QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXI, NUMBER 4
DECEMBER 2003, PAGES 639–655

ON UNIQUE SOLVABILITY AND REGULARITY IN THE LINEARIZED TWO-DIMENSIONAL WAVE RESISTANCE PROBLEM

By

DARIO PIEROTTI

Dipartimento di Matematica del Politecnico, Piazza L. da Vinci 32, 20133 Milano, Italy

Abstract. We discuss existence, uniqueness, and regularity of the solutions of a boundary value problem in a strip, which is obtained by linearization of the equations of the wave-resistance problem for a cylinder semisubmerged in a heavy fluid of constant depth H and moving at uniform velocity c in the direction orthogonal to its generators. We show that the problem has a unique solution, rapidly decreasing at infinity, for every $c > \sqrt{gH}$, where g is the acceleration of gravity. For $c < \sqrt{gH}$, we prove unique solvability provided $c \neq c_k$, where c_k is a known sequence monotonically decreasing to zero. In this case, the related flow has in general nontrivial oscillations at infinity downstream.

The appearance of the singular values c_k can be interpreted in terms of a "nonresonance condition" between the length of the cylinder's section and the gravitational wave bifurcating from the free parallel flow at the same velocity c.

1. Introduction. In the present work, we conclude the analysis started in two preceding papers [1], [2], of a boundary value problem which is obtained by linearization of the *wave-resistance problem* for the steady-state motion of a semisubmerged cylinder in an ideal, incompressible, heavy fluid.

Let us briefly recall the formulation of this problem. The cylinder is assumed to be infinitely long and moving at a uniform speed c in the direction orthogonal to its generators. The unperturbed fluid, which is at rest, has finite constant depth H; the fluid motion is assumed to be irrotational. Because of the geometry of the problem, the flow can be completely described in the vertical plane containing the direction of the motion. We want to find the steady two-dimensional flow generated by the cylinder's motion. This is a boundary value problem for the Laplace equation, characterized by the presence of a nonlinear condition (Bernoulli condition) on a free boundary (the free surface of the fluid); moreover, the free boundary is the union of two disconnected curves ending on the cylinder's profile at unknown points.

Received January 10, 2001.
2000 *Mathematics Subject Classification.* Primary 35J25.

Because of the above features, most of the mathematical work on the wave resistance problem introduces some kind of linearization, which avoids the treatment of a nonlinear condition on a free boundary. The linearization proposed in [1], [2], relies on the assumption that the body is "slender", in the sense that the piercing part of the cylinder is small compared to its length; more precisely, we assume that the cylinder's cross section depends on a small positive parameter ϵ in such a way that for $\epsilon \to 0$ it reduces to a *beam* parallel to the unperturbed flow (the same kind of approximation is discussed in [3] for a completely submerged cylinder). For $\epsilon = 0$, the trivial parallel flow is a solution of the problem; by assuming that all the relevant quantities of the problem admit an expansion in powers of ϵ, the wave-resistance problem is linearized around the solution at $\epsilon = 0$ by retaining the first order terms. In particular, the Bernoulli condition is replaced by a linear condition on a fixed boundary. Thus, one obtains a boundary value problem in a *strip*, which can be formulated either in terms of the vertical component of the fluid's velocity field, or in terms of the velocity potential. As discussed in [1], [2], the former statement of the problem represents an important difference between the above linearization and other existing approaches [4]–[6], which lead to boundary value problems for the potential function in different domains. Actually, the problem for the velocity field, besides being simpler than the problem for the potential, has solutions which are continuous and bounded up to the strip boundary; as was shown in [7], [8], this is a crucial property for the proof of the solvability of the nonlinear problem (in the case of supercritical velocities, see below). We refer to [1], [2], for the detailed description of the two dimensional wave-resistance problem and of the linearization procedure. We recall here the problem for the velocity field:

Let us denote by S_H the strip $\{(x,y) \in \mathbf{R}^2 : -H < y < 0\}$; we will call $B = \mathbf{R}\times\{-H\}$ the bottom boundary $\mathbf{R}\times\{0\}$. We denote by I the open interval $(-x_0, x_0)\times\{0\}$ (representing the beam) and set $F = \{\mathbf{R}\backslash[-x_0, x_0]\} \times \{0\}$. We consider the following problem:

PROBLEM $\mathcal{P}_\nu$. Given a function $\mathcal{F}$ defined on I and a number $\nu \in \mathbf{R}$, find $v \in H^1_{loc}(S_H)$ such that

$$\Delta v = 0 \qquad \text{in} \quad S_H, \tag{1.1}$$

$$v = \mathcal{F} \qquad \text{on} \quad I, \tag{1.2}$$

$$v_y - \nu v = 0 \qquad \text{on} \quad F, \tag{1.3}$$

$$v = 0 \qquad \text{on} \quad B, \tag{1.4}$$

$$\lim_{x\to-\infty} v(x,y) = 0, \tag{1.5}$$

uniformly with respect to $y \in [-H, 0]$.

$$\sup_{(x,y)\in S_H\backslash A} |v(x,y)| < \infty, \tag{1.6}$$

where A is any neighborhood of I.

Some comments and remarks on the above equations are in order. In the linearized wave-resistance problem, the function v represents the vertical component of the perturbed velocity field; in that case, assuming that the piercing part of the cylinder is

described by the equation $y = \epsilon f(x)$, one has $\mathcal{F} = cf'$ in condition (1.2) (see [1], [2]). Similarly, (1.3) becomes the limit of the Bernoulli condition by setting

$$\nu = g/c^2, \tag{1.7}$$

where g is the acceleration of gravity. Finally, (1.4) indicates that the fluid bottom is a streamline and the asymptotic conditions (1.5)–(1.6) state that the perturbed field vanishes at infinity upstream and is bounded outside any neighborhood of the beam. We remark that the solutions of problem $\mathcal{P}_\nu$ are locally in $H^1(S_H)$; hence, we will necessarily take $\mathcal{F} \in H^{1/2}(I)$ in condition (1.2), which implies (by Sobolev embedding theorems) that the cylinder's profile must be a Hölder continuous Cartesian curve. As we will show below (see also [1], [2]) for sufficiently regular data $\mathcal{F}$ the velocity field is continuous up to the strip boundary. We point out that this regularity result can not be achieved by requiring only local finiteness of kinetic energy as in [4], [5].

A relevant parameter for the discussion of the wave resistance problem is the Froude number F_r, defined by $F_r = c^2/gH$. We say that the motion of the cylinder is supercritical if $F_r > 1$ and subcritical if $F_r < 1$. Correspondingly, by (1.7), one has $\nu < 1/H$ in problem $\mathcal{P}_\nu$ in the case of a supercritical motion, and $\nu > 1/H$ for subcritical velocities. In the former case, we proved in reference [1] that the problem (1.1)–(1.4) is uniquely solvable in the Sobolev space $H^1(S_H)$; the result was extended in [2] to the subcritical motion, provided the datum $\mathcal{F}$ in condition (1.2) satisfies certain linear conditions and the parameter ν in (1.3) does not belong to a discrete subset of $(1/H, +\infty)$. In both cases, the results follow by a suitable variational formulation of the problem (1.1)–(1.4).

In this paper, we discuss existence, uniqueness, and regularity of the solutions of problem $\mathcal{P}_\nu$, i.e. of (1.1)–(1.6). We rely on the results obtained in [1], [2] with the variational approach, which are summarized in the next section. By assuming $\mathcal{F} \in H^{1/2}(I)$ in (1.2), we prove in § 3 that for supercritical velocities the problem has unique solution, which coincides with the solution in $H^1(S_H)$ found in [1]; hence, such solution has vanishing limit also for $x \to +\infty$. The treatment of the subcritical motion is more delicate; in this case, with the same assumptions on $\mathcal{F}$, we prove unique solvability provided the values of ν do not belong to a known discrete set, which depends on x_0 and H (see § 4). In addition, the flow has in general nontrivial oscillations at infinity downstream (the same holds for an infinite-depth fluid; see [4]). Furthermore, we give sufficient conditions on $\mathcal{F}$ for the continuity of the solutions of problem $\mathcal{P}_\nu$ (both in the supercritical and subcritical cases) up to the boundary of the strip S_H. Finally, we discuss the relevance of our results for the solvability of the nonlinear wave resistance problem.

2. A related variational problem. We now briefly recall some results of [1], [2] which will be useful in subsequent sections. The weak form of (1.1)–(1.4) can be stated in $H^1(S_H)$ equipped with the equivalent norm

$$||v||^2 = \int_{S_H} |\nabla v|^2 dxdy + \int_B |v|^2 dx,$$

which reduces to the Dirichlet integral on any subspace of functions vanishing on B. We denote with H^1_* the subspace of the functions vanishing on $I \cup B$. Then, we look for $v \in H^1(S_H)$ satisfying $v|_I = \mathcal{F}$, $v|_B = 0$, and such that

$$\int_{S_H} \nabla v \nabla w \, dxdy - \nu \int_F vw \, dx = 0 \tag{2.1}$$

for every $w \in H^1_*$

Now, if $\nu H < 1$, one readily verifies that the bilinear form at the left hand side of (2.1) is coercive in H^1_* (see [1], Eq. (3.3)). Then, we have

THEOREM 2.1. For every $\nu < 1/H$ and $\mathcal{F} \in H^{1/2}(I)$, there is a unique v satisfying (2.1) and the boundary conditions (1.2), (1.4). Moreover, v is harmonic in S_H, smooth in the closed strip outside any neighborhood of I, and (1.3) holds.

The details of the proof, together with further regularity results, can be found in [1], § 3.

Let us now review the results for the subcritical case. If $\nu H > 1$, the bilinear form in (2.1) is no longer coercive in H^1_*; nevertheless, one can restore coercivity by restricting to the subspace

$$V_* = \left\{ w \in H^1_*(S_H); \ \int_{-H}^{0} \sinh[\nu_0(y+H)] w(x,y) \, dy = 0, \ \text{for } |x| \geq x_0 \right\}, \tag{2.2}$$

where ν_0 is the positive solution of the equation

$$\frac{\nu_0}{\nu} = \tanh(\nu_0 H), \tag{2.3}$$

see [2], Proposition 4.1. We stress that this equation has real solutions only if $\nu H > 1$, i.e., when the velocity is subcritical. Then, one can prove that for every $\nu > 1/H$ and $\mathcal{F} \in H^{1/2}(I)$, there is $v \in H^1(S_H)$ satisfying the boundary conditions (1.2), (1.4), and such that (2.1) holds for every $w \in V_*$ ([2], theorem 4.4). Moreover, v is uniquely determined if one requires the additional condition

$$\int_{-H}^{0} \sinh[\nu_0(y+H)] v(x,y) \, dy = 0, \ \text{for } |x| \geq x_0, \tag{2.4}$$

([2], theorem 4.5 and Remark 4.6). However, the function v is not harmonic in general, since Eq. (2.1) holds for $w \in V_* \subset H^1_*$. Actually, we have the following result:

THEOREM 2.2. Let $\nu > 1/H$, $\mathcal{F} \in H^{1/2}(I)$ and let $v \in H^1(S_H)$ be the solution of (1.2), (1.4), (2.4), and of (2.1) for every $w \in V_*$. Then, there exist real constants λ_+, λ_-, such that

$$\Delta v = [\lambda_+ \delta(x - x_0) + \lambda_- \delta(x + x_0)] \sinh[\nu_0(y+H)], \tag{2.5}$$

where $\delta(\cdot)$ is the 'Dirac delta function'. Moreover, v is smooth in the closed strip outside any neighborhood of the set $I \cup \{(x_0, y), -H \leq y \leq 0\} \cup \{(-x_0, y), -H \leq y \leq 0\}$, and satisfies (1.3). Finally, v is uniquely determined by (1.2)-(1.4) and (2.5), and the

following relations hold:

$$\frac{\lambda_+ - \lambda_-}{2} \sin(\nu_0 x_0) = C(\nu_0) \int_{-x_0}^{x_0} [v_y(x,0) - \nu \mathcal{F}(x)] \sin(\nu_0 x)\, dx; \tag{2.6}$$

$$\frac{\lambda_+ + \lambda_-}{2} \cos(\nu_0 x_0) = C(\nu_0) \int_{-x_0}^{x_0} [v_y(x,0) - \nu \mathcal{F}(x)] \cos(\nu_0 x)\, dx, \tag{2.7}$$

where

$$C(\nu_0) = \frac{\nu_0 \sinh(\nu_0 H)}{\sinh(\nu_0 H)\cosh(\nu_0 H) - \nu_0 H}. \tag{2.8}$$

For the proof of the above theorem we refer to [2], Theorem 4.7. Here we make some comments and remarks on the results obtained. The uniqueness of the solution in $H^1(S_H)$ of (1.2)–(1.4) and (2.5) implies that problem (1.1)–(1.4) has no variational solutions for $\nu > 1/H$, except when $\lambda_+ = \lambda_- = 0$. In [2], we proved the following result: provided the parameter ν_0 in Eq. (2.3) is different from $k\pi/2x_0$, $k = 1, 2, ...$, a variational solution exists if and only if the datum $\mathcal{F}$ is orthogonal (in the space $L^2(I)$) to a given two-dimensional subspace, depending on ν_0, x_0, and H. Clearly, such conditions impose some restrictions on the profile of the cylinder's hull and on the values of the velocity c; if they are fulfilled, we obtain wave-free solutions of the linearized wave-resistance problem [2]. On the other hand, one can show that in general the subcritical flow produces oscillations of wavenumber ν_0 at downstream infinity (see [6], [10], and § 4 below); as we will show in § 4, the amplitudes of the above oscillations are related to the quantities $\lambda_\pm$ defined in theorem 2.2.

3. Unique solvability in the supercritical case. In the case of supercritical velocities, unique solvability of problem $\mathcal{P}_\nu$ can be easily proved by the results of the variational problem and by the following *a priori estimate* of the solutions:

PROPOSITION 3.1. Let $v \in H^1_{loc}(S_H)$ be a solution of (1.1)–(1.6), with $\nu < 1/H$. Then, for every neighborhood A of I, v is smooth in $\overline{S_H} \backslash A$ and the following bound holds:

$$\sup_{(x,y)\in S_H \backslash A} e^{\mu_1 |x|} |v(x,y)| < \infty, \tag{3.1}$$

where μ_1 is the first positive solution of

$$\tan(\mu H) = \frac{\mu}{\nu}. \tag{3.2}$$

Proof. The smoothness of v up to the boundaries F and B follows by standard regularity results for weak solutions of elliptic problems. Then, by defining $Q_R = [-R, R] \times [-H, 0]$, we have that v is bounded in $Q_R \backslash A$ for every $R > x_0$. Let us now consider the restriction of v to the domain $(R, +\infty) \times (-H, 0)$. Clearly v is harmonic in this domain and satisfies the conditions (1.3) and (1.4) on the upper and lower bound respectively; furthermore, $v(\xi, \cdot)$ is smooth and bounded in $[-H, 0]$ for every $\xi \geq R$. We will write a series expansion for v in $(R, +\infty) \times (-H, 0)$; let us fix $R' > R$ and observe that v is H^1 in the rectangle $(R, R') \times (-H, 0)$. Then, by (1.3), (1.4) and by standard results on elliptic problems in polygons [9], v is uniquely determined in the rectangle by the boundary values $v(R, \cdot)$ and $v(R', \cdot)$. On the other hand, we can solve the problem for

v in the rectangle by separation of variables; by elementary calculations, we are reduced to the (self-adjoint) eigenvalue problem: $-\psi'' = \mu\psi$, $\psi(-H) = 0$, $\psi'(0) = \nu\psi(0)$, with $\mu \in \mathbf{R}$ and ψ smooth function on $[-H, 0]$. Since we have $\nu H < 1$, we get the eigenvalues μ_n^2, $n = 1, 2, ...$, where $0 < \mu_1 < ... < \mu_n < ...$ are the positive solutions of Eq. (3.2); then, we can write the expansion

$$v(x, y) = \sum_{n=1}^{+\infty} \Big[a_n e^{-\mu_n x} + b_n \sinh(\mu_n(x - R))\Big] \sin(\mu_n(y + H)), \tag{3.3}$$

for $(x, y) \in (R, R') \times (-H, 0)$, with uniquely determined coefficients a_n, b_n. Note that the coefficients a_n depend only on $v(R, y)$, $y \in [-H, 0]$. Since for any fixed x the function $v(x, \cdot)$ is independent of R, R', it follows that also the coefficients b_n must be independent of R'. On the other hand, by condition (1.6), the function

$$v(R', y) = \sum_{n=1}^{+\infty} \Big[a_n e^{-\mu_n R'} + b_n \sinh(\mu_n(R' - R))\Big] \sin(\mu_n(y + H)),$$

is uniformly bounded in $L^2(-H, 0)$ with respect to R'; thus, we easily get the bound

$$|b_n| \leq C e^{-\mu_n R'},$$

with C independent of R'. By the arbitrariness of R', we obtain $b_n = 0$ for every n. Therefore, we have in $(R, +\infty) \times (-H, 0)$:

$$v(x, y) = \sum_{n=1}^{+\infty} a_n e^{-\mu_n x} \sin(\mu_n(y + H)), \tag{3.4}$$

so that $|v(x, y)| \leq C e^{-\mu_1 x}$ for $x \geq R$. Clearly, a similar conclusion holds for $x \leq -R$. Hence, the bound (3.1) follows. □

Then, we have at once

THEOREM 3.2. For every $\nu < 1/H$ and for every $\mathcal{F} \in H^{1/2}(I)$, problem $\mathcal{P}_\nu$ is uniquely solvable; the solution v satisfy the estimate (3.1). Moreover, if $\mathcal{F} \in H^{3/2}(I)$, the function v is continuous and bounded in the closed strip $\overline{S_H}$.

Proof. By (3.1), we have in particular that every solution of problem $\mathcal{P}_\nu$ with $\nu < 1/H$ belongs to $H^1(S_H)$. Now, unique solvability follows by theorem 2.1 of the previous section, and regularity by proposition 3.2 of [1]. □

REMARK 3.3. It is worthwhile to recall that the crucial point for the regularity of the solutions of $\mathcal{P}_\nu$ is the behaviour of v in the neighborhood of the points $(\pm x_0, 0)$, where the two different boundary conditions (1.2) and (1.3) meet. In proposition 3.2 of [1], it is shown that for $\mathcal{F} \in H^{3/2}(I)$ one can write, in a neighborhood $\mathcal{B} \subset S_H$ of $(x_0, 0)$,

$$v(x, y) = C r^{1/2} \sin(\theta/2) + v_1(x, y), \tag{3.5}$$

where C is a constant, $v_1 \in H^2(\mathcal{B})$, and r, θ are the polar coordinates of (x, y) around the point $(x_0, 0)$, with $\theta = 0$ on I and $\theta = \pi$ on F. A similar statement holds in the neighborhood of $(-x_0, 0)$. Then, the continuity of v in $\overline{\mathcal{B}}$ follow by Sobolev embedding theorems. Furthermore, if (3.5) is true, one can also show that any harmonic conjugate u of v extends to a continuous function in $\overline{S_H}$.

REMARK 3.4. Clearly, all of the other results proved in [1] for the variational solutions also hold for the solutions of $\mathcal{P}_\nu$ in the supercritical case. In particular, any harmonic conjugate u of a solution v has finite limits $\lim_{x\to\pm\infty} u(x,y) = c_\pm$ uniformly with respect to y; moreover, by assuming $\mathcal{F} \in H^{3/2}(I)$ and by suitable application of Green's theorem (see [1], Proposition 3.3), one has the relation

$$u(x_0,0) - u(-x_0,0) + \nu \int_I \mathcal{F}\, dx - (1 - H\nu)(c_+ - c_-) = 0. \tag{3.6}$$

The arbitrary constant in the definition of u is fixed by the limit condition $\lim_{x\to-\infty} u(x,y) = 0$ of the linearized wave resistance problem (see [1], Eq. (2.16)). Then, we have $c_- = 0$ in (3.6) and the perturbed velocity field $u - iv$ has finite limit, for $x \to +\infty$, given by

$$c_+ = \frac{1}{(1 - H\nu)}\Big[u(x_0,0) - u(-x_0,0) + \nu \int_I \mathcal{F}\, dx\Big].$$

4. The subcritical case. In this section we discuss unique solvability of problem $\mathcal{P}_\nu$ for $\nu > 1/H$. We start by describing the asymptotic properties of the solutions.

PROPOSITION 4.1. Let $v \in H^1_{loc}(S_H)$ be a solution of (1.1)–(1.6), with $\nu > 1/H$. Then, for every neighborhood A of I, v is smooth in $\overline{S_H}\backslash A$ and there are real constants $\mathcal{A}$, $\mathcal{B}$ such that the following bound holds:

$$\sup_{(x,y)\in S_H\backslash A} e^{\mu_1|x|}\Big|v(x,y) - \theta(x)[\mathcal{A}\sin(\nu_0 x) + \mathcal{B}\cos(\nu_0 x)]\sinh(\nu_0(y+H))\Big| < \infty, \tag{4.1}$$

where μ_1 is the first positive solution of (3.2), ν_0 is the positive solution of (2.3), and θ is the characteristic function of $(0,+\infty)$.

Proof. The smoothness properties of v follow as in the proof of Proposition 3.1; furthermore, we can as well repeat the arguments leading to the series expansions for v in the regions $(R,+\infty)\times(-H,0)$ and $(-\infty,-R)\times(-H,0)$, $R > x_0$. In this case, by solving the eigenvalue problem, we obtain a sequence of positive eigenvalues μ_n^2, $n = 1, 2...$ as before, and a negative eigenvalue $-\nu_0^2$, with ν_0 the positive solution of (2.3); then, taking account of the asymptotic condition (1.5), we easily get the expansions:

$$\begin{aligned} v(x,y) = {} & [\mathcal{A}\sin(\nu_0 x) + \mathcal{B}\cos(\nu_0 x)]\sinh(\nu_0(y+H)) \\ & + \sum_{n=1}^{+\infty} a_n e^{-\mu_n x}\sin(\mu_n(y+H)) \end{aligned} \tag{4.2}$$

for $(x,y) \in (R,+\infty)\times(-H,0)$, and

$$v(x,y) = \sum_{n=1}^{+\infty} b_n e^{+\mu_n x}\sin(\mu_n(y+H)) \tag{4.3}$$

for $(x,y) \in (-\infty,-R)\times(-H,0)$. Then, the bound (4.1) follows. □

REMARK 4.2. As can be checked by taking the Fourier transform with respect to x (see also [10], § 5), the homogeneous problem

$$\begin{aligned} \Delta v &= 0 && \text{in} \quad S_H, \\ v_y - \nu v &= 0 && \text{on} \quad \mathbf{R} \times \{0\}, \\ v &= 0 && \text{on} \quad B, \\ \sup_{(x,y)\in S_H} |v(x,y)| &< \infty \end{aligned}$$

has nontrivial solutions in $H^1_{loc}(S_H)$ for $\nu > 1/H$, which have the form

$$[C_1 \sin(\nu_0 x) + C_2 \cos(\nu_0 x)] \sinh(\nu_0(y+H)),$$

with C_1, C_2 arbitrary constants. We point out that the above problem is obtained by linearization of the equations of the problem of *periodic water waves* [11] in a fluid with finite depth (with no obstacles).

We now turn to unique solvability of problem $\mathcal{P}_\nu$; our strategy for its solution relies on the following straightforward consequence of theorem 2.2:

PROPOSITION 4.3. Given $\nu > 1/H$ and $\mathcal{F} \in H^{1/2}(I)$, let $v \in H^1(S_H)$ be the solution of (1.2)–(1.4) and (2.5), according to theorem 2.2; furthermore, let θ be the characteristic function of $(0, +\infty)$. Then, the function

$$\begin{aligned} \tilde{v}(x,y) = v(x,y) &- \frac{\lambda_+}{\nu_0}\theta(x-x_0)\sin(\nu_0(x-x_0))\sinh[\nu_0(y+H)] \\ &+ \frac{\lambda_-}{\nu_0}\theta(-(x+x_0))\sin(\nu_0(x+x_0))\sinh[\nu_0(y+H)] \end{aligned} \tag{4.4}$$

satisfies the conditions (1.1)–(1.4) and (1.6) of problem $\mathcal{P}_\nu$ (with the same $\mathcal{F}$ in (1.2)). The asymptotic condition (1.5) holds only if $\lambda_- = 0$. Moreover, by assuming $\mathcal{F} \in H^{3/2}(I)$ in condition (1.2) the function $\tilde{v}$ is continuous and bounded in the closed strip $\overline{S_H}$, together with any harmonic conjugate $\tilde{u}$.

Proof. By direct computation and by (2.5), the function $\tilde{v}$ is harmonic in S_H and satisfies the boundary conditions (1.2)–(1.4); furthermore, it is readily verified that $\tilde{v} \in H^1_{loc}(S_H)$ and that (1.6) holds. Moreover, by recalling that every $v \in H^1(S_H)$ is vanishing for $|x| \to +\infty$, we get that (1.5) is satisfied only if $\lambda_- = 0$. Finally, one can verify that the regularity arguments quoted in remark 3.3 also apply to $\tilde{v}$, so that the proposition follows. □

The second step is the construction of nontrivial, continuous and bounded solutions of the *homogeneous problem* (1.1)–(1.4) (i.e., with $\mathcal{F} = 0$ in (1.2)). To this aim, we introduce, for every fixed $\nu > 1/H$, a pair of functions v^s, $v^c \in H^1(S_H)$ in the following way (see also [2, § 4]):

Given $\nu_0 > 0$ satisfying (2.3), $v^s \in H^1(S_H)$ is the solution of (1.2)–(1.4) and (2.5), with $\mathcal{F}(x) = \sin(\nu_0 x)$ in condition (1.2). Similarly, $v^c \in H^1(S_H)$ is the solution of (1.2)–(1.4) and (2.5), with $\mathcal{F}(x) = \cos(\nu_0 x)$ in condition (1.2).

REMARK 4.4. By the symmetry properties of the data and by the uniqueness statement of theorem 2.2, we have $v^s(-x,y) = -v^s(x,y)$ and $v^c(-x,y) = v^c(x,y)$; as a consequence, v^s and v^c satisfy (2.5) with $\lambda_+ = -\lambda_- \equiv \lambda^s$ and $\lambda_+ = \lambda_- \equiv \lambda^c$ respectively.

By recalling (2.6), (2.7), we now have the relations

$$\lambda^s \sin(\nu_0 x_0) = C(\nu_0) \int_{-x_0}^{x_0} [v_y^s(x,0) - \nu \sin(\nu_0 x)] \sin(\nu_0 x)\, dx, \tag{4.5}$$

$$\lambda^c \cos(\nu_0 x_0) = C(\nu_0) \int_{-x_0}^{x_0} [v_y^c(x,0) - \nu \cos(\nu_0 x)] \cos(\nu_0 x)\, dx. \tag{4.6}$$

We also remark that the conditions for the existence of variational solutions of (1.1)–(1.4) (in the subcritical case) mentioned at the end of § 2 are expressed in terms of the functions v^s, v^c (see [2], theorem 4.9). Furthermore, these functions are explicitly known for special values of ν_0; actually we have

$$v^s(x,y) = \begin{cases} \frac{1}{\sinh(\nu_0 H)} \sin(\nu_0 x) \sinh[\nu_0(y+H)], & \text{if} \quad |x| < x_0, \\ 0 \quad \text{if} \quad |x| > x_0, \end{cases} \tag{4.7}$$

for $\nu_0 = n\pi/x_0$, $n = 1, 2, ...$;

$$v^c(x,y) = \begin{cases} \frac{1}{\sinh(\nu_0 H)} \cos(\nu_0 x) \sinh[\nu_0(y+H)], & \text{if} \quad |x| < x_0, \\ 0 \quad \text{if} \quad |x| > x_0, \end{cases} \tag{4.8}$$

for $\nu_0 = (n - \frac{1}{2})\pi/x_0$, $n = 1, 2, ...$ (see [2], proposition 4.10).

Clearly, the corresponding values of λ^s, λ^c, are easily calculated:

$$\lambda^s(n\pi/x_0) = (-1)^{n+1} \frac{n\pi/x_0}{\sinh(n\pi H/x_0)}, \tag{4.9}$$

$$\lambda^c((n-1/2)\pi/x_0) = (-1)^{n+1} \frac{(n-\frac{1}{2})\pi/x_0}{\sinh\Big((n-\frac{1}{2})\pi H/x_0\Big)}, \tag{4.10}$$

with $n = 1, 2....$

Now, we can define the functions with the required properties:

PROPOSITION 4.5. For every $\nu > 1/H$ and $\nu_0 > 0$ solution of (2.3), we set

$$\begin{aligned} \tilde{v}^s(x,y) = v^s(x,y) - \frac{\lambda^s}{\nu_0} \Big[& \theta(x-x_0) \sin(\nu_0(x-x_0)) \\ & + \theta(-(x+x_0)) \sin(\nu_0(x+x_0)) \Big] \sinh[\nu_0(y+H)], \end{aligned} \tag{4.11}$$

$$\begin{aligned} \tilde{v}^c(x,y) = v^c(x,y) - \frac{\lambda^c}{\nu_0} \Big[& \theta(x-x_0) \sin(\nu_0(x-x_0)) \\ & - \theta(-(x+x_0)) \sin(\nu_0(x+x_0)) \Big] \sinh[\nu_0(y+H)], \end{aligned} \tag{4.12}$$

where λ^s, λ^c are defined in remark 4.4. Then, the functions

$$\zeta^s(x,y) = \tilde{v}^s(x,y) - \frac{1}{\sinh(\nu_0 H)} \sin(\nu_0 x) \sinh[\nu_0(y+H)], \tag{4.13}$$

$$\zeta^c(x,y) = \tilde{v}^c(x,y) - \frac{1}{\sinh(\nu_0 H)} \cos(\nu_0 x) \sinh[\nu_0(y+H)], \tag{4.14}$$

solve the homogeneous problem (1.1)–(1.4), are continuous and bounded in the closed strip $\overline{S_H}$ and satisfy $\zeta^s(-x,y) = -\zeta^s(x,y)$, $\zeta^c(-x,y) = \zeta^c(x,y)$.

Proof. By (4.11), (4.12) and by proposition 4.3, $\tilde{v}^s$, $\tilde{v}^c$, are harmonic and satisfy $\tilde{v}^s(x,0) = \sin(\nu_0 x)$, $\tilde{v}^c(x,0) = \cos(\nu_0 x)$ together with the other boundary conditions (1.3), (1.4); continuity and boundedness follow as well from proposition 4.3. Furthermore, $\tilde{v}^s$, $\tilde{v}^c$ have the same symmetry properties as v^s and v^c (see remark 4.4). Then, the proposition follows by the definitions (4.13), (4.14). □

We now show that a condition for unique solvability of problem $\mathcal{P}_\nu$ can be deduced from propositions 4.3, 4.5. All that we need is an asymptotic formula for ζ^s, ζ^c as $|x| \to \infty$.

COROLLARY 4.6. Let ζ^s, ζ^c be defined by (4.13), (4.14). Then, we have the following asymptotic representation as $x \to \pm\infty$:

$$\zeta^s(x,y) = [\mathcal{A}_s \sin(\nu_0 x) \pm \mathcal{B}_s \cos(\nu_0 x)] \sinh[\nu_0(y+H)] + \zeta_0^s(x,y), \quad \pm x > 0, \tag{4.15}$$

$$\zeta^c(x,y) = [\pm\mathcal{A}_c \sin(\nu_0 x) + \mathcal{B}_c \cos(\nu_0 x)] \sinh[\nu_0(y+H)] + \zeta_0^c(x,y), \quad \pm x > 0, \tag{4.16}$$

where ζ_0^s, ζ_0^c, are harmonic and rapidly decreasing as $|x| \to \infty$ and the following relations hold:

$$\mathcal{A}_s = -\Big(\frac{\lambda^s}{\nu_0}\cos(\nu_0 x_0) + \frac{1}{\sinh(\nu_0 H)}\Big); \qquad \mathcal{B}_s = \frac{\lambda^s}{\nu_0}\sin(\nu_0 x_0); \tag{4.17}$$

$$\mathcal{A}_c = -\frac{\lambda^c}{\nu_0}\cos(\nu_0 x_0); \qquad \mathcal{B}_c = \Big(\frac{\lambda^c}{\nu_0}\sin(\nu_0 x_0) - \frac{1}{\sinh(\nu_0 H)}\Big). \tag{4.18}$$

Proof. By elementary calculation from (4.11)–(4.14), we find that (4.15), (4.16) hold for $|x| > x_0$, with $\zeta_0^s(x,y) = v^s(x,y)$ and $\zeta_0^c(x,y) = v^c(x,y)$. On the other hand, v^s, v^c are harmonic for $|x| > x_0$ and vanish at infinity; then, we can apply the same arguments as in proposition 4.1 and find that $e^{\mu_1|x|}|\zeta_0^s(x,y)|$ and $e^{\mu_1|x|}|\zeta_0^c(x,y)|$ are bounded, where μ_1 is defined as in proposition 3.1. □

We can now state the promised condition of unique solvability:

THEOREM 4.7. Let $\nu > 1/H$ be given and suppose that the positive solution ν_0 of $\nu_0/\nu = \tanh(\nu_0 H)$ is such that

$$\lambda^c(\nu_0)\sin(\nu_0 x_0) - \lambda^s(\nu_0)\cos(\nu_0 x_0) \neq \frac{\nu_0}{\sinh(\nu_0 H)}, \tag{4.19}$$

where λ^s, λ^c are defined in remark 4.4. Then, for every $\mathcal{F} \in H^{1/2}(I)$, problem $\mathcal{P}_\nu$ is uniquely solvable.

Proof. We first prove uniqueness of the solution. Assume that v_0 is a solution of problem $\mathcal{P}_\nu$ with $\mathcal{F} = 0$; let $\mathcal{A}_0$, $\mathcal{B}_0$ be the constants in the asymptotic formula 4.1 for v_0. We now apply Green's formula to v_0 and to each of the harmonic functions ζ^s, ζ^c given by (4.13), (4.14) in a bounded rectangle $(-R,R) \times (-H,0)$ with $R > x_0$; then,

letting $R \to \infty$ and taking account of corollary 4.6 we get:

$$0 = \lim_{R\to+\infty} \int_{-H}^{0} \Big[\zeta^s(R,y)\partial_x v_0(R,y) - v_0(R,y)\partial_x \zeta^s(R,y)\Big] dy$$
$$= \nu_0(\mathcal{A}_0\mathcal{B}_s - \mathcal{B}_0\mathcal{A}_s) \int_{-H}^{0} \sinh^2[\nu_0(y+H)]dy,$$
$$0 = \lim_{R\to+\infty} \int_{-H}^{0} \Big[\zeta^c(R,y)\partial_x v_0(R,y) - v_0(R,y)\partial_x \zeta^c(R,y)\Big] dy$$
$$= \nu_0(\mathcal{A}_0\mathcal{B}_c - \mathcal{B}_0\mathcal{A}_c - \int_{-H}^{0} \sinh^2[\nu_0(y+H)]dy.$$

Hence, we get the relations

$$\mathcal{A}_0\mathcal{B}_s - \mathcal{B}_0\mathcal{A}_s = \mathcal{A}_0\mathcal{B}_c - \mathcal{B}_0\mathcal{A}_c = 0,$$

which are equivalent to $\mathcal{A}_0 = \mathcal{B}_0 = 0$ if the condition

$$\mathcal{A}_s\mathcal{B}_c - \mathcal{B}_s\mathcal{A}_c \neq 0$$

holds; by (4.17), (4.18), this condition is equivalent to (4.19). In this case, v_0 is rapidly decreasing as $|x| \to \infty$, so that $v_0 \in H^1(S_H)$. Thus, by the results of § 2, we have $v_0 = 0$.

We now show that the same condition (4.19) assures existence of the solution. Given $\mathcal{F} \in H^{1/2}(I)$, consider the harmonic function $\tilde{v}$ defined by (4.4) and set:

$$v^{\alpha,\beta} = \tilde{v} - \alpha\zeta^s - \beta\zeta^c, \tag{4.20}$$

where ζ^s, ζ^c are given by (4.13), (4.14) and α, β are real constants. Then, by propositions 4.3, 4.5 and by corollary 4.6, one verifies that $v^{\alpha,\beta}$ satisfies all the relations of problem $\mathcal{P}_\nu$, including the asymptotic condition (1.5), if the pair α, β solve the linear system:

$$\begin{aligned} \mathcal{A}_s\alpha - \mathcal{A}_c\beta &= \frac{\lambda_-}{\nu_0}\cos(\nu_0 x_0), \\ -\mathcal{B}_s\alpha + \mathcal{B}_c\beta &= \frac{\lambda_-}{\nu_0}\sin(\nu_0 x_0). \end{aligned} \tag{4.21}$$

Clearly, the condition for unique solvability of the system (4.21) is again (4.19). □

It is now crucial to check the validity of the condition (4.19) as ν_0 varies in the interval $(0, +\infty)$; by recalling the relations (4.9), (4.10), we readily see that (4.19) fails for $\nu_0 = k\pi/2x_0$, $k = 1, 2, \ldots$. We can prove that there are no other "singular values" of ν_0; in fact we have:

PROPOSITION 4.8. For every $\nu_0 > 0$ the following relation holds:

$$\lambda^c(\nu_0)\sin(\nu_0 x_0) - \lambda^s(\nu_0)\cos(\nu_0 x_0) = \frac{\nu_0}{\sinh(\nu_0 H)} + K(\nu_0)\sin(\nu_0 x_0)\cos(\nu_0 x_0), \tag{4.22}$$

where $K(\nu_0) < 0$.

The proof is reported in the appendix, together with some additional remarks on the functions $\nu_0 \mapsto \lambda^s(\nu_0)$, $\nu_0 \mapsto \lambda^c(\nu_0)$.

We can now state the main result of this section:

COROLLARY 4.9. For any given $\mathcal{F} \in H^{1/2}(I)$, problem $\mathcal{P}_\nu$ is uniquely solvable for $\nu > 1/H$, provided the positive solution ν_0 of $\nu_0/\nu = \tanh(\nu_0 H)$ is different from $k\pi/2x_0$, $k = 1, 2, \ldots$. Furthermore, if $\mathcal{F} \in H^{3/2}(I)$, the solution is continuous and bounded in the closed strip $\overline{S_H}$.

Proof. The condition for unique solvability is obtained from theorem 4.7 taking account of equation (4.22); moreover, the regularity properties of the solution (4.20) in the case $\mathcal{F} \in H^{3/2}(I)$ follow by propositions 4.3 and 4.5. □

The solvability of problem $\mathcal{P}_\nu$ in correspondence with the "singular values" $k\pi/2x_0$ of ν_0 remains an open problem. By the relations (4.7)–(4.14), we see that in these cases one has $\zeta^s = 0$ for k even and $\zeta^c = 0$ for k odd; hence, both uniqueness and existence proofs of theorem 4.7 fail. Nevertheless, we can still get an existence result, for a particular class of data, from proposition 4.3 and from (4.7), (4.8); let us denote by $g \mapsto \lambda_-^g$ the linear map which associates to each $g \in H^{1/2}(I)$ the real number λ_- according to Eq. (2.5) of theorem 2.2. Moreover, by recalling (4.9) and (4.10), we set $\lambda_n^s = \lambda^s(n\pi/x_0)$, $\lambda_n^c = \lambda^c((n-1/2)\pi/x_0)$, $n = 1, 2\ldots$. Then, we have

PROPOSITION 4.10. Let T_n^s, T_n^c, be the bounded linear operators on $H^{1/2}(I)$ defined by

$$T_n^s g = g + \frac{\lambda_-^g}{\lambda_n^s} \sin\Big(\frac{n\pi}{x_0} x\Big), \tag{4.23}$$

$$T_n^c g = g - \frac{\lambda_-^g}{\lambda_n^c} \cos\Big(\frac{(n-1/2)\pi}{x_0} x\Big). \tag{4.24}$$

Then, if $\nu_0 = n\pi/x_0$ and $\mathcal{F} \in Ran\, T_n^s$, problem $\mathcal{P}_\nu$ has a solution; similarly, problem $\mathcal{P}_\nu$ has a solution for $\nu_0 = (n-1/2)\pi/x_0$ and $\mathcal{F} \in Ran\, T_n^c$.

Proof. Let $\nu_0 = n\pi/x_0$ for a given n; by linearity and by remark 4.4, if $\mathcal{F}$ is in the range of the operator (4.23), we have $\lambda_-^{\mathcal{F}} = 0$. Then, the existence of the solution follows by proposition 4.3. A similar conclusion follows for $\nu_0 = (n-1/2)\pi/x_0$. □

5. Final remarks. We make some final comment on the meaning of the results obtained in the previous sections from the point of view of the wave resistance problem. By considering the linearization discussed in the introduction, we have found, for any cylinder's profile that is smooth enough, a unique solution with continuous and bounded velocity field for every value of the cylinder's velocity above the critical value $\sqrt{gH}$; moreover, the flow vanishes at infinity both upstream and downstream. In the case of subcritical velocities, we have unique solvability (and regularity) if the assumptions of corollary 4.9 hold, with solutions which (in general) oscillate at infinity downstream with a wavenumber defined by (2.3). Thus, by recalling (1.7), we get a sequence of singular values for the cylinder's velocity given by

$$c_k = \Big[g\frac{2x_0}{k\pi}\tanh\Big(\frac{k\pi H}{2x_0}\Big)\Big]^{1/2}, \quad k = 1, 2, \ldots. \tag{5.1}$$

The sequence $\{c_k\}_{k=1}^\infty$ decreases monotonically to zero. Note that for large values of the ratio x_0/H, the highest singular value c_1 of the velocity approaches the critical value $\sqrt{gH}$; for $x_0/H << 1$, the value of c_1 is small compared to the critical velocity.

We also remark that the same relation (5.1) gives the values of the critical velocities for the existence of nontrivial water waves (in the fluid without obstacles) bifurcating from the trivial parallel flow and with wave lengths $\lambda_k \approx 4x_0/k$ (see [11], chapter 71); by recalling that the length of the beam (i.e., of the cylinder's section as $\epsilon \to 0$, see § 1) is $L = 2x_0$, we get the values $2L/k$ for the wave lengths at the bifurcation points c_k. Thus, the condition $c \neq c_k$ in the assumptions of corollary 4.9 appears as a "nonresonance condition" between the length of the cylinder's section (in the limit $\epsilon \to 0$) and the gravitational wave bifurcating from the free parallel flow at the same velocity.

The results of the present work suggest the possibility of proving the solvability of the nonlinear problem, at least for subcritical velocities bounded away from the singular values c_k, by following the same strategy (hodograph transformation and implicit function theorem) adopted in [7], [8], for the supercritical case.

Appendix. In this appendix, we prove proposition 4.8 and discuss further properties of the functions v^s, v^c, and of the corresponding parameters λ^s, λ^c, defined in § 4. We start from the relations (4.5), (4.6) which we report below:

$$\lambda^s \sin(\nu_0 x_0) = C(\nu_0) \int_{-x_0}^{x_0} [v_y^s(x,0) - \nu \sin(\nu_0 x)] \sin(\nu_0 x)\, dx, \tag{A.1}$$

$$\lambda^c \cos(\nu_0 x_0) = C(\nu_0) \int_{-x_0}^{x_0} [v_y^c(x,0) - \nu \cos(\nu_0 x)] \cos(\nu_0 x)\, dx, \tag{A.2}$$

where

$$C(\nu_0) = \frac{\nu_0 \sinh(\nu_0 H)}{\sinh(\nu_0 H)\cosh(\nu_0 H) - \nu_0 H}. \tag{A.3}$$

By the definition of v^s, v^c and by Green's theorem (see the appendix of [2]), one can prove the following identities:

$$\int_{-x_0}^{x_0} v_y^s(x,0)\sin(\nu_0 x)dx = \int_{S_H} |\nabla v^s|^2 dxdy - \nu \int_F |v^s|^2 dx, \tag{A.4}$$

$$\int_{-x_0}^{x_0} v_y^c(x,0)\cos(\nu_0 x)dx = \int_{S_H} |\nabla v^c|^2 dxdy - \nu \int_F |v^c|^2 dx. \tag{A.5}$$

Then, we can write (A.1), (A.2) in the form:

$$\lambda^s \sin(\nu_0 x_0) = C(\nu_0)\Big\{\int_{S_H} |\nabla v^s|^2 dxdy - \nu \int_F |v^s|^2 dx - \nu \int_{-x_0}^{x_0} \sin^2(\nu_0 x)\, dx\Big\} \tag{A.6}$$

$$\lambda^c \cos(\nu_0 x_0) = C(\nu_0)\Big\{\int_{S_H} |\nabla v^c|^2 dxdy - \nu \int_F |v^c|^2 dx - \nu \int_{-x_0}^{x_0} \cos^2(\nu_0 x)\, dx\Big\}. \tag{A.7}$$

REMARK A.1. By the variational characterizations of v^s, v^c (see § 2), it follows that they are the minimum points of the functionals at the right hand sides of (A.6), (A.7) in the classes of the $H^1(S_H)$ functions satisfying the conditions (1.4), (2.4), and (1.2) with $\mathcal{F} = \sin(\nu_0 x)$ and $\mathcal{F} = \cos(\nu_0 x)$, respectively. Moreover, by the regularity of these data and recalling proposition 4.3, it can be shown that the relation (3.5) holds for v^s and v^c in the neighborhood of the points $(\pm x_0, 0)$. In particular, the functions v^s and v^c are bounded and Hölder continuous (in the closed strip $\overline{S_H}$) and the same is true for the traces $v^s(\pm x_0, \cdot)$, $v^c(\pm x_0, \cdot)$ in the interval $[-H, 0]$.

Let us decompose the integrals on S_H in the above expressions as the sum of two integrals, one extended to $Q_0 = (-x_0, x_0) \times (-H, 0)$, and the other to $S_H/\overline{Q_0}$. By recalling that both v^s and v^c are harmonic in the above regions, we can use Green's theorem to transform both integrals; taking account of the boundary conditions, of the asymptotic properties and of the symmetries, we get

$$\begin{aligned}\int_{S_H/Q_0} |\nabla v|^2 dxdy &= \int_F v_y(x,0)v(x,0)dx \\ -\int_{-H}^0 v_x(x_0^+, y)v(x_0, y)dy &+ \int_{-H}^0 v_x(-x_0^-, y)v(-x_0, y)dy \\ &= \nu \int_F |v(x,0)|^2 dx - 2\int_{-H}^0 v_x(x_0^+, y)v(x_0, y)dy, \end{aligned} \tag{A.8}$$

where in the above relation v denotes either v^s or v^c and $v_x(x_0^+, \cdot)$, $v_x(-x_0^-, \cdot)$ are the traces of their x-derivatives as $x \to x_0$ from the right and $x \to -x_0$ from the left. We recall that v^s, v^c are continuous in the closed strip by remark A.1.

In order to transform the integrals on Q_0, we set

$$v^s = w^s + z^s, \qquad v^c = w^c + z^c, \tag{A.9}$$

where

$$\begin{aligned} w^s(x,y) &= \frac{1}{\sinh(\nu_0 H)} \sin(\nu_0 x) \sinh[\nu_0(y+H)]; \\ w^c(x,y) &= \frac{1}{\sinh(\nu_0 H)} \cos(\nu_0 x) \sinh[\nu_0(y+H)]. \end{aligned}$$

The functions z^s, z^c are harmonic in Q_0 and vanish for $y = 0$ and $y = -H$; furthermore, they satisfy

$$z^s(\pm x_0, y) = v^s(\pm x_0, y) \mp \frac{\sin(\nu_0 x_0)}{\sinh(\nu_0 H)} \sinh[\nu_0(y+H)]. \tag{A.10}$$

$$z^c(\pm x_0, y) = v^c(\pm x_0, y) - \frac{\cos(\nu_0 x_0)}{\sinh(\nu_0 H)} \sinh[\nu_0(y+H)]. \tag{A.11}$$

Now, by explicit calculations as in the proof of proposition A.1 of [2], we have

$$\begin{aligned}\int_{Q_0} |\nabla v^s|^2 dxdy &= \int_{Q_0} |\nabla w^s|^2 dxdy + 2\int_{Q_0} \nabla w^s \nabla z^s dxdy + \int_{Q_0} |\nabla z^s|^2 dxdy \\ &= \nu \int_{-x_0}^{x_0} \sin^2(\nu_0 x)dx - \frac{\nu_0}{C(\nu_0)} \frac{\sin(2\nu_0 x_0)}{2\sinh(\nu_0 H)} + 2\int_{-H}^0 z_x^s(x_0^-, y) z^s(x_0, y)dy. \end{aligned} \tag{A.12}$$

$$\begin{aligned}\int_{Q_0} |\nabla v^c|^2 dxdy &= \int_{Q_0} |\nabla w^c|^2 dxdy + 2\int_{Q_0} \nabla w^c \nabla z^c dxdy + \int_{Q_0} |\nabla z^c|^2 dxdy \\ &= \nu \int_{-x_0}^{x_0} \cos^2(\nu_0 x)dx + \frac{\nu_0}{C(\nu_0)} \frac{\sin(2\nu_0 x_0)}{2\sinh(\nu_0 H)} + 2\int_{-H}^0 z_x^c(x_0^-, y) z^c(x_0, y)dy, \end{aligned} \tag{A.13}$$

where $z_x^s(x_0^-,\cdot)$ $z_x^c(x_0^-,\cdot)$ denote the traces of z_x^s, z_x^c as $x \to x_0$ from the left. By using (A.8), (A.12), (A.13) in the equations (A.6), (A.7), we find

$$\lambda^s \sin(\nu_0 x_0) + \nu_0 \frac{\sin(2\nu_0 x_0)}{2\sinh(\nu_0 H)}$$
$$= 2C(\nu_0)\Big[\int_{-H}^0 z_x^s(x_0^-,y) z^s(x_0,y)dy - \int_{-H}^0 v_x^s(x_0^+,y) v^s(x_0,y)dy\Big], \tag{A.14}$$

$$\lambda^c \cos(\nu_0 x_0) - \nu_0 \frac{\sin(2\nu_0 x_0)}{2\sinh(\nu_0 H)}$$
$$= 2C(\nu_0)\Big[\int_{-H}^0 z_x^c(x_0^-,y) z^c(x_0,y)dy - \int_{-H}^0 v_x^c(x_0^+,y) v^c(x_0,y)dy\Big]. \tag{A.15}$$

We now write the functionals at the right hand sides of (A.14), (A.15), in a form which is suitable for proving proposition 4.8. To this aim, by recalling (2.4) and the proof of proposition 4.1, we can write in the region $(x_0,+\infty)\times(-H,0)$ the following series expansions:

$$v^s(x,y) = \sin(\nu_0 x_0)\sum_{n=1}^\infty a_n^s e^{-\mu_n(x-x_0)}\sin[\mu_n(y+H)], \tag{A.16}$$

$$v^c(x,y) = \cos(\nu_0 x_0)\sum_{n=1}^\infty a_n^c e^{-\mu_n(x-x_0)}\sin[\mu_n(y+H)], \tag{A.17}$$

where the coefficients a_n^s, a_n^c are uniquely determined by the functions $v^s(x_0,\cdot)$, $v^c(x_0,\cdot)$; by the regularity properties of remark A.1 and by the relation

$$\mu_n = (n+\frac{1}{2})\frac{\pi}{H} + O(\frac{1}{n}),$$

which follows from (3.2), it is not difficult to check that the above coefficients satisfy the bound

$$\sum_{n=1}^\infty n^{2-\epsilon}|a_n|^2 < \infty, \tag{A.18}$$

for every $\epsilon > 0$, where a_n denotes either a_n^s or a_n^c. Then, the series (A.16), (A.17) are uniformly convergent in $[x_0,+\infty)\times[-H,0]$; by recalling that $v^s(x_0,0) = \sin(\nu_0 x_0)$ and $v^c(x_0,0) = \cos(\nu_0 x_0)$, we now get the condition

$$\sum_{n=1}^\infty a_n^s \sin(\mu_n H) = \sum_{n=1}^\infty a_n^c \sin(\mu_n H) = 1. \tag{A.19}$$

By (A.16), (A.17), we readily obtain

$$-2\int_{-H}^0 v_x^s(x_0^+,y) v^s(x_0,y)dy = \sin^2(\nu_0 x_0)\sum_{n=1}^\infty \gamma_n \mu_n |a_n^s|^2, \tag{A.20}$$

$$-2\int_{-H}^0 v_x^c(x_0^+,y) v^c(x_0,y)dy = \cos^2(\nu_0 x_0)\sum_{n=1}^\infty \gamma_n \mu_n |a_n^c|^2, \tag{A.21}$$

where

$$\gamma_n = H\big(1 - \frac{\cos^2(\mu_n H)}{\nu H}\big).$$

Since $\mu_n \sim (n+1/2)\pi/H$ and $0 < \gamma_n < H$, we have that the series at the right hand sides of (A.20), (A.21) are convergent by (A.18).

We want to transform in a similar way the first integrals in the square brackets of (A.14), (A.15). By (A.9)–(A.11), we can write in the domain Q_0 the series expansions

$$z^s(x,y) = \sin(\nu_0 x_0) \sum_{n=1}^{\infty} b_n^s \frac{\sinh(n\pi x/H)}{\sinh(n\pi x_0/H)} \sin(n\pi y/H), \tag{A.22}$$

$$z^c(x,y) = \cos(\nu_0 x_0) \sum_{n=1}^{\infty} b_n^c \frac{\cosh(n\pi x/H)}{\cosh(n\pi x_0/H)} \sin(n\pi y/H), \tag{A.23}$$

where the coefficients b_n^s, b_n^c, satisfy the bound (A.18). Hence, we obtain

$$2\int_{-H}^{0} z_x^s(x_0^-, y) z^s(x_0, y) dy = \sin^2(\nu_0 x_0) \sum_{n=1}^{\infty} n\pi \coth(n\pi x_0/H) |b_n^s|^2, \tag{A.24}$$

$$2\int_{-H}^{0} z_x^c(x_0^-, y) z^c(x_0, y) dy = \cos^2(\nu_0 x_0) \sum_{n=1}^{\infty} n\pi \tanh(n\pi x_0/H) |b_n^c|^2. \tag{A.25}$$

We finally observe that from (A.10), (A.11), it follows:

$$\sum_{n=1}^{\infty} b_n \sin(n\pi y/H) = \sum_{n=1}^{\infty} a_n \sin[\mu_n(y+H)] - \frac{\sinh[\nu_0(y+H)]}{\sinh(\nu_0 H)}, \tag{A.26}$$

where a_n, b_n stand for a_n^s, b_n^s or a_n^c, b_n^c. Hence, by the usual orthogonality relations we can write

$$b_n = \sum_{m=1}^{\infty} T_{nm} a_m + \mathcal{S}_n, \tag{A.27}$$

with uniquely determined coefficients T_{nm} and $\mathcal{S}_n$. It is not difficult to verify that the operator T defined by the first term at the right hand side of (A.27) is bounded in the Hilbert space l^2.

We are now prepared for the *proof of proposition 4.8*: Let us consider the Hilbert space of the sequences $\{a_n\}_{n=1}^{\infty}$ satisfying the bound (A.18) and let A be the closed subspace of the sequences satisfying (A.19). Let us define the following positive functionals:

$$J^s(\{a_n\}) = \sum_{n=1}^{\infty} (\gamma_n \mu_n |a_n|^2 + n\pi \coth(n\pi x_0/H) |b_n|^2) \tag{A.28}$$

$$J^c(\{a_n\}) = \sum_{n=1}^{\infty} (\gamma_n \mu_n |a_n|^2 + n\pi \tanh(n\pi x_0/H) |b_n|^2), \tag{A.29}$$

where the sequence $\{b_n\}$ is given by (A.27). We note that, by (A.26) and by the orthogonality of $\sinh[\nu_0(y+H)]$ to the subspace generated by the functions $\sin[\mu_n(y+H)]$, the coefficients b_n can not be all zero; hence, we have

$$J^s(\{a_n\}) > J^c(\{a_n\}) \tag{A.30}$$

for every $\{a_n\} \in A$.

Now, from the identities (A.20), (A.21) and (A.24), (A.25), we find that the right hand sides of (A.14),(A.15) are equal to $C(\nu_0)\sin^2(\nu_0 x_0) J^s(\{a_n^s\})$ and $C(\nu_0)\cos^2(\nu_0 x_0) J^c(\{a_n^c\})$, respectively; on the other hand, the functionals J^s and J^c are strictly convex, coercive

and lower semicontinuous (as limits of increasing sequences of continuous functionals) and therefore they assume their minimum values at unique points on the subspace A. By recalling remark A.1, the minimum points are necessarily the sequences $\{a_n^s\}$ and $\{a_n^c\}$, so that by (A.30) we get

$$J^s(\{a_n^s\}) > J^c(\{a_n^c\}). \tag{A.31}$$

Let us now go back to (A.14), (A.15). If $\nu_0 x_0 \neq n\pi$, $n = 1, 2, ...$, we can divide both terms of (A.14) by $\sin(\nu_0 x_0)$ and obtain:

$$\lambda^s + \nu_0 \frac{\cos(\nu_0 x_0)}{\sinh(\nu_0 H)} = C(\nu_0)\sin(\nu_0 x_0) J^s(\{a_n^s\}). \tag{A.32}$$

Similarly, if $\nu_0 x_0 \neq (n - 1/2)\pi$, $n = 1, 2, ...$, we get from (A.15)

$$\lambda^c - \nu_0 \frac{\sin(\nu_0 x_0)}{\sinh(\nu_0 H)} = C(\nu_0)\cos(\nu_0 x_0) J^c(\{a_n^c\}). \tag{A.33}$$

Now, if $\nu_0 x_0 \neq k\pi/2$, $k = 1, 2...$, Eq. (4.22) follows by (A.32), (A.33) with $K(\nu_0) = C(\nu_0)[J^c(\{a_n^c\}) - J^s(\{a_n^s\})]$. Then, proposition 4.8 is proved by the bound (A.31) and by recalling that (4.22) also holds for $\nu_0 x_0 = k\pi/2$, due to the equations (4.9), (4.10).

REMARK A.2. From (A.32), (A.33), we get in particular the relations:

$$(-1)^{n+1}\lambda^s((n - 1/2)\pi/x_0) > 0, \quad (-1)^n \lambda^c(n\pi/x_0) > 0, \quad n = 1, 2, ... \tag{A.34}$$

Moreover, it can be shown that the maps $\nu_0 \mapsto \lambda^s(\nu_0)$, $\nu_0 \mapsto \lambda^c(\nu_0)$ are real analytic functions in $(0, +\infty)$ ([2], corollary A.3). Then, from (4.9), (4.10) and (A.34), we find that $\lambda^s(\nu_0)$ and $\lambda^c(\nu_0)$ must vanish at some point in every interval $(n\pi/x_0, (n+1/2)\pi/x_0)$ and $((n - 1/2)\pi/x_0, n\pi/x_0)$, $(n = 1, 2, ...)$, respectively.

REFERENCES

[1] C. D. Pagani and D. Pierotti, The Neumann-Kelvin problem for a beam, *J. Math. Anal. Appl.* **240** (1999), 60–79.

[2] D. Pierotti, The subcritical motion of a surface-piercing cylinder: existence and regularity of waveless solutions of the linearized problem, *Advances in Differential Equations* **7** (2002), 385–418.

[3] C. D. Pagani and D. Pierotti, Exact solution of the wave resistance problem for a submerged cylinder. I. Linearized theory, *Arch. Rational Mech. Anal.* **149** (1999), 271–288.

[4] N. G. Kuznetsov and V. G. Maz'ya, *On unique solvability of the plane Neumann-Kelvin problem*, Math. USSR-Sb., **63** (1989), 425–446.

[5] N. G. Kuznetsov, On uniqueness and solvability in the linearized two-dimensional problem of a supercritical stream about a surface-piercing body, *Proc. Roy. Soc. London Ser. A*, **450** (1995), 233–253.

[6] C. D. Pagani and D. Pierotti, The Neumann-Kelvin problem in a bounded domain, *J. Math. Analysis and Appl.* **192** (1995), 41–62.

[7] C. D. Pagani and D. Pierotti, On solvability of the nonlinear wave resistance problem for a surface-piercing symmetric cylinder, *SIAM J. Math. Anal.* **32** (2000), 214–233.

[8] C. D. Pagani and D. Pierotti, The forward motion of an unsymmetric surface-piercing cylinder: the solvability of the nonlinear problem in the supercritical case, *Q. Jl Mecc. Appl. Math.* **54** (2000), 1–22.

[9] P. Grisvard, "Elliptic problems in nonsmooth domains." Pitman Adv. Publ. Program, London (1985).

[10] B. R. Vainberg and V. G. Maz'ya, *On the plane problem of the motion of a body immersed in a fluid*, Trans. Moscow Math. Soc., **28** (1973), 33–55.

[11] E. Zeidler, "Nonlinear functional analysis and its applications IV." Springer-Verlag, New York (1988).