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QUASISTATIC VISCOELASTIC CONTACT WITH FRICTION AND WEAR DIFFUSION

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Abstract. We consider a quasistatic problem of frictional contact between a deformable body and a moving foundation. The material is assumed to have nonlinear viscoelastic behavior. The contact is modeled with normal compliance and the associated law of dry friction. The wear takes place on a part of the contact surface and its rate is described by the Archard differential condition. The main novelty in the model is the diffusion of the wear particles over the potential contact surface. Such phenomena arise in orthopaedic biomechanics where the wear debris diffuse and influence the properties of joint prosthesis and implants. We derive a weak formulation of the model which is given by a coupled system with an evolutionary variational inequality and a nonlinear evolutionary variational equation. We prove that, under a smallness assumption on some of the data, there exists a unique weak solution for the model.

1. Introduction. Frictional contact between deformable bodies can be frequently found in industry and everyday life. The contact between a train wheel and the rails, a shoe and the floor, the car's braking pad and the wheel, or contact between tectonic plates are only a few examples. Considerable progress has been made in modeling and analyzing static contact problems and the literature on this topic is extensive. Only recently, however, have the quasistatic and dynamic problems been considered in the mathematical literature. The reason lies in the considerable difficulties that the process of frictional contact presents in the modeling and analysis because of the complicated nonlinear surface phenomena involved.

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Quasistatic elastic contact problems with normal compliance and friction have been considered in [4] and [15], where the existence of weak solutions has been proven. The existence of a weak solution to the, technically very complicated, problem with Signorini's contact condition has been established in [7]. General models for thermoelastic frictional contact were derived from thermodynamical principles in [12, 29, 30]. Quasistatic frictional contact problems for viscoelastic materials can be found in [19, 23] and those for elastoviscoplastic materials in [2, 3, 26]. Dynamic problems with normal compliance were first considered in [16]. The existence of weak solutions to dynamic thermoelastic contact problems with frictional heat generation has been proven in [5] and, when wear is taken into account, in [6]. Models and problems with wear can be found in [5, 20, 21, 29, 30, 32].

The mathematical, mechanical, and numerical state of the art in Contact Mechanics can be found in the proceedings [17, 18], in the special issue [22], and in the recent monographs [11] and [24]. In the latter, a more comprehensive literature on problems with wear is provided.

In this work, we consider the process of contact with friction and wear between a viscoelastic body and a moving foundation. We assume that the forces and tractions change slowly in time so that accelerations in the system are negligible. This leads to the quasistatic approximation for the process. The material is assumed to be nonlinearly viscoelastic. The contact is modeled with a normal compliance condition and friction with a general law of dry fiction. The wear takes place only on a part of the contact surface and the wear rate is described by the differential Archard condition. The main novelty in the model is that it takes into account the diffusion of the wear particles or debris over the whole of the contact surface. Such phenomena can be found in many engineering settings; however, in all mathematical publications on wear, it is assumed that the wear particles are removed from the surface once they are formed. Here, they are assumed to remain and diffuse on the contact surface.

This work is motivated by biomechanical applications. Indeed, such problems arise in joints after arthroplasty (knee, hip, shoulder, elbow, etc.), where debris are produced by articulating parts of the prosthesis and are transported to the bone-implant interface. These debris cause the deterioration of the interface and are believed to be an important factor leading to prosthesis loosening (see, e.g., [20, 21] and references therein). Hence there is a considerable interest in modeling such complex contact problems arising in implanted joints. This pertains to both cementless (the so-called "press-fit") and cemented implants. Our paper opens a new way to studying contact problems with friction and wear diffusion. In fact, for many contact problems, one should also take into account the process of adhesion that is coupled with friction and wear diffusion. For instance, clinical practice shows that adhesion plays an important role at the bone-implant interface, and for further details we refer to the references in [20, 21]. We hope to deal with contact problems with friction, adhesion, and wear diffusion in the near future.

Our aim here is threefold: we describe the mechanical model for the processes, derive its variational formulation, and prove an existence and uniqueness of the solution. These results form the background for the numerical treatment of the problem and represent a first step in the study of more complicated frictional contact problems with wear, with emphasis on applications in orthopaedic biomechanics. In later stages the assumption that the contacting surfaces are planar will be relaxed, leading to diffusion on manifolds. Other assumptions can and will be relaxed, too, to have the model better reflect reality. A related paper is [25], where this model has been announced.

The paper is organized as follows. In Sec. 2 we describe the classical model. In Sec. 3 we list the assumptions on the problem data and derive its variational formulation. It is in a form of a system coupling an evolutionary variational inequality with an evolutionary variational equation. Then, we present our main existence and uniqueness result in Theorem 3.1. It states that, under a smallness assumption on the normal compliance function and the coefficient of friction, there exists a unique weak solution for the model. The proof of the Theorem is presented in Sec. 4. It is based on arguments of parabolic evolutionary equations, elliptic variational inequalities, and a fixed point theorem. A short summary can be found in Sec. 5, where some open problems are mentioned.

2. The model. We are interested in the following process and setting. A viscoelastic body occupies a domain $\Omega \subset \mathbb{R}^3$ and is acted upon by volume forces and surface tractions, and consequently its mechanical state evolves. The body may come into frictional contact with a moving foundation and, as a result of friction, a part of the surface undergoes wear. The wear particles or debris produced in this manner diffuse on the whole of the contact surface. This is in contrast to the usual assumption that the wear debris is removed instantly from the surface (see, e.g., [5, 20, 21, 29, 30, 32] and references therein). The presence of these particles influences the process considerably. If the debris is made of a material that is harder than that of the body, it may produce grooves and cause damage to the contacting surface; if it is softer, it may act as a lubricant.

To proceed we introduce the following notation. $\mathbb{E}^3_s$ represents the space of second order symmetric tensors on $\mathbb{R}^3$ while "$\cdot$" and $\|\cdot\|$ denote the inner product and the Euclidean norm, respectively, on the spaces $\mathbb{R}^2$, $\mathbb{R}^3$, or $\mathbb{E}^3_s$. Also, $\boldsymbol{\nu}$ denotes the outward unit normal to Ω and $[0,T]$ is the time interval of interest, for $T>0$.

Let Γ denote the boundary of Ω. It is assumed to be Lipschitz, and is divided into three disjoint measurable parts Γ_D, Γ_N, and Γ_C, such that $\text{meas}\Gamma_D > 0$ and $\text{meas}\Gamma_C > 0$. The body is clamped on Γ_D, prescribed surface tractions of density $\boldsymbol{f}_N$ act on Γ_N, and volume forces of density $\boldsymbol{f}_0$ act on Ω. An initial gap g exists between the potential contact surface Γ_C and the foundation, and is measured along the outward normal $\boldsymbol{\nu}$. To simplify the model we assume that the coordinate system is such that Γ_C occupies a regular domain in the Ox_1x_2 plane and the foundation is moving with velocity $\boldsymbol{v}^*$ in the Ox_1x_2 plane.

The wear resulting from friction happens on a part of Γ_C, and the wear particles or debris diffuse on the whole Γ_C. To describe this process it is assumed that Γ_C is divided into two subdomains D_d and D_w by a smooth curve γ^*, and wear takes place only on the part D_w, while the diffusion of the particles takes place in the whole of Γ_C. The boundary $\gamma = \partial\Gamma_C$ of Γ_C is assumed Lipschitz and is composed of two parts γ_d and γ_w. Then, $\partial D_w = \gamma_w \cup \gamma^*$ and $\partial D_d = \gamma_d \cup \gamma^*$. The setting is depicted schematically in Figs. 1 and 2.

We denote by $\boldsymbol{u}$ the displacement vector, $\boldsymbol{\sigma}$ the stress tensor field, and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ the linearized strain tensor field. On the boundary Γ, u_ν and $\boldsymbol{u}_\tau$ represent the *normal*

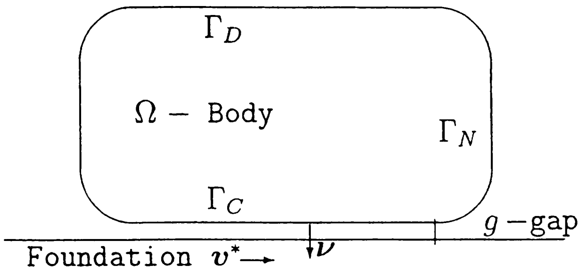

FIG. 1. The physical setting; Γ_C is the contact surface.

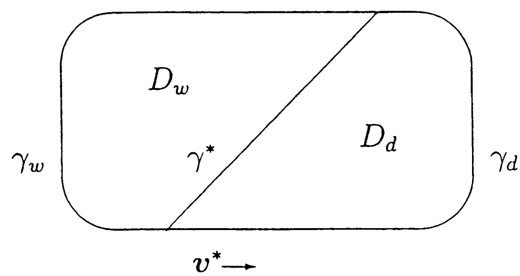

FIG. 2. The contact surface Γ_C; wear debris is produced on D_w.

and *tangential* displacements, respectively, while σ_ν and $\boldsymbol{\sigma}_\tau$ represent the *normal* and *tangential* stresses, respectively; a dot above a variable represents the time derivative and, for the sake of simplicity, we do not show explicitly the dependence of various functions on $\boldsymbol{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$.

The viscoelastic constitutive law of the material is assumed to be

$$\boldsymbol{\sigma} = \mathcal{A}(\varepsilon(\dot{\boldsymbol{u}})) + \mathcal{G}(\varepsilon(\boldsymbol{u})) \quad \text{in } \Omega \times (0, T), \tag{2.1}$$

in which $\mathcal{A}$ is the viscosity tensor function and $\mathcal{G}$ is the elasticity one; both are given nonlinear constitutive functions. We recall that in linear viscoelasticity the stress tensor $\boldsymbol{\sigma} = (\sigma_{ij})$ is given by

$$\sigma_{ij} = a_{ijkl}\varepsilon_{kl}(\dot{\boldsymbol{u}}) + g_{ijkl}\varepsilon_{kl}(\boldsymbol{u}), \tag{2.2}$$

where $\mathcal{A} = (a_{ijkl})$ is the viscosity tensor and $\mathcal{G} = (g_{ijkl})$ the elasticity tensor, for $i, j, k, l = 1, 2, 3$.

We turn to model the process of surface wear and the diffusion of the wear particles or debris. In this work we use a rather "simple" model; more sophisticated and elaborate models will be considered in the future. Our interest lies in the case when the wear of the surface resulting from material removal takes place only on D_w, while the wear particles diffuse on the whole of the contact surface Γ_C. This choice is motivated by the biomechanical applications, as mentioned in the Introduction (see [8, 20, 21] for details). We describe the wear of the surface in terms of the *wear function* $w = w(\boldsymbol{x}, t)$, which is defined on D_w and the diffusion of the wear particles by the *wear particle surface density function* $\zeta = \zeta(\boldsymbol{x}, t)$, which is defined on Γ_C. Notice that here $\boldsymbol{x} = (x_1, x_2, 0)$, since Γ_C belongs to the plane Ox_1x_2. The wear function w measures the volume density of material removed per unit surface area; thus, it describes the average depth of the grooves on D_w and the corresponding change in the surface geometry. The function ζ measures the surface density of the diffusing wear particles.

In this work we assume that $w = \eta\zeta$ in D_w, where η is a conversion factor from wear debris surface density to wear depth, which we assume to be a positive constant. This assumption simplifies the model, since it allows for the elimination of the wear function w. However, it would be of interest to investigate a model without this assumption. For the sake of convenience we extend w by zero to the whole of Γ_C, and below, when confusion is unlikely, we use the same symbol for the function and its extension. Thus,

$$w = \eta\zeta\chi_{[D_w]} \quad \text{on } \Gamma_C \times (0,T), \tag{2.3}$$

where $\chi_{[D_w]}$ is the characteristic function of the set D_w (i.e., $\chi_{[D_w]}(\boldsymbol{x}) = 1$ when $\boldsymbol{x} \in D_w$ and $\chi_{[D_w]}(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \notin D_w$).

The *wear diffusion coefficient* k is given by

$$k = k(\mathbf{x}) = \begin{cases} k_w & \text{in } D_w, \\ k_d & \text{in } D_d, \end{cases}$$

to allow for the different surface characteristics in D_w and D_d. Then, the diffusion of the particles or debris is described by the diffusion equation,

$$\dot{\zeta} - \operatorname{div}(k\nabla\zeta) = \kappa\|\boldsymbol{\sigma}_\tau\|\,\|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|\chi_{[D_w]} \quad \text{in } \Gamma_C \times (0,T). \tag{2.4}$$

We use $\chi_{[D_w]}$ on the right-hand side of (2.4) since the debris is produced only in D_w. Here ∇ and "div" denote the gradient and the divergence operators in the variables x_1 and x_2, respectively, and κ is the wear rate coefficient. We note that (2.4) contains the rate form of Archard's law, which expresses the fact that the wear rate is proportional to the intensity of the friction traction and the relative slip rate. Indeed, when the diffusion of the wear particles is negligible, (2.4) may be written as

$$\dot{\zeta} = \kappa\|\boldsymbol{\sigma}_\tau\|\,\|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|\chi_{[D_w]},$$

which is the differential form of Archard's law of wear (see, e.g., [5, 29, 30] and references therein).

Now, to avoid some mathematical difficulties which arise when the slip rate is very large, we replace the term $\|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|$ in (2.4) by the term $R^*(\|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|)$ where $R^* : \mathbb{R}_+ \to \mathbb{R}_+$ is the truncation operator

$$R^*(r) = \begin{cases} r & \text{if } r \le R, \\ R & \text{if } r > R, \end{cases} \tag{2.5}$$

R being a fixed positive constant. We note that from the applied point of view this does not cause any real change in the model, since in practice the slip velocity is bounded and no smallness assumption is imposed on R, thus it may be chosen as large as necessary in each application. To conclude, wear diffusion is described by the following nonlinear diffusion equation

$$\dot{\zeta} - \operatorname{div}(k\nabla\zeta) = \kappa\|\boldsymbol{\sigma}_\tau\| R^*(\|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|)\chi_{[D_w]} \quad \text{in } \Gamma_C \times (0,T). \tag{2.6}$$

We assume that once a wear particle reaches the boundary $\gamma = \partial\Gamma_C$ it disappears; i.e., we assume an absorbing boundary condition,

$$\zeta = 0 \quad \text{on } \gamma \times (0, T). \tag{2.7}$$

Next, we describe the process of frictional contact on the surface Γ_C. We use a version of the *normal compliance* condition to model the contact and a general *law of dry friction* to model friction. We recall that in the case without wear, a general version of the normal compliance condition is given by

$$-\sigma_\nu = p_\nu(u_\nu - g), \tag{2.8}$$

where p_ν is a prescribed positive function, such that $p_\nu(r) = 0$ for $r \leq 0$; moreover, the quantity $u_\nu - g$, when positive, represents the interpenetration of the body's surface asperities into those of the foundation. Such contact condition was proposed in [16] and used in a number of publications; see, e.g., [5, 13, 14, 15] and references therein. In this condition the interpenetration is allowed but penalized. In [14, 16], the following form of the normal compliance function was employed,

$$p_\nu(r) = c_\nu(r)_+^{m_\nu}, \tag{2.9}$$

where c_ν is a positive constant, m_ν is a positive exponent, and $r_+ = \max\{0, r\}$. Formally, Signorini's nonpenetration condition is obtained in the limit $c_\nu \to \infty$.

Since our process involves the wear of the contacting surfaces we need to take into account the change in the geometry by replacing the initial gap function g with $g + w$ during the process. Therefore, keeping in mind (2.8) and (2.3), we obtain

$$-\sigma_\nu = p_\nu(u_\nu - \eta\zeta\chi_{[D_w]} - g) \quad \text{on } \Gamma_C \times (0, T). \tag{2.10}$$

The precise assumptions on p_ν will be given below. The associated friction law is chosen as

$$\begin{gathered} \|\boldsymbol{\sigma}_\tau\| \leq \mu|\sigma_\nu|, \quad \text{on } \Gamma_C \times (0, T), \\ \text{if } \dot{\boldsymbol{u}}_\tau \neq \boldsymbol{v}^* \quad \text{then } \boldsymbol{\sigma}_\tau = -\mu|\sigma_\nu|\frac{\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*}{\|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|}. \end{gathered} \tag{2.11}$$

Here, μ is the coefficient of friction which is assumed to depend on the density of the wear particles and on the slip rate, that is

$$\mu = \mu(\zeta, \|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|),$$

and will be described below.

We note that this is a novelty to have the friction coefficient depend on the wear.

To conclude, keeping in mind (2.1), (2.6), (2.7), (2.10), and (2.11), the classical formulation of the problem of *frictional contact of a viscoelastic body with wear diffusion* is as follows.

PROBLEM P. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^3$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{E}_s^d$, *and a surface particle density field* $\zeta : \Gamma_C \times [0, T] \to \mathbb{R}$, *such that*

$$\boldsymbol{\sigma} = \mathcal{A}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})) + \mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in } \Omega \times (0, T), \tag{2.12}$$

$$\mathrm{div}\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega \times (0, T), \tag{2.13}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D \times (0, T), \tag{2.14}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_N \quad \text{on } \Gamma_N \times (0, T), \tag{2.15}$$

$$-\sigma_\nu = p_\nu \quad \text{on } \Gamma_C \times (0, T), \tag{2.16}$$

$$\left.\begin{aligned} &\|\boldsymbol{\sigma}_\tau\| \le \mu p_\nu, \\ &\boldsymbol{\sigma}_\tau = -\mu p_\nu \frac{\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*}{\|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|} \text{ if } \dot{\boldsymbol{u}}_\tau \neq \boldsymbol{0} \end{aligned}\right. \quad \text{on } \Gamma_C \times (0, T), \tag{2.17}$$

$$\dot{\zeta} - \mathrm{div}(k\nabla\zeta) = \kappa\mu p_\nu R^*(\|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|)\chi_{[D_w]} \quad \text{on } \Gamma_C \times (0, T), \tag{2.18}$$

$$\zeta = 0 \quad \text{on } \gamma \times (0, T), \tag{2.19}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \zeta(0) = \zeta_0 \quad \text{in } \Omega. \tag{2.20}$$

Here, $\mu = \mu(\zeta, \|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|)$ and $p_\nu = p_\nu(u_\nu - \eta\zeta\chi_{[D_w]} - g)$; (2.13) is the equilibrium equation, since the process is assumed to be quasistatic; (2.14) and (2.15) are the displacement and traction boundary conditions, respectively; and (2.20) are the initial conditions, in which $\boldsymbol{u}_0$ and ζ_0 are given.

3. Variational formulation. To obtain a variational formulation for problem P we need additional notation and some preliminaries. We use the standard notation for L^p and Sobolev spaces associated with the domains $\Omega \subset \mathbb{R}^3$ and $\Gamma_C \subset \mathbb{R}^2$ (see, e.g., [1]). Moreover, we let

$$\begin{aligned} H &= \{\boldsymbol{v} = (v_i) \mid v_i \in L^2(\Omega)\} = L^2(\Omega)^3, \\ H_1 &= \{\boldsymbol{v} = (v_i) \mid v_{i,j} \in L^2(\Omega)\} = H^1(\Omega)^3, \\ Q &= \{\boldsymbol{\tau} = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega)\} = L^2(\Omega)_s^{3\times3}, \\ Q_1 &= \{\boldsymbol{\tau} \in Q \mid \tau_{ij,j} \in H\}. \end{aligned}$$

Here and throughout this paper, $i, j \in \{1, 2, 3\}$, the summation convention over repeated indices is employed, and an index following a comma indicates a partial derivative with respect to the corresponding variable.

The spaces H, Q, H_1, and Q_1 are real Hilbert spaces endowed with inner products given by

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i \, dx, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \sigma_{ij}\tau_{ij} \, dx,$$

$$(\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{Q_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q + (\mathrm{div}\boldsymbol{\sigma}, \mathrm{div}\boldsymbol{\tau})_H,$$

respectively. Here $\boldsymbol{\varepsilon} : H_1 \to \mathcal{H}$ and div : $\mathcal{H}_1 \to H$ are the *deformation* and *divergence* operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad (\mathrm{div}\boldsymbol{\sigma})_i = (\sigma_{ij,j}).$$

The associated norms on the spaces H, H_1, Q, and Q_1 are denoted by $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_Q$, and $\|\cdot\|_{Q_1}$, respectively.

For an element $\boldsymbol{v} \in H_1$ we denote by $\boldsymbol{v}$ its trace on Γ and by $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$ its *normal* component and *tangential* part on the boundary. We also denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the *normal* and *tangential* traces of $\boldsymbol{\sigma} \in Q_1$. If $\boldsymbol{\sigma}$ is a regular function (e.g., C^1), then $\sigma_\nu(\boldsymbol{\sigma\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu}$. Moreover, the following Green formula holds:

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\mathrm{div}\boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma\nu}\cdot\boldsymbol{v}\, dS \qquad \forall \boldsymbol{v} \in H_1, \tag{3.1}$$

where dS is the surface measure on Γ.

Let V be the closed subspace of H_1 given by

$$V = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \mathbf{0} \quad \text{on } \Gamma_D\},$$

and denote

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \qquad \forall \boldsymbol{u},\ \boldsymbol{v} \in V. \tag{3.2}$$

Since $\mathrm{meas}(\Gamma_D) > 0$ it follows from Korn's inequality that $(V, (\cdot,\cdot)_V)$ is a real Hilbert space, and the associated norm is denoted by $\|\cdot\|_V$. By the Sobolev trace theorem there exists a constant $c_\Gamma > 0$, which depends on Ω, Γ_D, and Γ_C, such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_C)^3} \le c_\Gamma \|\boldsymbol{v}\|_V \qquad \forall \boldsymbol{v} \in V. \tag{3.3}$$

Recall that Γ_C is assumed to be a regular domain in the Ox_1, x_2 plane with Lipschitz boundary γ. Keeping in mind the boundary condition (2.19), for the surface particle density function we shall use the space

$$H_0^1(\Gamma_C) = \{\xi \in H^1(\Gamma_C) \mid \xi = 0 \quad \text{on } \gamma\}.$$

This is a real Hilbert space endowed with the inner product

$$(\zeta, \xi)_{H_0^1(\Gamma_C)} = (\nabla\zeta, \nabla\xi)_{L^2(\Gamma_C)^2},$$

where $\nabla : H_0^1(\Gamma_C) \to L^2(\Gamma_C)^2$ denotes the gradient operator, that is $\nabla\xi = (\xi_{,x_1}, \xi_{,x_2})$. Note that by the Friedrichs-Poincaré inequality there exists a constant $\tilde{c}_\Gamma > 0$, which depends on Γ_C, such that

$$\|\zeta\|_{L^2(\Gamma_C)} \le \tilde{c}_\Gamma \|\zeta\|_{H_0^1(\Gamma_C)} \qquad \forall \zeta \in H_0^1(\Gamma_C). \tag{3.4}$$

We use the notation $H^{-1}(\Gamma_C)$ for the dual of the space $H_0^1(\Gamma_C)$. Identifying $L^2(\Gamma_C)$ with its own dual we can write $H_0^1(\Gamma_C) \subset L^2(\Gamma_C) \subset H^{-1}(\Gamma_C)$. Below, $\langle\cdot,\cdot\rangle$ represents the duality pairing between $H^{-1}(\Gamma_C)$ and $H_0^1(\Gamma_C)$, and $\|\cdot\|_{H^{-1}(\Gamma_C)}$ denotes the norm on the dual space $H^{-1}(\Gamma_C)$. Also, $\langle\zeta, \xi\rangle = (\zeta, \xi)_{L^2(\Gamma_C)}$ for $\zeta \in L^2(\Gamma_C)$ and $\xi \in H_0^1(\Gamma_C)$.

Finally, if $(X, (\|\cdot\|)_X)$ is a real Banach space and $T > 0$, we denote by $C([0,T]; X)$ and $C^1([0,T]; X)$ the spaces of continuous and continuously differentiable functions from $[0,T]$ to X, with norms

$$\|\varphi\|_{C([0,T];X)} = \max_{t\in[0,T]} \|\varphi(t)\|_X, \quad \|\varphi\|_{C^1([0,T];X)} = \max_{t\in[0,T]} \|\varphi(t)\|_X + \max_{t\in[0,T]} \|\dot{\varphi}(t)\|_X.$$

Moreover, we use the Lebesgue space $L^2(0,T;X)$ with the usual norm

$$\|\varphi\|_{L^2(0,T;X)} = \left(\int_0^T \|\varphi(t)\|_X^2\, dt\right)^{\frac{1}{2}}.$$

To study the mechanical problem P we make the following assumptions on the problem data.

The *viscosity operator* $\mathcal{A} : \Omega \times \mathbb{E}^3_s \to \mathbb{E}^3_s$ satisfies the following: there exist two positive constants $L_{\mathcal{A}}$ and $m_{\mathcal{A}}$ such that

$$\begin{aligned}
&(a) \quad \|\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \qquad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{E}^3_s, \text{a.e. } \boldsymbol{x} \in \Omega;\\
&(b) \quad (\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2\\
&\qquad\qquad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{E}^3_s, \ \text{a.e. } \boldsymbol{x} \in \Omega;\\
&(c) \quad \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega \qquad \forall \boldsymbol{\varepsilon} \in \mathbb{E}^3_s;\\
&(d) \quad \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x},\boldsymbol{0}) \in Q.
\end{aligned} \tag{3.5}$$

The *elasticity operator* $\mathcal{G} : \Omega \times \mathbb{E}^3_s \to \mathbb{E}^3_s$ satisfies the following: there exists a positive constant $L_{\mathcal{G}}$ such that

$$\begin{aligned}
&(a) \quad \|\mathcal{G}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{G}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \qquad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{E}^3_s, \ \text{a.e. } \boldsymbol{x} \in \Omega;\\
&(b) \quad \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega \qquad \forall \boldsymbol{\varepsilon} \in \mathbb{E}^3_s;\\
&(c) \quad \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x},\boldsymbol{0}) \in Q.
\end{aligned} \tag{3.6}$$

The *normal compliance function* $p_\nu : \Gamma_C \times \mathbb{R} \to \mathbb{R}_+$ satisfies the following: there exist two positive constants L_ν and p^*_ν such that

$$\begin{aligned}
&(a) \quad |p_\nu(\boldsymbol{x},u_1) - p_\nu(\boldsymbol{x},u_2)| \le L_\nu|u_1 - u_2| \qquad \forall u_1, u_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_C;\\
&(b) \quad \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x},u) \text{ is Lebesgue measurable on } \Gamma_C \qquad \forall u \in \mathbb{R};\\
&(c) \quad \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x},u) = 0 \text{ for } u \le 0, \ \text{a.e. } \boldsymbol{x} \in \Gamma_C;\\
&(d) \quad p_\nu(\boldsymbol{x},u) \le p^*_\nu \qquad \forall u \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_C.
\end{aligned} \tag{3.7}$$

The *coefficient of friction* $\mu : \Gamma_C \times \mathbb{R}^2 \to \mathbb{R}_+$ satisfies the following: there exist two positive constants L_μ and μ^* such that

$$\begin{aligned}
&(a) \quad |\mu(\boldsymbol{x},a_1,b_1) - \mu(\boldsymbol{x},a_2,b_2)| \le L_\mu(|a_1 - a_2| + |b_1 - b_2|)\\
&\qquad\qquad \forall a_1, a_2, b_1, b_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_C;\\
&(b) \quad \boldsymbol{x} \mapsto \mu(\boldsymbol{x},a,b) \text{ is Lebesgue measurable on } \Gamma_C \qquad \forall a, b \in \mathbb{R};\\
&(c) \quad \mu(\boldsymbol{x},a,b) \le \mu^* \qquad \forall a, b \in \mathbb{R}, \text{a.e. } \boldsymbol{x} \in \Gamma_C.
\end{aligned} \tag{3.8}$$

The assumptions (3.5) on the viscosity operator are rather routine, and effectively follow from the linear case (2.2), and so are the assumptions (3.6) on the elasticity operator $\mathcal{G}$. The main restriction on p_ν in (3.7) is its boundedness. Although the function p_ν in (2.9) does not satisfy condition (3.7), the truncated function

$$p^\rho_\nu(r) = \begin{cases} c_\nu(r_+)^{m_\nu} & \text{if } r \le \rho, \\ c_\nu(r_+)^{m_\nu} & \text{if } r > \rho, \end{cases}$$

does, for a given $\rho > 0$, and p^ρ_ν coincides with p_ν on $(-\infty, \rho]$. Since the interpenetration of body's surface asperities into those of the foundation are supposed to be small, replacing the normal compliance function (2.9) with the regularized normal compliance function p^ρ_ν does not represent a practical restriction of the model. A similar comment could be made on the assumptions (3.8) on the coefficient of friction which is assumed to be Lipschitz continuous and bounded.

The forces and tractions are assumed to satisfy

$$\boldsymbol{f}_0 \in C([0,T];H), \qquad \boldsymbol{f}_N \in C([0,T];L^2(\Gamma_N)^d); \tag{3.9}$$

the initial gap function satisfies

$$g \in L^2(\Gamma_C), \qquad g \geq 0 \text{ a.e. on } \Gamma_C; \tag{3.10}$$

the wear diffusion coefficient satisfies

$$k \in L^\infty(\Gamma_C), \qquad k \geq k^* > 0 \text{ a.e. on } \Gamma_C; \tag{3.11}$$

and the wear rate coefficient satisfies

$$\kappa \in L^\infty(\Gamma_{D_w}), \qquad \kappa \geq 0 \text{ a.e. on } \Gamma_{D_w}. \tag{3.12}$$

Finally, we assume that the initial displacements and the initial surface particle density satisfy

$$\boldsymbol{u}_0 \in V, \qquad \zeta_0 \in L^2(\Gamma_C). \tag{3.13}$$

Next, we define the vector valued function $\boldsymbol{f} : [0,T] \to V$ as

$$(\boldsymbol{f}(t),\boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{v}\,dx + \int_{\Gamma_N} \boldsymbol{f}_N(t)\cdot\boldsymbol{v}\,dS, \tag{3.14}$$

for all $\boldsymbol{v} \in V$, $t \in [0,T]$. We also define the functional $j : L^2(\Gamma_C)\times V^3 \to \mathbb{R}$ by

$$\begin{aligned} j(\zeta,\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) &= \int_{\Gamma_C} p_\nu(u_\nu - \eta\zeta\chi_{[D_w]} - g)w_\nu\,dS \\ &\quad + \int_{\Gamma_C} \mu(\zeta,\|\boldsymbol{v}_\tau - \boldsymbol{v}^*\|)p_\nu(u_\nu - \eta\zeta\chi_{[D_w]} - g)\|\boldsymbol{w}_\tau - \boldsymbol{v}^*\|\,dS, \end{aligned} \tag{3.15}$$

for all $\zeta \in L^2(\Gamma_C)$, $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in V$. The bilinear form $a : H_0^1(\Gamma_C)\times H_0^1(\Gamma_C) \to \mathbb{R}$ is defined as

$$a(\zeta,\xi) = \int_{\Gamma_C} k\nabla\zeta\cdot\nabla\xi\,dS \tag{3.16}$$

for all $\zeta,\xi \in H_0^1(\Gamma_C)$. Finally, the operator $F : H_0^1(\Gamma_C)\times V^3 \to H^{-1}(\Gamma_C)$ is given by

$$\begin{aligned} &\langle F(\zeta,\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}),\xi\rangle \\ &\quad = \int_{D_w} \kappa\mu(\zeta,\|\boldsymbol{v}_\tau - \boldsymbol{v}^*\|)p_\nu(u_\nu - \eta\zeta - g)R^*(\|\boldsymbol{w}_\tau - \boldsymbol{v}^*\|)\xi\,dS, \end{aligned} \tag{3.17}$$

for all $\zeta,\xi \in H_0^1(\Gamma_C)$, $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in V$.

We note that by conditions (3.7)–(3.11) the integrals in (3.14)–(3.17) are well defined. Moreover, we used the Riesz representation theorem to define the vector valued function $\boldsymbol{f}$.

We now turn to derive a variational formulation of the mechanical problem P. To that end we assume that $\{\boldsymbol{u},\boldsymbol{\sigma},\zeta\}$ is a triplet of regular functions satisfying (2.12)–(2.20) and let $\boldsymbol{v} \in V$, $\xi \in H_0^1(\Gamma_C)$, and $t \in [0,T]$. Using (3.1) and (2.13) we have

$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q = \int_\Omega \boldsymbol{f}_0(t)\cdot(\boldsymbol{v} - \dot{\boldsymbol{u}}(t))\,dx + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot(\boldsymbol{v} - \dot{\boldsymbol{u}}(t))\,dS,$$

and by (2.14), (2.15), and (3.14) we find

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q = (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V + \int_{\Gamma_C} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \dot{\boldsymbol{u}}(t))\, dS. \tag{3.18}$$

Using now (2.16) and (2.17), it follows that

$$\begin{aligned} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \dot{\boldsymbol{u}}(t)) &\geq -p_\nu(u_\nu(t) - \eta\zeta(t)\chi_{[D_w]} - g)(v_\nu - \dot{u}_\nu(t)) \\ &\quad - \mu(\zeta(t), \|\dot{\boldsymbol{u}}_\tau(t) - \boldsymbol{v}^*\|)p_\nu(u_\nu(t) - \eta\zeta(t)\chi_{[D_w]} - g)\|\boldsymbol{v}_\tau - \boldsymbol{v}^*\| \\ &\quad + \mu(\zeta(t), \|\dot{\boldsymbol{u}}_\tau(t) - \boldsymbol{v}^*\|)p_\nu(u_\nu(t) - \eta\zeta(t)\chi_{[D_w]} - g)\|\dot{\boldsymbol{u}}_\tau(t) - \boldsymbol{v}^*\|, \end{aligned}$$

a.e. on $\Gamma_C \times (0, T)$ and, keeping in mind (3.15), we find

$$\int_{\Gamma_C} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \dot{\boldsymbol{u}}(t))\, dS \geq j(\zeta(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \dot{\boldsymbol{u}}(t)) - j(\zeta(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \boldsymbol{v}). \tag{3.19}$$

Combining (3.18) and (3.19) yields

$$\begin{aligned} &(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q + j(\zeta(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \boldsymbol{v}) \\ &\qquad - j(\zeta(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V. \end{aligned} \tag{3.20}$$

On the other hand, multiplying (2.18) with ξ, integrating the result on Γ_C, and using the equality

$$\int_{\Gamma_C} \operatorname{div}(k\nabla\zeta(t))\xi\, dS = -\int_{\Gamma_C} k\nabla\zeta(t) \cdot \nabla\xi\, dS,$$

since $\xi \in H_0^1(\Gamma_C)$, we find

$$\begin{aligned} &\int_{\Gamma_C} \dot{\zeta}(t)\xi\, dS + \int_{\Gamma_C} k\nabla\zeta(t) \cdot \nabla\xi\, dS \\ &\quad = \int_{D_w} \kappa\mu(\zeta(t), \|\dot{\boldsymbol{u}}_\tau(t) - \boldsymbol{v}^*\|)p_\nu(u_\nu - \eta\zeta(t) - g)R^*(\|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|)\xi\, dS. \end{aligned}$$

We use now (3.16) and (3.17) in the previous equality to obtain

$$\langle\dot{\zeta}(t), \xi\rangle + a(\zeta(t), \xi) = \langle F(\zeta(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \dot{\boldsymbol{u}}(t)), \xi\rangle. \tag{3.21}$$

To conclude, we obtain from (3.20), (3.21), (2.12), and (2.20) the following variational formulation of problem P.

PROBLEM P_V. Find a displacement field $\boldsymbol{u} : [0, T] \to V$ and a surface particle density field $\zeta : [0, T] \to H_0^1(\Gamma_C)$ such that

$$\begin{aligned} &(\mathcal{A}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q + (\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q \\ &\qquad + j(\zeta(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \boldsymbol{v}) - j(\zeta(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \dot{\boldsymbol{u}}(t)) \\ &\quad \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \qquad \forall \boldsymbol{v} \in V,\ t \in [0, T], \end{aligned} \tag{3.22}$$

$$\begin{aligned} &\langle\dot{\zeta}(t), \xi\rangle + a(\zeta(t), \xi) = \langle F(\zeta(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \dot{\boldsymbol{u}}(t)), \xi\rangle \\ &\qquad \forall \xi \in H_0^1(\Gamma_C), \quad \text{a.e. } t \in (0, T), \end{aligned} \tag{3.23}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \zeta(0) = \zeta_0. \tag{3.24}$$

Our main result concerning the well-posedness of problem P_V is stated next and established in Sec. 4.

THEOREM 3.1. Assume that (3.5)–(3.13) hold. Then, there exists a constant $c^* > 0$, which depends on c_Γ, $\tilde{c}_\Gamma$, $m_\mathcal{A}$, L_ν, L_μ, $\|\kappa\|_{L^\infty(D_w)}$, η, and R such that, if $p_\nu^* < c^*$ and $\mu^* < c^*$, then there exists a unique solution of problem P_V. Moreover, the solution satisfies

$$\boldsymbol{u} \in C^1([0,T];V), \tag{3.25}$$

$$\zeta \in L^2(0,T;H_0^1(\Gamma_C)) \cap C([0,T];L^2(\Gamma_C)), \quad \dot{\zeta} \in L^2(0,T;H^{-1}(\Gamma_C)). \tag{3.26}$$

Let now $\{\boldsymbol{u}, \zeta\}$ denote a solution of Problem P_V and let $\boldsymbol{\sigma}$ be the stress field given by (2.12). Using (3.5) and (3.6) it follows that $\boldsymbol{\sigma} \in C([0,T];Q)$ and, using (3.22), (3.14), and standard arguments, we find that $\operatorname{div}\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0}$, $\forall t \in [0,T]$. It follows now from (3.9) that $\operatorname{div}\boldsymbol{\sigma} \in C([0,T];H)$, which implies

$$\boldsymbol{\sigma} \in C([0,T];Q_1). \tag{3.27}$$

A triplet of functions $\{\boldsymbol{u}, \boldsymbol{\sigma}, \zeta\}$ which satisfies (2.12), (3.22)–(3.24) is called a *weak solution* of the mechanical problem P. We conclude by Theorem 3.1 that, under the assumptions (3.5)–(3.13), if the normal compliance function p_ν and the coefficient of friction μ are small enough, then problem P has a unique weak solution which satisfies (3.25)–(3.27).

We now comment on the variational problem P_V. The following features make P_V a rather difficult mathematical problem and make the strong assumption discussed above necessary:

- the dependence of the nonlinear and nondifferentiable functional j on the solution $\{\boldsymbol{u}, \zeta\}$, as well as on the derivative $\dot{\boldsymbol{u}}$;
- the dependence of the nonlinear operator F on the solution $\{\boldsymbol{u}, \zeta\}$ and on the derivative $\dot{\boldsymbol{u}}$;
- the strong coupling between the evolutionary variational inequality (3.22) and the evolutionary variational equation (3.23).

Clearly, the problem of frictional contact of a viscoelastic body with wear diffusion leads to a new and interesting mathematical model. We notice, however, that in the case when the wear of the contact surface Γ_C is taken into account but there is no diffusion of the wear particles, then the mechanical problem leads to a simplified mathematical model for which the existence of a unique weak solution has been proved in [19].

We end this section with the remark that the viscosity term has a regularization effect in the study of the problem P_V. Indeed, the study of the corresponding inviscid problem (i.e., problem (3.22)–(3.24) in which the viscosity tensor $\mathcal{A}$ vanishes) seems to lead to severe mathematical difficulties; we have a good reason to believe that additional smallness assumptions would be needed to prove the existence of a solution of the inviscid problem, while the uniqueness of the solution seems to be an open problem.

4. Proof.

The proof of Theorem 3.1 will be carried out in several steps, by using arguments of evolutionary equations, time-dependent elliptic variational inequalities, and a fixed point theorem. Similar arguments have been already used in [9, 10, 11, 19, 27] and therefore, when the modifications are straightforward, we omit the details.

We assume in what follows that (3.5)–(3.13) hold and, moreover,

$$c_\Gamma^2 p_\nu^* L_\mu < m_{\mathcal{A}}. \tag{4.1}$$

In the first step we solve the parabolic equation (3.23) under the assumption that F is given. More precisely, let $\theta \in L^2(0,T;H^{-1}(\Gamma_C))$ and consider the problem of finding $\zeta_\theta : [0,T] \to H_0^1(\Gamma_C)$ such that

$$\langle \dot{\zeta}_\theta(t), \xi \rangle + a(\zeta_\theta(t), \xi) = \langle \theta(t), \xi \rangle \qquad \forall \xi \in H_0^1(\Gamma_C), \text{ a.e. } t \in (0,T), \tag{4.2}$$

$$\zeta_\theta(0) = \zeta_0. \tag{4.3}$$

LEMMA 4.1. There exists a unique solution of problem (4.2)–(4.3). Moreover, it satisfies

$$\zeta_\theta \in L^2(0,T;H_0^1(\Gamma_C)) \cap C([0,T];L^2(\Gamma_C)), \quad \dot{\zeta}_\theta \in L^2(0,T;H^{-1}(\Gamma_C)). \tag{4.4}$$

Proof. The lemma follows from a well-known result for evolutionary equations with linear continuous operators and may be found in [31, pp. 424–425]. □

In the next step we solve the variational inequality (3.22) when $\zeta = \zeta_\theta$. To that end, let $\boldsymbol{z} \in C([0,T];V)$ and $\boldsymbol{w} \in C([0,T];V)$ be given and consider the following auxiliary variational inequality of finding $\boldsymbol{v}_{\theta z w} : [0,T] \to V$ such that

$$\begin{aligned} &(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_{\theta zw}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_{\theta zw}(t)))_Q + (\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{z}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_{\theta zw}(t)))_Q \\ &\quad + j(\zeta_\theta(t), \boldsymbol{z}(t), \boldsymbol{w}(t), \boldsymbol{v}) - j(\zeta_\theta(t), \boldsymbol{z}(t), \boldsymbol{w}(t), \boldsymbol{v}_{\theta zw}(t)) \\ &\quad \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{v}_{\theta zw}(t))_V \qquad \forall \boldsymbol{v} \in V, t \in [0,T]. \end{aligned} \tag{4.5}$$

LEMMA 4.2. There exists a unique solution $\boldsymbol{v}_{\theta zw} \in C([0,T];X)$ of problem (4.5).

Proof. It follows from standard arguments of variational inequalities (see for instance [11]) that there exists a unique element $\boldsymbol{v}_{\theta zw}(t)$ which solves (4.5) for each $t \in [0,T]$. Let us show that $\boldsymbol{v}_{\theta zw} : [0,T] \to V$ is continuous. Let $t_1, t_2 \in [0,T]$, and for the sake of simplicity we employ the notation $\boldsymbol{v}_{\theta zw}(t_i) = \boldsymbol{v}_i$, $\zeta_\theta(t_i) = \zeta_i$, $\boldsymbol{z}(t_i) = \boldsymbol{z}_i$ and $\boldsymbol{w}(t_i) = \boldsymbol{w}_i$ for $i = 1, 2$. Using (4.5) we easily derive the relation

$$\begin{aligned} &(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_1)) - \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_2)), \boldsymbol{\varepsilon}(\boldsymbol{v}_1) - \boldsymbol{\varepsilon}(\boldsymbol{v}_2))_Q \leq (\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{z}_1)) - \mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{z}_2)), \boldsymbol{\varepsilon}(\boldsymbol{v}_2) - \boldsymbol{\varepsilon}(\boldsymbol{v}_1))_Q \\ &\quad + j(\zeta_1, \boldsymbol{z}_1, \boldsymbol{w}_1, \boldsymbol{v}_2) - j(\zeta_1, \boldsymbol{z}_1, \boldsymbol{w}_1, \boldsymbol{v}_1) + j(\zeta_2, \boldsymbol{z}_2, \boldsymbol{w}_2, \boldsymbol{v}_1) - j(\zeta_2, \boldsymbol{z}_2, \boldsymbol{w}_2, \boldsymbol{v}_2) \\ &\quad + (\boldsymbol{f}_1 - \boldsymbol{f}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2)_V. \end{aligned}$$

Then, we use conditions (3.5)–(3.8) to obtain

$$\begin{aligned} m_{\mathcal{A}} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V &\leq (L_{\mathcal{G}} + c_\Gamma^2(L_\nu + \mu^* L_\nu)) \|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_V \\ &\quad + c_\Gamma(L_\nu \eta + \mu^* L_\nu \eta + p_\nu^* L_\mu) \|\zeta_1 - \zeta_2\|_{L^2(\Gamma_C)} \\ &\quad + c_\Gamma^2 p_\nu^* L_\mu \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_V + \|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_V. \end{aligned} \tag{4.6}$$

We deduce that $\boldsymbol{v}_{\theta zw} : [0,T] \to V$ is a continuous function. □

We now consider an operator $\Lambda_{\theta z} : C([0,T];V) \to C([0,T];V)$ defined by

$$\Lambda_{\theta z} \boldsymbol{w} = \boldsymbol{v}_{\theta zw}. \tag{4.7}$$

We have the following result.

LEMMA 4.3. The operator $\Lambda_{\theta z}$ has a unique fixed-point $\boldsymbol{w}_{\theta z} \in C([0,T];V)$.

Proof. Let $\boldsymbol{w}_1, \boldsymbol{w}_2 \in C([0,T];V)$ and let $\boldsymbol{v}_i$ denote the solution of (4.6) for $\boldsymbol{w} = \boldsymbol{w}_i$, i.e., $\boldsymbol{v}_i = \boldsymbol{v}_{\theta z w_i}$, $i = 1,2$. From the definition (4.7) we have

$$\|\Lambda_{\theta z}\boldsymbol{w}_1(t) - \Lambda_{\theta z}\boldsymbol{w}_2(t)\|_V = \|\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\|_V \qquad \forall t \in [0,T].$$

An argument similar to the one used in the proof of (4.6) shows that

$$m_{\mathcal{A}}\|\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\|_V \leq c_\Gamma^2 p_\nu^* L_\mu \|\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\|_V \qquad \forall t \in [0,T].$$

Keeping in mind (4.1), the two inequalities show that the operator $\Lambda_{\theta z}$ is a contraction on the Banach space $C([0,T];V)$, which concludes the proof of the lemma. $\square$

In what follows we denote by $\boldsymbol{w}_{\theta z}$ the fixed-point function stated in Lemma 4.3 and let $\boldsymbol{v}_{\theta z} \in C([0,T];X)$ be the function defined by

$$\boldsymbol{v}_{\theta z} = \boldsymbol{v}_{\theta z w_{\theta z}}. \tag{4.8}$$

We have $\Lambda_{\theta z}\boldsymbol{w}_{\theta z} = \boldsymbol{w}_{\theta z}$ and

$$\boldsymbol{v}_{\theta z} = \boldsymbol{w}_{\theta z} \tag{4.9}$$

by (4.7) and (4.8). Therefore, choosing $\boldsymbol{w} = \boldsymbol{w}_{\theta z}$ in (4.5) and using (4.8) and (4.9), we see that $\boldsymbol{v}_{\theta z}$ satisfies

$$\begin{aligned}(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_{\theta z}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_{\theta z}(t)))_Q + (\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{z}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_{\theta z}(t)))_Q \\ + j(\zeta_\theta(t), \boldsymbol{z}(t), \boldsymbol{v}_{\theta z}(t), \boldsymbol{v}) - j(\zeta_\theta(t), \boldsymbol{z}(t), \boldsymbol{v}_{\theta z}(t), \boldsymbol{v}_{\theta z}(t)) \\ \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{v}_{\theta z}(t))_V \qquad \forall \boldsymbol{v} \in V,\ t \in [0,T].\end{aligned} \tag{4.10}$$

We denote by $\boldsymbol{u}_{\theta z} \in C^1([0,T];V)$ the function

$$\boldsymbol{u}_{\theta z}(t) = \int_0^t \boldsymbol{v}_{\theta z}(s)ds + \boldsymbol{u}_0 \qquad \forall t \in [0,T], \tag{4.11}$$

and define the operator $\Lambda_\theta : C([0,T];V) \to C([0,T];V)$ by

$$\Lambda_\theta \boldsymbol{z} = \boldsymbol{u}_{\theta z}. \tag{4.12}$$

We have the following fixed-point result.

LEMMA 4.4. The operator Λ_θ has a unique fixed-point $\boldsymbol{z}_\theta \in C([0,T];V)$.

Proof. Let $\boldsymbol{z}_1, \boldsymbol{z}_2 \in C([0,T];V)$ and denote $\boldsymbol{v}_i = \boldsymbol{v}_{\theta_{z_i}}$, $\boldsymbol{u}_i = \boldsymbol{u}_{\theta_{z_i}}$ for $i = 1,2$. Using (4.10) and the estimates in the proof of Lemma 4.2 yield

$$(m_{\mathcal{A}} - c_\Gamma^2 p_\nu^* L_\mu)\|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_V \leq (L_{\mathcal{G}} + c_\Gamma^2(L_\nu + \mu^* L_\nu))\|\boldsymbol{z}_1(s) - \boldsymbol{z}_2(s)\|_V, \tag{4.13}$$

for all $s \in [0,T]$. Using now (4.11) (4.13) we obtain

$$\|\Lambda_\theta \boldsymbol{z}_1(t) - \Lambda_\theta \boldsymbol{z}_2(t)\|_V \leq \frac{L_{\mathcal{G}} + c_\Gamma^2 L_\nu(1+\mu^*)}{m_{\mathcal{A}} - c_\Gamma^2 p_\nu^* L_\mu}\int_0^t \|\boldsymbol{z}_1(s) - \boldsymbol{z}_2(s)\|_V\, ds,$$

for all $t \in [0,T]$. By reiterating this inequality we obtain that a power of Λ_θ is a contraction mapping on $C([0,T];V)$, which concludes the proof of the lemma. $\square$

We are now ready to prove the unique solvability of the variational problem

$$\begin{aligned}
&(\mathcal{A}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta(t)))_Q + (\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\theta(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta(t)))_Q \qquad (4.14)\\
&\qquad + j(\zeta_\theta(t), \boldsymbol{u}_\theta(t), \dot{\boldsymbol{u}}_\theta(t), \boldsymbol{v}) - j(\zeta_\theta(t), \boldsymbol{u}_\theta(t), \dot{\boldsymbol{u}}_\theta(t), \dot{\boldsymbol{u}}_\theta(t))\\
&\quad \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\theta(t))_V \qquad \forall \boldsymbol{v} \in V,\ t \in [0,T],\\
&\boldsymbol{u}_\theta(0) = \boldsymbol{u}_0. \qquad (4.15)
\end{aligned}$$

LEMMA 4.5. There exists a unique function $\boldsymbol{u}_\theta \in C^1([0,T];V)$ which satisfies (4.14) and (4.15).

Proof. Let $\boldsymbol{z}_\theta \in C([0,T];V)$ be the fixed point stated in Lemma 4.4 and let $\boldsymbol{u}_\theta \in C^1([0,T];V)$ be the function defined by (4.11) for $\boldsymbol{z} = \boldsymbol{z}_\theta$. We have $\dot{\boldsymbol{u}}_\theta = \boldsymbol{v}_{\theta_{z_\theta}}$ and, writing (4.10) for $\boldsymbol{z} = \boldsymbol{z}_\theta$, we find

$$\begin{aligned}
&(\mathcal{A}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta(t)))_Q + (\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{z}_\theta(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta(t)))_Q\\
&\qquad + j(\zeta_\theta(t), \boldsymbol{z}_\theta(t), \dot{\boldsymbol{u}}_\theta(t), \boldsymbol{v}) - j(\zeta_\theta(t), \boldsymbol{z}_\theta(t), \dot{\boldsymbol{u}}_\theta(t), \dot{\boldsymbol{u}}_\theta(t)) \qquad (4.16)\\
&\quad \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\theta(t))_V \qquad \forall \boldsymbol{v} \in V,\ t \in [0,T].
\end{aligned}$$

The inequality (4.14) follows now from (4.16) and (4.12) since $\boldsymbol{u}_\theta = \boldsymbol{z}_\theta$. Moreover, (4.15) results from (4.11). We conclude that $\boldsymbol{u}_\theta$ is a solution of (4.14) and (4.15).

To prove the uniqueness of the solution, let $\boldsymbol{u}_\theta$ be the solution of (4.14), (4.15) obtained above and let $\boldsymbol{u}_\theta^*$ be any other solution such that $\boldsymbol{u}_\theta^* \in C^1([0,T];V)$. Let $\boldsymbol{v}_\theta^* = \dot{\boldsymbol{u}}_\theta^*$. Using (4.14) we obtain that $\boldsymbol{v}_\theta^*$ satisfies

$$\begin{aligned}
&(\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}_\theta^*(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_\theta^*(t)))_Q + (\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\theta^*(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_\theta^*(t)))_Q\\
&\qquad + j(\zeta_\theta(t), \boldsymbol{u}_\theta^*(t), \boldsymbol{v}_\theta^*(t), \boldsymbol{v}) - j(\zeta_\theta(t), \boldsymbol{u}_\theta^*(t), \boldsymbol{v}_\theta^*(t), \boldsymbol{v}_\theta^*(t))\\
&\quad \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{v}_\theta^*(t))_V \qquad \forall \boldsymbol{v} \in V,\ t \in [0,T].
\end{aligned}$$

Clearly, this is an inequality of the form (4.10) with $\boldsymbol{z} = \boldsymbol{u}_\theta^*$ and, therefore, it follows from (4.13) that it has a unique solution, already denoted by $\boldsymbol{v}_{\theta u_\theta^*}$. We conclude that $\boldsymbol{v}_\theta^* = \boldsymbol{v}_{\theta u_\theta^*}$. Since $\boldsymbol{v}_\theta^* = \dot{\boldsymbol{u}}_\theta^*$, it follows from (4.15) that

$$\boldsymbol{u}_\theta^*(t) = \int_0^t \boldsymbol{v}_{\theta u_\theta^*}(s)ds + \boldsymbol{u}_0 \qquad \forall t \in [0,T]. \qquad (4.17)$$

Comparing (4.11) and (4.17) we obtain $\boldsymbol{u}_\theta^* = \boldsymbol{u}_{\theta u_\theta^*}$, which shows that $\boldsymbol{u}_\theta^*$ is a fixed point of the operator Λ_θ, defined by (4.12). Using now Lemma 4.4 we find

$$\boldsymbol{u}_\theta^* = \boldsymbol{z}_\theta. \qquad (4.18)$$

The uniqueness of the solution of problem (4.14) and (4.15) is now a consequence of the fact that $\boldsymbol{u}_\theta = \boldsymbol{z}_\theta$ and equality (4.18). □

To use the Banach fixed-point theorem again, we need to investigate the properties of the operator $F : H_0^1(\Gamma_C) \times V^3 \to H^{-1}(\Gamma_C)$ given by (3.17). To that end, let

$$L_F = \tilde{c}_\Gamma \|\kappa\|_{L^\infty(D_w)} \max\{\mu^* p_\nu^* c_\Gamma, \mu^* L_\nu R c_\Gamma, \mu^*(L_\mu + \eta L_\nu) R \tilde{c}_\Gamma, p_\nu^* L_\mu R c_\Gamma\}.$$

LEMMA 4.6. The following inequality holds true:

$$\begin{aligned}
&\|F(\zeta_1, \boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{w}_1) - F(\zeta_2, \boldsymbol{u}_2, \boldsymbol{v}_2, \boldsymbol{w}_2)\|_{H^{-1}(\Gamma_C)} \\
&\quad \le L_F(\|\zeta_1 - \zeta_2\|_{H_0^1(\Gamma_C)} + \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V + \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V + \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_V) \\
&\qquad\qquad \forall \zeta_1, \zeta_2 \in H_0^1(\Gamma_C), \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{w}_1, \boldsymbol{w}_2 \in V.
\end{aligned} \tag{4.19}$$

Proof. Inequality (4.19) is obtained from (3.17) by an elementary but tedious computation, based on (3.7)(a) and (d), (3.8)(a) and (c), on inequalities (3.3), (3.4), and on the definition of the truncation operator (2.5). □

Notice that it follows from (3.17) and (3.7)(c) that $F(0, \mathbf{0}, \mathbf{0}, \mathbf{0}) = 0$. Therefore, keeping in mind that $\zeta \in L^2(0, T; H_0^1(\Gamma_C))$, $\boldsymbol{u}_\theta \in C^1([0, T]; V)$ and Lemma (4.6), we find that $F(\zeta_\theta, \boldsymbol{u}_\theta, \dot{\boldsymbol{u}}_\theta, \dot{\boldsymbol{u}}_\theta) \in L^2(0, T; H^{-1}(\Gamma_C))$. This result allows us to consider the operator $\Lambda : L^2(0, T; H^{-1}(\Gamma_C)) \to L^2(0, T; H^{-1}(\Gamma_C))$ defined by

$$\Lambda\theta = F(\zeta_\theta, \boldsymbol{u}_\theta, \dot{\boldsymbol{u}}_\theta, \dot{\boldsymbol{u}}_\theta). \tag{4.20}$$

We now introduce the following positive constants:

$$C_1 = \frac{L_{\mathcal{G}} + c_\Gamma^2 L_\nu (1 + \mu^*)}{m_{\mathcal{A}} - c_\Gamma^2 p_\nu^* L_\mu}, \tag{4.21}$$

$$C_2 = \frac{c_\Gamma (L_\nu \eta + \mu^* L_\nu \eta + p_\nu^* L_\mu)}{m_{\mathcal{A}} - c_\Gamma^2 p_\nu^* L_\mu}, \tag{4.22}$$

$$K = 2L_F^2 (1 + 2\tilde{c}_\Gamma C_2)^2, \tag{4.23}$$

$$C = 2\tilde{c}_\Gamma T L_F^2 C_2^2 e^{2C_1 T} (1 + 2C_1)^2. \tag{4.24}$$

We have the following result.

LEMMA 4.7. Let $\theta_1, \theta_2 \in L^2(0, T; H^{-1}(\Gamma_C))$ and let ζ_{θ_i} denote the functions obtained in Lemma 4.1, for $i = 1, 2$. Then, the following inequalities hold:

$$\begin{aligned}
\|\Lambda\theta_1(t) - \Lambda\theta_2(t)\|_{H^{-1}(\Gamma_C)}^2 &\le K \|\zeta_{\theta_1}(t) - \zeta_{\theta_2}(t)\|_{H_0^1(\Gamma_C)}^2 \\
&+ C \int_0^t \|\zeta_{\theta_1}(s) - \zeta_{\theta_2}(s)\|_{H_0^1(\Gamma_C)}^2 \, ds \text{ a.e. } t \in (0, T),
\end{aligned} \tag{4.25}$$

$$(k^*)^2 \int_0^t \|\zeta_{\theta_1}(s) - \zeta_{\theta_2}(s)\|_{H_0^1(\Gamma_C)}^2 \, ds \le \int_0^t \|\theta_1(s) - \theta_2(s)\|_{H^{-1}(\Gamma_C)}^2 \, ds \qquad \forall t \in [0, T]. \tag{4.26}$$

Proof. Let $\boldsymbol{u}_{\theta_i}$ be the functions obtained in Lemma 4.5 and, for the sake of simplicity, denote $\zeta_{\theta_i} = \zeta_i$, $\boldsymbol{u}_{\theta_i} = \boldsymbol{u}_i$, for $i = 1, 2$. Using (4.20) and (4.19) we obtain

$$\begin{aligned}
\|\Lambda\theta_1(t) - \Lambda\theta_2(t)\|_{H^{-1}(\Gamma_C)} &\le L_F(\|\zeta_1(t) - \zeta_2(t)\|_{H_0^1(\Gamma_C)} \\
&+ \|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V + 2\|\dot{\boldsymbol{u}}_1(t) - \dot{\boldsymbol{u}}_2(t)\|_V) \text{ a.e. } t \in (0, T).
\end{aligned} \tag{4.27}$$

On the other hand, using in (4.16) an argument similar to that used in (4.6), we obtain

$$\|\dot{\boldsymbol{u}}_1(t) - \dot{\boldsymbol{u}}_2(t)\|_V \le C_1 \|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V + C_2 \|\zeta_1(t) - \zeta_2(t)\|_{L^2(\Gamma_C)} \text{ a.e. } t \in (0, T), \tag{4.28}$$

where C_1 and C_2 are given by (4.21) and (4.22), respectively. We use now (4.28) and (4.15) to obtain

$$\begin{aligned}\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V &\le \int_0^t \|\dot{\boldsymbol{u}}_1(s)-\dot{\boldsymbol{u}}_2(s)\|_V\,ds\\ &\le C_1\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V\,ds + C_2\int_0^t \|\zeta_1(s)-\zeta_2(s)\|_{L^2(\Gamma_C)}\,ds \qquad \forall t\in[0,T],\end{aligned}$$

and then Gronwall's inequality yields

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V \le C_2 e^{C_1 t}\int_0^t \|\zeta_1(s)-\zeta_2(s)\|_{L^2(\Gamma_C)}\,ds \qquad \forall t\in[0,T]. \tag{4.29}$$

Inequality (4.25) is now a consequence of (4.27)–(4.29), keeping in mind (3.4), (4.23), and (4.24).

It follows from a standard procedure and (4.2) that

$$\begin{aligned}&\langle\dot\zeta_1(s)-\dot\zeta_2(s),\zeta_1(s)-\zeta_2(s)\rangle + a(\zeta_1(s)-\zeta_2(s),\zeta_1(s)-\zeta_2(s))\\ &\quad = \langle\theta_1(s)-\theta_2(s),\zeta_1(s)-\zeta_2(s)\rangle \text{ a.e. } s\in(0,T).\end{aligned}$$

Let $t\in[0,T]$. We integrate the previous inequality on $[0,t]$ and use (4.3), (3.16), and (3.11) to obtain

$$\begin{aligned}&\frac12\|\zeta_1(t)-\zeta_2(t)\|^2_{L^2(\Gamma_C)} + k^*\int_0^t \|\zeta_1(s)-\zeta_2(s)\|^2_{H_0^1(\Gamma_C)}\,ds\\ &\quad\le \int_0^t \|\theta_1(s)-\theta_2(s)\|_{H^{-1}(\Gamma_C)}\|\zeta_1(s)-\zeta_2(s)\|_{H_0^1(\Gamma_C)}\,ds.\end{aligned} \tag{4.30}$$

We now use the inequality $ab\le a^2/2k^* + b^2k^*/2$ and (4.30) to obtain (4.26). □

LEMMA 4.8. Assume that $K\le e^{-1}$. Then, there exists a unique element $\theta^*\in L^2(0,T;H^{-1}(\Gamma_C))$ such that $\Lambda\theta^*=\theta^*$.

Proof. We use arguments similar to those in [28]. Denote

$$I_0(t) = \|\zeta_1(t)-\zeta_2(t)\|^2_{H_0^1(\Gamma_C)}, \tag{4.31}$$

$$I_1(t) = \int_0^t \|\zeta_1(s)-\zeta_2(s)\|^2_{H_0^1(\Gamma_C)}\,ds, \tag{4.32}$$

and, for $j=2,3\ldots$,

$$I_j(t) = \int_0^t\int_0^{s_{j-1}}\int_0^{s_j-2}\cdots\int_0^{s_1}\|\zeta_1(r)-\zeta_2(r)\|^2_{H_0^1(\Gamma_C)}\,dr\,ds_1\,ds_2\ldots ds_{j-1}. \tag{4.33}$$

Let $p\in\mathbb{N}$, and we denote by C_p^j the binomial coefficients. Reiterating (4.25), using the well-known recurrence identity $C_p^j + C_p^{j-1} = C_{p+1}^j$, using (4.31)–(4.33), and integrating over $[0,T]$, yield

$$\int_0^T \|\Lambda^p\theta_1(t)-\Lambda^p\theta_2(t)\|^2_{H^{-1}(\Gamma_C)}\,dt \le \sum_{j=0}^p C_p^j K^{p-j} C^j \int_0^T I_j(t)\,dt.$$

It follows from (4.26) that

$$\int_0^T I_j(t)\,dt \le \frac{1}{(k^*)^2}\cdot\frac{T^2}{j!}\int_0^T \|\theta_1(s)-\theta_2(s)\|^2_{H^{-1}(\Gamma_C)}\,ds,$$

and the last two inequalities imply

$$\begin{aligned}\int_0^T &\|\Lambda^p\theta_1(t)-\Lambda^p\theta_2(t)\|^2_{H^{-1}(\Gamma_C)}\,dt\\ &\le \frac{1}{(k^*)^2}\left(\sum_{j=0}^{p} C_p^j K^{p-j}\frac{C^jT^j}{j!}\right)\int_0^T \|\theta_1(s)-\theta_2(s)\|^2_{H^{-1}(\Gamma_C)}\,ds.\end{aligned}\tag{4.34}$$

Now, $p!K^p/j!K^j \le (Kp)^{p-j}$, and it is easy to check that

$$\sum_{j=0}^{p} C_p^j K^{p-j}\frac{C^jT^j}{j!} \le \frac{(Kp+CT)^p}{p!}.\tag{4.35}$$

From (4.34) and (4.35) we deduce that

$$\begin{aligned}\|\Lambda^p\theta_1 &- \Lambda^p\theta_2\|^2_{L^2(0,T;H^{-1}(\Gamma_C))}\\ &\le \frac{1}{(k^*)^2}\cdot\frac{(Kp+CT)^p}{p!}\|\theta_1-\theta_2\|^2_{L^2(0,T;H^{-1}(\Gamma_C))}.\end{aligned}\tag{4.36}$$

Next, when $K<e^{-1}$, the series $\sum_{p=0}^{\infty}\frac{(Kp+CT)^p}{p!}$ converges, and consequently, its general term converges to zero, thus,

$$\lim_{p\to\infty}\frac{(Kp+CT)^p}{p!}=0.\tag{4.37}$$

We use now (4.36) and (4.37) and find that, for a sufficiently large p, the mapping Λ^p is a contraction on the Banach space $L^2(0,T;H^{-1}(\Gamma_C))$, which concludes the proof of the lemma. □

We now have all of the ingredients needed to prove our main result.

Proof of Theorem 3.1

Choice of c^*. It follows from (4.23), (4.19), and (4.22) that the constant K depends on c_Γ, $\tilde{c}_\Gamma$, $m_{\mathcal{A}}$, L_ν, L_μ, $\|\kappa\|_{L^\infty(D_w)}$, η, R, p_ν^*, and μ^*. Moreover,

$$K\to 0 \quad\text{as}\quad p_\nu^*\to 0 \quad\text{and}\quad \mu^*\to 0,$$

when c_Γ, $\tilde{c}_\Gamma$, $m_{\mathcal{A}}$, L_ν, L_μ, $\|\kappa\|_{L^\infty(D_w)}$, η, and R are fixed. We conclude that there exists $c^*>0$ which depends on c_Γ, $\tilde{c}_\Gamma$, $m_{\mathcal{A}}$, L_ν, L_μ, $\|\kappa\|_{L^\infty(D_w)}$, η, and R such that, if $p_\nu^*<c^*$ and $\mu^*<c^*$, then $K<e^{-1}$. We may also assume that

$$c^*\le\frac{m_{\mathcal{A}}}{c_\Gamma^2 L_\mu},$$

and therefore, if $p_\nu^*<c^*$, then (4.1) holds. With this choice of c^* we assume in the sequel that $p_\nu^*<c^*$ and $\mu^*<c^*$ and we prove the existence of a unique solution of problem P_V which satisfies (3.25), (3.26).

Existence. Let θ^* be the fixed point obtained in Lemma 4.8. W denote by $\boldsymbol{u}_{\theta^*}$ the solution of problem (4.14) and (4.15) for $\theta=\theta^*$ (see Lemma 4.5) and by ζ_{θ^*} the solution of problem (4.2) and (4.3) for $\theta=\theta^*$ (see Lemma 4.1). Then, the pair $\{\boldsymbol{u}_{\theta^*},\zeta_{\theta^*}\}$ satisfies

(3.22), (3.24)–(3.26) and, by using (4.20) and (4.2), we obtain that it satisfies Eq. (3.23), too. This concludes the existence part.

Uniqueness. Let $\{\boldsymbol{u}_{\theta^*}, \zeta_{\theta^*}\}$ be the solution of problem P_V obtained above and let $\{\boldsymbol{u}, \zeta\}$ be any other solution of problem P_V satisfying (3.25) and (3.26). Define the element $\theta \in L^2(0,T; H^{-1}(\Gamma_C))$ by

$$\theta = F(\zeta, \boldsymbol{u}, \dot{\boldsymbol{u}}, \dot{\boldsymbol{u}}). \tag{4.38}$$

Then, by (3.23) and (3.24) it follows that ζ is a solution of problem (4.2) and (4.3) and, by the uniqueness part in Lemma 4.1, we obtain that $\zeta = \zeta_\theta$. We use this, (3.22), and (3.24) to conclude that $\boldsymbol{u}$ is a solution of problem (4.14), (4.15); therefore, from the uniqueness part in Lemma 4.5, we obtain that $\boldsymbol{u} = \boldsymbol{u}_\theta$. We deduce from these and (4.20) that θ is a fixed point of the operator Λ and, therefore, it follows by Lemma (4.8) that $\theta = \theta^*$.

We conclude that $\boldsymbol{u} = \boldsymbol{u}_{\theta^*}$ and $\zeta = \zeta_{\theta^*}$, which shows that the solution of problem P_V is unique.

5. Conclusions. The process of quasistatic wear of contacting bodies, resulting from frictional contact, was modelled, allowing for the diffusion of the wear debris on the contact surface. The model was in the form of a coupled system which includes the diffusion equation for the wear particles and an evolution inequality for the frictional contact. A variational formulation for the model was derived and the existence of the unique weak solution established, under smallness assumptions on a part of the problem data. The proof used various results from the theory of evolution inequalities and repeated fixed-point arguments.

This is the first result in the mathematical literature in which the diffusion of the wear debris was taken into account, and moreover, the coefficient of friction was assumed to depend on the wear, taking into account the changes due to wear of the contacting surfaces.

The problem is strongly nonlinear and the result was proved under the assumption of the smallness of the friction coefficient and the normal compliance function. Whether the size restriction on these two coefficients is due to the mathematical method of investigation, or there is a physical underlying reason for such an assumption, is an important open question. Indeed, it is of interest to find out if c^* is essential and whether we need estimates for it or it is a mathematical byproduct of the method of proof and a different method may remove it.

For the sake of simplicity, it was assumed in the model that the wear depth was proportional to the surface debris density. It may be of interest to relax this assumption. Also, a model and its analysis of the dynamic problem remain open and important problems.

Problems in which wear particles or debris remain on the contacting surfaces abound in engineering systems, and this line of research is likely to attract increased attention, because of its practical applications. The applications to biomechanics, which motivated this study and were mentioned in the Introduction, are also of considerable interest, and there is a need to add adhesion to the process. However, it is clear from the result in this work that such problems are intrinsically very difficult and some new mathematical tools will have to be developed to address them.

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