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SHRINKING SIMILAR SOLUTIONS OF A CONVECTION DIFFUSION EQUATION

By

JINGXUE YIN AND CHUNPENG WANG

Dept. of Math., Jilin Univ., Changchun, Jilin 130012, P. R. C.

Abstract. In this paper we study the convection diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u^m - x \cdot \nabla u^q, \quad (x,t) \in \mathbb{R}^n \times (0, +\infty),$$

where $m > 1$, $1 < q \leq m$.

We are interested in similar solutions with the properties of finite speed propagation of perturbations and with shrinking or unchanged supports. We establish the existence and uniqueness, and then discuss some properties of similar solutions.

1. Introduction. It is well-known that for the porous media equation

$$\frac{\partial u}{\partial t} = \Delta u^m, \quad (m > 1)$$

the solutions have the property of finite speed of propagation of perturbations; namely, if the initial datum $u_0(x)$ has compact support, then any solution u of the porous media equation with $u(x,0) = u_0(x)$ has the following properties: for any $t > 0$, $\mathrm{supp}u(\cdot, t)$ remains compact and increases with respect to t, see [1].

However, if convection or absorption is considered, then the solution may behave quite differently; see, for example, [2]–[4].

The present paper is devoted to the porous media equation with a strong convection of the form

$$\frac{\partial u}{\partial t} = \Delta u^m - x \cdot \nabla u^q, \quad (x,t) \in \mathbb{R}^n \times (0, +\infty), \tag{1.1}$$

where $m > 1$ and $1 < q \leq m$. The purpose is to show some different aspects for the solution of (1.1), comparing these to the solution of the porous medium equation.

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E-mail address: yjx@mail.jlu.edu.cn

Precisely, we are seeking solutions of the form

$$u(x,t) = \frac{1}{(t+1)^{\beta}} w((t+1)^{\alpha}|x|^2), \tag{1.2}$$

where $\alpha = \frac{m-q}{q-1} \geq 0$, $\beta = \frac{1}{q-1} > 0$, and $w(\xi)$ has the following properties: that $w(\xi) > 0$ in $[0, a)$ and $w(\xi) = 0$ in $[a, +\infty)$. It is obvious that these types of solutions possess the properties of finite speed propagation of perturbations. Furthermore, the support of $u(\cdot, t)$ remains unchanged if $q = m$, while the support of $u(\cdot, t)$ shrinks as t increases if $1 < q < m$. Furthermore, for $1 < q < m$, we also show that the support of $u(\cdot, t)$ shrinks to a single point as t tends to infinity. We also discuss the singularities of $\frac{1}{u(x,t)}$ at this point as t tends to infinity and compare $\frac{1}{u(x,t)}$ with the Dirac function as t tends to infinity in some sense.

This paper is arranged as follows. In §2 we first deduce the ordinary differential equation that the similar solutions should satisfy and then present the main results of this paper. The proof of the main results is given in §3.

2. Preliminaries and main results. To discuss the shrinking similar solutions, we should first deduce the equation satisfied by the function $w(\xi)$ in (1.2) used for the definition of similar solutions. A direct calculation shows that $w = w(\xi)$ should satisfy the following ordinary differential equation

$$4\xi(w^m)'' + 2n(w^m)' - 2\xi(w^q)' - \alpha\xi w' + \beta w = 0, \tag{2.1}$$

where n is the spatial dimension.

Let $v = \xi(w^m)'$. Then we have

$$4m\xi w^m v' + \{2m(n-2)w^m - \alpha\xi w - 2q\xi w^q\} v + \beta m\xi w^{m+1} = 0$$

and Eq. (2.1) is transformed to the system

$$\begin{cases} w' = \frac{v}{m\xi w^{m-1}}, \\ v' = \left(\frac{2-n}{2\xi} + \frac{\alpha}{4mw^{m-1}} + \frac{q}{2mw^{m-q}}\right) v - \frac{\beta}{4} w. \end{cases} \tag{2.2}$$

Consider the system (2.2) with the following initial value conditions:

$$\begin{cases} w(0) = A, \\ v(0) = 0, \end{cases} \tag{2.3}$$

where $A > 0$.

DEFINITION. A pair (w, v) is called a solution with compact support of the initial value problem (2.2),(2.3), if there exists a constant $a > 0$, such that

(i) $w(a) = 0$ and

$$w(\xi) > 0, \qquad \forall \xi \in [0, a);$$

(ii) w is monotone nonincreasing on $[0, a]$;

(iii) w and v are continuously differentiable in $(0, a)$ and satisfy (2.2). w and v are right continuous at 0 and satisfy (2.3). In addition, w is left continuous at a.

The main results of this paper are the following theorems.

THEOREM 1 (Existence). For $\alpha = \frac{m-q}{q-1}$, $\beta = \frac{1}{q-1}$, and any $A > 0$, there exists at least one solution with compact support of the initial value problem (2.2),(2.3).

THEOREM 2 (Uniqueness). The initial value problem (2.2),(2.3) has at most one solution with compact support.

THEOREM 3. Assume $1 < q < m$. Let (w, v) be the solution with compact support of the initial value problem (2.2),(2.3). Then $\frac{1}{w(\xi)}$ is integrable on $[0, a]$. Thus $\frac{s^{n-1}}{w(s^2)}$ is integrable on $[\,0, \sqrt{a}\,]$. We denote $\lambda = \int_0^{\sqrt{a}} \frac{s^{n-1}}{w(s^2)} ds$. In addition, let $u(x,t) = \frac{1}{(t+1)^\beta} w((t+1)^\alpha |x|^2)$ be the shrinking similar solution of Eq. (1.1) and $\phi(x)$ be a continuous function on $\mathbb{R}^n$. Then

$$\lim_{t\to+\infty} (t+1)^{(nm-nq-2)/(2(q-1))} \int_{\operatorname{supp} u(\cdot,t)} \frac{\phi(x)}{u(x,t)} dx = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \lambda \phi(0),$$

where Γ is the standard Gamma function. In particular, when $n(m-q) = 2$, we get that

$$\lim_{t\to+\infty} \int_{\operatorname{supp} u(\cdot,t)} \frac{\phi(x)}{u(x,t)} dx = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \lambda \phi(0).$$

3. Proof of the main results. Because the initial value problem (2.2),(2.3) is singular, we first study an approximate problem. For small $\varepsilon > 0$, consider the system (2.2) with the following initial value conditions:

$$\begin{cases} w(\varepsilon) = A, \\ v(\varepsilon) = 0. \end{cases} \tag{3.1}$$

PROPOSITION 1. Let $(w_\varepsilon, v_\varepsilon)$ be the classical solution of the initial value problem (2.2),(3.1). Then there exists a constant $a_\varepsilon > 0$, such that

$$w_\varepsilon(a_\varepsilon) = 0$$

and

$$w_\varepsilon(\xi) > 0, \qquad w'_\varepsilon(\xi) < 0, \qquad \forall \xi \in (\varepsilon, a_\varepsilon).$$

In addition, the estimate

$$a_\varepsilon \le 2\varepsilon + \frac{4m(n+1)A^{m-1}}{\beta(m-1)}$$

holds.

Proof. It follows easily from the theory for ordinary differential equations that $(w_\varepsilon, v_\varepsilon)$ exists locally. So there exists $\delta > \varepsilon$ such that $(w_\varepsilon, v_\varepsilon)$ exists in (ε, δ), and for any $\xi \in (\varepsilon, \delta)$, we have

$$w_\varepsilon(\xi) > 0, \qquad v_\varepsilon(\xi) < 0, \qquad w'_\varepsilon(\xi) < 0,$$

and

$$\begin{cases} w_\varepsilon^m(\xi) = A^m + \int_\varepsilon^\xi \frac{v_\varepsilon(t)}{t} dt, \\ v_\varepsilon(\xi) = \quad -\exp\left\{ \int_\varepsilon^\xi \left(\frac{2-n}{2t} + \frac{\alpha}{4mw_\varepsilon^{m-1}(t)} + \frac{q}{2mw_\varepsilon^{m-q}(t)} \right) dt \right\} \\ \qquad\qquad \int_\varepsilon^\xi \frac{\beta w_\varepsilon(s)}{4} \exp\left\{ -\int_\varepsilon^s \left(\frac{2-n}{2t} + \frac{\alpha}{4mw_\varepsilon^{m-1}(t)} + \frac{q}{2mw_\varepsilon^{m-q}(t)} \right) dt \right\} ds. \end{cases} \tag{3.2}$$

The extension theorem implies that one and only one of the following two conclusions holds:

(i) There exists $a_\varepsilon > 0$ such that $w_\varepsilon(a_\varepsilon) = 0$, and for any $\xi \in (\varepsilon, a_\varepsilon)$, we have

$$w_\varepsilon(\xi) > 0, \qquad v_\varepsilon(\xi) < 0, \qquad w'_\varepsilon(\xi) < 0;$$

(ii) For any $\xi \in (\varepsilon, +\infty)$, we have

$$w_\varepsilon(\xi) > 0, \qquad v_\varepsilon(\xi) < 0, \qquad w'_\varepsilon(\xi) < 0.$$

Now we show that conclusion (ii) does not hold. In fact, if conclusion (ii) were valid, then it would follow from (3.2) that

$$\begin{aligned}
&-v_\varepsilon(\xi)\\
=\ & \exp\left\{\int_\varepsilon^\xi \left(\frac{2-n}{2t} + \frac{\alpha}{4mw_\varepsilon^{m-1}(t)} + \frac{q}{2mw_\varepsilon^{m-q}(t)}\right) dt\right\}\\
& \int_\varepsilon^\xi \frac{\beta w_\varepsilon(s)}{4} \exp\left\{-\int_\varepsilon^s \left(\frac{2-n}{2t} + \frac{\alpha}{4mw_\varepsilon^{m-1}(t)} + \frac{q}{2mw_\varepsilon^{m-q}(t)}\right) dt\right\} ds\\
=\ & \int_\varepsilon^\xi \frac{\beta w_\varepsilon(s)}{4} \exp\left\{\int_s^\xi \left(\frac{2-n}{2t} + \frac{\alpha}{4mw_\varepsilon^{m-1}(t)} + \frac{q}{2mw_\varepsilon^{m-q}(t)}\right) dt\right\} ds.
\end{aligned}$$

We will distinguish two cases to estimate the inferior bound of $-v_\varepsilon$.

(1) When $n = 1, 2$, we have

$$\begin{aligned}
-v_\varepsilon(\xi) &= \int_\varepsilon^\xi \frac{\beta w_\varepsilon(s)}{4} \exp\left\{\int_s^\xi \left(\frac{2-n}{2t} + \frac{\alpha}{4mw_\varepsilon^{m-1}(t)} + \frac{q}{2mw_\varepsilon^{m-q}(t)}\right) dt\right\} ds\\
&\geq \int_\varepsilon^\xi \frac{\beta w_\varepsilon(s)}{4} ds \geq \frac{\beta}{4} w_\varepsilon(\xi)(\xi - \epsilon).
\end{aligned}$$

(2) When $n \geq 3$, we have

$$\begin{aligned}
-v_\varepsilon(\xi) &= \int_\varepsilon^\xi \frac{\beta w_\varepsilon(s)}{4} \exp\left\{\int_s^\xi \left(\frac{2-n}{2t} + \frac{\alpha}{4mw_\varepsilon^{m-1}(t)} + \frac{q}{2mw_\varepsilon^{m-q}(t)}\right) dt\right\} ds\\
&\geq \int_\varepsilon^\xi \frac{\beta w_\varepsilon(s)}{4} \exp\left\{\int_s^\xi \frac{2-n}{2t} dt\right\} ds\\
&= \int_\varepsilon^\xi \frac{\beta w_\varepsilon(s)}{4} \left(\frac{s}{\xi}\right)^{n/2-1} ds\\
&\geq \frac{\beta}{2n} w_\varepsilon(\xi)\xi^{1-n/2}(\xi^{n/2} - \varepsilon^{n/2})\\
&\geq \frac{\beta}{2n} w_\varepsilon(\xi)(\xi - \varepsilon).
\end{aligned}$$

Therefore, for each $n \geq 1$, we get that

$$-v_\varepsilon(\xi) \geq \frac{\beta}{2(n+1)} w_\varepsilon(\xi)(\xi - \varepsilon).$$

Hence when $\xi \geq 2\varepsilon$,

$$-(w_\varepsilon^{m-1})'(\xi) \geq \frac{\beta(m-1)}{4m(n+1)}$$

holds. This contradicts conclusion (ii). So only conclusion (i) holds and from the mean value theorem we get the estimate

$$a_\varepsilon \leq 2\varepsilon + \frac{4m(n+1)A^{m-1}}{\beta(m-1)}.$$

The proof is complete.

Now we establish the comparison principle. We give two initial value conditions

$$\begin{cases} w(\varepsilon) = A_1, \\ v(\varepsilon) = B_1, \end{cases} \tag{3.3}$$

and

$$\begin{cases} w(\varepsilon) = A_2, \\ v(\varepsilon) = B_2. \end{cases} \tag{3.4}$$

PROPOSITION 2 (Comparison Principle). Let $0 < A_1 < A_2$, $B_1 < B_2 \leq 0$, and (w_1, v_1) be the solution of the initial value problem (2.2),(3.3), and let (w_2, v_2) be the solution of the initial value problem (2.2),(3.4). If w_1 and w_2 are both positive on $[\varepsilon, a_0]$ for some $a_0 > \varepsilon$, then

$$w_1(\xi) > w_2(\xi), \qquad \frac{v_1(\xi)}{w_1(\xi)} < \frac{v_2(\xi)}{w_2(\xi)}, \qquad \forall \xi \in [\varepsilon, a_0].$$

Proof. Since $0 < A_1 < A_2$, $B_1 < B_2 \leq 0$, there exists $\delta > \varepsilon$ such that $0 < w_1 < w_2$ and $v_1 < v_2 \leq 0$ on $[\varepsilon, \delta]$. From (2.2), we see that

$$\left(\frac{v}{w}\right)' + \frac{vw'}{w^2} = \left(\frac{2-n}{2\xi} + \frac{\alpha}{4mw^{m-1}} + \frac{q}{2mw^{m-q}}\right)\frac{v}{w} - \frac{\beta}{4},$$

namely,

$$\left(\frac{v}{w}\right)' = \left(\frac{2-n}{2\xi} + \frac{\alpha}{4mw^{m-1}} + \frac{q}{2mw^{m-q}} - \frac{w'}{w}\right)\frac{v}{w} - \frac{\beta}{4}.$$

Thus

$$\begin{aligned}
&\frac{v_1(\xi)}{w_1(\xi)}\\
=\ &\frac{B_1}{A_1}\exp\left\{\int_\varepsilon^\xi\left(\frac{2-n}{2t}+\frac{\alpha}{4mw_1^{m-1}(t)}+\frac{q}{2mw_1^{m-q}(t)}-\frac{w_1'}{w_1}\right)dt\right\}\\
&-\exp\left\{\int_\varepsilon^\xi\left(\frac{2-n}{2t}+\frac{\alpha}{4mw_1^{m-1}(t)}+\frac{q}{2mw_1^{m-q}(t)}-\frac{w_1'}{w_1}\right)dt\right\}\\
&\int_\varepsilon^\xi\frac{\beta}{4}\exp\left\{-\int_\varepsilon^s\left(\frac{2-n}{2t}+\frac{\alpha}{4mw_1^{m-1}(t)}+\frac{q}{2mw_1^{m-q}(t)}-\frac{w_1'}{w_1}\right)dt\right\}ds\\
=\ &\frac{B_1}{A_1}\exp\left\{\int_\varepsilon^\xi\left(\frac{2-n}{2t}+\frac{\alpha}{4mw_1^{m-1}(t)}+\frac{q}{2mw_1^{m-q}(t)}-\frac{w_1'}{w_1}\right)dt\right\}\\
&-\int_\varepsilon^\xi\frac{\beta}{4}\exp\left\{\int_s^\xi\left(\frac{2-n}{2t}+\frac{\alpha}{4mw_1^{m-1}(t)}+\frac{q}{2mw_1^{m-q}(t)}-\frac{w_1'}{w_1}\right)dt\right\}ds
\end{aligned}$$

and

$$\begin{aligned}
&\frac{v_2(\xi)}{w_2(\xi)}\\
=\ &\frac{B_2}{A_2}\exp\left\{\int_\varepsilon^\xi\left(\frac{2-n}{2t}+\frac{\alpha}{4mw_2^{m-1}(t)}+\frac{q}{2mw_2^{m-q}(t)}-\frac{w_2'}{w_2}\right)dt\right\}\\
&-\exp\left\{\int_\varepsilon^\xi\left(\frac{2-n}{2t}+\frac{\alpha}{4mw_2^{m-1}(t)}+\frac{q}{2mw_2^{m-q}(t)}-\frac{w_2'}{w_2}\right)dt\right\}\\
&\int_\varepsilon^\xi\frac{\beta}{4}\exp\left\{-\int_\varepsilon^s\left(\frac{2-n}{2t}+\frac{\alpha}{4mw_2^{m-1}(t)}+\frac{q}{2mw_2^{m-q}(t)}-\frac{w_2'}{w_2}\right)dt\right\}ds\\
=\ &\frac{B_2}{A_2}\exp\left\{\int_\varepsilon^\xi\left(\frac{2-n}{2t}+\frac{\alpha}{4mw_2^{m-1}(t)}+\frac{q}{2mw_2^{m-q}(t)}-\frac{w_2'}{w_2}\right)dt\right\}\\
&-\int_\varepsilon^\xi\frac{\beta}{4}\exp\left\{\int_s^\xi\left(\frac{2-n}{2t}+\frac{\alpha}{4mw_2^{m-1}(t)}+\frac{q}{2mw_2^{m-q}(t)}-\frac{w_2'}{w_2}\right)dt\right\}ds.
\end{aligned}$$

Since $0<A_1<A_2$ and $B_1<B_2\le 0$, thus $\frac{B_1}{A_1}>\frac{B_2}{A_2}$. From $0<w_1<w_2$ and $v_1<v_2\le 0$ on $[\epsilon,\delta]$, we see that for all $t\in[\epsilon,\delta]$,

$$\frac{w_1'}{w_1}=\frac{v_1}{mtw_1^m}<\frac{v_2}{mtw_2^m}=\frac{w_2'}{w_2}\le 0.$$

Thus for all $t\in[\epsilon,\delta]$,

$$\frac{\alpha}{4mw_1^{m-1}(t)}+\frac{q}{2mw_1^{m-q}(t)}-\frac{w_1'}{w_1}>\frac{\alpha}{4mw_2^{m-1}(t)}+\frac{q}{2mw_2^{m-q}(t)}-\frac{w_2'}{w_2}>0.$$

So $\frac{v_1}{w_1}<\frac{v_2}{w_2}\le 0$ on $[\varepsilon,\delta]$, namely $(w_1^{m-1})'-(w_2^{m-1})'<0$ on $[\varepsilon,\delta]$. Therefore,

$$0<w_1(\delta)<w_2(\delta),\qquad \frac{v_1(\delta)}{w_1(\delta)}<\frac{v_2(\delta)}{w_2(\delta)}\le 0$$

and

$$w_2^{m-1}(\varepsilon) - w_1^{m-1}(\varepsilon) < w_2^{m-1}(\delta) - w_1^{m-1}(\delta).$$

Here the proof is complete if $a_0 \le \delta$.

If $a_0 > \delta$, we repeat the above arguments. Noticing that w_1', v_1' and w_2', v_2' are bounded on $[\varepsilon, a_0]$ and

$$w_2^{m-1}(\varepsilon) - w_1^{m-1}(\varepsilon) < w_2^{m-1}(\delta) - w_1^{m-1}(\delta)$$

holds, we see that the above results hold for some $\delta \ge a$. The proof is complete.

PROPOSITION 3. There exists a constant $C_0 > 0$ independent of ε, such that

$$a_\varepsilon > \varepsilon + C_0 > 2\varepsilon$$

holds when $0 < \varepsilon < C_0$. Furthermore, for $0 < \delta < a_\varepsilon - 2\varepsilon$, we have the estimate

$$w_\varepsilon(\xi) \ge \frac{\beta(m-1)\delta}{4m(n+1)}, \qquad \forall \xi \in [\varepsilon, a_\varepsilon - \delta].$$

Proof. We will distinguish two cases to estimate the superior bound of $-v_\varepsilon$.

(1) When $n = 1$, for any $\xi \in [\varepsilon, a_\varepsilon)$, we have

$$w_\varepsilon(\xi) > 0, \qquad v_\varepsilon(\xi) \le 0$$

and

$$\begin{aligned}
-v_\varepsilon(\xi) &= \int_\varepsilon^\xi \frac{\beta w_\varepsilon(s)}{4} \exp\left\{\int_s^\xi \left(\frac{1}{2t} + \frac{\alpha}{4mw_\varepsilon^{m-1}(t)} + \frac{q}{2mw_\varepsilon^{m-q}(t)}\right) dt\right\} ds \\
&\le \frac{\beta A}{4} \int_\varepsilon^\xi \exp\left\{\int_s^\xi \left(\frac{\alpha}{4mw_\varepsilon^{m-1}(t)} + \frac{q}{2mw_\varepsilon^{m-q}(t)}\right) dt\right\} \left(\frac{\xi}{s}\right)^{1/2} ds \\
&\le \frac{\beta A \xi^{1/2}}{4} \int_\varepsilon^\xi \exp\left\{\left(\frac{\alpha}{4mw_\varepsilon^{m-1}(\xi)} + \frac{q}{2mw_\varepsilon^{m-q}(\xi)}\right)(\xi - s)\right\} s^{-1/2} ds \\
&\le \frac{\beta A \xi^{1/2}}{4} \exp\left\{\xi\left(\frac{\alpha}{4mw_\varepsilon^{m-1}(\xi)} + \frac{q}{2mw_\varepsilon^{m-q}(\xi)}\right)\right\} \int_\varepsilon^\xi s^{-1/2} ds \\
&= \frac{\beta A \xi^{1/2}}{2} \exp\left\{\xi\left(\frac{\alpha}{4mw_\varepsilon^{m-1}(\xi)} + \frac{q}{2mw_\varepsilon^{m-q}(\xi)}\right)\right\} (\xi^{1/2} - \varepsilon^{1/2}) \\
&\le \frac{\beta A \xi}{2} \exp\left\{\xi\left(\frac{\alpha}{4mw_\varepsilon^{m-1}(\xi)} + \frac{q}{2mw_\varepsilon^{m-q}(\xi)}\right)\right\}.
\end{aligned}$$

(2) When $n \ge 2$, for any $\xi \in [\varepsilon, a_\varepsilon)$, we have

$$w_\varepsilon(\xi) > 0, \qquad v_\varepsilon(\xi) \le 0$$

and

$$\begin{aligned}
-v_\varepsilon(\xi) &= \int_\varepsilon^\xi \frac{\beta w_\varepsilon(s)}{4}\exp\left\{\int_s^\xi \left(\frac{2-n}{2t}+\frac{\alpha}{4mw_\varepsilon^{m-1}(t)}+\frac{q}{2mw_\varepsilon^{m-q}(t)}\right)dt\right\}ds \\
&\le \frac{\beta A}{4}\int_\varepsilon^\xi \exp\left\{\int_s^\xi \left(\frac{\alpha}{4mw_\varepsilon^{m-1}(t)}+\frac{q}{2mw_\varepsilon^{m-q}(t)}\right)\right\}ds \\
&\le \frac{\beta A}{4}\int_\varepsilon^\xi \exp\left\{\left(\frac{\alpha}{4mw_\varepsilon^{m-1}(\xi)}+\frac{q}{2mw_\varepsilon^{m-q}(\xi)}\right)(\xi-s)\right\}ds \\
&\le \frac{\beta A}{4}\exp\left\{\xi\left(\frac{\alpha}{4mw_\varepsilon^{m-1}(\xi)}+\frac{q}{2mw_\varepsilon^{m-q}(\xi)}\right)\right\}(\xi-\epsilon) \\
&\le \frac{\beta A\xi}{4}\exp\left\{\xi\left(\frac{\alpha}{4mw_\varepsilon^{m-1}(\xi)}+\frac{q}{2mw_\varepsilon^{m-q}(\xi)}\right)\right\}.
\end{aligned}$$

Therefore, for each $n \ge 1$, we get that

$$-v_\varepsilon(\xi) \le \frac{\beta A\xi}{2}\exp\left\{\xi\left(\frac{\alpha}{4mw_\varepsilon^{m-1}(\xi)}+\frac{q}{2mw_\varepsilon^{m-q}(\xi)}\right)\right\}, \qquad \forall \xi \in [\varepsilon, a_\varepsilon).$$

Hence

$$(w_\varepsilon^m)'(\xi) \ge -\frac{\beta A}{2}\exp\left\{\xi\left(\frac{\alpha}{4mw_\varepsilon^{m-1}(\xi)}+\frac{q}{2mw_\varepsilon^{m-q}(\xi)}\right)\right\}, \qquad \forall \xi \in [\varepsilon, a_\varepsilon).$$

The comparison principle implies $a_\varepsilon \le a_1$ for $0<\varepsilon\le 1$. Thus for $0<\varepsilon\le 1$, we have

$$(w_\varepsilon^m)'(\xi) \ge -\frac{\beta A}{2}\exp\left\{a_1\left(\frac{\alpha}{4mw_\varepsilon^{m-1}(\xi)}+\frac{q}{2mw_\varepsilon^{m-q}(\xi)}\right)\right\}, \qquad \forall \xi \in [\varepsilon, a_\varepsilon).$$

Let $\xi_0 \in (\varepsilon, a_\varepsilon)$ such that $w_\varepsilon(\xi_0) = \frac{A}{2}$. Then

$$(w_\varepsilon^m)'(\xi) \ge -\frac{\beta A}{2}\exp\left\{a_1\left(\frac{\alpha}{4m\,(A/2)^{m-1}}+\frac{q}{2m\,(A/2)^{m-q}}\right)\right\}, \qquad \forall \xi \in [\varepsilon, \xi_0].$$

From the mean value theorem, we get that

$$\begin{aligned}
a_\varepsilon > \xi_0 &\ge \varepsilon + (A^m-(A/2)^m)\frac{2}{\beta A}\exp\left\{-a_1\left(\frac{\alpha}{4m\,(A/2)^{m-1}}+\frac{q}{2m\,(A/2)^{m-q}}\right)\right\} \\
&\ge \varepsilon + \frac{A^{m-1}}{\beta}\exp\left\{-a_1\left(\frac{\alpha}{4m\,(A/2)^{m-1}}+\frac{q}{2m\,(A/2)^{m-q}}\right)\right\}.
\end{aligned}$$

Let

$$C_0 = \min\left\{1, \frac{A^{m-1}}{\beta}\exp\left\{-a_1\left(\frac{\alpha}{4m\,(A/2)^{m-1}}+\frac{q}{2m\,(A/2)^{m-q}}\right)\right\}\right\}.$$

Then for $0<\varepsilon<C_0$, we have

$$a_\varepsilon > \varepsilon + C_0 > 2\varepsilon.$$

We now show the latter of the proposition correctly. Let $0<\delta<a_\varepsilon-2\varepsilon$. From the proof of Proposition 1, we get that

$$-(w_\varepsilon^{m-1})'(\xi) \ge \frac{\beta(m-1)}{4m(n+1)}, \qquad \forall \xi \ge 2\varepsilon.$$

Noticing that $a_\varepsilon - \delta > 2\varepsilon$ and $w_\varepsilon(a_\varepsilon) = 0$, we get from the mean value theorem that

$$w_\varepsilon^{m-1}(a_\varepsilon - \delta) \geq \frac{\beta(m-1)\delta}{4m(n+1)}.$$

Therefore,

$$w_\varepsilon(\xi) \geq w_\varepsilon(a_\varepsilon - \delta) \geq \frac{\beta(m-1)\delta}{4m(n+1)}, \qquad \forall \xi \in [\varepsilon, a_\varepsilon - \delta].$$

The proof is complete.

LEMMA 1. Let $0 < \varepsilon < C_0$, where C_0 is the constant given in Proposition 3. For any $0 < \tau < 1$, there exists a constant $\delta > 0$ independent of ε such that

$$A^m - \tau^m < w_\varepsilon^m(\xi) \leq A^m, \qquad -\tau < v_\varepsilon(\xi) \leq 0, \qquad \forall \xi \in [\varepsilon, \varepsilon + \delta).$$

Proof. Let $0 < \varepsilon < C_0$. From the proof of Proposition 3, we get that

$$v_\varepsilon'(\xi) \geq -\frac{\beta A(\xi - \epsilon)}{2} \exp\left\{a_1\left(\frac{\alpha}{4mw_\varepsilon^{m-1}(\xi)} + \frac{q}{2mw_\varepsilon^{m-q}(\xi)}\right)\right\}, \qquad \forall \xi \in [\varepsilon, a_\varepsilon)$$

and

$$(w_\varepsilon^m)'(\xi) \geq -\frac{\beta A}{2} \exp\left\{a_1\left(\frac{\alpha}{4mw_\varepsilon^{m-1}(\xi)} + \frac{q}{2mw_\varepsilon^{m-q}(\xi)}\right)\right\}, \qquad \forall \xi \in [\varepsilon, a_\varepsilon).$$

Noticing that $w_\varepsilon(\varepsilon + C_0) \geq \frac{A}{2}$, we get that

$$-v_\varepsilon'(\xi) \leq \frac{\beta A(\xi - \epsilon)}{2} \exp\left\{a_1\left(\frac{2^{m-1}\alpha}{4mA^{m-1}} + \frac{2^{m-q}q}{2mA^{m-q}}\right)\right\}, \qquad \forall \xi \in [\varepsilon, \varepsilon + C_0]$$

and

$$-(w_\varepsilon^m)'(\xi) \leq \frac{\beta A}{2} \exp\left\{a_1\left(\frac{2^{m-1}\alpha}{4mA^{m-1}} + \frac{2^{m-q}q}{2mA^{m-q}}\right)\right\}, \qquad \forall \xi \in [\varepsilon, \varepsilon + C_0].$$

For any $0 < \tau < 1$, we let $\delta = \min\left\{C_0, \frac{2\tau^m}{\beta A} \exp\left\{-a_1\left(\frac{2^{m-1}\alpha}{4mA^{m-1}} + \frac{2^{m-q}q}{2mA^{m-q}}\right)\right\}\right\}$. Then

$$A^m - \tau^m < w_\varepsilon^m(\xi) \leq A^m, \qquad -\tau < v_\varepsilon(\xi) \leq 0, \qquad \forall \xi \in [\varepsilon, \varepsilon + \delta).$$

The proof is complete.

LEMMA 2. Let $0 < \varepsilon < C_0$. For any $\tau > 0$, there exists a constant $\delta > 0$ independent of ε such that

$$0 < w_\varepsilon(\xi) < \tau, \qquad \xi \in (a_\varepsilon - \delta, a_\varepsilon).$$

Proof. Let $0 < \varepsilon < C_0$. The result is trivial if $\tau \geq A$. We assume $0 < \tau < A$ in the following proof.

From the proof of Proposition 3, we get that

$$(w_\varepsilon^m)'(\xi) \geq -\frac{\beta A}{2} \exp\left\{a_1\left(\frac{\alpha}{4mw_\varepsilon^{m-1}(\xi)} + \frac{q}{2mw_\varepsilon^{m-q}(\xi)}\right)\right\}, \qquad \forall \xi \in [\varepsilon, a_\varepsilon).$$

Let $\xi_1, \xi_2 \in (\varepsilon, a_\varepsilon)$ such that $w_\varepsilon(\xi_1) = \frac{\tau}{2}$ and $w_\varepsilon(\xi_2) = \tau$. Then

$$(w_\varepsilon^m)'(\xi) \geq -\frac{\beta A}{2} \exp\left\{a_1\left(\frac{\alpha}{4m(\tau/2)^{m-1}} + \frac{q}{2m(\tau/2)^{m-q}}\right)\right\}, \qquad \forall \xi \in [\varepsilon, \xi_1].$$

The mean value theorem implies that

$$\begin{aligned}\xi_2 &< \xi_1 - \frac{(2^m-1)\tau^m}{2^m}\frac{2}{\beta A}\exp\left\{-a_1\left(\frac{\alpha}{4m(\tau/2)^{m-1}}+\frac{q}{2m(\tau/2)^{m-q}}\right)\right\}\\ &< a_\varepsilon - \frac{(2^m-1)\tau^m}{2^{m-1}\beta A}\exp\left\{-a_1\left(\frac{\alpha}{4m(\tau/2)^{m-1}}+\frac{q}{2m(\tau/2)^{m-q}}\right)\right\}.\end{aligned}$$

Let $\delta = \frac{(2^m-1)\tau^m}{2^{m-1}\beta A}\exp\left\{-a_1\left(\frac{\alpha}{4m(\tau/2)^{m-1}}+\frac{q}{2m(\tau/2)^{m-q}}\right)\right\}$. Then

$$0 < w_\varepsilon(\xi) < w_\varepsilon(\xi_2) = \tau, \qquad \forall \xi \in (a_\varepsilon - \delta, a_\varepsilon).$$

The proof is complete.

PROPOSITION 4. Let the conditions of Proposition 2 hold. In addition, we assume that $0 < \varepsilon < C_0$ and $B_2 = 0$, where C_0 is the constant given in Proposition 3. Then there exist three constants $M_1, M_2, M_3 > 0$ depending only on a_0, $w_1(a_0)$, and A_2 but independent of ε such that for any $\xi \in [\varepsilon, a_0]$, the estimates

$$|w_1(\xi) - w_2(\xi)| < M_3(A_2 - A_1 - a_0M_2B_1)\exp\{a_0M_3M_1\}$$

and

$$|v_1(\xi) - v_2(\xi)| < a_0M_3M_1(A_2 - A_1 - a_0M_2B_1)\exp\{a_0M_3M_1\} - a_0M_2B_1$$

hold.

Proof. We see from the comparison principle that

$$0 < w_1(a_0) \le w_1(\xi) \le w_2(\xi) \le A_2, \qquad v_1(\xi) \le v_2(\xi) \le 0, \qquad \forall \xi \in [\varepsilon, a_0],$$

and for $\varepsilon \le \xi_1 \le \xi_2 \le a_0$, we have

$$w_2^{m-1}(\xi_2) - w_1^{m-1}(\xi_2) \ge w_2^{m-1}(\xi_1) - w_1^{m-1}(\xi_1) \ge 0.$$

Thus for any $\xi \in [\varepsilon, a_0]$, we get that

$$\begin{aligned}&\left|\int_\varepsilon^\xi \frac{\beta w_1(s)}{4}\exp\left\{\int_s^\xi\left(\frac{2-n}{2t}+\frac{\alpha}{4mw_1^{m-1}(t)}+\frac{q}{2mw_1^{m-q}(t)}\right)dt\right\}ds\right.\\ &\left.-\int_\varepsilon^\xi \frac{\beta w_2(s)}{4}\exp\left\{\int_s^\xi\left(\frac{2-n}{2t}+\frac{\alpha}{4mw_2^{m-1}(t)}+\frac{q}{2mw_2^{m-q}(t)}\right)dt\right\}ds\right|\\ \le&\left|\int_\varepsilon^\xi \frac{\beta(w_1(s)-w_2(s))}{4}\exp\left\{\int_s^\xi\left(\frac{2-n}{2t}+\frac{\alpha}{4mw_1^{m-1}(t)}+\frac{q}{2mw_1^{m-q}(t)}\right)dt\right\}ds\right|\\ &+\left|\int_\varepsilon^\xi \frac{\beta w_2(s)}{4}\left(\exp\left\{\int_s^\xi\left(\frac{2-n}{2t}+\frac{\alpha}{4mw_1^{m-1}(t)}+\frac{q}{2mw_1^{m-q}(t)}\right)dt\right\}\right.\right.\\ &\left.\left.-\exp\left\{\int_s^\xi\left(\frac{2-n}{2t}+\frac{\alpha}{4mw_2^{m-1}(t)}+\frac{q}{2mw_2^{m-q}(t)}\right)dt\right\}\right)ds\right|\\ \le&\ M_1\xi|w_1(\xi)-w_2(\xi)|\\ =&\ M_1\xi(w_2(\xi)-w_1(\xi)),\end{aligned}$$

and

$$\exp\left\{\int_\epsilon^\xi \left(\frac{2-n}{2t} + \frac{\alpha}{4mw_1^{m-1}(t)} + \frac{q}{2mw_1^{m-q}(t)}\right) dt\right\} < M_2\xi,$$

where $M_1, M_2 > 0$ depending only on a_0, $w_1(a_0)$, and A_2 but independent of ε. Therefore, for any $\xi \in [\varepsilon, a_0]$,

$$v_2(\xi) - v_1(\xi) \le M_1\xi(w_2(\xi) - w_1(\xi)) - M_2\xi B_1$$

and

$$\begin{aligned} w_2^m(\xi) - w_1^m(\xi) &= A_2 - A_1 + \int_\varepsilon^\xi ((w_2^m)'(t) - (w_1^m)'(t))dt \\ &\le A_2 - A_1 + \int_\varepsilon^\xi (M_1(w_2(t) - w_1(t)) - M_2B_1)dt \\ &= A_2 - A_1 - M_2B_1(\xi - \varepsilon) + M_1\int_\varepsilon^\xi (w_2(t) - w_1(t))dt \\ &< A_2 - A_1 - a_0M_2B_1 + M_1\int_\varepsilon^\xi (w_2(t) - w_1(t))dt \end{aligned}$$

hold.

Noticing that

$$0 < w_1(a_0) \le w_1(\xi) \le w_2(\xi) \le A_2, \qquad \forall \xi \in [\varepsilon, a_0],$$

we see that there exists a constant $M_3 > 0$ depending only on $w_1(a_0)$ and A_2 but independent of ε such that

$$w_2(\xi) - w_1(\xi) < M_3(A_2 - A_1 - a_0M_2B_1) + M_3M_1\int_\varepsilon^\xi (w_2(t) - w_1(t))dt, \qquad \forall \xi \in [\varepsilon, a_0].$$

Gronwall's Inequality implies that

$$\begin{aligned} |w_1(\xi) - w_2(\xi)| &= w_2(\xi) - w_1(\xi) \\ &< M_3(A_2 - A_1 - a_0M_2B_1)\exp\{M_3M_1(\xi - \varepsilon)\} \\ &< M_3(A_2 - A_1 - a_0M_2B_1)\exp\{a_0M_3M_1\}. \end{aligned}$$

Therefore, for any $\xi \in [\varepsilon, a_0]$, we get that

$$\begin{aligned} |v_1(\xi) - v_2(\xi)| &= v_2(\xi) - v_1(\xi) \\ &\le M_1\xi(w_2(\xi) - w_1(\xi)) - M_2\xi B_1 \\ &< a_0M_3M_1(A_2 - A_1 - a_0M_2B_1)\exp\{a_0M_3M_1\} - a_0M_2B_1. \end{aligned}$$

The proof is complete.

Now we prove the theorems.

Proof of Theorem 1. Let $0 < \varepsilon < C_0$ and $(w_\varepsilon, v_\varepsilon)$ be the solution of the initial value problem (2.2),(3.1). From Proposition 1, there exists $a_\varepsilon > 0$ such that

$$w_\varepsilon(\xi) > 0, \qquad \forall \xi \in [0, a_\varepsilon), \qquad w_\varepsilon(a_\varepsilon) = 0.$$

Let

$$a = \lim_{\varepsilon \to 0^+} a_\varepsilon.$$

From the comparison principle, we see that $0 \le a_{\varepsilon_1} \le a_{\varepsilon_2}$ for any $0 < \varepsilon_1 \le \varepsilon_2 < C_0$. It follows from Proposition 3 that $a_\varepsilon > C_0$ for any $0 < \varepsilon < C_0$. Thus a exists and $a > 0$. For any $\xi \in (0, a)$, the comparison principle implies that $w_\varepsilon(\xi)$ and $v_\varepsilon(\xi)$ are bounded and monotone for $0 < \varepsilon < C_0$. Thus $w_\varepsilon(\xi)$ and $v_\varepsilon(\xi)$ converge when $\varepsilon \to 0^+$. For $\xi \in (0, a)$, let

$$w(\xi) = \lim_{\varepsilon \to 0^+} w_\varepsilon(\xi), \qquad v(\xi) = \lim_{\varepsilon \to 0^+} v_\varepsilon(\xi).$$

In addition, we let

$$w(0) = A, \qquad v(0) = 0, \qquad w(a) = 0.$$

We see that w and v are right continuous at 0 from Lemma 1 and w is left continuous at a from Lemma 2.

It follows from Proposition 3 that

$$w(\xi) > 0, \qquad \forall \xi \in [0, a).$$

Since w_ε is monotone nonincreasing on $[0, a_\varepsilon]$, w is also monotone nonincreasing on $[0, a]$.

For any $0 < \widetilde{\varepsilon} < C_0$ and $\widetilde{a} < a$, Proposition 4 implies that w_ε and v_ε are uniformly convergent on $[\widetilde{\varepsilon}, \widetilde{a}]$. Thus w_ε' and v_ε' are uniformly convergent on $[\widetilde{\varepsilon}, \widetilde{a}]$. Hence w and v are continuously differentiable in $(0, a)$ and satisfy (2.2).

Thus (w, v) is a solution with the compact support of the initial value problem (2.2),(2.3). The proof is complete.

Proof of Theorem 2. Let (w_1, v_1) and (w_2, v_2) be two solutions with compact support of the initial value problem (2.2),(2.3). The constant a satisfying the definition are written a_1 and a_2, respectively. Without loss of generality, we assume $a_1 \le a_2$. Let $0 < a_0 < a_1$. For any $\tau > 0$, the right continuity of (w_1, v_1) and (w_2, v_2) at 0 implies that there exists a constant $0 < \delta < \min\{C_0, a_0\}$ such that

$$A - \tau < w_1(\xi) \le A, \qquad A - \tau < w_2(\xi) \le A, \qquad \forall \xi \in [0, \delta),$$

and

$$0 \le -v_1(\xi) < \tau, \qquad 0 \le -v_2(\xi) < \tau, \qquad \forall \xi \in [0, \delta).$$

For any $\varepsilon \in (0, \delta)$, we see that $(w_1(\xi), v_1(\xi))$ $(\xi \in [\varepsilon, a_0])$ and $(w_2(\xi), v_2(\xi))$ $(\xi \in [\varepsilon, a_0])$ are the solutions of the system (2.2) with the initial value conditions

$$\begin{cases} w(\varepsilon) = w_1(\varepsilon), \\ v(\varepsilon) = v_1(\varepsilon), \end{cases}$$

and

$$\begin{cases} w(\varepsilon) = w_2(\varepsilon), \\ v(\varepsilon) = v_2(\varepsilon), \end{cases}$$

respectively.

Let $(w_0(\xi), v_0(\xi))$ be the solution of the system (2.2) with the initial value conditions

$$\begin{cases} w(\varepsilon) = A, \\ v(\varepsilon) = 0. \end{cases}$$

The comparison principle implies that

$$w_1(\xi) \le w_0(\xi), \qquad v_1(\xi) \le v_0(\xi), \qquad \forall \xi \in [\varepsilon, a_0],$$

and

$$w_2(\xi) \le w_0(\xi), \qquad v_2(\xi) \le v_0(\xi), \qquad \forall \xi \in [\varepsilon, a_0].$$

Proposition 4 implies that there exist three constants $M_1, M_2, M_3 > 0$ independent of ε such that when $\xi \in [\varepsilon, a_0]$, we have

$$|w_1(\xi) - w_0(\xi)| < M_3(\tau + a_0 M_2 \tau)\exp\{a_0 M_3 M_1\},$$
$$|v_1(\xi) - v_0(\xi)| < a_0 M_3 M_1(\tau + a_0 M_2 \tau)\exp\{a_0 M_3 M_1\} + a_0 M_2 \tau,$$

and

$$|w_2(\xi) - w_0(\xi)| < M_3(\tau + a_0 M_2 \tau)\exp\{a_0 M_3 M_1\},$$
$$|v_2(\xi) - v_0(\xi)| < a_0 M_3 M_1(\tau + a_0 M_2 \tau)\exp\{a_0 M_3 M_1\} + a_0 M_2 \tau.$$

Thus when $\xi \in [\varepsilon, a_0]$, we get that

$$|w_1(\xi) - w_2(\xi)| < 2\tau M_3(1 + a_0 M_2)\exp\{a_0 M_3 M_1\},$$

and

$$|v_1(\xi) - v_2(\xi)| < 2\tau(a_0 M_3 M_1(1 + a_0 M_2)\exp\{a_0 M_3 M_1\} + a_0 M_2).$$

Owing to the arbitrariness of $\tau > 0$ and $\varepsilon \in (0, \delta)$, we get that

$$w_1(\xi) = w_2(\xi), \qquad v_1(\xi) = v_2(\xi), \qquad \forall \xi \in (0, a_0].$$

Owing to the arbitrariness of $a_0 \in (0, a_1)$, we get that

$$w_1(\xi) = w_2(\xi), \qquad v_1(\xi) = v_2(\xi), \qquad \forall \xi \in (0, a_1).$$

The left continuity of w_1 and w_2 at a_1 implies

$$w_1(a_1) = w_2(a_1) = 0.$$

So

$$a_1 = a_2.$$

In addition,

$$w_1(0) = w_2(0) = A, \qquad v_1(0) = v_2(0) = 0.$$

Thus (w_1, v_1) and (w_2, v_2) are the same solution with compact support of the initial value problem (2.2),(2.3). The proof is complete.

Proof of Theorem 3. From Theorem 1,

$$w(\xi) > 0, \quad v(\xi) < 0, \quad \forall \xi \in (0, a).$$

Since $\lim\limits_{\xi \to a^-} w(\xi) = 0$ and $\alpha > 0$, there exists $\delta \in (0, a)$ such that

$$v'(\xi) < 0, \quad \forall \xi \in (\delta, a).$$

Thus $\lim\limits_{\xi \to a^-} v(\xi)$ exists and is negative, or $\lim\limits_{\xi \to a^-} v(\xi) = -\infty$. From the L'Hospital rule, we see that

$$\lim_{\xi \to a^-} \frac{a - \xi}{w^m(\xi)} = \lim_{\xi \to a^-} \frac{-\xi}{v(\xi)} = -\frac{a}{\lim\limits_{\xi \to a^-} v(\xi)} < +\infty.$$

Noticing that $m > 1$ we see that $\frac{1}{w(\xi)}$ is integrable on $[0, a]$. So $\frac{s^{n-1}}{w(s^2)}$ is integrable on $[\,0, \sqrt{a}\,]$.

Let $\phi(x)$ be a continuous function on $\mathbb{R}^n$. From the replacement formula of variables and the integral mean value theorem, we get that

$$\begin{aligned}
&\int_{\mathrm{supp}u(\cdot,t)} \frac{\phi(x)}{u(x,t)} dx \\
= &\ (t+1)^{\beta} \int_{\mathrm{supp}u(\cdot,t)} \frac{\phi(x)}{w((t+1)^{\alpha}|x|^2)} dx \\
= &\ \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} (t+1)^{\beta - n\alpha/2} \phi(\zeta) \int_0^{\sqrt{a}} \frac{s^{n-1}}{w(s^2)} ds \\
= &\ \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} (t+1)^{(2-nm+nq)/(2(q-1))} \phi(\zeta) \int_0^{\sqrt{a}} \frac{s^{n-1}}{w(s^2)} ds,
\end{aligned}$$

where $\zeta \in \mathbb{R}^n$ and $|\zeta| \le (t+1)^{-\alpha/2}\sqrt{a}$. The proof is complete.

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