of (1) belongs to $L^\infty(0, +\infty; L^p(\mathbb{R}))$ for every $p \in (1, \infty)$. Classical parabolic regularity results [LSU] then entail that $u \in W_p^{1,2}((0,T) \times \mathbb{R})$ for each $p \in (1, \infty)$ and $T \in (0, +\infty)$. Consequently, u is a strong solution to (1).

We next notice that the right-hand side of (1)–(2) is quasi-monotone which ensures that the mapping S_t is an order-preserving contraction in $L^1(\mathbb{R}; \mathbb{R}^2)$ (see, e.g., [Na, Sec. 5]). That is, if (u_0, v_0) and $(\hat{u}_0, \hat{v}_0)$ are two pairs of functions satisfying (4), there holds

$$|(u - \hat{u})_+(t)|_{L^1} + |(v - \hat{v})_+(t)|_{L^1} \leq |(u_0 - \hat{u}_0)_+|_{L^1} + |(v_0 - \hat{v}_0)_+|_{L^1} \tag{15}$$

for $t \in [0, +\infty)$, with the obvious notation $(u, v)(t) = S_t(u_0, v_0)$, $(\hat{u}, \hat{v})(t) = S_t(\hat{u}_0, \hat{v}_0)$, and $r_+ = \max\{r, 0\}$. As a consequence of (14)–(15) and [CT], we have

$$|(u - \hat{u})(t)|_{L^1} + |(v - \hat{v})(t)|_{L^1} \leq |(u_0 - \hat{u}_0)|_{L^1} + |(v_0 - \hat{v}_0)|_{L^1} \tag{16}$$

for every $t \in [0,+\infty)$ and pairs $(u_0,v_0),(\hat{u}_0,\hat{v}_0)$ satisfying (4). Observe finally that, thanks to (16), we may extend S_t by density to the whole positive cone of $L^1(\mathbb{R};\mathbb{R}^2)$.

We now fix a couple (u_0,v_0) of functions satisfying (4). For $t \in [0,+\infty)$ we put $(u(t),v(t)) = S_t(u_0,v_0)$ and

$$M := \int (u_0+v_0)(x)\,dx. \tag{17}$$

We first recall that (14) and the nonnegativity of u and v entail

$$|u(t)|_{L^1} + |v(t)|_{L^1} \le M, \quad t \in [0,+\infty). \tag{18}$$

The remainder of this section is devoted to the analysis of some properties enjoyed by (u,v).

LEMMA 3. There is a constant C_1 depending only on q and M such that, for each $p \in [0,+\infty)$, $t_1 \in (0,+\infty)$, and $t_2 \in (t_1,+\infty)$, there holds

$$\begin{aligned} &|u(t_2)|_{L^{1+pq}}^{1+pq} + \frac{1+pq}{1+p}|v(t_2)|_{L^{1+p}}^{1+p} + \frac{1}{C_1}\frac{pq}{1+pq}\int_{t_1}^{t_2} |u(s)|_{L^\infty}^{2+pq}\,ds \\ &\quad\le |u(t_1)|_{L^{1+pq}}^{1+pq} + \frac{1+pq}{1+p}|v(t_1)|_{L^{1+p}}^{1+p}. \end{aligned} \tag{19}$$

Proof. Let $p \in [0,+\infty)$. We multiply (1) by u^{pq}, (2) by v^p, add the resulting identities and integrate over $(t_1,t_2)\times\mathbb{R}$; we obtain

$$\begin{aligned} &|u(t_2)|_{L^{1+pq}}^{1+pq} + \frac{1+pq}{1+p}|v(t_2)|_{L^{1+p}}^{1+p} \\ &+ \int_{t_1}^{t_2}\int (v-u^q)(v^p-u^{pq})\,dx\,ds + \frac{4pq}{1+pq}\int_{t_1}^{t_2} |(u^{(1+pq)/2})_x|^2\,dx\,ds \\ &\le |u(t_1)|_{L^{1+pq}}^{1+pq} + \frac{1+pq}{1+p}|v(t_1)|_{L^{1+p}}^{1+p}. \end{aligned} \tag{20}$$

By the Gagliardo-Nirenberg inequality and (18), we have

$$\begin{aligned} |u|_{L^\infty}^{(1+pq)/2} &\le C_1|(u^{(1+pq)/2})_x|_{L^2}^{1/2}|(u^{(1+pq)/2})|_{L^2}^{1/2} \\ &\le C_1|u|_{L^1}^{1/4}|u|_{L^\infty}^{pq/4}|(u^{(1+pq)/2})_x|_{L^2}^{1/2}, \\ |u|_{L^\infty}^{2+pq} &\le C_1|(u^{(1+pq)/2})_x|_{L^2}^2. \end{aligned} \tag{21}$$

The inequality (19) then follows at once from (20) and (21). □

We next exploit further (19) to obtain a differential inequality involving various L^p-norms of u and v which will lead to the expected temporal decay estimates (see Lemma 6 and Lemma 8 below).

LEMMA 4. There are positive constants C_2 and C_3 depending only on q and M such that, if $p \ge (q-3)_+/q$, the mapping

$$t \mapsto E_p(t) := |u(t)|_{L^{1+pq}}^{1+pq} + \frac{1+pq}{1+p}|v(t)|_{L^{1+p}}^{1+p} + \frac{C_2pq}{1+pq}|v(t)|_{L^{p+3/q}}^{p+3/q} \tag{22}$$

is nonincreasing and satisfies

$$\frac{pq}{1+pq}\int_{t_1}^{t_2} D_p(s)\,ds \le C_3 E_p(t_1), \quad 0 \le t_1 \le t_2, \tag{23}$$

with

$$D_p(s) = |u(s)|_{L^\infty}^{2+pq} + |v(s)|_{L^{p+3/q}}^{p+3/q}. \tag{24}$$

Proof. We multiply (2) by $v^{p-1+3/q}$ and integrate over $(t_1, t_2) \times \mathbb{R}$; using the Young inequality and (18), we obtain

$$|v(t_2)|_{L^{p+3/q}}^{p+3/q} + \int_{t_1}^{t_2} |v(s)|_{L^{p+3/q}}^{p+3/q}\,ds \le |v(t_1)|_{L^{p+3/q}}^{p+3/q} + M\int_{t_1}^{t_2} |u(s)|_{L^\infty}^{2+pq}\,ds.$$

Multiplying the above inequality by $pq/(2C_1M(1+pq))$ and adding the result to (19) yield (23) with

$$C_2 = \frac{1}{2C_1M} \text{ and } C_3 = \frac{2C_1M}{\min\{1, M\}}.$$

□

We next prove that the L^∞-norms of u and v converge to zero as time increases to infinity.

LEMMA 5. There holds

$$\lim_{t\to+\infty}(|u(t)|_{L^\infty} + |v(s)|_{L^\infty}) = 0, \tag{25}$$

$$\lim_{t\to+\infty}|u(t)|_{L^1} = M \text{ and } \lim_{t\to+\infty}|v(t)|_{L^1} = 0. \tag{26}$$

Proof. We first claim that the quasi-monotonicity of the right-hand side of (1)–(2) ensures that

$$t \mapsto |u(t)|_{L^\infty} + |v(t)|_{L^\infty} \text{ is nonincreasing.} \tag{27}$$

Indeed consider $t_0 \in [0, +\infty)$ and denote by (X, Y) the solution to the system of ordinary differential equations

$$\frac{dX}{dt} = Y - X^q, \frac{dY}{dt} = X^q - Y, \quad t \ge t_0, \tag{28}$$

$$X(t_0) = |u(t_0)|_{L^\infty}, Y(t_0) = |v(t_0)|_{L^\infty}. \tag{29}$$

Then $(u - X, v - Y)$ satisfies

$$(u - X)_t + (u - X)_x - (u - X)_{xx} = (v - Y) - (u^q - X^q), \tag{30}$$

$$(v - Y)_t = (u^q - X^q) - (v - Y). \tag{31}$$

We put $\varrho(x) = (1 + x^2)^{-1}$ for $x \in \mathbb{R}$. We multiply (30) by $\varrho\,\text{sign}_+(u - X)$, (31) by $\varrho\,\text{sign}_+(v - Y)$, add the resulting identities and integrate over $(t_0, t) \times \mathbb{R}$ to obtain, with the help of the Kato inequality,

$$\begin{aligned}\int (u-X)_+(t)\varrho\,dx + \int (v-Y)_+(t)\varrho\,dx &\\ \leq \int_{t_0}^t \int (u-X)_+(\varrho_x+\varrho_{xx})\,dx\,ds &\\ + \int_{t_0}^t \int ((v-Y)_+ - (u^q - X^q)_+)\varrho\,dx\,ds &\\ + \int_{t_0}^t \int ((u^q - X^q)_+ - (v-Y)_+)\varrho\,dx\,ds &\\ \leq 3\int (u-X)_+(t)\varrho\,dx. &\end{aligned}$$

It then follows from the Gronwall lemma that

$$\int (u-X)_+(t)\varrho\,dx + \int (v-Y)_+(t)\varrho\,dx = 0$$

for $t \geq t_0$; hence

$$u(t,x) \leq X(t) \text{ and } v(t,x) \leq Y(t)$$

for $t \geq t_0$ and almost every $x \in \mathbb{R}$. Consequently, if $t \geq t_0$,

$$|u(t)|_{L^\infty} + |v(t)|_{L^\infty} \leq X(t) + Y(t).$$

But the right-hand side of the previous inequality is obviously equal to $X(t_0)+Y(t_0)$ by (28)–(29) and the proof of the claim is complete.

As $t \mapsto |u(t)|_{L^\infty} + |v(t)|_{L^\infty}$ is bounded from below, it follows from (27) that there is $\ell \in [0,+\infty)$ such that

$$\lim_{t\to+\infty} (|u(t)|_{L^\infty} + |v(t)|_{L^\infty}) = \ell. \tag{32}$$

Now, on the one hand, we infer from (13) and the Hölder inequality that

$$|v(t)|_{L^\infty} \leq |v_0|_{L^\infty} e^{-t} + \left(\int_0^t e^{s-t} |u(s)|_{L^\infty}^{2+q}\,ds \right)^{q/(2+q)}. \tag{33}$$

On the other hand, it follows from (19) with $p=1$ and $t_1 = 0$ that

$$\int_0^t |u(s)|_{L^\infty}^{2+q}\,ds \leq 2C_1(|u_0|_{L^\infty}^q |u_0|_{L^1} + |v_0|_{L^\infty}|u_0|_{L^1})$$

for each $t > 0$, which entails that $t \mapsto |u(t)|_{L^\infty}^{2+q}$ belongs to $L^1(0,+\infty)$. It is then straightforward to check that

$$\lim_{t\to+\infty} \int_0^t e^{s-t}|u(s)|_{L^\infty}^{2+q}\,ds = 0.$$

Recalling (33) we conclude that

$$\lim_{t\to+\infty} |v(t)|_{L^\infty} = 0.$$

The above assertion and (32) next yield that $|u(t)|_{L^\infty}$ converges to ℓ as time increases to infinity. But $t \mapsto |u(t)|_{L^\infty}^{2+q}$ belongs to $L^1(0,+\infty)$ which implies $\ell = 0$ and we have proved (25).

In order to prove (26) we first notice that (13) yields

$$|v(t)|_{L^1} \leq |v_0|_{L^1} e^{-t} + \int_0^t e^{s-t} |u(s)|_{L^\infty}^{q-1} |u(s)|_{L^1}\, ds.$$

As $q > 1$ it is easy to check that (18) and (25) ensure that the right-hand side of the above inequality converges to zero as time increases to infinity. Consequently,

$$\lim_{t \to +\infty} |v(t)|_{L^1} = 0,$$

and we use (14) to complete the proof of (26). □

3. Temporal decay estimates. We keep the notations of Sec. 2 and assume further that

$$u_{0,x}, v_{0,x} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}). \tag{34}$$

It follows from (18) and (27) that

$$|u(t)|_{L^\infty} + |v(t)|_{L^\infty} + |u(t)|_{L^1} + |v(t)|_{L^1} \leq C_0 \text{ for } t \geq 0, \tag{35}$$

where

$$C_0 = M + |u_0|_{L^\infty} + |v_0|_{L^\infty}.$$

We now handle separately the cases $q \geq 3$ and $q \in [2,3)$. In the following, we denote by C any positive constant which depends only on q, u_0, and v_0. The dependence of C upon additional parameters will be indicated explicitly.

Remark 4. Though the large time behaviour of solutions to (1)–(2) is the same for $q \geq 3$ and $q \in (2,3)$, the proof of Theorem 1 differs in the two cases. Though we have no clear explanation for this point, let us recall that $q = 3$ is known to be a critical exponent for some one-dimensional parabolic equations. More precisely, $q = 3$ is a critical exponent for the equation $u_t - u_{xx} + u^q = 0$ with respect to the existence of solutions with measures as initial data [BF] and to the large time behaviour; see, e.g., [Va] and the references therein. Also, the second term in the asymptotic development of solutions of the scalar convection-diffusion equation $u_t + (u^q)_x - u_{xx} = 0$ differs when $q > 3$ and when $q \in (2,3)$ [Zu].

3.1. *The case* $q \geq 3$.

Lemma 6. For each $p \in [1, \infty)$ and $t \in (0, +\infty)$ there holds

$$t^{(1-1/p)/2} |u(t)|_{L^p} + t^{(q-1/p)/2} |v(t)|_{L^p} \leq C(p). \tag{36}$$

Proof. Consider $p \geq 1$. Then $p + 1 \geq p + 3/q$ and it follows from (35) that

$$|v(t)|_{L^{1+p}}^{1+p} \leq C_0^{(q-3)/q} |v(t)|_{L^{p+3/q}}^{p+3/q}, \quad t \geq 0. \tag{37}$$

We now estimate D_p defined by (24) from below by a power of E_p defined by (22). More precisely it follows from (37) and (35) that

$$\begin{aligned} E_p &\leq |u|_{L^1} |u|_{L^\infty}^{pq} + (qC_0^{(q-3)/q} + C_2) |v|_{L^{p+3/q}}^{p+3/q} \\ &\leq C D_p^{(pq)/(2+pq)} + C |v|_{L^{p+3/q}}^{(p+3/q)(pq)/(2+pq)} \\ E_p &\leq C D_p^{(pq)/(2+pq)}. \end{aligned} \tag{38}$$

We then infer from (38) and (23) that

$$\int_{t_1}^{t_2} E_p(s)^{(2+pq)/pq}\, ds \le C(p) E_p(t_1), \quad 0 \le t_1 \le t_2.$$

Since E_p is a nonincreasing function on $(0,+\infty)$, the above inequality and [Ko, Theorem 9.1] yield

$$E_p(t) \le C(p) t^{-(pq)/2}, \quad t > 0.$$

Therefore

$$|u(t)|_{L^{1+pq}} \le C(p) t^{-(1-1/(1+pq))/2}, \quad t > 0,$$

which gives (36) for u and $p \ge 1+q$. The assertion (36) for u and $p \in [1, 1+q)$ then follows by interpolation with (35).

Consider next $p \ge 0$ and $t \in (0,+\infty)$. We infer from (13) and (35) that

$$|v(t)|_{L^{1+p}} \le |v(t/2)|_{L^{1+p}} e^{-t/2} + \int_{t/2}^{t} e^{s-t} |u(s)|_{L^{q(p+1)}}^q\, ds$$
$$|v(t)|_{L^{1+p}} \le C e^{-t/2} + \sup_{s \in [t/2,t]} |u(s)|_{L^{q(p+1)}}^q.$$

We now use (36) for u in the above inequality to deduce (36) for v. □

The constant $C(p)$ obtained in Lemma 6 blows up as p increases to infinity. Thus, we cannot obtain L^∞-decay estimates for u and v by letting $p \to +\infty$ in (36). We rather make use of Lemma 6 together with the regularizing properties of the semigroup $\widetilde{G}(t)$ to obtain temporal decay estimates in $L^\infty(\mathbb{R})$ and in $H^1(\mathbb{R})$.

LEMMA 7. For $t \in (0,+\infty)$ there holds

$$t^{1/2} |u(t)|_{L^\infty} + t^{q/2} |v(t)|_{L^\infty} \le C, \tag{39}$$

$$t^{3/4} (|u_x(t)|_{L^2} + |v_x(t)|_{L^2}) \le C. \tag{40}$$

Proof. Let $t \in (0,+\infty)$. The Duhamel formula (12) yields

$$\begin{aligned} |u(2t)|_{L^\infty} &\le |\tilde{G}(t) u(t)|_{L^\infty} + \int_0^t |\widetilde{G}(t-s)(v-u^q)(s+t)|_{L^\infty}\, ds \\ &\le C |u(t)|_{L^1} t^{-1/2} \\ &\quad + C \int_0^t (t-s)^{-1/2} (|v(s+t)|_{L^1} + |u(s+t)|_{L^q}^q)\, ds. \end{aligned}$$

Owing to (35) and (36) we further obtain

$$|u(2t)|_{L^\infty} \le C t^{-1/2} (1 + t^{(3-q)/2}).$$

As $q \ge 3$ we conclude that

$$|u(t)|_{L^\infty} \le C t^{-1/2} \text{ for } t \ge 1.$$

As $|u(t)|_{L^\infty} \le C_0$ by (35) we finally obtain

$$t^{1/2} |u(t)|_{L^\infty} \le C, \quad t \in (0,+\infty). \tag{41}$$

We next infer from (13) and (35) that

$$|v(2t)|_{L^\infty} \leq C_0 e^{-t} + \sup_{s\in[t,2t]} |u(s)|_{L^\infty}^q,$$

which easily yields (39) by (41).

We next claim that

$$(|u_x(t)|_{L^2} + |v_x(t)|_{L^2}) \leq C, \quad t \in [0,1]. \tag{42}$$

Indeed, putting $\xi = u_x$ and $\eta = v_x$, we infer from (1)–(2) that

$$\xi_t + \xi_x - \xi_{xx} = \eta - qu^{q-1}\xi,$$
$$\eta_t = qu^{q-1}\xi - \eta.$$

By (35) we have

$$\frac{d}{dt}\int (|\xi|^2 + |\eta|^2)\,dx \leq \int (|\eta\xi| + q\xi^2 - \eta^2 + q|\eta\xi|)\,dx$$

for $t \in [0,1]$, from which the claim (42) follows by the Gronwall lemma.

We next use again the Duhamel formula (12) to obtain

$$\begin{aligned}|u_x(2t)|_{L^2} &\leq |\widetilde{G}_x(t)u(t)|_{L^2} + \int_0^t |\widetilde{G}_x(t-s)(v-u^q)(s+t)|_{L^2}\,ds \\ &\leq C|u(t)|_{L^1} t^{-3/4} \\ &\quad + C\int_0^t (t-s)^{-3/4}(|v(s+t)|_{L^1} + |u(s+t)|_{L^q}^q)\,ds.\end{aligned}$$

The above inequality, (35) and (36) next entail

$$|u_x(2t)|_{L^2} \leq Ct^{-3/4}(1 + t^{(3-q)/2}),$$

and (40) for u follows directly from (42) and the above inequality. We next proceed as before to prove that (40) for u, (35) and (13) imply that (40) for v also holds true. □

3.2. *The case* $q \in [2,3)$. We first show that the estimate (36) still holds true, its proof being slightly different.

LEMMA 8. For each $p \in [1,\infty)$ and $t \in (0,+\infty)$, there holds

$$t^{(1-1/p)/2}|u(t)|_{L^p} + t^{(q-1/p)/2}|v(t)|_{L^p} \leq C(p). \tag{43}$$

Proof. Consider $p \geq 1$. Then $p+1 \leq p+3/q$ and (35) and the Hölder inequality yield

$$|v(t)|_{L^{1+p}}^{1+p} \leq C(p)|v(t)|_{L^{p+3/q}}^{(p+3/q)(pq)/(pq+3-q)}, \quad t \geq 0. \tag{44}$$

Owing to (35) and (44), we may estimate E_p from above as follows.

$$\begin{aligned}E_p &\leq |u|_{L^1}|u|_{L^\infty}^{pq} + C(p)|v|_{L^{p+3/q}}^{(p+3/q)(pq)/(pq+3-q)} \\ &\quad + C_2|v|_{L^{p+3/q}}^{p+3/q} \\ &\leq CD_p^{(pq)/(2+pq)} + C(p)|v|_{L^{p+3/q}}^{(p+3/q)(pq)/(2+pq)},\end{aligned}$$

since $pq + 3 - q < pq + 2$. Consequently,

$$E_p \leq C(p)D_p^{(pq)/(2+pq)},$$

and we argue as in the proof of Lemma 6 to conclude that (43) holds true. □

REMARK 5. Lemma 8 is actually valid for $q \in (1,3)$. Moreover it is clear from the proofs of Lemma 6 and Lemma 8 that the constants $C(p)$ in (36) and (43) only depend on $q, M, |u_0|_{L^\infty}, |v_0|_{L^\infty}$, and p.

The proof of Lemma 7 relies heavily on the assumption $q \geq 3$ and clearly does not extend to the present case. Still, we have the following (weaker) result.

LEMMA 9. For $t \in (0,+\infty)$, there holds

$$t^{1/2}|u(t)|_{L^\infty} + t^{q/2}|v(t)|_{L^\infty} \leq C. \tag{45}$$

In addition, for each $\tau \in (0,+\infty)$ and $t \in (\tau,+\infty)$, we have

$$t^{1/2}|(u+v)(t)|_{H^{1/2}} \leq C(\tau). \tag{46}$$

Proof. The proof is divided into three steps.

Step 1. We first claim that, if $p \in [1,\infty)$ and $t \in (0,+\infty)$, there holds

$$|u_x(t)|_{L^p} \leq C(p)t^{(1-1/p)/2}(t^{-1/2} + t^{(2-q)/2}). \tag{47}$$

Indeed, the Duhamel formula (12) yields

$$\begin{aligned}|u_x(2t)|_{L^p} &\leq |\widetilde{G}_x(t)u(t)|_{L^p} + \int_0^t |\widetilde{G}_x(t-s)(v-u^q)(s+t)|_{L^p}\,ds\\ &\leq C|u(t)|_{L^1}t^{-(2-1/p)/2}\\ &\quad + C\int_0^t (t-s)^{-(2-1/p)/2}(|v(s+t)|_{L^1} + |u(s+t)|_{L^q}^q)\,ds,\end{aligned}$$

and (47) follows at once from (35), (43), and the above inequality.

Step 2. We next deduce from (47) some temporal decay estimates for v_x. More precisely, if $p \in [1,\infty)$ and $t \in (0,+\infty)$, we infer from (13), the Hölder inequality, (34), (35), (43), and (47) that

$$\begin{aligned}|v_x(t)|_{L^p} &\leq e^{-t}|v_{0,x}|_{L^p} + q\int_0^{t/2} e^{s-t}|u(s)|_{L^\infty}^{q-1}|u_x(s)|_{L^p}\,ds\\ &\quad + q\int_{t/2}^t e^{s-t}|u(s)|_{L^{2p(q-1)}}^{q-1}|u_x(s)|_{L^{2p}}\,ds\\ &\leq Ce^{-t} + C(p)e^{-t/2}t^{1/2p}(1+t^{(3-q)/2})\\ &\quad + C(p)\int_{t/2}^t e^{s-t}s^{(q-1/p)/2}(s^{-1/2} + s^{(2-q)/2})\,ds\\ &\leq C(p)(e^{-t/4} + t^{(q-1/p)/2}(t^{-1/2} + t^{(2-q)/2})).\end{aligned}$$

$$|v_x(t)|_{L^p} \leq C(p)t^{(q-1/p)/2}(1 + t^{-1/2} + t^{(2-q)/2}), \quad t > 0. \tag{48}$$

Step 3. We put $w = u + v$. By (1)–(2), w is the solution to

$$w_t + w_x - w_{xx} = v_x - v_{xx} \text{ in } (0,+\infty) \times \mathbb{R}$$

with initial datum $w(0) = u_0 + v_0$. We infer from the Duhamel formula that

$$w(2t) = \widetilde{G}(t)w(t) + \int_0^t \widetilde{G}_x(t-s)(v - v_x)(s+t)\,ds, \quad t \geq 0. \tag{49}$$

It first follows from (35), (43), (48), and (49) that

$$\begin{aligned}|w(2t)|_{L^\infty} &\le Ct^{-1/2}|w(t)|_{L^1}\\ &\quad + C\int_0^t (t-s)^{-3/4}(|v(s+t)|_{L^2} + |v_x(s+t)|_{L^2})\,ds\\ &\le Ct^{-1/2} + Ct^{-(q-1)/2}(1+t^{-1/2}+t^{(2-q)/2})\\ &\le Ct^{-1/2}(1+t^{-(q-2)/2}(1+t^{-1/2}+t^{(2-q)/2})).\end{aligned}$$

Since $q \ge 2$, we obtain

$$|w(t)|_{L^\infty} \le Ct^{-1/2} \text{ for } t \ge 1.$$

Recalling (35), the above inequality and the nonnegativity of u and v yield

$$|u(t)|_{L^\infty} \le Ct^{-1/2} \text{ for } t \ge 0.$$

We now argue as in the proof of Lemma 7 to show that (45) follows from the previous estimate.

Using again the Duhamel formula (49), we obtain

$$\begin{aligned}|w(2t)|_{H^{1/2}} &\le Ct^{-1/2}|w(t)|_{L^1}\\ &\quad + C\int_0^t (t-s)^{-3/4}(|v(s+t)|_{L^2} + |v_x(s+t)|_{L^2})\,ds.\end{aligned}$$

Proceeding as above, we deduce from (35), (43), and (48) that

$$|w(2t)|_{H^{1/2}} \le Ct^{-1/2}(1+t^{-(q-2)/2}(1+t^{-1/2}+t^{(2-q)/2})),$$

hence (46) since $q \ge 2$. □

4. Proof of Theorem 1. We now proceed with the proof of Theorem 1. Let (u_0, v_0) be two measurable functions satisfying (4) and (34). We put $(u,v)(t) = S_t(u_0,v_0)$ for $t \ge 0$ and

$$M := \int (u_0+v_0)(x)\,dx.$$

Following the analysis of [GvDD], we introduce the new unknown functions (U,V) defined by

$$(U,V)(t,x) = (u,v)(t,x+t) \text{ for } (t,x) \in (0,+\infty)\times\mathbb{R}. \tag{50}$$

Then (U,V) solves

$$U_t - U_{xx} = V - U^q \text{ in } (0,+\infty)\times\mathbb{R}, \tag{51}$$

$$V_t - V_x = U^q - V \text{ in } (0,+\infty)\times\mathbb{R}, \tag{52}$$

with initial data $(U,V)(0) = (u_0,v_0)$. We next define

$$U_\lambda(t,x) = \lambda U(\lambda^2 t, \lambda x) \text{ and } V_\lambda(t,x) = \lambda^q V(\lambda^2 t,\lambda x) \tag{53}$$

for $(t,x) \in (0,+\infty)\times\mathbb{R}$ and $\lambda \ge 1$. We also put

$$W_\lambda(t,x) = \lambda(U+V)(\lambda^2 t,\lambda x) = U_\lambda(t,x) + \lambda^{1-q}V_\lambda(t,x). \tag{54}$$

It follows from (51)–(52) that (U_λ, V_λ) solves

$$U_{\lambda,t} - U_{\lambda,xx} = \lambda^{2-q} V_{\lambda,x} - \lambda^{1-q} V_{\lambda,t} \text{ in } (0,+\infty) \times \mathbb{R}, \tag{55}$$

$$V_\lambda - U_\lambda^q = \lambda^{-1} V_{\lambda,x} - \lambda^{-2} V_{\lambda,t} \text{ in } (0,+\infty) \times \mathbb{R}. \tag{56}$$

Observe that an equivalent formulation of (55) is

$$W_{\lambda,t} - U_{\lambda,xx} = \lambda^{2-q} V_{\lambda,x} \text{ in } (0,+\infty) \times \mathbb{R}. \tag{57}$$

We now make use of the analysis of the previous section to derive estimates on $(U_\lambda, V_\lambda, W_\lambda)$ which are uniform with respect to $\lambda \geq 1$. In the following, we denote by C any positive constant which only depends on q, M, u_0, and v_0. The dependence of C upon additional parameters will be indicated explicitly.

LEMMA 10. Consider $p \in [1,\infty]$ and $t \in (0,+\infty)$. There holds

$$t^{(1-1/p)/2} |U_\lambda(t)|_{L^p} + t^{(q-1/p)/2} |V_\lambda(t)|_{L^p} + \frac{t^{(1-1/p)/2}}{1+t^{(1-q)/2}} |W_\lambda(t)|_{L^p} \leq C(p). \tag{58}$$

In addition, if $q \geq 3$, we have

$$t^{3/4} |W_{\lambda,x}(t)|_{L^2} \leq C \text{ for } t \geq 0, \tag{59}$$

while for $q \in [2,3)$ and $\tau > 0$, there holds

$$t^{1/2} |W_\lambda(t)|_{H^{1/2}} \leq C(\tau) \text{ for } t \geq \tau. \tag{60}$$

Proof. Let us first consider $p \in [1,\infty]$ and $t \in (0,+\infty)$. By (50), we have

$$|U(t)|_{L^p} = |u(t)|_{L^p} \text{ and } |V(t)|_{L^p} = |v(t)|_{L^p}.$$

The assertion (58) then follows from (36), (39), (43), and (45) by elementary computations. Assuming next that $q \geq 3$, (59) is a straightforward consequence of (40). We finally consider $q \in [2,3)$. Let $t \in (0,+\infty)$. As $\lambda \geq 1$ we have

$$|W_\lambda(t)|_{H^{1/2}} \leq \lambda |W_\lambda(\lambda^2 t)|_{H^{1/2}}.$$

For $\tau > 0, \lambda \geq 1$, and $t \in (\tau,+\infty)$, we have $\lambda^2 t \geq \tau$ and we infer from (46) that (60) holds true. □

LEMMA 11. For $t_1 \in (0,+\infty)$ and $t_2 \in (t_1,+\infty)$, there holds

$$|W_{\lambda,t}|_{L^2(t_1,t_2;H^{-1})} \leq C(t_1,t_2). \tag{61}$$

Proof. Consider $t_1 \in (0,+\infty), t_2 \in (t_1,+\infty)$, and $\varphi \in L^2(t_1,t_2;H^1(\mathbb{R}))$. We first infer from (20) (with $p = 1/q$) that

$$\int_{\lambda^2 t_1}^{\lambda^2 t_2} \int |u_x(s,x)|^2 \, dx \, ds \leq C(|u(\lambda^2 t_1)|_{L^2}^2 + |v(\lambda^2 t_1)|_{L^{(1+q)/q}}^{(1+q)/q}).$$

Consequently

$$\int_{t_1}^{t_2} \int |U_{\lambda,x}(s,x)|^2 \, dx \, ds \leq \lambda \int_{\lambda^2 t_1}^{\lambda^2 t_2} \int |u_x(s,x)|^2 \, dx \, ds$$
$$\leq C\lambda(|u(\lambda^2 t_1)|_{L^2}^2 + |v(\lambda^2 t_1)|_{L^{(1+q)/q}}^{(1+q)/q}),$$

and Lemma 6 and Lemma 8 entail

$$\int_{t_1}^{t_2} \int |U_{\lambda,x}(s,x)|^2 \, dx \, ds \leq C(t_1^{-1/2} + \lambda^{1-q} t_1^{-q/2}) \leq C(t_1) \tag{62}$$

(recall that $\lambda \geq 1$ and $q \geq 2$). We next infer from (57), (58), and (62) that

$$\begin{aligned} &\left| \int_{t_1}^{t_2} \int W_{\lambda,t} \varphi \, dx \, ds \right| \\ &\quad \leq (|U_{\lambda,x}|_{L^2(t_1,t_2;L^2)} + \lambda^{2-q} |V_\lambda|_{L^2(t_1,t_2;L^2)}) |\varphi_x|_{L^2(t_1,t_2;L^2)} \\ &\quad \leq C(t_1,t_2) |\varphi|_{L^2(t_1,t_2;H^1)}, \end{aligned}$$

from which (61) follows. □

LEMMA 12. For $T \in (0,+\infty)$, there holds

$$\lim_{R \to +\infty} \sup_{\lambda \geq 1, t \in [0,T]} \int_{\{|x| \geq R\}} W_\lambda(t,x) \, dx = 0. \tag{63}$$

Proof. Let ϑ be a smooth function in $\mathcal{C}^\infty(\mathbb{R})$ satisfying $0 \leq \vartheta \leq 1$ and

$$\vartheta(x) = 0 \text{ if } |x| \leq 1/2 \text{ and } \vartheta(x) = 1 \text{ if } |x| \geq 1.$$

For $R \geq 1$ and $x \in \mathbb{R}$, we put $\vartheta_R(x) = \vartheta(x/R)$. Let $t \in [0,T]$. It follows from (57) that

$$\begin{aligned} \int \vartheta_R(x) W_\lambda(t,x) \, dx \leq &\int (u_0 + v_0)(x) \vartheta_R(x/\lambda) \, dx \\ &+ \frac{|\vartheta_{xx}|_{L^\infty}}{R^2} \int_0^t |U_\lambda(s)|_{L^1} \, ds \\ &+ \frac{\lambda^{2-q} |\vartheta_x|_{L^\infty}}{R} \int_0^t |V_\lambda(s)|_{L^1} \, ds. \end{aligned}$$

On the one hand, it follows from (58) that

$$\int_0^t |U_\lambda(s)|_{L^1} \, ds \leq CT.$$

On the other hand, using Lemma 6 and Lemma 8, we obtain

$$\begin{aligned} \lambda^{2-q} \int_0^t |V_\lambda(s)|_{L^1} \, ds &= \lambda \int_0^t |V(\lambda^2 s)|_{L^1} \, ds \\ &\leq \lambda \int_0^{T/\lambda^2} M \, ds + C\lambda^{2-q} \int_{T/\lambda^2}^T s^{-(q-1)/2} \, ds \\ &\leq C(T) + C \begin{cases} \lambda^{2-q} T^{(3-q)/2} & \text{if } q \in [2,3), \\ \lambda^{-1} \ln \lambda & \text{if } q = 3, \\ \lambda^{-1} T^{(3-q)/2} & \text{if } q > 3 \end{cases} \\ &\leq C(T). \end{aligned}$$

Combining the above estimates yields, since $\lambda \geq 1$,

$$\int \vartheta_R(x) W_\lambda(t,x)\, dx \leq \int_{\{|x|\geq R\}} (u_0+v_0)(x)\, dx + \frac{C(T,\vartheta)}{R}.$$

As u_0+v_0 belongs to $L^1(\mathbb{R})$, Lemma 12 follows at once from the above inequality. □

Summarizing the outcome of Lemma 10 and Lemma 11, we have proved that $(W_\lambda)_{\lambda\geq 1}$ is bounded in $L^\infty(t_1,t_2;H^s(\mathbb{R}))$ and in $W^{1,2}(t_1,t_2;H^{-1}(\mathbb{R}))$ for every $t_1>0$ and $t_2>t_1$, the exponent s being equal to 1 if $q\geq 3$ and to 1/2 if $q\in[2,3)$. Owing to the compactness of the embedding of $H^s_{\text{loc}}(\mathbb{R})$ in $L^2_{\text{loc}}(\mathbb{R})$, we infer from [Si, Corollary 4] that (W_λ) is relatively compact in $\mathcal{C}([t_1,t_2];L^2_{\text{loc}}(\mathbb{R}))$ for every $t_1>0$ and $t_2>t_1$. Furthermore Lemma 12 provides the uniform integrability of (W_λ) for large values of x and we finally conclude that

$$(W_\lambda) \text{ is relatively compact in } \mathcal{C}([t_1,t_2];L^1(\mathbb{R})) \text{ for every } t_1>0 \text{ and } t_2>t_1.$$

Therefore there is a sequence (λ_n), $\lambda_n\to+\infty$, and a function

$$U_\infty \in \mathcal{C}((0,+\infty);L^1(\mathbb{R}))$$

such that

$$\lim_{n\to+\infty} \sup_{t\in[t_1,t_2]} |W_{\lambda_n}(t)-U_\infty(t)|_{L^1} = 0 \tag{64}$$

for every $t_1>0$ and $t_2>t_1$. Notice first that (64) and the nonnegativity of W_λ ensure that

$$U_\infty(t)\geq 0 \text{ a.e. in } \mathbb{R} \text{ for } t>0. \tag{65}$$

Since $q\geq 2$, we next infer from (58) that

$$\lim_{\lambda\to+\infty} \sup_{t\in[t_1,t_2]} \lambda^{1-q}|V_\lambda(t)|_{L^1} = 0 \tag{66}$$

for every $t_1>0$ and $t_2>t_1$, which yields, together with (64),

$$\lim_{n\to+\infty} \sup_{t\in[t_1,t_2]} |U_{\lambda_n}(t)-U_\infty(t)|_{L^1} = 0. \tag{67}$$

We now identify the behaviour of $U_\infty(t)$ as $t\to 0$. Consider $\varphi\in\mathcal{D}(\mathbb{R})$, $t\in(0,1)$, and $s\in(0,t)$. It follows from (57) that

$$\left|\int W_{\lambda_n}(t,x)\varphi(x)\, dx - \int W_{\lambda_n}(s,x)\varphi(x)\, dx\right| \leq C(\varphi)\int_s^t \left(|U_{\lambda_n}(\sigma)|_{L^1} + \lambda_n^{2-q}|V_{\lambda_n}(\sigma)|_{L^1}\right) d\sigma.$$

Thanks to (58), we further obtain

$$\left|\int W_{\lambda_n}(t,x)\varphi(x)\, dx - \int W_{\lambda_n}(s,x)\varphi(x)\, dx\right| \leq C(\varphi)\left(t+\lambda_n^{2-q}\int_s^t \sigma^{-(q-1)/2}\, d\sigma\right). \tag{68}$$

We first treat the case $q \in [2,3)$. On the one hand, $(q-1)/2 < 1$ and we may let $s \to 0$ in (68) to obtain

$$\left| \int W_{\lambda_n}(t,x)\varphi(x)\,dx - \int W_{\lambda_n}(0,x)\varphi(x)\,dx \right| \le C(\varphi) \quad (t + t^{(3-q)/2}).$$

On the other hand, it readily follows from (53) that

$$\lim_{\lambda \to +\infty} \int W_\lambda(0,x)\varphi(x)\,dx = M\varphi(0). \tag{69}$$

Recalling (64) we may pass to the limit as $n \to +\infty$ in the previous inequality and obtain

$$\left| \int U_\infty(t,x)\varphi(x)\,dx - M\varphi(0) \right| \le C(\varphi) \quad (t + t^{(3-q)/2}).$$

We then let $t \to 0$ and conclude that

$$\lim_{t \to 0} \int U_\infty(t,x)\varphi(x)\,dx = M\varphi(0). \tag{70}$$

We next handle the case $q \ge 3$. We fix $\alpha \in (2, 2(q-2)/(q-3))$ and take $s = \lambda_n^{-\alpha}$ in (68). We thus obtain

$$\begin{aligned} &\left| \int W_{\lambda_n}(t,x)\varphi(x)\,dx - \int W_{\lambda_n}(\lambda_n^{-\alpha},x)\varphi(x)\,dx \right| \\ &\quad \le C(\varphi) \begin{cases} (t + \lambda_n^{2-q-\alpha(3-q)/2}) & \text{if } q > 3, \\ (t + \lambda_n^{-1} \ln(\lambda_n^{-\alpha} t)) & \text{if } q = 3. \end{cases} \end{aligned}$$

Owing to the choice of α we may let $n \to +\infty$ in the above inequality and use (64) to conclude that

$$\limsup_{n \to +\infty} \left| \int U_\infty(t,x)\varphi(x)\,dx - \int W_{\lambda_n}(\lambda_n^{-\alpha},x)\varphi(x)\,dx \right| \le C(\varphi) t. \tag{71}$$

We next infer from (54) that

$$\begin{aligned} &\left| \int W_{\lambda_n}(0,x)\varphi(x)\,dx - \int W_{\lambda_n}(\lambda_n^{-\alpha},x)\varphi(x)\,dx \right| \\ &\quad \le C(\varphi) |W(\lambda_n^{2-\alpha}) - W(0)|_{L^1}. \end{aligned}$$

Since $\alpha > 2$ and $W \in \mathcal{C}([0,+\infty); L^1(\mathbb{R}))$, we finally obtain

$$\lim_{n \to +\infty} \int W_{\lambda_n}(0,x)\varphi(x)\,dx - \int W_{\lambda_n}(\lambda_n^{-\alpha},x)\varphi(x)\,dx = 0. \tag{72}$$

Combining (71), (72), and (69) then yields that (70) also holds for $q \ge 3$. A standard density argument and Lemma 12 further warrant that (70) holds true for every $\varphi \in \mathcal{BC}(\mathbb{R})$, that is,

$$\lim_{t \to 0} \int U_\infty(t,x)\varphi(x)\,dx = M\varphi(0) \text{ for } \varphi \in \mathcal{BC}(\mathbb{R}). \tag{73}$$

Here $\mathcal{BC}(\mathbb{R})$ denotes the space of bounded and continuous functions on $\mathbb{R}$.

We finally identify the equation satisfied by U_∞. We consider $\varphi \in \mathcal{D}((0,+\infty)\times\mathbb{R})$ and infer from (56)–(57) that

$$\int_0^{+\infty}\int (W_{\lambda_n}\varphi_t + U_{\lambda_n}\varphi_{xx})\,dx\,ds = \lambda_n^{2-q}\int_0^{+\infty}\int V_{\lambda_n}\varphi_x\,dx\,ds, \tag{74}$$

$$\int_0^{+\infty}\int (V_{\lambda_n} - U_{\lambda_n}^q)\varphi\,dx\,ds = \int_0^{+\infty}\int (\lambda_n^{-2}\varphi_t - \lambda_n^{-1}\varphi_x)V_{\lambda_n}\,dx\,ds. \tag{75}$$

Since (V_{λ_n}) is bounded in $L^1_{\mathrm{loc}}((0,+\infty);L^1(\mathbb{R}))$ by (58), (58) and (67) allows to pass to the limit as $n\to+\infty$ in (75) to obtain that

$$(V_{\lambda_n}) \text{ converges to } U_\infty^q \text{ in } \mathcal{D}'((0,+\infty)\times\mathbb{R}). \tag{76}$$

We may then pass to the limit as $n\to+\infty$ in (74) and use (64), (67), and (76) to conclude that U_∞ is a solution in $\mathcal{D}'((0,+\infty)\times\mathbb{R})$ to

$$U_{\infty,t} - U_{\infty,xx} = 0 \text{ if } q>2, \tag{77}$$

and

$$U_{\infty,t} - U_{\infty,xx} = (U_\infty^2)_x \text{ if } q=2. \tag{78}$$

In addition, it follows from (58) and (67) that

$$\sup_{t\in(0,+\infty)} \left(|U_\infty(t)|_{L^1} + t^{1/2}|U_\infty(t)|_{L^\infty}\right) < \infty. \tag{79}$$

We have thus proved that U_∞ is a nonnegative solution to either (77) or (78) with initial datum $M\delta$ (recall (73)). Such a solution being unique (see [EVZ] for (78)) we conclude that U_∞ is the fundamental solution to the linear heat equation (77) with mass M if $q>2$ and the source-type solution to the viscous Burgers equation (78) with mass M if $q=2$. Therefore $U_\infty = S_M$ with the notation in Theorem 1. Furthermore, the previous arguments guarantee that (W_λ) has only one cluster point S_M as $\lambda\to+\infty$. The relative compactness of (W_λ) then entails that the whole family (W_λ) converges to S_M in $\mathcal{C}([t_1,t_2];L^1(\mathbb{R}))$ as $\lambda\to+\infty$ for every $t_1>0$ and $t_2>t_1$. Recalling (66), so does (U_λ); that is,

$$\lim_{\lambda\to+\infty} |U_\lambda(t) - S_M(t)|_{L^1} = 0 \text{ for } t\in(0,+\infty). \tag{80}$$

As S_M has a self-similar form, (80) may also be written as

$$\lim_{t\to+\infty} |U(t) - S_M(t)|_{L^1} = 0,$$

hence (5) for $p=1$. As $t\mapsto t^{1/2}|U(t)|_{L^\infty}$ is bounded by (39) and (45), the assertion (5) for $p\in(1,\infty)$ easily follows by interpolation and the proof of Theorem 1 is complete for initial data enjoying the additional regularity (34). The case of initial data satisfying only (4) then follows by a density argument combined with the contraction property (16), keeping in mind Remark 5.

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