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STABILITY OF CONSTANT EQUILIBRIUM STATE FOR DISSIPATIVE BALANCE LAWS SYSTEM WITH A CONVEX ENTROPY

By

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Abstract. For a one-dimensional system of dissipative balance laws endowed with a convex entropy, we prove, under natural assumptions, that a constant equilibrium state is asymptotically L^2-stable under a zero-mass initial disturbance. The technique is based on the construction of an appropriate Liapunov functional involving the entropy and a so-called compensation term.

1. Introduction. Recently, nonequilibrium theories and, in particular, the Extended Thermodynamics [16] have generated a new interest in quasi-linear hyperbolic systems of balance laws with dissipation due to the presence of production terms (systems with relaxation). On this subject, it is very important to find connections between properties of the full system and the associated subsystem obtained when certain parameters (relaxation coefficients) are just equal to zero. Mathematical examples on this topic were developed in the linear case by Whitham [21] and in the nonlinear case by Liu [14] and by Chen, Levermore, and Liu [5]. The requirement that the system of balance laws satisfies an entropy principle with a convex entropy density gives several strong restrictions. In fact, as is well known, starting from the observation of Godunov [9], it was shown that the entire system of balance laws can be put in a very special hyperbolic symmetric system providing the introduction of *main field* variables [2], [17]. As was observed by Boillat and Ruggeri [3], the main field also permits us to recognize that these nonequilibrium systems have a structure of *nesting theories.* In fact, it is possible to define the *principal subsystems* that are obtained by *freezing* some components of the main field that have the properties that preserve the existence of a convex entropy law and the spectrum of the

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characteristic eigenvalues is contained into the one of the full system (sub-characteristic conditions). A particular subsystem is the equilibrium one.

Our goal in this paper is to prove that, under natural assumptions, a constant equilibrium state of such balance laws is asymptotically L^2-stable. Though the technique employed here may look rather classical, involving an "energy" (actually entropy) estimate, plus a compensation term as introduced by Kawashima et al. [13] for other purposes, it has the nice feature that it applies to weak entropy solutions. It is therefore valid in the presence of shock waves. For some natural reason, due to the finite propagation velocity of the support of a solution, it is natural to assume that the total mass of the conserved components of the unknown vanish. Under this condition, we find a $t^{-1/2}$ decay rate of the L^2-norm of the solution, though the decay could not be better than $t^{-1/4}$ in general, that is, when a non-zero mass is present at initial time. Though we were able to get rid of this zero-mass assumption in the linear case, to the price of the loss of the decay rate, we did not succeed in overcoming this restriction in the nonlinear case.

The plan is as follows. The next section recalls a few important facts about dissipative balance laws endowed with a convex entropy. In particular, we emphasize the solution of equilibrium subsystem. Then we turn to the analysis of the linear case, where we construct an appropriate quadratic Liapunov function whose dissipation rate is positive definite. In the last section, we deal with the more involved nonlinear case, where a few additional ideas are needed. Our main result is Theorem 8.

We should mention here a companion paper [19], where the L^2-stability is proved for the Jin-Xin relation model (see [11]) in the presence of a positively invariant domain. There, the linearity of the principal part allows a treatment by compensated compactness, which yields a similar but weaker result, without the zero-mass assumption.

At last, we must emphasize our smallness hypothesis, which constrained not only the data but the solution itself. We feel that this is reasonable, especially in the light of the global existence of smooth solutions for small data, obtained recently by Hanouzet and Natalini [10].

2. Balance laws, systems, entropy, and generators. Let us consider a general hyperbolic system of N balance laws:

$$\partial_\alpha F^\alpha(u) = F(u), \tag{1}$$

where the *densities* F^o, the *fluxes* F^i and the *productions* F are $\mathbb{R}^N$-column vectors depending on the space variables x^i $(i = 1, 2, 3)$ and the time $t = x^o$, $(\alpha = 0, 1, 2, 3; \partial_\alpha = \partial/\partial x^\alpha)$ through the field $u \equiv u(x^\alpha) \in \mathbb{R}^N$.

Now we suppose, as is usual in the applications, that the system (1) satisfies an entropy principle; i.e., there exists an entropy density $h = h^o(u)$ and an entropy flux $h^i(u)$ such that for every solution of (1) we have a nonpositive-entropy production[1] $\Sigma(u)$:

$$\partial_\alpha h^\alpha = \Sigma \leq 0. \tag{2}$$

[1]In thermodynamics, $-h^0$, $-h^i$, and $-\Sigma$ have the physical meaning of entropy density, entropy flux, and entropy production, respectively. But, by mathematical tradition, it is common to use the word convex entropy density instead of concave entropy density and improperly call entropy the negative entropy.

The compatibility between (1) and (2) implies the existence of a *main field* u' of Lagrange multipliers such that

$$\partial_\alpha h^\alpha - \Sigma \equiv u' \cdot (\partial_\alpha F^\alpha - F). \tag{3}$$

As a consequence of this identity, we have

$$dh^\alpha = u' \cdot dF^\alpha, \quad \Sigma = u' \cdot F \leq 0. \tag{4}$$

As it has been well known since the pioneering papers of Godunov [9] and Friedrichs and Lax [8], it is possible to show [2, 17] that, because of $(4)_1$, there exist four potentials h'^α:

$$h'^\alpha = u' \cdot F^\alpha - h^\alpha, \tag{5}$$

such that

$$F^\alpha = \frac{\partial h'^\alpha}{\partial u'}. \tag{6}$$

It follows that, upon selecting the main field as the field variables, the original system (1) can be written with Hessian matrices in the symmetric form

$$\partial_\alpha \left(\frac{\partial h'^\alpha}{\partial u'} \right) = F \Leftrightarrow \frac{\partial^2 h'^\alpha}{\partial u' \partial u'} \partial_\alpha u' = F \tag{7}$$

provided that $h = h^o$ is a convex function of $u \equiv F^o$ (or, equivalently, the Legendre transform $h' = h'^o$ is a convex function of the dual field u'). This form is called *canonical* by Dafermos [6].

3. Principal subsystems. We split the main field $u' \in \mathbb{R}^N$ into two parts: $u' \equiv (v', w'), v' \in \mathbb{R}^M, w' \in \mathbb{R}^{N-M}$ $(0 < M < N)$; and the system (7) with $F \equiv (f, \pi)$, reads

$$\partial_\alpha \left(\frac{\partial h'^\alpha}{\partial v'} \right) = f(v', w'), \tag{8}$$

$$\partial_\alpha \left(\frac{\partial h'^\alpha}{\partial w'} \right) = \pi(v', w'). \tag{9}$$

Given some assigned value $w'_*(x^\alpha)$ of w' (in the usual case $w'_* =$ const.), Boillat and Ruggeri [3] call a *principal subsystem* of (8) and (9) the system

$$\partial_\alpha \left(\frac{\partial}{\partial v'} h'^\alpha(v', w'_*) \right) = f(v', w'_*). \tag{10}$$

By construction, the solutions of the principal subsystems satisfy also a supplementary entropy law with convex density and the spectrum of the characteristic eigenvalues is included in the spectrum of the full system (sub-characteristic conditions) [3].

4. Equilibrium subsystem. In examples from physics involving nonequilibrium processes such as extended thermodynamics [16], $M < N$ equations of (1) represent *conservation laws* while the remaining ones contain productions responsible for the dissipative mechanism. In this case, $f = 0$ in (8). Recalling that a *thermodynamical equilibrium state* is such that the production of the entropy Σ in (2) vanishes and attains its maximum value:

$$\Sigma|_E = 0, \quad \left(\frac{\partial \Sigma}{\partial u'} \right)_E = 0, \quad \left(\frac{\partial^2 \Sigma}{\partial u' \partial u'} \right)_E \text{ is negative semi-definite,} \tag{11}$$

it is easy to prove [4]:

THEOREM 1. In an equilibrium state, under the assumption of dissipative productions, i.e., if

$$D = \frac{1}{2}\left\{\frac{\partial \pi}{\partial w'} + \left(\frac{\partial \pi}{\partial w'}\right)^T\right\}\Bigg|_E \quad \text{is negative definite,} \tag{12}$$

then the production vanishes and the main field components vanish except for the first M ones. Thus

$$\pi|_E = 0, \quad w'|_E = 0. \tag{13}$$

These results reveal another privilege of the main field. In fact, while the field of densities u are in general all different from zero in equilibrium, the components of the main field are all zero except for the first M ones. Therefore the infinite equilibrium states are the ones belonging to the M-dimensional manifold

$$w'(v, w) = 0 \tag{14}$$

as solution of the $N - M$ equations (14) (we recall the global univalence between w and w' from the concavity assumption), it is possible to write $w|_E$ as function of v. The principal system of M conservation law with $w = w|_E$ is the *equilibrium principal subsystem* associated to the system (8) and (9).

Another important characteristic property of the equilibrium state is put in evidence from the following theorem:

THEOREM 2. In equilibrium the entropy density h is minimal; i.e.,

$$h > h|_E \quad \forall u \neq u|_E,$$

where

$$h|_E = h(v, w|_E(v)).$$

The proof follows immediately, taking into account the convexity of h:

$$W = h|_E - h + u'|_E \cdot (u - u|_E) < 0 \quad \forall u \neq u|_E, \tag{15}$$

and therefore choosing

$$u \equiv \begin{pmatrix} v \\ w \end{pmatrix}, u|_E \equiv \begin{pmatrix} v \\ w|_E(v) \end{pmatrix}. \tag{16}$$

The last term in (15) is null due to the fact that $u'|_E$ and $u - u|_E$ are orthogonal (see (16) and $(13)_2$) and therefore

$$W = h|_E - h < 0 \quad \forall u \neq u|_E. \tag{17}$$

In this manner we have confirmed that the entropy density reaches its minimum value in the equilibrium state.

We would stress the fact that these results are a consequence not only of the convexity of the entropy density but also of the dissipation condition (12). In the mathematical literature there exist several definitions of dissipation. The one most used is due to Dafermos [7].

Now we consider the one-dimensional case and, taking into account (13), we assume, without loss of generality, that $\pi = -gw'$; i.e.:

$$\begin{cases} \partial_t h'_{v'} + \partial_x k'_{v'} = 0 \\ \partial_t h_{w'} + \partial_x k_{w'} = -g(v', w')w', \end{cases}$$

with $D^2 h' > 0$ and $\Sigma := {}^t w' g(v', w')w'$.

5. The linear problem. In a first instance, we restrict to the linear case:

- h' is a positive definite quadratic form,
- k' is a quadratic form,
- g is a constant matrix.

LEMMA 3. Without loss of generality, we may suppose that $h = \frac{1}{2}(|v'|^2 + |w'|^2)$. Therefore $v = v'; w = w'$, and the system rewrites

$$\partial_t v + \partial_x (K_0 v + K_1 w) = 0,$$
$$\partial_t w + \partial_x (K_1^T v + K_2 w) = -Gw,$$

where K_o, K_2 are symmetric, and ${}^t wGw > 0$ for $w \neq 0$.

Proof. We have

$$h' = \frac{1}{2} {}^t v' H_0 v' + {}^t v' H_1 w' + \frac{1}{2} {}^t w' H_2 w'$$

and

$$h'_{v'} = H_0 v' + H_1 w'.$$

Since $H_0 > 0$, we introduce the square root $h_0 = \sqrt{H_0}$ and define $y := h_0 v' + h_0^{-1} H_1 w' = h_0^{-1} h'_{v'}$. Then we have

$$h' = \frac{1}{2}|y|^2 + \frac{1}{2} {}^t w' (H_2 - H_1^T H_0^{-1} H_1) w'.$$

By Schur's complement theorem (see, for instance, [20]), $H_2 - H_1^T H_0^{-1} H_1$ is positive. Let h_1 be its square root, and define $z := h_1 w'$. Then

$$h = \frac{1}{2}(|y|^2 + |z|^2).$$

In variables w, z, the system still writes

$$\partial_t y + \partial_x (\text{linear flux}) = 0,$$
$$\partial_t z + \partial_x (\text{linear flux}) = -\tilde{g} z.$$

Lastly, since h is an entropy of the problem, the fluxes must be given by a potential, obviously a quadratic one. The general form of this potential is

$$\frac{1}{2} {}^t y K_0 y + {}^t y K_1 z + \frac{1}{2} {}^t z K_2 z.$$

□

5.1. *A necessary condition for stability.* From now on, we consider a linear system in its canonical form, as given in Lemma 3. We wish that finite energy data give rise to asymptotic strong stability, that is

$$\lim_{t\to\infty}(\|v(t)\|^2_{L^2}+\|w(t)\|^2_{L^2})=0.$$

LEMMA 4. Strong asymptotic stability needs that $\ker(K_1^T)$ (which is $R(K_1)^\perp$) does not contain any eigenvector of K_0.

We shall call this property *genuine coupling*, and we shall use the acronym (GC). Our motivation is that under (GC), a slight disturbance on the component w always has an influence on the full component v. This property closely resembles Kawashima's condition for hyperbolic-parabolic systems [12] and thus is often called *Kawashima's condition.*

Proof. Let $e\in\ker(K_1^T)$ satisfy

$$K_0e=\lambda e.$$

Let $a:\mathbb{R}\to\mathbb{R}$ be smooth and compactly supported. Then

$$w\equiv 0 \text{ and } v(x,t):=a(x-\lambda t)e$$

is a solution of finite energy, which does not decay to zero, except if $e=0$. □

We now show by an energy method that (GC) is sufficient. For this, we assume that the domain is either $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$ (the latter case concerns periodic data) and that

$$\int v_0(x)dx=0. \tag{18}$$

Obviously, v_0 is given in $L^1\cap L^2$ and w_0 in L^2. Condition (18) is restrictive only if the domain is $\mathbb{R}$, since in the periodic case we expect the stability towards the mean value of v_0.

From $\partial_t v+\partial_x(K_0v+K_1w)=0$, we may introduce a potential p by

$$p_x=v,\quad p_t=-(K_0v+K_1w).$$

We assume at last that $p_0\in L^2$. As we shall see, we have $p(t)\in L^2$ for every t. Note that the properties $v_0\in L^1, p_0\in L^2$ follow from $v_0\in L^2$ in the periodic case.

We now use Liapunov functions. The first one is the entropy

$$\frac{d}{dt}\int\eta(v,w)dx+\int {}^t wGw dx=0, \tag{19}$$

where $\eta=h=\frac{1}{2}(|y|^2+|w|^2)$. Next, we choose

$$L(v,w,p):=\frac{1}{2}(|v|^2+|w|^2)-\varepsilon\,{}^tpK_1G^{-1}w-\frac{\varepsilon}{2}{}^tpAv+\frac{\varepsilon}{2}|p|^2,$$

with A skew-symmetric, to be chosen. Note that the symmetric part in A is meaningless, since ${}^tpSv$ equals $\partial_x(\frac{1}{2}{}^tpSp)$ for a symmetric matrix S, and this term may be absorbed by the entropy flux. This is why there is no loss of generality in considering a skew-symmetric A. Using the system, we obtain

$$\begin{aligned}\partial_t L + \partial_x M = &-{}^t wGw + \varepsilon {}^t pK_1 w \\ &+ \varepsilon {}^t(K_0 v + K_1 w)\left(K_1 G^{-1} w + \frac{1}{2}Av - p\right) \\ &- \varepsilon {}^t vK_1 G^{-1}(K_1^T v + K_2 w) - \frac{1}{2}\varepsilon {}^t vA(K_0 v + K_1 w).\end{aligned}$$

The terms ${}^t pK_1 w$ cancel, while ${}^t(K_0 v)p$ may be incorporated in $\partial_x M$ since K_0 is symmetric. Hence there remains

$$\partial_t L + \partial_x \widetilde{M} + Q(v,w) = 0,$$

where

$$\begin{aligned}Q = {}^t wGw &- \varepsilon {}^t(K_0 v + K_1 w)\left(K_1 G^{-1} w + \frac{1}{2}Av\right) \\ &+ \varepsilon {}^t vK_1 G^{-1}(K_1^T v + K_2 w) + \frac{\varepsilon}{2} {}^t vA(K_0 v + K_1 w).\end{aligned}$$

We shall apply several times the following obvious result.

LEMMA 5. Let q_0, q_1 be two quadratic forms on $\mathbb{R}^d$, with the properties that q_0 is positive semi-definite and that the restriction of q_1 to $\ker q_0$ is definite positive. Then the quadratic form $q_0 + \varepsilon q_1$ is definite positive for every small enough $\varepsilon > 0$.

With this lemma, $L(v,w,p)$ is positive definite whenever $\varepsilon > 0$ is small enough. Likewise, $Q(v,w)$ will be definite positive for small enough $\varepsilon > 0$, provided the quadratic form

$$q(v) := -{}^t vK_0 Av + {}^t vK_1 G^{-1} K_1^T v$$

is positive definite.

We now invoke Theorem 9 in the Appendix. There exists a skew-symmetric $\widehat{A}$ such that $\widehat{A}K_0 - K_0\widehat{A}$ be positive definite on $\ker(K_1^T)$, because this subspace does not contain any eigenvector of K_0. Since besides

$$v \to {}^t vK_1 G^{-1} K_1^T v$$

is nonnegative, and is zero only on $\ker(K_1^T)$, $A = \alpha\widehat{A}$ is convenient for $\alpha > 0$ small enough, using again Lemma 5. Thus a positive definite q exists, and therefore it is possible to have both $L(v,w,p)$ and $Q(v,w)$ positive definite.

Integrating the identity $\partial_t L + \partial_x \widetilde{M} + Q = 0$, we therefore have

$$\frac{d}{dt}\int L(v,w,p)dx + \int Q(v,w)dx = 0. \tag{20}$$

Since L is positive definite, as well as Q, this implies that $t \to \int Q(v,w)dx$ is integrable on $\mathbb{R}^+$. Therefore $t \to \int \eta(v,w)dx$ is integrable on $\mathbb{R}^+$. Since it is also nonincreasing (from (19)), this implies that $\int \eta(v,w)dx \le cst \cdot t^{-1}$, and therefore

$$\lim_{t\to\infty}(v(t), w(t)) = (0,0) \text{ in } L^2(\mathbb{R}).$$

More precisely, with $Q \geq \omega\eta$ and $\omega > 0$, we have

$$\int \eta(v(t), w(t))dx \leq \frac{1}{\omega t} \int L(v_0, w_0, p_0)dx. \tag{21}$$

When the domain is $\mathbb{R}$, we now consider the general case, where we only assume that $v_0, w_0 \in L^2(\mathbb{R})$, but neither $v_0 \in L^1$ nor $\int v_0 dx = 0$. Since the subspace

$$X = \left\{ y \in L^1 \cap L^2; \int y dx = 0 \right\}$$

is dense in $L^2(\mathbb{R})$, there exists a sequence $(v_0^m)_{m \in N}$ in X, which tends to v_0 in $L^2(\mathbb{R})$. Let us also take $w_0^m = w_0$. From the above analysis, we know that the corresponding solutions (v^m, w^m) of the Cauchy problem satisfy

$$\lim_{t \to +\infty} \int \eta(v^m(t), w^m(t))dx = 0.$$

On the other hand, because the problem is linear, we also know that

$$\int \eta(v^m(t) - v(t), w^m(t) - w(t))dx \leq \int \eta(v_0^m - v_0, 0)dx \xrightarrow{m \to \infty} 0.$$

Now, let ε be a positive number, and let us choose m such that $\|v_0^m - v_0\|_2 < \frac{\varepsilon}{2}$. There holds

$$\|(v(t) - v^m(t), w(t) - w^m(t))\|_2 \leq \frac{\varepsilon}{2}.$$

From above, there is a T such that

$$t > T \Rightarrow \|(v^m(t), w^m(t))\|_2 \leq \frac{\varepsilon}{2}.$$

We conclude that

$$t > T \Rightarrow \|(v(t), w(t))\|_2 \leq \varepsilon.$$

We have thus proved:

THEOREM 6. Under condition (GC), the constant equilibria are strongly asymptotically stable:

- if $(v_0, w_0) \in L^2(\mathbb{R})$, then

$$(v(t), w(t)) \xrightarrow{t \to +\infty} (0, 0) \text{ in } L^2(\mathbb{R}),$$

- if $(v_0, w_0) \in L^2(\mathbb{R}/\mathbb{Z})$, then

$$(v(t), w(t)) \xrightarrow{t \to +\infty} (\overline{v}, 0) \text{ in } L^2(\mathbb{R}/\mathbb{Z}),$$

where $\overline{v}$ stands for the average of v_0 over a period.

REMARK 1. Condition (GC) also reads as follows: the matrix

$$K = \begin{pmatrix} K_0 & K_1 \\ K_1^T & K_2 \end{pmatrix}$$

does not have any eigenvector in block form $(x, 0)^T$.

Obviously, a given system is usually not in the canonical form (that is with $v' = v, w' = w$). To check (GC), we only have to verify that travelling waves are trivial. Here is an example, taken from nonequilibrium gas dynamics in Lagrangian variables:

$$\begin{cases} \tau_t - v_x = 0, \\ v_t + p_x = 0, \\ p_t + E^2 v_x = -a\tau - p, \end{cases}$$

where $a > 0$ (hyperbolicity of the subsystem) and $E^2 > a$ (dissipativeness). Travelling waves which depend on $x - ct$ satisfy

$$c\tau + v = 0, \; p = cv, \; E^2 v = cp, \; p + a\tau = 0.$$

The matrix of this system,

$$M := \begin{pmatrix} c & 1 & 0 \\ 0 & c & -1 \\ 0 & E^2 & -c \\ a & 0 & 1 \end{pmatrix}$$

always has rank 3, since 3×3 minors contain $a(E^2 - c^2)$ and $c(E^2 - a)$, which cannot vanish simultaneously. Hence we obtain $\tau = v = p = 0$.

WARNING. The decay rate $\int \eta(v(t), w(t))dx = O(t^{-1})$ shown above is always true for periodic data. However, when $v_0, w_0 \in L^2(\mathbb{R})$, it strongly relies on the assumption that $v_0 \in L^1$ and $\int v_0 dx = 0$. As a matter of fact, let us assume that v_0, w_0 are compactly supported, say in $[-l, l]$, and that $m := \int v_0 dx$ is nonzero. Then, from mass conservation, we have

$$|m| = \left| \int v(t)dx \right| \leq \|v(t)\|_1.$$

Also, $v(t)$ is supported in $[-l, -\Lambda t, l + \Lambda t]$, where Λ is the maximal wave velocity in the full system. Using Cauchy-Schwarz, we therefore have

$$\|v(t)\|_2 \geq \frac{|m|}{\sqrt{2(l + \Lambda t)}}. \tag{22}$$

This shows that, though w' always belongs to $L^2(\mathbb{R}_x \times \mathbb{R}_t^+)$, this is not the case for v in general.

Mind also that the lower bound (22) is not sharp, in general. An asymptotic analysis shows that v behaves as the solution of

$$\partial_t v + \partial_x(K_0 v + K_1 w) = 0, \quad \partial_x(K_1^T v) = -Gw,$$

which is a diffusion equation. If $v_0 \in L^1(\mathbb{R})$ and $m \neq 0$, then $v(x, t)$ behaves as the sum of diffusion waves

$$\frac{1}{\sqrt{t}} \phi \left(\frac{x - \lambda t}{\sqrt{t}} \right) \quad (\lambda \in SpK_0).$$

Thus $\|v(t)\|_2$ is of order $t^{-1/4}$, while $\|w(t)\|_2$ is of order $t^{-3/4}$.

We remark that in the diffusion equation, written as

$$\partial_t v + K_0 \partial_x v = D\partial_x^2 v, \quad D := K_1 G^{-1} K_1^T,$$

the diffusion matrix D need not be positive definite, but only satisfies

$$ {}^tXDX \geq cst|K_1^TX|^2. $$

Condition (GC) says that it is positive on the eigenvectors of K_0. Thus, in transient times, the hyperbolic part $\partial_t v + K_0\partial_x v$ separates the modes (this is a "polarization" process). Then each mode $a(x,t)r$ (with $K_0 r = \lambda r$) satisfies approximately the scalar diffusion equation

$$ \partial_t a + \lambda\partial_x a = d\partial_x^2 a, $$

with

$$ d := {}^trK_1G^{-1}K_1^Tr/|r|^2, $$

which is a strictly positive number since r does not belong to $\ker K_1^T$. More information about incompletely parabolic diffusion equations may be found in the Memoir by Liu and Zeng [15].

5.1.1. *Dependence on the relaxation time* τ. Taking into account a relaxation time $\tau > 0$, the system reads as

$$ \begin{cases} \partial_t v + \partial_x(K_0 v + K_1 w) = 0, \\ \partial_t w + \partial_x(K_1^T v + K_2 w) = -\frac{1}{\tau}Gw. \end{cases} $$

The previous calculations are valid with

$$ \eta^\tau(v,w) = \eta(v,w) = \frac{1}{2}(|v|^2 + |w|^2), $$

$$ \frac{d}{dt}\int \eta(v,w)dx + \frac{1}{\tau}\int {}^twGwdx = 0, $$

and

$$ L^\tau(v,w,p) = L\left(v,w,\frac{1}{\tau}p\right), \quad Q^\tau(v,w) = \frac{1}{\tau}Q(v,w). $$

Hence, when $v_0 \in L^1 \cap L^2$ and $\int v_0 dx = 0$, the estimate becomes

$$ \int \eta(v(t),w(t))dx \leq cst\frac{\tau}{t}\int L\left(v_0,w_0,\frac{1}{\tau}p_0\right)dx, $$

which gives smallness only for times t beyond τ^{-1}.

6. The nonlinear case. Besides natural assumptions:

- strictly convex h' (or h as well),
- strict dissipation ($\sigma \geq \omega|w'|^2$ with $\omega > 0$),

we assume that the weak admissible solution u that we deal with takes values in a small enough neighborhood $\mathcal{U}$ of the origin (without loss of generality, $u = 0$ is an equilibrium of the system). We prove in this section that, if the mass at initial time vanishes, then the solution converges to the equilibrium in L^2 as time goes to infinity.

COMMENT. Such an existence result, global in time, has not been yet proved in full generality. It could be attacked using Glimm's scheme or front tracking. We note, however, that Amadori and Guerra have some intermediate result [1] when the dissipation is stronger. For small and smooth data, estimates of derivatives up to the order 2 (order $1 + d/2$ in space dimension d), in the same spirit as in the present paper, yield a global smooth and small solution [10]. Notice also that the smallness of weak solutions for small data may be guaranteed by the presence of a small positively invariant domain. This kind of maximum principle has been analysed in [18], in the context of the Jin-Xin relaxation of a system of conservation laws:

$$\partial_t v + \partial_x w = 0,\ \partial_t w + a^2 \partial_x v = f(v) - w. \tag{23}$$

When the equilibrium subsystem $\partial_t v + \partial_x f(u) = 0$ admits a convex compact positively invariant domain K, and when

$$a > \sup_{v \in K} \rho(df(v)),$$

(sub-characteristic condition), then (23) admits a compact positively invariant domain. Under the same assumptions, it can be written in the canonical form (1) and is compatible with an equality of the form (2). For such a system, the maximal principle guarantees the existence and uniqueness of a global solution, because the principal part (the first order terms in the system) is linear.

THE CONTEXT. The system is

$$\partial_t h'_{u'} + \partial_x k'_{u'} = \begin{pmatrix} 0 \\ -g(u')w' \end{pmatrix}, u' = \begin{pmatrix} v' \\ w' \end{pmatrix}. \tag{24}$$

By definition

$$u = \begin{pmatrix} v \\ w \end{pmatrix} = h'_{u'}.$$

By assumption, h' is strictly convex, and its Legendre transform h is such that

$$h(u) \geq \omega |u|^2$$

in $\mathcal{U}$. Here and below, ω denotes a positive constant. Additionally, a weak admissible solution satisfies[2]

$$\partial_t h(u) + \partial_x k(u) + \Sigma(u) \leq 0, \tag{25}$$

where $\Sigma(u) := {}^t w' g(u') w' \geq \omega |w'|^2$ for small u', and $k(u) := u' \cdot k'_{u'} - k'$. Inequality (25) is not sufficient to prove that $\|u(t)\|_2 \to 0$ as $t \to +\infty$, because Σ does not dominate h. To go further, we assume that the initial data u_0 has compact support (therefore L^2 implies L^1) and that v_0 has zero-mass:

$$\int v_0(x) dx = 0. \tag{26}$$

Therefore the primitive

$$p_0(x) = \int_{-\infty}^{x} v_0(y) dy$$

has compact support and belongs to H^1.

[2]The inequality $\leq$, instead of equality, takes into account the possible shock waves.

Thanks to hyperbolicity, the support of $u(t)$ extends with finite velocity. Because u stays in the compact set $\mathcal{U}$, the waves velocities remain bounded. Thus there exists a constant Λ such that

$$\operatorname{Supp} u(t) \subset [-l-\Lambda t, l+\Lambda t]. \tag{27}$$

Also, conservation of mass implies that

$$\int v(x,t)dx \equiv 0,$$

so that

$$p(x,t) := \int_{-\infty}^{x} v(y,t)dy$$

still has compact support in space. On the other hand, we have[3]

$$p(x,0) = p_0(x),\ p_x = v,\ p_t = -k'_{v'}.$$

A Liapunov function. We now look for a Liapunov function of the following form (we follow the analysis of the linear case):

$$L_\varepsilon(u,p) = h(u) + \varepsilon\left\{\frac{1}{2}|p|^2 - \frac{1}{2}{}^tpAv - {}^tpBw\right\}, \tag{28}$$

where A is skew-symmetric. Both matrices A and B are constant.

What is really important in formula (28) is that as a function of u, $L_\varepsilon(\cdot,p)$ is a linear combination of conserved (v) or balanced (w) quantities, and of entropy h. Besides, the nonlinear dependence on p is harmless since p is Lipschitz. Thus we can establish an inequality for L_ε, using (24) and (25). This reads

$$\partial_t L_\varepsilon + \partial_x M_\varepsilon + Q_\varepsilon \le 0,$$

where

$$M_\varepsilon = k(u) - \varepsilon\left\{\frac{1}{2}{}^tpAk'_{v'} + {}^tpBk'_{w'}\right\}$$

and

$$Q_\varepsilon = \Sigma(u) + \varepsilon\{{}^tp(k'_{v'} - Bgw') + {}^tvBk'_{w'} - {}^twB^Tk'_{v'} - {}^tvAk'_{v'}\}.$$

As in the linear case, there is not a loss of generality to assume that $h(u) \sim \frac{1}{2}|u|^2$ near the origin, that is $u' = u + O(|u|^2)$, or

$$u = u' + O(|u'|^2).$$

With this in mind, we see that the leading terms in ${}^tp(k'_{v'} - Bgw')$ are quadratic:

$${}^tp(k'_{v'v'}(0)v + k'_{v'w'}(0)w - Bg(0)w).$$

Thus, our first choice will be

$$B = k'_{v'w'}(0)g(0)^{-1}. \tag{29}$$

The term

$$\varepsilon\, {}^tpk'_{v'v'}(0)v = \frac{\varepsilon}{2}\partial_x({}^tpk'_{v'v'}(o)p)$$

is as usual incorporated to $\partial_x M_\varepsilon$. Thus we can write instead

$$\partial_t L_\varepsilon + \partial_x M_\varepsilon + Q_\varepsilon \le \varepsilon\, {}^tp \cdot N(u),$$

[3]Here we have normalized $u = 0 \Leftrightarrow u' = 0, k' = O(|u'|^2)$.

where $N(u) = O(|u|^2)$ and

$$Q_\varepsilon = \Sigma(u) + \varepsilon\{{}^t vBk'_{w'} - {}^t wB^T k'_{v'} - {}^t vAk'_{v'}\}.$$

Our next goal is to choose $\varepsilon > 0$ in such a way that

1. $L_\varepsilon(u,p) \geq \omega(|u|^2 + |p|^2)$,
2. $Q_\varepsilon(u) \geq \omega|u|^2$,

in $\mathcal{U} \times \mathbb{R}^M$ and $\mathcal{U}$ respectively.

To do so, we use an argument analogous to Lemma 5. Actually, since both L and Q have quadratic leading order terms at the origin, they will satisfy the desired inequalities provided their quadratic parts do, and $\mathcal{U}$ is chosen small enough. The first inequality is achieved for small $\varepsilon > 0$, since $h(u) \geq \omega|u|^2$ and since the corrector

$$\frac{1}{2}|p|^2 - \frac{1}{2}{}^t pAv - {}^t pBw,$$

when restricted to $u = 0$, is $|p|^2$.

We argue similarly for Q_ε. Since $u \mapsto (v, w')$ is a change of variable, the second inequality will be satisfied by small $\varepsilon > 0$, provided that the restriction $R(v)$ of the expression ${}^t vBk'_{w'} - {}^t wB^T k'_{v'} - {}^t vAk'_{v'}$ to the equilibrium manifold $\{w' = 0\}$ satisfies $R(v) \geq \omega|v|^2$ for small v. However, this amounts to showing that $R_0(v) \geq \omega|v|^2$, where R_0 is the Hessian of R at the origin.

But considering R_0 means that we are back to the study of a linear problem. Thus, we make our last, natural, hypothesis of "genuine coupling": *the kernel of $k'_{v'w'}(0)$ does not contain any eigenvector of $k'_{v'v'}(0)$.* Under this assumption, we know (see the linear case analysis) that a skew-symmetric A exists, such that R_0 is positive definite. Therefore we have proved

PROPOSITION 7. Under natural hypotheses

- strongly convex entropy,
- strict dissipativeness,
- genuine coupling,

there exist a compact neighborhood $\mathcal{U}$ of the origin and smooth functions $L(u,p)$, $M(u,p)$, $Q(u)$, $N(u)$ such that

1. every admissible solution with compact support and zero-mass satisfies

$$\partial_t L(u,p) + \partial_x M(u,p) + Q(u) \leq p \cdot N(u), \tag{30}$$

2. for $u \in \mathcal{U}$ and $p \in \mathbb{R}^M$, we have
   - $\omega(|u|^2 + |p|^2) \leq L(u,p) \leq \Omega(|u|^2 + |p|^2)$,
   - $|M(u,p)| \leq \Omega(|u|^2 + |p|^2)$,
   - $\omega|u|^2 \leq Q(u) \leq \Omega|u|^2$,
   - $|N(u)| \leq \Omega|u|^2$,

   for some positive constants $0 < \omega < \Omega < +\infty$.

UNIFORM ESTIMATE. Let us assume now that a weak solution $u(x,t)$ takes values in $\mathcal{U}$ almost everywhere. We integrate (30) in the space variable and obtain

$$\frac{d}{dt}\int L dx + \int Q(u) dx \leq \|p\|_\infty \int N(u) dx \leq \|u\|_1 \int N(u) dx. \tag{31}$$

Using Cauchy-Schwarz inequality, there comes

$$\|u\|_1 \le \|u\|_2 \sqrt{2(l+\Lambda t)}.$$

Considering now the equivalence of the functions $\|u\|^2, Q(u)$, and $h(u)$ in $\mathcal{U}$, we derive from (31) the inequality

$$\frac{d}{dt}\int L(u,p)dx + \omega \int h(u)dx \le \Omega\sqrt{l+\Lambda t}\left(\int h(u)dx\right)^{3/2}, \tag{32}$$

where $0 < \omega, \Omega < \infty$ depend only on the compact set $\mathcal{U}$, as well as Λ, while l is the half-length of the support of the initial data. Let us denote by T the maximal time during which the expression

$$\Omega\sqrt{l+\Lambda t}y(t) \left(\text{with } y(t) := \left(\int h(u)dx\right)^{1/2}\right)$$

remains smaller than $\frac{\omega}{2}$. Since $y \le y(0)$, because of (25), we already know that T is larger than

$$T_0 = \Lambda^{-1}\left(\left(\frac{\omega}{2\Omega y(0)}\right)^2 - l\right).$$

We make here the smallness assumption that T_0 is positive, that is

$$l y(0)^2 = l\int h(u(x,0))dx < \left(\frac{\omega}{2\Omega}\right)^2. \tag{33}$$

We now integrate (32) on $(0,T)$ and receive

$$\int_0^T y(t)^2 dt \le \frac{2}{\omega}\int L(u_0,p_0)dx.$$

Since $y(t) \ge y(T)$ on $(0,T)$, there follows that

$$y(T)^2 \le \frac{2}{\omega T}\int L(u_0,p_0)dx. \tag{34}$$

Now reinforcing the smallness assumption (33) by

$$\frac{2\Lambda}{\omega}\int L(u_0,p_0)dx + l y(0)^2 < \left(\frac{\omega}{2\Omega}\right)^2,$$

we find that the right-hand side in (34) is strictly less than $(\frac{\omega}{2\Omega})^2(l+T\Lambda)^{-1}$, except if $T=+\infty$. Since T is maximal, this upper bound may not imply that $\Omega\sqrt{l+T\Lambda}y(T) < \omega/2$ and hence we conclude that $T=+\infty$. As a by-product we obtain

$$\int_0^{+\infty} y(t)^2 dt \le \frac{2}{\omega}\int L(u_0,p_0)dx,$$

from which (using monotonicity) we infer

$$y(t) \le \left(\frac{2}{\omega t}\int L(u_0,p_0)dx\right)^{1/2}.$$

We summarize our results as follows.

THEOREM 8. Under the same assumptions as in Proposition 7, there exists a number $\delta > 0$ such that, if the zero-mass initial data and the solution itself takes values in $\mathcal{U}$, and if

$$\int L(u_0, p_0)dx + |\operatorname{Supp} u_0| \int h(u_0)dx < \delta,$$

then the solution stabilizes:

$$\|u(t)\|_2 = O(t^{-1/2}).$$

Actually, one has

$$\int_0^{+\infty} \|u(t)\|_2^2 dt < +\infty.$$

7. Appendix.

THEOREM 9 (Kawashima et al. [13]). Let M be a real symmetric $m \times m$ matrix and Π be a subspace of $\mathbb{R}^m$ which does not contain any eigenvector of M. Then there exists an $m \times m$ skew-symmetric matrix A, such that the restriction of the symmetric matrix $AM - MA$ to Π is positive definite:

$${}^t xAMx > 0, \quad \forall x \in \Pi, \ x \neq 0.$$

Proof. We argue by induction on the number e of distinct eigenvalues of M. If $e = 1$, then $M = \lambda I_m$, and hence $\Pi = \{0\}$; there is nothing to prove.

If $e \geq 2$, we choose an eigenvalue λ and define $F = R(M - \lambda)$, which is a strict subspace, invariant by M. By the induction hypothesis, there exists an alternate bilinear form $a : F \times F \to \mathbb{R}$, such that

$$a(x, Mx) > 0, \quad \forall x \in \Pi \cap F, \ x \neq 0.$$

One then searches for an extension $\hat{a}$ of a to $\mathbb{R}^m \times \mathbb{R}^m$. Denoting by

$$x = x_\lambda + x_F$$

the decomposition in $\mathbb{R}^m = \ker(M - \lambda) \oplus F$, the bilinear form $\hat{a}$ will have read

$$\hat{a}(x, y) = a(x_F, y_F) + \frac{1}{\varepsilon}\{b(x_\lambda, y_F) - b(y_\lambda, x_F)\},$$

where $b : \ker(M - \lambda) \times F \to \mathbb{R}$ and ε are to be determined. Here above, b is bilinear and $\varepsilon > 0$ is small. Since $(Mx)_\lambda = \lambda x_\lambda$, we have

$$\hat{a}(x, Mx) = a(x_F, Mx_F) + \frac{1}{\varepsilon} b(x_\lambda, (M - \lambda)x_F).$$

By assumption, $\Pi \cap \ker(M - \lambda) = \{0\}$. Thus, let P be a subspace such that

$$\Pi \subset P, \ \mathbb{R}^m = P \oplus \ker(M - \lambda).$$

Then P has an equation of the form

$$x_\lambda = v(x_F)$$

for some linear map v. Since $M - \lambda : F \to F$ is invertible, it has an inverse, which we denote w. Then, choosing some scalar product on $\ker(M - \lambda)$, we define

$$b(x, y) = x \cdot v(w(y)), \quad x \in \ker(M - \lambda), \ y \in F.$$

By definition we have

$$b(x_\lambda, (M-\lambda)x_F) = x_\lambda \cdot v(x_F).$$

If moreover $x \in \Pi$, then

$$b(x_\lambda, (M-\lambda)x_F) = \|v(x_F)\|^2 = \|x_\lambda\|^2.$$

This shows that the restriction to Π of the quadratic form $x \to b(x_\lambda, (M-\lambda)x_F)$ is positive semi-definite, and vanishes only on $\Pi \cap F$. Since, by the induction hypothesis, that of $x \to a(x_F, Mx_F)$ is positive definite on $\Pi \cap F$, a last application of Lemma 5 shows that there exists a small $\varepsilon > 0$ such that the restriction of $x \to a(x, Mx)$ to Π is positive definite.

At last, A is the matrix of $\hat{a}$:

$$\hat{a}(x,y) = {}^t x A y.$$

In some papers, Kawashima et al. [13] call such a matrix a *compensating matrix.* □

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