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GLOBAL SOLUTIONS TO THE LAKE EQUATIONS WITH ISOLATED VORTEX REGIONS

By

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Abstract. The vorticity formulation for the lake equations in R^2 is studied. We assume that the initial vorticity has the form $\omega(x,0) = \omega_0(x)\chi_{\bar{\Omega}_0}$, where the initial vortex region Ω_0 is a $C^{1+\alpha}$ domain and $\omega_0 \in C^\alpha\left(\bar{\Omega}_0\right)$. It is shown that the Cauchy problem can be formulated as an integral system. Global existence and uniqueness of the $C^{1+\alpha}$ solution to the integral system are established. Consequently, the lake equation admits a unique weak solution, global in time, in the form of $\omega(x,t) = \omega_t(x)\chi_{\bar{\Omega}_t}$, where $\omega_t(x) \in C^\alpha_x\left(\bar{\Omega}_t\right)$ and $\partial\Omega_t \in C^\alpha$.

1. Introduction. Consider the circulation of an inviscid fluid in a very large shallow basin with a varying bottom. Let $x = (x_1, x_2) \in R^2$ be the horizontal position inside the basin and let t be the time; then the evolution of vertically averaged fluid velocity vector $u(x,t) = (u_1(x,t), u_2(x,t)) \in R^2$ is governed by the great lake equations [2, 11, 12]:

$$v_t = u^\perp \nabla \wedge v + \nabla\left(h - \frac{1}{2}|u|^2 + u\cdot v\right),\ \nabla\cdot(bu) = 0,$$
$$v = u + \delta^2\left((u\cdot\nabla b)\nabla b + \frac{1}{2}b(\nabla\cdot u)\nabla b - \frac{1}{2b}\nabla\left(b^2 u\cdot\nabla b\right) - \frac{1}{3b}\nabla\left(b^3\nabla\cdot u\right)\right),$$

where $h(x,t)$ is the surface height variation, $b(x)$ is the depth of the basin, $u^\perp = (u_2, -u_1)$, $\nabla\wedge u = \partial_1 u_2 - \partial_2 u_1$, and δ is the aspect ratio—the ratio of vertical to horizontal length scales. In the 0th order approximation in δ^2, we obtain $v = u$. The great lake equation then reduces to the lake equation

$$u_t = u^\perp\nabla\wedge v + \nabla\left(h - \frac{1}{2}|u|^2 + u\cdot v\right),\ \nabla\cdot(bu) = 0.$$

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Introducing the potential vorticity $\omega = b^{-1}\nabla \wedge u$ and applying the curl in the first equation above, we arrive at the following system:

$$\omega_t(x,t) + u(x,t) \cdot \nabla\omega(x,t) = 0, \tag{1.1}$$

$$b(x)\,\omega(x,t) = \nabla \wedge u(x,t), \tag{1.2}$$

$$\nabla \cdot (b(x)\,u(x,t)) = 0. \tag{1.3}$$

We refer to [12] for detailed description about the great lake equations and lake equations.

The system (1.1)–(1.3) is closely related to the 2D incompressible Euler equation. In fact, when $b(x) \equiv 1$, this system reduces to the vorticity equation for the 2D Euler equation. Hence, system (1.1)–(1.3) may be viewed as the vorticity formulation of the lake equation. The lake equation in a bounded domain Ω has been studied in [10, 11, 12, 14]. In those papers, the authors analyzed even more general systems in which (1.2) is replaced by $u = K\omega$ for certain linear operators K. They established global existence of solutions in the Sobolev spaces $W^{m,p}$ and in the space of analytical functions. For the studies of the 2D Euler equation, we refer to [13] and references therein.

We are interested in the situation when the basin is very large with an isolated initial vortex region. This case may be approximated by the vorticity formulation of the lake equation (1.1)–(1.3) in the whole space with the initial vorticity of the form

$$\omega(x,0) = \omega_0(x)\,\chi_{\bar{\Omega}_0}, \tag{1.4}$$

where $\chi_{\bar{\Omega}_0}$ is the characteristic function of $\bar{\Omega}_0$, the closure of a domain Ω_0, and $\omega_0(x) \geq 0$ is a given function. We assume that the initial vortex region Ω_0 is a bounded $C^{1+\alpha}$ domain, and that $\omega_0(x)$ is $\alpha-$Hölder continuous in $\bar{\Omega}_0$. Note that $\omega_0(x)$ may be extended to the whole space. Hence, the last assumption is equivalent to $\omega_0(x)$ being $\alpha-$Hölder continuous in R^2. We also note that when $\omega_0(x_0) > 0$ for some $x_0 \in \partial\Omega_0$, the initial vorticity $\omega(x,0)$ is discontinuous at x_0. Hence, the initial condition (1.4) includes, in particular, an interesting situation that the initial vorticity may be discontinuous across the boundary $\partial\Omega_0$ of Ω_0.

For the 2D Euler equation, existence of global weak solutions for general initial data was established in [15]. For the Cauchy problem with initial data (1.4), the problem of global regularity for weak solutions was proposed by Majda in [13]. In particular, when $\omega_0(x)$ is a constant, a weak solution is referred to as a constant vortex patch. Global regularity of constant vortex patches was established by Chemin [3]. Roughly speaking, Chemin asserts that the initial regularity of the vortex region for a constant vortex patch will persist for all time. A simpler proof of more general regularity results can be found in [1] and [7].

In this paper, we are concerned with global existence and stratified regularity of weak solutions to (1.1)–(1.4). The main purpose is to extend Chemin's regularity result to the lake equation. We shall present an integral equation approach to the Cauchy problem for the lake equation. In this approach, the vorticity equation will be formulated as a non-local integral system in $\bar{\Omega}_0 \times [0,\infty)$. We shall show that there exists a unique $C^{1+\alpha}$ solution $\Phi(x,t)$ to the integral system for all $t \geq 0$. This $C^{1+\alpha}$ solution $\Phi(x,t)$ will naturally produce a unique weak solution to the lake equation (1.1)–(1.4) by $\omega(x,t) = \omega_0\left(\Phi^{-1}(x,t)\right)\chi_{\bar{\Omega}_t}$, where $\Omega_t = \Phi(\Omega_0,t)$ is the vortex region. Clearly, this implies immediately that the vortex region is a $C^{1+\alpha}$ domain and the vorticity enjoy the stratified regularity of Hölder types, i.e., $\omega(\cdot,t) \in C^{\alpha}\left(\bar{\Omega}_t\right)$, for any fixed $t > 0$.

The integral equation method was previously used to study dynamics of charged particles in [5] and superconductor vortices in [8]. The method employed in [1] for 2D constant vortex patches is essentially an integral equation method, though the authors there did not explicitly deal with the integral equation for $\Phi(x,t)$. For local existence, we shall use a fixed point argument as in [5]. However, the integral system for the lake equation is structurally different from those in [5]. Some additional non-trivial technical treatments are necessary in order to establish existence and uniqueness of solutions of (1.1)–(1.4) for a short time. The treatment for global existence and regularity is inspired by an idea in [1] and [7]. As we demonstrated in [5], existence of global solutions for these types of integral systems depends on their structures and initial data. Fortunately, for the lake equation, the singular part of the integral kernel is similar to the one for the Euler system. Though the initial vorticity $\omega_0(x)$ is no longer a constant here, we are still able to establish various estimates that show, for some $0 < \beta < \alpha$, the logarithmically superlinear growth for the $\beta-$Hölder norms of the vorticity and the boundary of the vortex region. These estimates then lead to *á prióri* estimates for the $\alpha-$Hölder norms of the above quantities. The results on global existence and regularity follow immediately.

The paper is organized as follows. In the next section, we shall derive the integral formulation for the lake equation (1.1)–(1.4). We shall show that in a certain sense, a solution of the integral equation will produce naturally a weak solution to (1.1)–(1.4). In Sec. 3, we shall study short-time existence, uniqueness, and regularity of solutions to the integral equation. Finally in Sec. 4, we shall establish existence and regularity for all $t > 0$.

2. Integral System Formulation. In this section, we shall formulate the system (1.1)–(1.4) as a non-local integral system in the sense that any solution of the integral system defines a unique weak solution. To illustrate this integral formulation, we may assume that $\omega(x,t)$ is a smooth solution of (1.1)–(1.4) with the smooth velocity field $u(x,t)$. This is the case when $\omega_0(x)$ in (1.4), defined only in $\bar{\Omega}_0$, is smooth and has a compact support in Ω_0 (hence $\omega(x,0)$ is smooth), and when $b(x) \in C_b^{\infty}$ (the space of smooth functions with bounded derivatives) and $b(x) \geq b_0 > 0$ for some constants b_0. Since $\nabla \cdot (bu) = 0$, there exists a potential function $\psi(x,t)$ such that

$$b(x)\,u(x,t) = \nabla^{\perp}\psi(x,t) \stackrel{\text{def}}{=} \left(\partial_2\psi(x,t), -\partial_1\psi(x,t)\right). \tag{2.1}$$

By (1.2), this potential satisfies

$$\nabla \wedge u = \nabla \wedge \left(b^{-1}\nabla^{\perp}\psi\right) = b\omega,$$

or equivalently, the following elliptic equation

$$-\Delta\psi(x,t) + \nabla \log b(x) \cdot \nabla\psi(x,t) = b(x)^2\,\omega(x,t), \tag{2.2}$$

for fixed t, with the growth condition at infinity: $\psi(x,t) \to O(\log|x|)$ as $|x| \to \infty$. Suppose that the elliptic operator $-\Delta + \nabla \log b \cdot \nabla$ in (2.2) has the fundamental solution, say $K(x,y)$. Then

$$\psi(x,t) = \int_{R^n} K(x,z)\, b^2(z)\, \omega(z,t)\, dz. \tag{2.3}$$

Next, we define the fluid particle trajectory $\Phi(x,t)$, for fixed $x \in R^2$, by

$$\frac{\partial \Phi}{\partial t} = u(\Phi,t), \quad \Phi(x,0) = x. \tag{2.4}$$

It follows from (1.1) that $\omega(\Phi(x,t),t)$, for fixed x, solves the equation

$$\frac{d\omega}{dt} = 0.$$

Hence $\omega(\Phi(x,t),t) = \omega(x,0)$, or $\omega(x,t) = \omega\left(\Phi^{-1}(x,t),0\right)$. Set

$$\Omega_t = \Phi(\Omega_0,t) = \{\Phi(z,t) :\ z\in\Omega_0\}.$$

Then, we may write the vorticity in the form

$$\omega(x,t) = \omega_0\left(\Phi^{-1}(x,t)\right)\chi_{\bar{\Omega}_t}. \tag{2.5}$$

Substituting (2.1), (2.3), and (2.5) into (2.4), we obtain the dynamical system

$$\begin{aligned} \frac{\partial \Phi(x,t)}{\partial t} &= \frac{1}{b(\Phi(x,t))}\int_{\Omega_t} \nabla_x^{\perp} K(\Phi(x,t),z)\, b(z)^2\, \omega_0\left(\Phi^{-1}(z,t)\right) dz, \\ \Phi(x,0) &= x, \end{aligned} \tag{2.6}$$

where Φ^{-1} is the inverse of the mapping $x \mapsto \Phi(x,t)$ for any fixed t. This system is non-local in the sense that the value $\Phi(x_0,t)$ at x_0 depends on $\Phi(x,t)$ for all x. This equation can also be written equivalently as the integral system

$$\Phi(x,t) = x + \int_0^t \int_{\Omega_s} \frac{\nabla_x^{\perp} K(\Phi(x,s),z)\, b(z)^2\, \omega_0\left(\Phi^{-1}(z,s)\right)}{b(\Phi(x,s))}\, dz ds. \tag{2.7}$$

This integral system clearly depends only on $\Phi(x,t)$ in $\bar{\Omega}_0 \times [0,\infty)$. It is easy to see that the above procedure can be reversed. In other words, suppose that $\Phi(x,t)$, defined in $\bar{\Omega}_0 \times [0,\infty)$, is smooth and it solves (2.7). Define ω by (2.5), ψ by (2.3), and then u by (2.1). A direct computation shows that (ω,u) solve (1.1)–(1.4). Hence, for smooth initial data, the integral system (2.7) is equivalent to (1.1)–(1.4).

In general, for the initial vorticity $\omega(x,0)$ in (1.4) that has jump discontinuity along $\partial\Omega_0$, one expects from (2.5) that $\omega(x,t)$ also has jump discontinuity along $\partial\Omega_t$. Since the kernel $\nabla_x^{\perp} K(x,y)$ in (2.7) is singular as $x \to y$, the integral on the right-hand side of (2.7) may not be smooth as x crosses $\partial\Omega_0$. Hence, we shall look for solutions of (2.7) with less regularity. To this end, we need to introduce some notations.

For any family of open subsets $\{G_t\}_{t\geq 0}$ in R^2, integer $m \geq 0$, and $0 \leq \alpha < 1$, denote by $C_x^{m+\alpha}(G_t)$ the set of all functions $f(x,t)$ defined for $x \in G_t$ such that $f(\cdot,t)$ and all the spatial derivatives up to mth order are α–Hölder continuous in G_t. Denote by $C_x^{m+\alpha}\left(\bar{G}_t\right)$ the set of extensions to $\bar{G}_t$ of all functions in $C_x^{m+\alpha}(G_t)$. Denote by $|f(t)|_{m+\alpha}$ the Hölder semi-norm defined as

$$|f(t)|_{m+\alpha} = \sup_{x,y\in G_t,\, |\gamma|=m} \frac{|\partial_\gamma f(x,t) - \partial_\gamma f(y,t)|}{|x-y|^{\alpha}},$$

where $\gamma = (\gamma_1, \gamma_2)$ is a multi-index, $\partial_\gamma = \partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}$. The Hölder norm is denoted by

$$\|f(t)\|_{m+\alpha} = \sup_{x\in G_t,\, |\gamma|\le m} |\partial_\gamma f(x,t)| + |f(t)|_{m+\alpha}\,.$$

Sometime we shall use the notations $|f(t)|_{m+\alpha,G_t}$ and $\|f(t)\|_{m+\alpha,G_t}$ to specify the dependence on the domain G_t. For convenience, we introduce the notation

$$|f(t)|_{\inf,\partial G_t} = \inf_{x\in\partial G_t} |f(x,t)|\,.$$

We shall also adopt the notation $W_{loc}^{m,p}$ for the local Sobolev spaces.

Definition 2.1. We call $\Phi(x,t)$ a $C^{1+\alpha}$ solution of (2.7) in $\bar{\Omega}_0 \times [0,T)$ if $\Phi, \partial_t\Phi \in C_x^{1+\alpha}(\bar{\Omega}_0)$, and $\Phi^{-1} \in C_x^1(\bar{\Omega}_t)$ for $t<T$, and if (2.7) holds everywhere in $\bar{\Omega}_0 \times [0,T)$.

There are several ways to define weak solutions to (1.1)–(1.4). We shall adopt the following natural definition.

Definition 2.2. A weak solution to (1.1)–(1.4), for $t<T$, is defined as a couple $(\omega, u) \in L^2 \times W_{loc}^{1,2}$ such that (1.2) and (1.3) hold in the sense of distributions, and

$$\begin{aligned}&\int_0^T \int_{R^2} b(x)\,\omega(x,t)\,u(x,t)\cdot\nabla\left(b^{-1}\xi(x,t)\right)dxdt \\ &= -\int_0^T\int_{R^2}\omega(x,t)\,\xi_t(x,t)\,dxdt - \int_{\Omega_0}\omega_0(x)\,\xi(x,0)\,dx\end{aligned} \tag{2.8}$$

for any test function $\xi(x,t)$ that has a compact support in $R^2\times[0,T)$.

Proposition 2.1. Suppose that $\omega_0(x)$ is bounded, $b(x)\in C^1$ and $b_0 \le b(x) \le b_1$ for some positive constants b_0 and b_1. Let $\Phi(x,t)$ be a $C^{1+\alpha}$ solution of (2.7) in $\bar{\Omega}_0\times[0,T)$. Define $\omega(x,t)$ by (2.5), ψ by (2.3), and u by (2.1). Then (ω,u) is a weak solution of the Cauchy problem for the lake equation (1.1)–(1.4).

Proof. Since ω_0 is bounded, we know from (2.5) that $\omega(x,t)$ is bounded. We also know that $\Omega_t = \Phi(\Omega_0,t)$ is a domain that is bounded uniformly in $t\le T_1$ for any $T_1<T$. By the regularity theory for the elliptic equation (2.2), we have $\psi\in W_{loc}^{2,p}$ for any $p>1$, and consequently $u\in W_{loc}^{1,p}$. Obviously, the relations (1.2) and (1.3) hold in the sense of distributions. It suffices to show (2.8) with T being replaced by T_1. By (2.3) and (2.1), we may write (2.7) as

$$\Phi(x,t) = x + \int_0^t u(\Phi(x,s),s)\,ds.$$

It follows that $\Phi(x,t)$ solve (2.4), i.e.,

$$\frac{d\Phi(x,t)}{dt} = u(\Phi(x,t),t)\,.$$

Denote by $J(\Phi(x,t))$ the Jacobian of the mapping $\Phi(\cdot,t)$. By (2.5), $\omega(\Phi(x,t),t) = \omega_0(x)\,\chi_{\bar{\Omega}_0}$. Multiplying this equation by

$$J(\Phi(x,t))\,\frac{d\xi(\Phi(x,t),t)}{dt},$$

where ξ is a test function such that $\xi(x, T_1) = 0$, and then integrating over $\Omega_0 \times (0, T_1)$, we obtain

$$\begin{aligned}
&\int_0^{T_1} \int_{\Omega_0} \omega(\Phi(x,t),t) J(\Phi(x,t)) \frac{d\xi(\Phi(x,t),t)}{dt} dxdt \\
&= \int_0^{T_1} \int_{\Omega_0} \omega_0(x) J(\Phi(x,t)) \frac{d\xi(\Phi(x,t),t)}{dt} dxdt \\
&= -\int_0^{T_1} \int_{\Omega_0} \omega_0(x) \frac{dJ(\Phi(x,t))}{dt} \xi(\Phi(x,t),t) dxdt - \int_{\Omega_0} \omega_0(x) \xi(x,0) dx,
\end{aligned}$$

where we have performed integration by parts and have used the fact that $\Phi(x,0) = x$. By (2.4) and (1.3), we have (see [4, p. 25])

$$\begin{aligned}
\frac{dJ(\Phi(x,t))}{dt} &= (\nabla \cdot u)(\Phi(x,t),t) J(\Phi(x,t)) \\
&= b^{-1}(\nabla \cdot (bu) - (\nabla b) \cdot u)(\Phi(x,t),t) J(\Phi(x,t)) \\
&= -\left(b^{-1}(\nabla b) \cdot u\right)(\Phi(x,t),t) J(\Phi(x,t)).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_0^{T_1} \int_{\Omega_0} \omega(\Phi(x,t),t) J(\Phi(x,t)) \frac{d\xi(\Phi(x,t),t)}{dt} dxdt \\
&= \int_0^{T_1} \int_{\Omega_0} \omega_0(x) \left(b^{-1}\nabla b \cdot u\right) J(\Phi(x,t)) \xi(\Phi(x,t),t) dxdt \\
&- \int_{\Omega_0} \omega_0(x) \xi(x,0) dx \\
&= \int_0^{T_1} \int_{\Omega_t} \omega(x,t) \left(b^{-1}\nabla b \cdot u\right) \xi(x,t) dxdt - \int_{\Omega_0} \omega_0(x) \xi(x,0) dx. \qquad (2.9)
\end{aligned}$$

In deriving the last equality, we have changed variables $y = \Phi(x,t)$. On the other hand, the left-hand side of (2.9) can be expressed as, after performing the same changes of variables,

$$\begin{aligned}
&\int_0^{T_1} \int_{\Omega_0} \omega(\Phi(x,t),t) \frac{d\xi(\Phi(x,t),t)}{dt} J(\Phi(x,t)) dxdt \\
&= \int_0^{T_1} \int_{\Omega_0} \omega(\Phi(x,t),t) \xi_t(\Phi(x,t),t) J(\Phi(x,t)) dxdt \\
&+ \int_0^{T_1} \int_{\Omega_0} \omega(\Phi(x,t),t) (\nabla\xi)(\Phi(x,t),t) \cdot \frac{d\Phi(x,t)}{dt} J(\Phi(x,t)) dxdt \\
&= \int_0^{T_1} \int_{\Omega_0} (\omega\xi_t(\Phi(x,t),t) + (\omega u \cdot \nabla\xi)(\Phi(x,t),t)) J(\Phi(x,t)) dxdt \\
&= \int_0^{T_1} \int_{\Omega_t} \omega(x,t)(\xi_t(x,t) + (u \cdot \nabla\xi)(x,t)) dxdt.
\end{aligned}$$

From this and (2.9), we obtain

$$\int_0^{T_1}\int_0^T\int_{\Omega_t}\omega(x,t)\left(\xi_t(x,t)+(u\cdot\nabla\xi)(x,t)\right)dxdt$$
$$=-\int_{\Omega_0}\omega_0(x)\,\xi(x,0)\,dx+\int_0^{T_1}\int_{\Omega_t}\omega(x,t)\left(b^{-1}\nabla b\cdot u\right)(x,t)\,\xi(x,t)\,dxdt.$$

This is the same as (2.8). Hence, the couple (ω, u) is a weak solution to the lake equation. $\square$

3. Short-time Solution for Integral System. Throughout the paper, we assume that $b_0 \le b(x) \le b_1$, for some positive constants b_0, b_1, that $b(x) \in C^{1+\alpha}\left(R^2\right)$, and that the elliptic operator $-\Delta + \nabla \log b \cdot \nabla$ in (2.2) has the fundamental solution $K(x,y)$ of the form

$$K(x,y)=\hat{b}\log|x-y|+K_1(x,y)\,, \tag{3.1}$$

where $\hat{b}$ is a positive constant depending only on $b(x)$, and $K_1(x,y)$ is a positive C_b^3 function satisfying

$$|x-y|\,|K_1(x,y)|+\left|\nabla_x^2K_1(x,y)\right|+\left|\nabla_x^3K_1(x,y)\right|\le c_0, \tag{3.2}$$

where and throughout the paper the symbol c_0 is reserved for constants depending only on the given initial data. However, it may vary in different places. Existence of such fundamental solutions is discussed in detail in [9]. In particular, when $b(x) = b(|x|)$ is radially symmetric, the fundamental solution $K(x,y) = K(|x-y|)$ can be found simply by solving the ordinary differential equation $K'(r) = b_0 b(r) r^{-1}$, where b_0 is a constant. We also assume that $\omega_0 \in C^\alpha\left(\bar{\Omega}_0\right)$ for some $0 < \alpha < 1$, and that Ω_0 is a bounded $C^{1+\alpha}$ domain. For simplicity, we assume that there exists $\varphi_0 \in C^{1+\alpha}$ such that $\Omega_0 = \left\{x \in R^2 :\ \varphi_0(x) < 0\right\}$ and $\nabla\varphi_0 \ne 0$ on $\partial\Omega_0$.

Let Ω be a $C^{1+\alpha}$ domain such that $\Omega = \{x :\ \varphi(x) < 0\}$ for some $C^{1+\alpha}$ functions $\varphi(x)$ with $\nabla\varphi(x) \ne 0$ for $x \in \partial\Omega$. We define

$$\delta_\Omega=\frac{|\nabla\varphi|_{\alpha,\Omega}}{|\nabla\varphi|_{\inf,\partial\Omega}}. \tag{3.3}$$

This quantity δ_Ω may be understood as a $C^{1+\alpha}$ character of the $C^{1+\alpha}$ domain. Sometimes we shall use $\delta_{\Omega,\alpha} = \delta_\Omega$ to emphasize the dependence on α. Consider the Newtonian potential

$$V(x)=P_v\int_\Omega\frac{\sigma(x-y)}{|x-y|^2}\,dy,$$

where P_v means the principal value of the following singular integral,

$$\frac{\sigma(x)}{|x|^2}=\nabla^2\log|x|=\frac{1}{|x|^4}\begin{pmatrix}x_2^2-x_1^2 & -2x_1x_2\\ -2x_1x_2 & x_1^2-x_2^2\end{pmatrix},$$

and

$$G(x)=\int_\Omega\frac{\sigma(x-y)}{|x-y|^2}\,(g(x)-g(y))\,dy.$$

It is well-known that the principal value of the integral in defining $V(x)$ exists. We need the following lemma.

LEMMA 3.1. Suppose that $\partial\Omega \in C^{1+\alpha}$ and $g \in C^{\alpha}(\bar{\Omega})$. Then

$$|V|_{0,\Omega} \le c\log(2 + \delta_\Omega d(\Omega)), \tag{3.4}$$

$$|V|_{\alpha,\Omega} \le c\delta_\Omega \log(2 + \delta_\Omega d(\Omega)), \tag{3.5}$$

$$|G|_{\alpha,\Omega} \le c\,|g|_{\alpha,\Omega} \log(2 + \delta_\Omega d(\Omega)), \tag{3.6}$$

$$|G|_{0,\Omega} \le c\,|g|_{0,\Omega} \log\left(2 + \alpha^{-1} d(\Omega)\,|g|_{\alpha,\Omega}\right), \tag{3.7}$$

where $d(\Omega)$ is the diameter of Ω, and c is a constant independent of Ω, α, and g.

Proof. The proof of (3.4)–(3.6) can be found in [5, Lemma 3.1]. An estimate similar to (3.7) can also be found there. However, the dependence on α is not explicitly given in [5, Lemma 3.1]. Since this dependence plays an essential role in establishing global regularity in the next section, we need to show (3.7). Let $\varepsilon > 0$ and $B_\varepsilon(x)$ be the ball centered at x with radius ε. We write $G(x) = k_1(x) + k_2(x)$, where

$$k_1(x) = \int_{\Omega\setminus B_\varepsilon(x)} \frac{\sigma(x-y)}{|x-y|^2}\,(g(x) - g(y))\,dy,$$

$$k_2(x) = \int_{B_\varepsilon(x)} \frac{\sigma(x-y)}{|x-y|^2}\,(g(x) - g(y))\,dy.$$

Since $|\sigma(x)| \le c$, where c is a constant independent of Ω, α, and g, we have

$$\begin{aligned}
|k_1(x)| &= c \int_{\Omega\setminus B_\varepsilon(x)} \frac{|g(x) - g(y)|}{|x-y|^2}\,dy \\
&\le c\,|g|_{0,\Omega} \int_{\varepsilon}^{d(\Omega)} \frac{1}{r}\,dr = c\,|g|_{0,\Omega} \log\frac{d(\Omega)}{\varepsilon}, \\
|k_2(x)| &= c \int_{B_\varepsilon(x)} \frac{|g(x) - g(y)|}{|x-y|^2}\,dy \\
&\le c\,|g|_{\alpha,\Omega} \int_0^{\varepsilon} \frac{1}{r^{1-\alpha}}\,dr = c\alpha^{-1}\,|g|_{\alpha,\Omega}\,\varepsilon^\alpha.
\end{aligned}$$

The estimate (3.7) follows directly by choosing $\varepsilon^\alpha = \alpha / |g|_{\alpha,\Omega}$. □

LEMMA 3.2. Suppose that $\partial\Omega \in C^{1+\alpha}$, $\omega(x) \in C^\alpha(\bar{\Omega})$, and that ψ is the potential associated with ω through (2.3), i.e.,

$$\psi(x,t) = \int_{R^n} K(x,z)\,b^2(z)\,\omega(z)\,dz.$$

Then

$$|\nabla\psi|_{0,\Omega} \le c\,\|\omega\|_{0,\Omega}\,d(\Omega), \tag{3.8}$$

$$\left|\nabla^2\psi\right|_{0,\Omega} \le c\,\|\omega\|_{0,\Omega}\log\left(2+\delta_\Omega d(\Omega)+\alpha^{-1}\|\omega\|_{\alpha,\Omega}\,d(\Omega)\right), \tag{3.9}$$

$$\left|\nabla^2\psi\right|_{\alpha,\Omega} \le c\left(\|\omega\|_{\alpha,\Omega}+\|\omega\|_{0,\Omega}\,\delta_\Omega\right)\log\left(2+\delta_\Omega d(\Omega)\right), \tag{3.10}$$

where $d(\Omega)$ is the diameter of Ω, and c is a constant independent of Ω, α, and ω.

Proof. According to (3.1), we write

$$\psi(x)=\int_\Omega K(x,z)\,b(z)^2\,\omega(z)\,dz=\psi_0(x)+\psi_1(x), \tag{3.11}$$

where

$$\psi_0(x)=\hat{b}\int_\Omega \log|x-z|\,b(z)^2\,\omega(z)\,dz$$

$$\psi_1(x)=\int_\Omega K_1(x,z)\,b(z)^2\,\omega(z)\,dz.$$

Since $|\nabla K_1(x,y)|\le c_0|x-y|^{-1}$, we have

$$\begin{aligned}|\nabla\psi(x)| &= \int_\Omega\left|\nabla K(x,z)\,b(z)^2\,\omega(z)\right|dz\\ &\le c\,\|\omega\|_{0,\Omega}\int_\Omega |x-z|^{-1}\,dz\\ &\le c\,\|\omega\|_{0,\Omega}\int_0^{d(\Omega)} d\rho = c\,\|\omega\|_{0,\Omega}\,d(\Omega).\end{aligned}$$

This is the assertion (3.8). To establish the rest of the estimates (3.9) and (3.10), we note that the second derivatives for the Newtonian potential ψ_0 have the expression

$$\begin{aligned}\nabla^2\psi_0(x) &= \hat{b}\int_\Omega \frac{\sigma(x-z)}{|x-z|^2}\left(b(z)^2\,\omega(z)-b(x)^2\,\omega(x)\right)dz\\ &\quad+\pi\hat{b}b^2(x)\,\omega(x)\,I_2+\hat{b}b(x)^2\,\omega(x)\,V(x),\end{aligned} \tag{3.12}$$

where the function $V(x)$ is defined in Lemma 3.1 and I_2 is the identity matrix in R^2. The estimates (3.9) and (3.10) follow directly from Lemma 3.1 and (3.2). □

THEOREM 3.1. There exists a unique $C^{1+\alpha}$ solution $\Phi(x,t)$ of (2.7) for $0<t<T$, for some $T>0$. Consequently, if we define $\Omega_t=\Phi(\Omega_0,t)$, $\omega(x,t)=\omega_0\left(\Phi^{-1}(x,t)\right)\chi_{\bar{\Omega}_t}$, ψ by (2.3) and u by (2.1). Then (ω,u) is the weak solution to the lake equation. Furthermore, $\omega(\cdot,t)\in C^\alpha\left(\bar{\Omega}_t\right)$, $\Omega_t\in C^{1+\alpha}$, for $t<T$.

Proof. For any $M,T>0$ to be chosen later on, define a set $B(M,T)$ of R^2 value functions in $\bar{\Omega}_0\times[0,T)$ as follows:

$$\begin{aligned}B(M,T)=\{\Phi(x,t)\in R^2:\ &\|\Phi(t)\|_{1+\alpha,\Omega_0}\le M,\\ &\|\Phi(x,\cdot)\|_\alpha\le M,\ |\nabla\Phi(x,t)-I_2|\le 1/2,\ \Phi(x,0)=x\}.\end{aligned}$$

We then define a mapping F from $B(M,T)$ to a functional space by

$$F(\Phi)(x,t)=x+\int_0^t u(\Phi(x,s),s)\,ds, \tag{3.13}$$

where

$$u(x,s)=\frac{1}{b(x)}\int_{\Omega_s}\nabla_x^{\perp}K(x,y)\,\omega_0\left(\Phi^{-1}(y,s)\right)dy, \tag{3.14}$$

$\Omega_s=\Phi(\Omega_0,s)$, $K(x,y)$ is the fundamental solution in (3.1). By (3.1), we may decompose u as

$$u(x,s)=u_0(x,s)+u_1(x,s), \tag{3.15}$$

where

$$u_0(x,s)=\frac{\hat{b}}{b(x)}\int_{\Omega_s}\nabla_x^{\perp}\log|x-y|\,\omega_0\left(\Phi^{-1}(y,s)\right)dy, \tag{3.16}$$

$$u_1(x,s)=\frac{1}{b(x)}\int_{\Omega_s}\nabla_x^{\perp}K_1(x,y)\,\omega_0\left(\Phi^{-1}(y,s)\right)dy, \tag{3.17}$$

and K_1 satisfies (3.2). Since $|\nabla\Phi-I_2|\le 1/2$, $\Phi^{-1}(\cdot,t)$ exists and maps $\bar{\Omega}_t$ onto $\bar{\Omega}_0$. The mapping F thus is well defined. The rest of the proof is divided into four steps.

Step 1: The mapping F maps $B(M,T)$ into itself for some M,T.

For any $\Phi\in B(M,T)$, by Lemma 3.2, since ω_0 is bounded and $\omega_0\left(\Phi^{-1}(y,t)\right)\in C^{\alpha}\left(\bar{\Omega}_t\right)$, we have $u(\cdot,t)\in C^{1+\alpha}\left(\bar{\Omega}_t\right)$, and

$$|\nabla u(t)|_{0,\Omega_t}\le c_0\log\left(2+\delta_{\Omega_t}d(\Omega_t)+\alpha^{-1}\left\|\omega_0\circ\Phi^{-1}(t)\right\|_{\alpha,\Omega_t}d(\Omega_t)\right), \tag{3.18}$$

$$|\nabla u(t)|_{\alpha,\Omega_t}\le c_0\left(\left\|\omega_0\circ\Phi^{-1}\right\|_{\alpha,\Omega_t}+\delta_{\Omega_t}\right)\log\left(2+\delta_{\Omega_t}d(\Omega_t)\right). \tag{3.19}$$

Let φ_0 be such that $\Omega_0=\{\varphi_0<0\}$. Set $\varphi(x,t)=\varphi_0\left(\Phi^{-1}(x,t)\right)$. Then $\Omega_t=\{\varphi(x,t)<0\}$. From the definition (3.3), it is easy to see that, for $\Phi\in B(M,T)$, we have

$$\delta_{\Omega_t}\le c_0M^{4+\alpha},\ d(\Omega_t)\le 2|\Phi(t)|_0\le 2M$$

and

$$\left\|\omega_0\left(\Phi^{-1}(t)\right)\right\|_{\alpha,\Omega_t}\le\|\omega_0\|_{\alpha,\Omega_0}\left\|\Phi^{-1}\right\|_{1,\Omega_t}^{\alpha}\le c_0M^{2\alpha},$$

where c_0 is a constant independent of Φ, M, and T. It follows from (3.18) and (3.19) that

$$\begin{aligned}\|\nabla u(\Phi(\cdot,t),t)\|_{\alpha,\Omega_0}&\le|\nabla u(t)|_{0,\Omega_t}+|\nabla u(t)|_{\alpha,\Omega_t}|\Phi(t)|_{1,\Omega_0}^{\alpha}\\&\le c_0M^{4+2\alpha}\log(2+M).\end{aligned}$$

By (3.13), the gradient $\nabla F(\Phi)$ can be expressed as

$$\nabla F(\Phi)(x,t)=I_2+\int_0^t\nabla u(\Phi(x,s),s)\,\nabla\Phi(x,s)\,ds. \tag{3.20}$$

Hence, for $t<T$,

$$\|F(\Phi)(t)\|_{1+\alpha,\Omega_0}\le c_0+c_0TM^{5+2\alpha}\log(2+M).$$

Analogously we can derive, for fixed $x\in\bar{\Omega}_0$,

$$\|F(\Phi)(x,\cdot)\|_{\alpha}\le c_0T^{1-\alpha}M^{3+\alpha}$$

and

$$|\nabla F(\Phi)(x,t)-I_2|\le c_0TM^2\log(2+M).$$

It is now easy to see that if we choose $M = 1 + 2c_0$, $T = \left(M^{5+2\alpha} \log\left(2 + M\right)\right)^{-1}$, then $F\left(\Phi\right) \in B\left(M, T\right)$ for any $\Phi \in B\left(M, T\right)$. We have just proved that the mapping F maps $B\left(M, T\right)$ into itself.

Let M and T be fixed so that F maps $B\left(M, T\right)$ into itself, and let Φ and Φ^* be any two functions in $B\left(M, T\right)$. In the rest of the proof, c_0 is a constant depending only on the initial data and the chosen M, T.

Step 2: Estimates of $\left|\left(F\left(\Phi\right) - F\left(\Phi^*\right)\right)(t)\right|_{0,\Omega_0}$.

Set

$$\rho\left(x, t\right) = \Phi\left(x, t\right) - \Phi^*\left(x, t\right).$$

By (3.13), we have

$$F\left(\Phi\right)\left(x, t\right) - F\left(\Phi^*\right)\left(x, t\right) = \int_0^t u\left(\Phi\left(x, s\right), s\right) ds - \int_0^t u^*\left(\Phi^*\left(x, s\right), s\right) ds,$$

where u is defined in (3.14) and u^* is defined accordingly. By changes of variables $z = \Phi^{-1}\left(y, s\right)$ and $z = \Phi^{*-1}\left(y, s\right)$ in (3.14) for u and u^*, respectively, we have

$$u\left(x, s\right) = \frac{1}{b\left(x\right)} \int_{\Omega_0} \nabla^\perp K\left(x, \Phi\left(z, s\right)\right) b\left(\Phi\left(z, s\right)\right)^2 \omega_0\left(z\right) J\left(\Phi\left(z, s\right)\right) dz,$$

$$u^*\left(x, s\right) = \frac{1}{b\left(x\right)} \int_{\Omega_0} \nabla^\perp K\left(x, \Phi^*\left(z, s\right)\right) b\left(\Phi^*\left(z, s\right)\right)^2 \omega_0\left(z\right) J\left(\Phi^*\left(z, s\right)\right) dz,$$

where $J\left(\Phi\left(z, s\right)\right)$ is the Jacobian of the mapping $\Phi\left(\cdot, s\right)$. It follows that

$$\left|u\left(\Phi\left(x, s\right), s\right) - u^*\left(\Phi^*\left(x, s\right), s\right)\right| \le \left\|\rho\left(s\right)\right\|_{1,\Omega_0} + \left|m\left(x, s\right)\right|, \tag{3.21}$$

where

$$m\left(x, s\right) = c_0 \int_{\Omega_0} \left|\nabla^\perp K\left(\Phi\left(x, s\right), \Phi\left(z, s\right)\right) - \nabla^\perp K\left(\Phi^*\left(x, s\right), \Phi^*\left(z, s\right)\right)\right| dz.$$

We write $m = m_1 + m_2$, where, for small $\varepsilon > 0$ to be chosen later,

$$m_1 = c_0 \int_{\Omega_0 \backslash B_\varepsilon(x)} \left|\nabla^\perp K\left(\Phi\left(x, s\right), \Phi\left(z, s\right)\right) - \nabla^\perp K\left(\Phi^*\left(x, s\right), \Phi^*\left(z, s\right)\right)\right| dz,$$

$$m_2 = c_0 \int_{B_\varepsilon(x)} \left|\nabla^\perp K\left(\Phi\left(x, s\right), \Phi\left(z, s\right)\right) - \nabla^\perp K\left(\Phi^*\left(x, s\right), \Phi^*\left(z, s\right)\right)\right| dz.$$

By (3.2) and the fact that $\nabla\Phi^{-1}\left(x, s\right)$ and $\nabla\Phi^{*-1}\left(x, s\right)$ are bounded, we have

$$\left|m_1\left(x, s\right)\right| \le c_0 \int_{\Omega_0 \backslash B_\varepsilon} \frac{\left\|\rho\left(s\right)\right\|_{1,\Omega_0}}{\left|x - z\right|^2} dz ds \le c_0 \left\|\rho\left(s\right)\right\|_{1,\Omega_0} \left(1 + \left|\log \varepsilon\right|\right)$$

and

$$\left|m_2\left(x, s\right)\right| \le c_0 \int_{B_\varepsilon} \left|x - y\right|^{-1} dz \le c_0 \varepsilon.$$

By taking $\varepsilon = \min\left(1, \|\rho(s)\|_{1,\Omega_0}\right)$, we obtain

$$\begin{aligned} |m(x,s)| &\le |m_1(x,s)| + |m_2(x,s)| \\ &\le c_0 \|\rho(s)\|_{1,\Omega_0}\left(1 + \left|\log \|\rho(s)\|_{1,\Omega_0}\right|\right) ds. \end{aligned}$$

Inserting this into (3.21), it follows that

$$\begin{aligned} &|u(\Phi(x,s), s) - u^*(\Phi^*(x,s), s)| \\ &\le c_0 \|\rho(s)\|_{1,\Omega_0}\left(1 + \left|\log \|\rho(s)\|_{1,\Omega_0}\right|\right). \end{aligned} \tag{3.22}$$

Therefore,

$$\begin{aligned} &|F(\Phi)(x,t) - F(\Phi^*)(x,t)| \\ &\le \int_0^t |u(\Phi(x,s), s) - u^*(\Phi^*(x,s), s)|\, ds \\ &\le c_0 \int_0^t \|\rho(s)\|_{1,\Omega_0}\left(1 + \left|\log \|\rho(s)\|_{1,\Omega_0}\right|\right) ds. \end{aligned}$$

Step 3: Estimates of $|(F(\Phi) - F(\Phi^*))(t)|_{1,\Omega_0}$.

From (3.16) and (3.12), we have

$$\begin{aligned} \nabla u_0(x,s) &= \hat{b}\nabla \int_{\Omega_s} \nabla^\perp \log|x-y|\,\omega(y,s)\, dy \\ &= \hat{b}\int_{\Omega_s} \frac{\sigma^\perp(x-y)}{|x-y|^2}\left(\omega(y,s) - \omega(x,s)\right) dy \\ &\quad + \pi\hat{b}\omega(x,s)\, I_2^\perp + \hat{b}\int_{\Omega_s} \frac{\sigma^\perp(x-y)}{|x-y|^2}\, dy\omega(x,s), \end{aligned} \tag{3.23}$$

where

$$\frac{\sigma^\perp(x)}{|x|^2} = \nabla\nabla^\perp \log|x| = \frac{1}{|x|^4}\begin{pmatrix} -2x_1x_2 & x_1^2 - x_2^2 \\ x_2^2 - x_1^2 & 2x_1x_2 \end{pmatrix},$$

and

$$\nabla u_1(x,s) = \int_{\Omega_s} \nabla\nabla^\perp K_1(x,y)\,\omega(y,s)\, dy.$$

Accordingly, $u^* = u_0^* + u_1^*$ and the derivatives have the analogous expressions. Using (3.2) and the same techniques as used in deriving (3.22), it follows that

$$|\nabla u_1(\Phi(x,s), s) - \nabla u_1^*(\Phi^*(x,s), s)| \le c_0 \|\rho(s)\|_{1,\Omega_0}. \tag{3.24}$$

By changes of variables, we have

$$\begin{aligned} &\int_{\Omega_s} \frac{\sigma^\perp(\Phi(x,s) - y)}{|\Phi(x,s) - y|^2}\left(\omega(y,s) - \omega(\Phi(x,s), s)\right) dy \\ &= \int_{\Omega_0} \frac{\sigma^\perp(\Phi(x,s) - \Phi(y,s))}{|\Phi(x,s) - \Phi(y,s)|^2}\left(\omega_0(y) - \omega_0(x)\right) J(\Phi(y,s))\, dy \end{aligned}$$

and

$$\int_{\Omega_s^*} \frac{\sigma^\perp\left(\Phi^*(x,s)-y\right)}{\left|\Phi^*(x,s)-y\right|^2}\left(\omega^*(y,s)-\omega^*\left(\Phi^*(x,s),s\right)\right)dy$$
$$=\int_{\Omega_0} \frac{\sigma^\perp\left(\Phi^*(x,s)-\Phi^*(y,s)\right)}{\left|\Phi^*(x,s)-\Phi^*(y,s)\right|^2}\left(\omega_0(y)-\omega_0(x)\right)J\left(\Phi^*(y,s)\right)dy.$$

Note that $\left|J\left(\Phi(y,s)\right)-J\left(\Phi^*(y,s)\right)\right|\leq c_0\left\|\rho(s)\right\|_{1,\Omega_0}$, and that

$$\left|\frac{\sigma^\perp\left(\Phi(x,s)-\Phi(y,s)\right)}{\left|\Phi(x,s)-\Phi(y,s)\right|^2}-\frac{\sigma^\perp\left(\Phi^*(x,s)-\Phi^*(y,s)\right)}{\left|\Phi^*(x,s)-\Phi^*(y,s)\right|^2}\right| \tag{3.25}$$
$$\leq c_0\left|\left(\Phi(x,s)-\Phi(y,s)\right)-\left(\Phi^*(x,s)-\Phi^*(y,s)\right)\right|\left|x-y\right|^{-3}$$
$$\leq c_0\left\|\rho(s)\right\|_{1,\Omega_0}\left|x-y\right|^{-2}.$$

It follows that the quantity

$$\int_{\Omega_s} \frac{\sigma^\perp\left(\Phi(x,s)-y\right)}{\left|\Phi(x,s)-y\right|^2}\left(\omega(y,s)-\omega\left(\Phi(x,s),s\right)\right)dy$$
$$-\int_{\Omega_s^*} \frac{\sigma^\perp\left(\Phi^*(x,s)-y\right)}{\left|\Phi^*(x,s)-y\right|^2}\left(\omega^*(y,s)-\omega^*\left(\Phi^*(x,s),s\right)\right)dy$$

is bounded by, since $\omega_0\in C^\alpha$,

$$c_0\left\|\rho(s)\right\|_{1,\Omega_0}\int_{\Omega_0}\frac{\left|\omega_0(y)-\omega_0(x)\right|}{\left|x-y\right|^2}dy$$
$$\leq c_0\left\|\rho(s)\right\|_{1,\Omega_0}\int_{\Omega_0}\frac{1}{\left|x-y\right|^{2-\alpha}}dy=c_0\left\|\rho(s)\right\|_{1,\Omega_0}.$$

Hence, we obtain from (3.23) that

$$\left|\nabla u_0\left(\Phi(x,s),s\right)-\nabla u_0^*\left(\Phi^*(x,s),s\right)\right| \tag{3.26}$$
$$\leq c_0\left\|\rho(s)\right\|_{1,\Omega_0}+\left|k_1(x,s)\right|+\left|k_2(x,s)\right|,$$

where

$$k_1(x,s)=\hat{b}\int_{\Omega_s\setminus\Phi(B_\varepsilon(x),s)}\frac{\sigma^\perp\left(\Phi(x,s)-y\right)}{\left|\Phi(x,s)-y\right|^2}dy$$
$$-\hat{b}\int_{\Omega_s^*\setminus\Phi^*(B_\varepsilon(x),s)}\frac{\sigma^\perp\left(\Phi^*(x,s)-y\right)}{\left|\Phi^*(x,s)-y\right|^2}dy,$$
$$k_2(x,s)=\hat{b}\int_{\Phi(\Omega_0\cap B_\varepsilon(x),s)}\frac{\sigma^\perp\left(\Phi(x,s)-y\right)}{\left|\Phi(x,s)-y\right|^2}dy$$
$$-\hat{b}\int_{\Phi^*(\Omega_0\cap B_\varepsilon(x),s)}\frac{\sigma^\perp\left(\Phi^*(x,s)-y\right)}{\left|\Phi^*(x,s)-y\right|^2}dy,$$

and ε is a positive constant to be determined later. By changes of variable, we have

$$k_1(x,s) = \hat{b} \int_{\Omega_0 \setminus B_\varepsilon(x)} \frac{\sigma^\perp(\Phi(x,s) - \Phi(y,s))}{|\Phi(x,s) - \Phi(y,s)|^2} J(\Phi(y,s))\, dy$$
$$- \hat{b} \int_{\Omega_0 \setminus B_\varepsilon(x)} \frac{\sigma^\perp(\Phi^*(x,s) - \Phi^*(y,s))}{|\Phi^*(x,s) - \Phi^*(y,s)|^2} J(\Phi^*(y,s))\, dy.$$

Hence, by (3.25),

$$\begin{aligned} |k_1(x,s)| &\le c_0 \|\rho(s)\|_{1,\Omega_0} \int_\varepsilon^{d(\Omega_0)} \frac{1}{r} dr \\ &\le c_0 \|\rho(s)\|_{1,\Omega_0} (1 + |\log \varepsilon|). \end{aligned} \tag{3.27}$$

To estimate $k_2(x,t)$, we consider first the case that $x \in \Omega_0$ and

$$\operatorname{dist}(x, \partial\Omega_0) \ge \varepsilon.$$

For this fixed x, $B_\varepsilon(x) \subset \Omega_0$. Since

$$\int_{B_r(0)} \frac{\sigma^\perp(y)}{|y|^2} dy = 0, \quad r > 0,$$

we have

$$\begin{aligned} k_2(x,s) &= \hat{b} \int_{\Phi(B_\varepsilon(x),s)} \frac{\sigma^\perp(\Phi(x,s) - y)}{|\Phi(x,s) - y|^2} dy - \hat{b} \int_{\Phi^*(B_\varepsilon(x),s)} \frac{\sigma^\perp(\Phi^*(x,s) - y)}{|\Phi^*(x,s) - y|^2} dy \\ &= \hat{b} \int_{(\Phi(B_\varepsilon(x),s) - \Phi(x,s)) \setminus B_{d(x)}(0)} \frac{\sigma^\perp(y)}{|y|^2} dy \\ &\quad - \hat{b} \int_{(\Phi^*(B_\varepsilon(x),s) - \Phi^*(x,s)) \setminus B_{d(x)}(0)} \frac{\sigma^\perp(y)}{|y|^2} dy \\ &= \hat{b} \int_{\Sigma_1} \frac{\sigma^\perp(y)}{|y|^2} dy - \hat{b} \int_{\Sigma_2} \frac{\sigma^\perp(y)}{|y|^2} dy, \end{aligned} \tag{3.28}$$

where

$$d(x) = \min\left(\operatorname{dist}(\Phi(x,s), \partial\Phi(B_\varepsilon(x),s)),\ \operatorname{dist}(\Phi^*(x,s), \partial\Phi^*(B_\varepsilon(x),s))\right),$$

and

$$\Sigma_1 = \left[(\Phi(B_\varepsilon(x),s) - \Phi(x,s)) \setminus B_{d(x)}(0)\right] \setminus \left[(\Phi^*(B_\varepsilon(x),s) - \Phi^*(x,s))\right],$$
$$\Sigma_2 = \left[(\Phi^*(B_\varepsilon(x),s) - \Phi^*(x,s)) \setminus B_{d(x)}(0)\right] \setminus \left[(\Phi(B_\varepsilon(x),s) - \Phi(x,s))\right].$$

Since $\Phi \in B(M,T)$, it is easy to see that there exist two positive constants c_1 and c_2 depending only on M, T such that

$$c_1 \varepsilon \le d(x) \le c_2 \varepsilon. \tag{3.29}$$

For any unit vector $\nu \in R^2$, set

$$\Lambda_i(\nu) = \sup\{r : r\nu \in \Sigma_i\}, \ \lambda_i(\nu) = \inf\{r : r\nu \in \Sigma_i\}, \ i = 1, 2.$$

By (3.29), $\lambda_i(\nu) \geq d(x) \geq c_1\varepsilon$. Let $x_1 \in \partial B_\varepsilon(x)$ such that

$$\Lambda_1(\nu)\nu = \Phi(x_1, s) - \Phi(x, s) = \nabla\Phi(y_1)(x_1 - x),$$

where y_1 is a point lying in the line segment connecting x and x_1. Without loss of generality, we may assume that there exists $x_1^* \in \partial B_\varepsilon(x)$ such that

$$\lambda_1(\nu)\nu = \Phi^*(x_1^*, s) - \Phi^*(x, s) = \nabla\Phi^*(y_1^*)(x_1^* - x),$$

where y_1^* is a point lying in the line segment connecting x and x_1^*. Hence,

$$\begin{aligned}
\Lambda_1(\nu) - \lambda_1(\nu) &= \left|\nabla\Phi(y_1)^{-1}\nu\right|^{-1}|x_1 - x| - \left|\nabla\Phi^*(y_1^*)^{-1}\nu\right|^{-1}|x_1^* - x| \\
&= \varepsilon\left(\left|\nabla\Phi(y_1)^{-1}\nu\right|^{-1} - \left|\nabla\Phi^*(y_1^*)^{-1}\nu\right|^{-1}\right) \\
&\leq c_0\varepsilon\left|\nabla\Phi(y_1)^{-1}\nu - \nabla\Phi^*(y_1^*)^{-1}\nu\right| \\
&\leq c_0\varepsilon\left|\nabla\Phi(y_1) - \nabla\Phi^*(y_1^*)\right| \\
&\leq c_0\varepsilon\left|y_1 - y_1^*\right|^\alpha \leq c_0\varepsilon^{1+\alpha}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left|\int\limits_{\Sigma_1}\frac{\sigma^\perp(y)}{|y|^2}dy\right| &\leq \left|\int_{\partial B_1(0)}\int\limits_{\lambda_1(\nu)}^{\Lambda_1(\nu)}\frac{1}{r^2}rdrd\nu\right| \leq \left|\int_{\partial B_1(0)}\log\left(\frac{\Lambda_1(\nu)}{\lambda_1(\nu)}\right)d\nu\right| \\
&\leq c_0\left|\int_{\partial B_1(0)}\frac{\Lambda_1(\nu) - \lambda_1(\nu)}{\lambda_1(\nu)}d\nu\right| \leq c_0\varepsilon^\alpha.
\end{aligned}$$

Analogously, we have

$$\left|\int\limits_{\Sigma_2}\frac{\sigma^\perp(y)}{|y|^2}dy\right| \leq c_0\varepsilon^\alpha.$$

Inserting this into (3.28), we obtain

$$|k_2(x, s)| \leq c_0\varepsilon^\alpha.$$

Substituting this and (3.27) into (3.26), we obtain, for $\operatorname{dist}(x, \partial\Omega_0) \geq \varepsilon$,

$$|\nabla u_0(\Phi(x, s), s) - \nabla u_0^*(\Phi^*(x, s), s)| \leq c_0\|\rho(s)\|_{1,\Omega_0}(1 + |\log\varepsilon|) + c_0\varepsilon^\alpha.$$

Since ∇u_0 and ∇u_0^* belong to C^α, the above estimate remains true for all $x \in \Omega_0$. We now choose $\varepsilon^\alpha = \|\rho(s)\|_{1,\Omega_0}$. The above inequality, together with (3.24) and (3.15), leads to

$$|\nabla u(\Phi(x, s), s) - \nabla u^*(\Phi^*(x, s), s)| \leq c_0\|\rho(s)\|_{1,\Omega_0}\left(1 + \left|\log\|\rho(s)\|_{1,\Omega_0}\right|\right).$$

Hence, by (3.20),

$$|\nabla F(\Phi)(x,t) - \nabla F(\Phi^*)(x,t)| \tag{3.30}$$
$$\leq c_0 \int_0^t \|\rho(s)\|_{1,\Omega_0} \left(1 + \left|\log \|\rho(s)\|_{1,\Omega_0}\right|\right) ds.$$

Step 4: F has a unique fixed point.

We can now define a sequence $\{\Phi_n\}$ by

$$\Phi_1(x,t) = x, \ \Phi_{n+1}(x,t) = F(\Phi_n)(x,t).$$

Set

$$\rho_n(t) = \|(\Phi_{n+1} - \Phi_n)(t)\|_{1,\Omega_0}.$$

By (3.22) and (3.30), we have

$$\rho_{n+1}(t) = \|(\nabla F(\Phi_{n+1}) - \nabla F(\Phi_n))(t)\|_{1,\Omega_0}$$
$$\leq c_0 \int_0^t \rho_n(s) \log(2 + \rho_n(s))\, ds. \tag{3.31}$$

By [6, Sec. 9], this implies that there exists a positive number $T^* \leq T$ such that the sequence is convergent in $C_x^1(\Omega_0)$ norm, for $t \leq T^*$. By (3.22), this limit is a fixed point of the mapping F. The above inequality also implies that the fixed point is unique. This fixed point is clearly the solution of (2.7). The proof is complete. □

REMARK 3.1. For the integral system studied in [5], it was shown that the sequence $\{\hat{\rho}_n(t)\}_{n=1}$ defined by

$$\hat{\rho}_n(t) = \|(\Phi_{n+1} - \Phi_n)(t)\|_{0,\Omega_0}$$

satisfies inequality (3.31). Hence *Step 3, i.e., the estimate of* $\nabla F(\Phi) - \nabla F(\Phi^*)$ is not needed in proving local existence. For the 2D Euler system, the authors of [1] actually assumed local existence based on a result in [3].

From the proof of Theorem 3.1, we can see that the existence time T depends only on the Hölder norms of the initial data. Hence, *à priòri* bounds on $\|\Phi(t)\|_{1+\alpha,\Omega_0}$ and $\left\|\Phi^{-1}(t)\right\|_{1,\Omega_t}$ for $t < T$ will guarantee that the solution can be extended beyond T. In fact, we have

COROLLARY 3.1. Let $\Phi(x,t)$ be the $C^{1+\alpha}$ solution of (2.7) for $t < T$. If

$$\theta = \sup_{t<T} \|\Phi(t)\|_{1+\alpha,\Omega_0} + \sup_{t<T} \left\|\Phi^{-1}(t)\right\|_{1,\Omega_t}$$

is finite, then the solution $\Phi(x,t)$ may be extended for $t < T + T_0$ for some positive constants T_0 depending only on θ.

Proof. Let $T^* < T$ be a constant, and let $\omega^*(x) = \omega_0\left(\Phi^{-1}(x,T^*)\right)$, where $\Phi(x,t)$ is the $C^{1+\alpha}$ solution of (2.7) for $t < T$. Consider the mapping

$$F^*(\Phi^*)(x,t) = \Phi(x,T^*)$$
$$+ \int_0^t \int_{\Omega_s^*} \frac{1}{b(\Phi^*(x,s))} \nabla^\perp K(\Phi^*(x,s),z)\, b(z)^2 \omega^*\left(\Phi^{*-1}(z,s)\right) dz ds,$$

defined in the set

$$B^*(M,T_1) = \{\Phi^*(x,t) \in R^2 : \|\Phi^*(t)\|_{1+\alpha,\Omega_0}, \ \|\Phi^*(x,\cdot)\|_\alpha \le M,$$
$$\Phi^*(x,0) = \Phi(x,T^*), \ |\nabla\Phi^*(x,t) - \nabla\Phi(x,T^*)| \le \frac{1}{2\theta}\}.$$

For any $\Phi^* \in B^*(M,T)$, we have

$$\begin{aligned} &\left|\nabla\Phi^*(x,t)\left(\nabla\Phi(x,T^*)\right)^{-1} - I_2\right| \\ &\le \left|\nabla\Phi^{-1}(\Phi(x,T^*),T^*)\right| |\nabla\Phi^*(x,t) - \nabla\Phi(x,T^*)| \le 1/2. \end{aligned}$$

Therefore, $\Phi^*(\cdot,t)$ is invertible and $\left\|\Phi^{*-1}(t)\right\|_{1,\Omega_t} \le c_0 M^2$, where c_0 is a constant independent of T and M. Using the same method as in the proof of Theorem 3.1, one may show that there exists an M and T_1 depending on θ (but not on the particular choice of T^*) such that F^* maps $B^*(M,T)$ into itself and that F^* admits a unique fixed point $\Phi^*(x,t)$. Now we select $T_0 = 3T_1/4$ and $T^* = \max(T - T_1/4, 0)$. The mapping $\Phi^*(x,t)$ is then the extension of $\Phi(x,t)$. □

4. Global Existence of the $C^{1+\alpha}$ Solution. In this section, we shall show that the $C^{1+\alpha}$ solution obtained in the previous section can be extended to all $t > 0$.

LEMMA 4.1. For any $T > 0$, there exist positive constants $\eta(T)$ and $\beta(T)$ with $\beta(T) \le \alpha$, depending only on T and the initial data $\delta_{\Omega_0,\alpha}$, $\|\omega_0\|_{\alpha,\Omega_0}$ and $b(x)$ such that for any $C^{1+\alpha}$ solution $\Phi(x,t)$ of (2.7) for $t < T_0$, we have

$$d(\Omega_t) + \|\omega(t)\|_{\beta(T),\Omega_t} \le \eta(T) \quad (\text{for } t < \min(T_0,T)),$$

where $\omega(x,t) = \omega_0\left(\Phi^{-1}(x,t)\right)$ and $d(\Omega_t)$ is the diameter of the region Ω_t.

Proof. Since $\omega(x,t) = \omega_0\left(\Phi^{-1}(x,t)\right)$, it follows $|\omega(x,t)| \le \|\omega_0\|_{0,\Omega_0}$. By (2.7) and (3.15), we write

$$\Phi(x,t) = x + \int_0^t u_0(\Phi(x,s),t)\,ds + \int_0^t u_1(\Phi(x,s),t)\,ds,$$

where u_0 and u_1 have the expressions (3.16) and (3.17). It is easy to see that

$$|u_0(x,t)| \le c_0 \int_{\Omega_t} \left|\nabla_x^\perp \log|x-y|\right| dy \le c_0 \int_0^{d(\Omega_t)} ds = c_0 d(\Omega_t)$$

for a constant c_0 dependent of the initial data only. By (3.2), we have

$$|u_1(x,t)| \le c_0 \int_{\Omega_t} \left|\nabla_x^\perp K_1(x,y)\right| dy \le c_0 (d(\Omega_t) + 1).$$

Hence, for $x \in \Omega_0$,

$$|\Phi(x,t)| \le |x| + \int_0^t |u(\Phi(x,s),t)|\,ds \le c_0 + c_0 \int_0^t (d(\Omega_t) + 1)\,ds.$$

It follows that

$$d(\Omega_t) = \sup_{x,y\in\Omega_0} |\Phi(x,t) - \Phi(y,t)| \leq c_0 + c_0 \int_0^t (d(\Omega_t) + 1)\, ds. \tag{4.1}$$

The Gronwall lemma guarantees that $d(\Omega_t) \leq \eta(T)$, where $\eta(T) = c_0 e^{c_0 T}$ depends only on T and the initial data. In what follows, $\eta(T)$ may vary. But it depends only on T and initial data.

By the well-known results in [15] (see also [13]), u_0 is quasi-Lipschitz, i.e.,

$$|u_0(x,t) - u_0(y,t)| \leq c_0 |x-y| \left|\log|x-y|\right|,$$

for $|x-y| \leq 1/2$. By (3.2), we have, for $|x-y| \leq 1/2$,

$$\begin{aligned} |u_1(x,t) - u_1(y,t)| &\leq \|\omega_0\|_{0,\Omega_0} \left|b^{-1}(x) - b^{-1}(y)\right| \int_{\Omega_t} \left|\nabla_x^\perp K_1(x,z)\, dz\right| \\ &\quad + \|\omega_0\|_{0,\Omega_0} \left|b^{-1}(y)\right| \int_{\Omega_t} \left|\nabla_x^\perp K_1(x,z) - \nabla_x^\perp K_1(y,z)\, dz\right| \\ &\leq \eta(T)|x-y|. \end{aligned}$$

It follows that

$$\begin{aligned} \left|\frac{d|\Phi(x,t) - \Phi(y,t)|}{dt}\right| &\leq |u(\Phi(x,t),t) - u(\Phi(y,t),t)| \\ &\leq \eta(T)|\Phi(x,t) - \Phi(y,t)| \left|\log|\Phi(x,t) - \Phi(y,t)|\right|, \end{aligned}$$

for $|\Phi(x,t) - \Phi(y,t)| \leq 1/2$. By the Gronwall lemma, we deduce that

$$\eta(T)|x-y|^{1/\mu(t)} \leq |\Phi(x,t) - \Phi(y,t)| \leq \eta(T)|x-y|^{\mu(t)},$$

for $|x-y|^{\mu(t)} \leq (2\eta(T))^{-1}$, where $\mu(t) = \exp(-\eta(T)t)$. Since $\Phi^{-1}(x,t) \in \Omega_0$ is bounded, this inequality implies that $\left\|\Phi^{-1}(t)\right\|_{\mu(t),\Omega_t} \leq \eta(T)$. Combining this with the fact that $\Phi(x,t)$ is bounded (due to (4.1)), since $\mu(T) \leq \mu(t)$ for $t < T$, we obtain $\left\|\Phi^{-1}(t)\right\|_{\mu(T),\Omega_t} \leq \eta(T)$. We now choose

$$\beta(T) = \alpha\mu(T).$$

Since $\omega(x,t) = \omega_0\left(\Phi^{-1}(x,t)\right)$, it follows that

$$\|\omega(t)\|_{\beta(T),\Omega_t} \leq c_0 \|\omega_0\|_{\alpha,\Omega_0} \left\|\Phi^{-1}(t)\right\|_{\mu(T),\Omega_t}^{\alpha} \leq \eta(T).$$

The assertion follows from this inequality and (4.1). □

LEMMA 4.2. Let $\Phi(x,t)$ be the $C^{1+\alpha}$ solution (2.7) for $t<T$, and $\omega(x,t)=\omega_0\left(\Phi^{-1}(x,t)\right)$. Suppose that there exist constants $\eta > 0$ and $0 < \beta \leq \alpha$ such that for $t < T$,

$$d(\Omega_t) + \|\omega(t)\|_{\beta,\Omega_t} \leq \eta < \infty. \tag{4.2}$$

Then

$$\left\|\Phi^{-1}(t)\right\|_{1,\Omega_t} + \delta_{\Omega_t,\beta} \leq c(\eta,\beta,T), \tag{4.3}$$

where $\delta_{\Omega_t,\beta}$ is defined as δ_{Ω_t} in (3.3) in which the α-Hölder seminorm $|\cdot|_\alpha$ is replaced by β-Hölder seminorm $|\cdot|_\beta$, and $c(\eta,\beta,T)$ is a constant depending only on η,β,T, and the initial data.

Proof. Let $\Omega_0=\{\varphi_0<0\}$ for some $C^{1+\alpha}$ function φ_0 such that $\nabla\varphi_0\neq 0$ on $\partial\Omega_0$. We recall that $\Omega_t=\{\varphi(\cdot,t)<0\}$ where $\varphi(x,t)=\varphi_0\left(\Phi^{-1}(x,t)\right)$. Let $u=u_0+u_1$ be the velocity as in (3.15)–(3.17). By (3.23), we have

$$\begin{aligned}\nabla u_0(x,t)\nabla^\perp\varphi(x,t) &= \hat{b}\nabla\int_{\Omega_t}\nabla^\perp\log|x-y|\,\omega(y,t)\,dy\nabla^\perp\varphi(x,t)\\ &= \hat{b}\int_{\Omega_t}\frac{\sigma^\perp(x-y)}{|x-y|^2}\left(\omega(y,t)-\omega(x,t)\right)dy\nabla^\perp\varphi(x,t)\\ &\quad+\pi\hat{b}\omega(x,t)\nabla\varphi(x,t)+\hat{b}\int_{\Omega_t}\frac{\sigma^\perp(x-y)}{|x-y|^2}\,dy\nabla^\perp\varphi(x,t)\,\omega(x,t)\\ &= k_1+k_2+k_3.\end{aligned}$$

Note that $\nabla^\perp\varphi(x,t)$ is weakly divergence free in Ω_t and tangent to $\partial\Omega_t$. We obtain from [1, Corollary 1] that

$$k_3=\hat{b}\int_{\Omega_t}\frac{\sigma^\perp(x-y)}{|x-y|^2}\left(\nabla^\perp\varphi(x,t)-\nabla^\perp\varphi(y,t)\right)dy\omega(x,t),$$

and consequently that, by Lemma 3.1 and (4.2),

$$\begin{aligned}|k_3|_\beta &\le c_0\left(|\omega(t)|_\beta|\nabla\varphi(t)|_0+|\omega(t)|_0|\nabla\varphi(t)|_\beta\right)\\ &\quad\cdot\log\left(2+\beta^{-1}|\nabla\varphi(t)|_\beta d(\Omega_t)+d(\Omega_t)\delta_{\Omega_t,\beta}\right)\\ &\le c(\eta,\beta)\|\nabla\varphi(t)\|_\beta\log\left(2+|\nabla\varphi(t)|_\beta+\delta_{\Omega_t,\beta}\right),\end{aligned}$$

where $c(\eta,\beta)$ depends only on η,β, and the initial data. In particular, $c(\eta,\beta)$ is bounded by $c(\eta)\log\beta^{-1}$, for a constant $c(\eta)$ depending only on η and the initial data. In the above inequality, as well as in what follows, we may omit the subscript: $\|\cdot\|_\beta=\|\cdot\|_{\beta,\Omega_t}$. Analogously, we have

$$\begin{aligned}|k_1|_\beta &\le c_0\left(|\omega(t)|_\beta|\nabla\varphi(t)|_0+|\omega(t)|_0|\nabla\varphi(t)|_\beta\right)\\ &\quad\cdot\log\left(2+\beta^{-1}|\omega(t)|_\beta d(\Omega_t)+d(\Omega_t)\delta_{\Omega_t,\beta}\right)\\ &\le c(\eta,\beta)\|\nabla\varphi(t)\|_\beta\log\left(2+\delta_{\Omega_t,\beta}\right).\end{aligned}$$

It follows that

$$\left|\nabla u_0\nabla^\perp\varphi(t)\right|_\beta\le c(\eta,\beta)\|\nabla\varphi(t)\|_\beta\log\left(2+\|\nabla\varphi(t)\|_\beta+\delta_{\Omega_t,\beta}\right). \tag{4.4}$$

By (3.2), it is easy to see that

$$|\nabla u_1(t)|_\beta\le c(\eta,\beta).$$

Hence

$$\left|\nabla u_1 \nabla^{\perp} \varphi (t)\right|_{\beta} \leq c(\eta, \beta) \|\nabla \varphi (t)\|_{\beta},$$

and consequently, this and (4.4) lead to

$$\left|\nabla u \nabla^{\perp} \varphi (t)\right|_{\beta} \leq c(\eta, \beta) \|\nabla \varphi (t)\|_{\beta} \log \left(2 + \|\nabla \varphi (t)\|_{\beta} + \delta_{\Omega_t, \beta}\right).$$

The same argument also shows that $\left|\nabla u \nabla^{\perp} \varphi (t)\right|_0$ is bounded by the right-hand side of the above inequality. Hence,

$$\left\|\nabla u \nabla^{\perp} \varphi (t)\right\|_{\beta} \leq c(\eta, \beta) \|\nabla \varphi (t)\|_{\beta} \log \left(2 + \|\nabla \varphi (t)\|_{\beta} + \delta_{\Omega_t, \beta}\right). \tag{4.5}$$

Next, by differentiating $\varphi(\Phi(x,t), t) = \varphi_0(x)$ with respect to t, we obtain

$$\frac{\partial \varphi (x,t)}{\partial t} + u(x,t) \cdot \nabla \varphi (x,t) = 0.$$

Hence $\nabla \varphi$ solves

$$\frac{\partial \nabla \varphi}{\partial t} + (u \cdot \nabla) \nabla \varphi = -(\nabla u)^{\top} \nabla \varphi, \tag{4.6}$$

where $(\nabla u)^{\top}$ is the transpose matrix of ∇u. It follows that

$$\nabla \varphi (\Phi(x,t), t) = \nabla \varphi_0 (x) - \int_0^t \left((\nabla u)^{\top} \nabla \varphi\right) (\Phi(x,s), s)\, ds$$

or equivalently,

$$\nabla \varphi (x,t) = \nabla \varphi_0 \left(\Phi^{-1}(x,t)\right) - \int_0^t \left((\nabla u)^{\top} \nabla \varphi\right) \left(\Phi\left(\Phi^{-1}(x,t), s\right), s\right) ds. \tag{4.7}$$

Note that (4.6) holds only in the distribution sense since $\varphi \in C^{1+\alpha}$. By using test functions as we did in the proof of Proposition 2.1, one may verify that (4.7) holds. Since

$$\frac{\partial \nabla \Phi (x,t)}{\partial t} = \nabla u (\Phi(x,t), t) \nabla \Phi,$$

it follows that, for $0 < s < t$,

$$\begin{aligned} \exp\left(-\int_s^t |\nabla u(\tau)|_0\, d\tau\right) &\leq \left|\nabla\left(\Phi\left(\Phi^{-1}(x,t), s\right)\right)\right| \\ &\leq \exp\left(\int_s^t |\nabla u(\tau)|_0\, d\tau\right). \end{aligned} \tag{4.8}$$

Therefore

$$
\begin{aligned}
|\nabla\varphi(t)|_\beta &\le |\nabla\varphi_0|_\beta \left|\nabla\Phi^{-1}(t)\right|_0^\beta \\
&\quad + \int_0^t \left|(\nabla u)^\top \nabla\varphi(s)\right|_\beta \left|\nabla\left(\Phi\left(\Phi^{-1}(\cdot,t),s\right)\right)\right|_0^\beta ds \\
&\le \left|\nabla^\perp\varphi_0\right|_\beta \exp\left(\beta\int_0^t |\nabla u(\tau)|_0 \, d\tau\right) \\
&\quad + \int_0^t \left|(\nabla u)^\top \nabla\varphi(s)\right|_\beta \exp\left(\beta\int_s^t |\nabla u(\tau)|_0 \, d\tau\right) ds.
\end{aligned}
$$

It is obvious that, by (4.7),

$$
|\nabla\varphi(t)|_0 \le |\nabla\varphi_0|_0 + \int_0^t \left|(\nabla u)^\top \nabla\varphi(s)\right|_0 ds.
$$

We have arrived at

$$
\begin{aligned}
\|\nabla\varphi\|_\beta &\le \|\nabla\varphi_0\|_\beta \exp\left(\beta\int_0^t |\nabla u(\tau)|_0 \, d\tau\right) \\
&\quad + \int_0^t \left\|(\nabla u)^\top \nabla\varphi(s)\right\|_\beta \exp\left(\beta\int_s^t |\nabla u(\tau)|_0 \, d\tau\right) ds.
\end{aligned}
\tag{4.9}
$$

Introducing the notation $(a,b)^\perp = (b,a)$, a direct computation leads to

$$
\left((\nabla u)^\top \nabla\varphi\right)^\perp = (\nabla\cdot u)\,\nabla^\perp\varphi - \nabla u\nabla^\perp\varphi.
$$

Since $\nabla\cdot(bu) = 0$, we have

$$
\nabla\cdot u = b^{-1}\nabla\cdot(bu) + bu\cdot\nabla b^{-1} = bu\cdot\nabla b^{-1}.
$$

Hence,

$$
\left((\nabla u)^\top \nabla\varphi\right)^\perp = \left(bu\cdot\nabla b^{-1}\right)\nabla^\perp\varphi - \nabla u\nabla^\perp\varphi. \tag{4.10}
$$

It follows from Lemma 3.2 that

$$
\begin{aligned}
\left\|\left(bu\cdot\nabla b^{-1}\right)\nabla^\perp\varphi(t)\right\|_\beta &\le c(\eta,\beta)\left(\|u(t)\|_\beta \|\nabla\varphi(t)\|_\beta\right) \\
&\le c(\eta,\beta)\|\nabla\varphi(t)\|_\beta \log\left(2+\delta_{\Omega_t,\beta}\right).
\end{aligned}
$$

Combining this with (4.5) and (4.10), we obtain

$$
\left\|(\nabla u)^\top \nabla\varphi(t)\right\|_\beta \le c(\eta,\beta)\|\nabla\varphi(t)\|_\beta \log\left(2+\|\nabla\varphi(t)\|_\beta+\delta_{\Omega_t,\beta}\right).
$$

Substituting this into (4.9), it follows that

$$\|\nabla\varphi(t)\|_\beta \le \|\nabla\varphi_0\|_\beta \exp\left(\beta\int_0^t |\nabla u(\tau)|_0\, d\tau\right)$$
$$+ c(\eta,\beta)\int_0^t \|\nabla\varphi(s)\|_{\beta,\Omega_s} \log\left(2+\|\nabla\varphi(s)\|_{\beta,\Omega_s}+\delta_{\Omega_s,\beta}\right)$$
$$\cdot \exp\left(\beta\int_s^t |\nabla u(\tau)|_0\, d\tau\right) ds.$$

Multiplying by $\exp\left(-\beta\int_0^t |\nabla u(\tau)|_0\, d\tau\right)$, the above inequality leads to

$$\|\nabla\varphi(t)\|_\beta \exp\left(-\beta\int_0^t |\nabla u(\tau)|_0\, d\tau\right) \le \|\nabla\varphi_0\|_\beta$$
$$+ c(\eta,\beta)\int_0^t \exp\left(-\beta\int_0^s |\nabla u(\tau)|_0\, d\tau\right) \|\nabla\varphi(s)\|_{\beta,\Omega_s} \tag{4.11}$$
$$\cdot \log\left(2+\|\nabla\varphi(s)\|_{\beta,\Omega_s}+\delta_{\Omega_s,\beta}\right) ds.$$

Recall that, by definition (3.3) and (4.8),

$$\delta_{\Omega_t,\beta} = \frac{|\nabla\varphi(t)|_\beta}{|\nabla\varphi(t)|_{\inf,\partial\Omega_t}} \le \frac{|\nabla\varphi(t)|_\beta}{|\nabla\Phi^{-1}\nabla\varphi_0(\Phi^{-1}(t))|_{\inf,\partial\Omega_t}}$$
$$\le c_0 \exp\left(\int_0^t |\nabla u(\tau)|_0\, d\tau\right) |\nabla\varphi(t)|_\beta . \tag{4.12}$$

Introduce the functions

$$h(t) = \beta\int_0^t |\nabla u(\tau)|_0\, d\tau,$$
$$f(t) = \|\nabla\varphi(t)\|_{\beta,\Omega_t} \exp(-h(t)) + 1.$$

From (4.11) and (4.12), we thus arrive at the following inequality:

$$
\begin{aligned}
f(t) &\leq c_0 + c(\eta,\beta)\int_0^t f(s)\log\left(2+\|\nabla\varphi(s)\|_{\beta,\Omega_s}+\delta_{s,\beta}\right)ds \\
&\leq c_0 + c(\eta,\beta)\int_0^t f(s)\log\left(2+f(s)\exp\left(\left(1+\beta^{-1}\right)h(s)\right)\right)ds \\
&\leq c_0 + c(\eta,\beta)\int_0^t f(s)\left(1+h(s)+\log f(s)\right)ds.
\end{aligned}
$$

Denote by $g(t)$ the function on the right-hand side of the above inequality:

$$
g(t) = c_0 + c(\eta,\beta)\int_0^t f(s)\left(1+h(s)+\log f(s)\right)ds.
$$

Then

$$
\begin{aligned}
g'(t) &= c(\eta,\beta)\, f(t)\left(1+h(t)+\log f(t)\right) \\
&\leq c(\eta,\beta)\, g(t)\left(1+h(t)+\log g(t)\right).
\end{aligned}
$$

It follows that

$$
\frac{d}{dt}\log g(t) \leq c(\eta,\beta)\left(1+h(t)\right) + c(\eta,\beta)\log g(t).
$$

Integrating this inequality, we obtain

$$
\log g(t) \leq c(\eta,\beta)\, e^{c(\eta,\beta)t}\left(1+\int_0^t \left(1+h(s)\right)e^{-c(\eta,\beta)s}\,ds\right).
$$

Since $f \leq g$ and $b(t)$ is increasing, it follows that, for $t < T$,

$$
\begin{aligned}
\log f(t) &\leq c(\eta,\beta,T)\left(1+h(t)\int_0^t e^{-c(\eta)s}\,ds\right) \\
&\leq c(\eta,\beta,T)\left(1+h(t)\right), \qquad (4.13)
\end{aligned}
$$

where $c(\eta,\beta,T)$ depends only on $c(\eta,\beta)$ and T. Using this inequality and recalling the definitions of $h(t)$ and $f(t)$ and applying Lemma (3.2) and (4.12), we arrive at

$$
\begin{aligned}
h(t) &= \beta\int_0^t |\nabla u(s)|_{0,\Omega_s}\,ds \leq c(\eta,\beta)\int_0^t \log\left(2+\delta_{\Omega_s,\beta}\right)ds \\
&\leq c(\eta,\beta)\int_0^t \left(\log f(s)+h(s)\right)ds \leq c(\eta,\beta,T)\int_0^t \left(1+h(s)\right)ds.
\end{aligned}
$$

The standard Gronwall's inequality implies

$$
h(t) \leq c(\eta,\beta,T).
$$

Therefore, by (4.13), $f(t) \leq c(\eta, \beta, T)$. Consequently, by the definitions of $f(t)$ and $h(t)$, and by (4.12), we find that $\delta_{\Omega_t, \beta}$ is bounded by $c(\eta, \beta, T)$, a constant depending on η, β and T. Taking $s = 0$ in (4.8), it follows that $\left|\Phi^{-1}(t)\right|_{1,\Omega_t} \leq \exp\left(\beta^{-1} h(t)\right) \leq c(\eta, \beta, T)$. The proof is complete. □

REMARK 4.1. For 2D constant vortex patches, since $\omega(x, t)$ is constant in $\bar{\Omega}_t$, Lemma 4.1 holds automatically, and Lemma 4.2 can be established directly for $\beta = \alpha$ using the incompressibility condition $\nabla \cdot u = 0$ (see [3]).

THEOREM 4.1. Then there exists a unique global $C^{1+\alpha}$ solution $\Phi(x, t)$ of (2.7) for all $t > 0$. Consequently, there exists a unique global weak solution (ω, u) for the lake equation. Furthermore, the solution has the regularity $\omega(\cdot, t) \in C^{\alpha}\left(\bar{\Omega}_t\right)$, $u(\cdot, t) \in C^{1+\alpha}\left(\bar{\Omega}_t\right)$, $\partial\Omega_t \in C^{1+\alpha}$ for all $t > 0$.

Proof. It suffices to show that for any $T > 0$, there exists a unique $C^{1+\alpha}$ solution $\Phi(x, t)$ of (2.7) for all $t < T$. By Theorem 3.1, we know that there exists a unique $C^{1+\alpha}$ solution $\Phi(x, t)$ of (2.7) for $t < T_0$ for some $T_0 > 0$. We assume $T_0 < T$, since otherwise the proof is complete. By Lemma 4.1 and 4.2, there exist $0 < \beta(T) \leq \alpha$ and $c(T) > 0$, depending only on the initial data and T such that

$$d(\Omega_t) + \|\omega(t)\|_{\beta(T), \Omega_t} + \left\|\Phi^{-1}(t)\right\|_{1,\Omega_t} + \delta_{\Omega_t, \beta(T)} \leq c(T).$$

Let u be the velocity defined in (3.15). By Lemma 3.2 with $\beta(T)$ replacing α, the above estimate implies $|\nabla u(t)|_{0,\Omega_t} \leq c(T)$. It follows that

$$\begin{aligned} |\nabla\Phi(t)|_{0,\Omega_0} &\leq c_0 + \int_0^t |\nabla u(s)|_{0,\Omega_s} |\nabla\Phi(s)|_{0,\Omega_0} \, ds \\ &\leq c_0 + c(T) \int_0^t |\nabla\Phi(s)|_{0,\Omega_0} \, ds. \end{aligned}$$

Hence, $|\nabla\Phi(t)|_{0,\Omega_0} \leq c(T)$ for $t < T_0 < T$. Using this and Lemma 3.2 again, we obtain

$$\begin{aligned} |\nabla\Phi(t)|_{\alpha,\Omega_0} &\leq c_0 + \int_0^t |\nabla u(\Phi(\cdot, s), s)|_{0,\Omega_0} |\nabla\Phi(s)|_{\alpha,\Omega_0} \, ds \\ &\quad + \int_0^t |\nabla u(\Phi(\cdot, s), s)|_{\alpha,\Omega_0} |\nabla\Phi(s)|_{0,\Omega_0} \, ds \\ &\leq c_0 + c(T) \int_0^t \left(|\nabla\Phi(s)|_{\alpha,\Omega_0} + |\nabla u(s)|_{\alpha,\Omega_s}\right) ds \\ &\leq c_0 + c(T) \int_0^t \left(\delta_{\Omega_s,\alpha} + \|\omega(s)\|_{\alpha,\Omega_s} + |\nabla\Phi(s)|_{\alpha,\Omega_0}\right) \\ &\quad \cdot \log\left(2 + \|\omega(s)\|_{\alpha,\Omega_s} + \delta_{\Omega_s,\alpha}\right) ds. \end{aligned}$$

Note that $\omega(x,t) = \omega_0\left(\Phi^{-1}(x,t)\right)$ and, by definition (3.3),

$$\delta_{\Omega_t,\alpha} = \frac{|\nabla\varphi(t)|_{\alpha,\Omega_t}}{|\nabla\varphi(t)|_{\inf,\partial\Omega_t}}, \ \varphi(x,t) = \varphi_0\left(\Phi^{-1}(x,t)\right).$$

By (4.8), we have

$$|\omega(t)|_{\alpha,\Omega_t} \le |\omega_0|_{\alpha,\Omega_0}\left|\nabla\Phi^{-1}(t)\right|^{\alpha}_{1,\Omega_t} \le c(T),$$
$$\delta_{\Omega_t,\alpha} \le |\nabla\varphi(t)|_{\alpha,\Omega_t}\exp\int_0^t |\nabla u(s)|_{0,\Omega_s}\,ds \le c(T)\,|\nabla\Phi(t)|_{\alpha,\Omega_0}.$$

Therefore,

$$|\nabla\Phi(t)|_{\alpha,\Omega_0} \le c_0 + c(T)\int_0^t \left(1 + |\nabla\Phi(s)|_{\alpha,\Omega_0}\right)\log\left(2 + |\nabla\Phi(s)|_{\alpha,\Omega_0}\right)ds.$$

This inequality implies that $|\nabla\Phi(t)|_{\alpha,\Omega_0} \le c(T)$. It then follows from Corollary 3.1 that there exists a $T_1 > 0$, depending only on $c(T)$ and the initial data such that the solution $\Phi(x,t)$ can be extended, still in the class of $C^{1+\alpha}$, to $t < T_0 + T_1$. We emphasize again that $c(T)$ depends only on T and the initial data. Since T_1 is independent of T_0, we may repeat this procedure to extend the solution to $t < T_0 + nT_1$, until $T_0 + nT_1$ exceeds the given T, where n is a positive integer. The proof is complete. □

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