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SMOOTH DOMAIN METHOD FOR CRACK PROBLEMS

BY

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Abstract. Equilibrium problems for elastic bodies in domains with cracks are considered. Inequality type boundary conditions are imposed at the crack describing a mutual nonpenetration between the crack faces. A new formulation for such problems is proposed in smooth geometrical domains for two-dimensional elasticity and Kirchhoff plates.

1. Introduction. A new approach to the crack theory for linear elastic bodies with inequality type boundary conditions prescribed on the crack faces is proposed in the paper. The resulting mathematical model allows us to solve the crack problem in a smooth domain. The problem under consideration is characterized by nonlinear boundary conditions imposed on nonsmooth parts of the boundary [8]. These conditions describe the mutual nonpenetration between the crack faces.

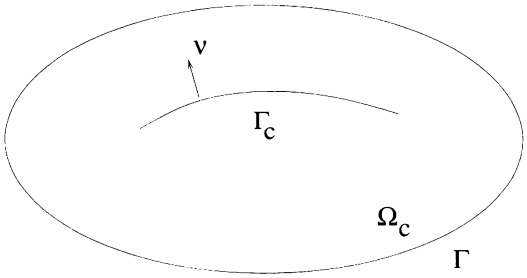

FIG. 1. Domain with a crack

It is well known that for a linear elastic body the frictionless contact problem is variational and can be formulated as the minimisation of the energy functional over

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the set of admissible displacements. Such an admissible set contains all displacement fields from the suitable function space, usually a Sobolev space, satisfying the unilateral nonpenetration condition on the crack faces. The boundary conditions for stresses on the crack faces follow directly from the variational formulation. In particular, normal stresses are nonpositive and the tangential stresses vanish. A different setting is proposed in the paper for the contact problem, with some inequality type conditions for admissible stress fields on crack faces. For such a setting, the nonpenetration conditions for the displacement field follow from the variational formulation and can be derived from the model, i.e., from the equations and the inequalities which form the mathematical model. This is a so-called mixed problem formulation. For domains with smooth boundaries and classical boundary conditions, mixed problem formulations are analysed in the book [3]. The peculiarity of the problem analysed in the paper is that the boundary conditions imposed on nonsmooth parts of the boundary are unilateral type relations. It turns out that the setting proposed in the paper is useful for the modelling and analysis of crack problems in smooth domains and results in a smooth domain method for solving the crack models with nonpenetration conditions on the boundary. In this case, restrictions imposed on the stress tensor components are considered to be internal restrictions, i.e., to be the relations prescribed on given subsets of the smooth domain. In fact, we extend the unknown functions to the crack surface and find the solution in the smooth domain. Note that the problem analysed in the paper is a free boundary problem. In particular, a specific boundary condition at a given point of the crack can be found after the problem is solved. It is said that the boundary conditions provide a possibility of contact between crack faces. Notice that the classical crack problem is characterized by equality type boundary conditions on the crack faces; we refer the reader to [4]–[6], [12], [15]–[19]. For the crack theory with possible contact between crack faces for different constitutive laws, the results can be found in [8]. We should remark that the smooth domain method can be applied to classical linear crack problems as well as to many other linear and nonlinear elliptic boundary value problems.

Throughout the paper we shall use the following notations for geometrical domains (see Fig. 1 and Fig. 2). Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ and let $\Gamma_c \subset \Omega$ be a smooth curve without selfintersections.

We assume that Γ_c can be extended up to a closed curve Σ without selfintersections of the class $C^{1,1}$ so that $\Sigma \subset \Omega$, and the domain Ω is divided into two subdomains Ω_1, Ω_2. In this case Σ is the boundary of the domain Ω_1, and the boundary of Ω_2 is $\Sigma \cup \Gamma$.

Assume that Γ_c does not contain the tip points, i.e., $\Gamma_c = \bar{\Gamma}_c \setminus \partial\Gamma_c$. Denote by $n = (n_1, n_2)$ the unit external normal vector to Γ and by $\nu = (\nu_1, \nu_2)$ a unit normal vector to Σ and therefore to Γ_c. Let $\Omega_c = \Omega \setminus \bar{\Gamma}_c$. In applications, Γ_c defines a crack in an elastic body in the reference domain configuration.

To demonstrate the idea of the smooth domain method, a simple example for the Poisson equation is discussed (see Fig. 3).

We prescribe the sign of the jump of a displacement on Γ_c for an elastic membrane, i.e., $[u] = u^+ - u^- \geq 0$. The following free boundary problem is considered in Ω_c (see [8], [10]).

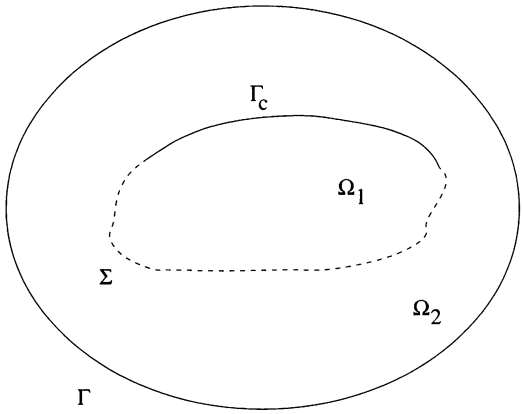

FIG. 2. Extension of the crack

Find a function u such that

$$-\Delta u = f \quad \text{in} \quad \Omega_c \,, \tag{1}$$

$$u = 0 \quad \text{on} \quad \Gamma \,, \tag{2}$$

$$[u] \geq 0, \qquad \left[\frac{\partial u}{\partial \nu}\right] = 0, \qquad [u] \cdot \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_c \,, \tag{3}$$

$$\frac{\partial u}{\partial \nu} \leq 0 \quad \text{on} \quad \Gamma_c^{\pm} \,. \tag{4}$$

It is clear that there exists a unique weak solution to the problem (1)–(4) that can be formulated as minimisation of the energy functional

$$\frac{1}{2}\int_{\Omega_c} |\nabla v|^2 - \int_{\Omega_c} fv$$

over the convex set in the Sobolev space $H^1(\Omega_c)$ with unilateral condition $[v] \geq 0$ on Γ_c and the condition $v = 0$ on Γ.

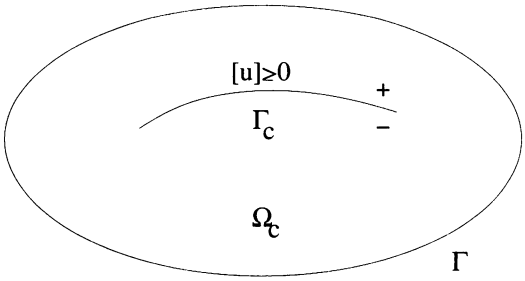

FIG. 3. Elastic membrane

For such a problem we can introduce the following smooth domain formulation.

In the domain Ω we have to find the functions $u, p = (p_1, p_2)$ such that

$$u \in L^2(\Omega), \quad p \in M \ , \tag{5}$$

$$-\mathrm{div} p = f \quad \text{in} \quad \Omega \ , \tag{6}$$

$$\int_\Omega p(\bar{p} - p) + \int_\Omega u(\mathrm{div}\bar{p} - \mathrm{div}p) \geq 0 \ \ \forall \ \ \bar{p} \in M \ , \tag{7}$$

where

$$M = \{p = (p_1, p_2) \in L^2(\Omega) \mid \mathrm{div}p \in L^2(\Omega), \ p\nu \leq 0 \text{ on } \ \Gamma_c\} \ .$$

The problem formulations (1)–(4) and (5)–(7) are equivalent. The advantage of the formulation (5)–(7) is that the solution is defined in the smooth domain Ω.

PROPOSITION 1.1. There exists a unique solution to the problem (5)–(7).

The proof is similar to the proofs of Theorem 2.2 and Theorem 3.1 below in more complicated settings of the elasticity problems.

1.1. *Main results.* We present two results which are proved in the paper. The smooth domain method is applied to the two-dimensional elasticity and the Kirchhoff plate model. As we can see from Theorem 2.2 and Theorem 3.1 below, the variational formulation of the crack contact problem is obtained in smooth domain Ω. Therefore, from a numerical point of view, the discretization is required in the domain Ω; however, the restriction imposed on the solution is considered on the curve Γ_c inside of Ω. It means that unknown functions are defined in the smooth domain Ω and should satisfy some inequality type constraints. We restrict ourselves to the two-dimensional elasticity; the same method can be applied to the three-dimensional elasticity with the contact on the crack faces along the lines of the paper [11].

1.1.1. *Two-dimensional elasticity.* The boundary value problem for frictionless contact on crack faces in two-dimensional elasticity is given in (15)–(19) below. The unilateral conditions (18)–(19) are imposed on Γ_c and $\Gamma_c^\pm$. The smooth domain formulation in this problem is considered in the smooth domain $\Omega = \Omega_c \cup \bar{\Gamma}_c$. It takes the following form.

Find $u = (u_1, u_2)$, $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$u \in L^2(\Omega), \quad \sigma \in N \ , \tag{8}$$

$$-\mathrm{div}\sigma = f \quad \text{in } \Omega \ , \tag{9}$$

$$(C\sigma, \bar{\sigma} - \sigma)_\Omega + (u, \mathrm{div}\bar{\sigma} - \mathrm{div}\sigma)_\Omega \geq 0 \quad \forall \bar{\sigma} \in N \ , \tag{10}$$

where

$$N = \{\sigma \in H \ |\sigma_\tau = 0 \ , \quad \sigma_\nu \leq 0 \quad \text{on } \Gamma_c\} \, ,$$

$$H = \{\sigma = \{\sigma_{ij}\} | \sigma, \mathrm{div}\sigma \in L^2(\Omega)\} \ .$$

Here σ_ν are normal stresses, σ_τ are tangential forces, and $(\cdot, \cdot)_\Omega$ is the scalar product in $L^2(\Omega)$. In the paper we prove the following statement.

THEOREM 1.1. There exists a unique solution to the problem (8)–(10).

The proof of Theorem 1.1 is given in Sec. 2.

1.1.2. *Kirchhoff plate.* The boundary value problem for the Kirchhoff plate with inequality type boundary condition imposed on Γ_c is given in (53)–(60) below. The smooth domain formulation for this problem is as follows.

We have to find functions u, w, σ, m such that

$$u = (u_1, u_2) \in L^2(\Omega),\ w \in L^2(\Omega),\ (\sigma, m) \in \mathcal{N}\ , \tag{11}$$

$$-\mathrm{div}\sigma = f \quad \text{in } \Omega\ , \tag{12}$$

$$-\nabla\nabla m = F \quad \text{in } \Omega\ , \tag{13}$$

$$(u, \mathrm{div}\bar{\sigma} - \mathrm{div}\sigma)_\Omega + (w, \nabla\nabla\bar{m} - \nabla\nabla m)_\Omega \tag{14}$$
$$+(C\sigma, \bar{\sigma} - \sigma)_\Omega + (Dm, \bar{m} - m)_\Omega \geq 0 \quad \forall(\bar{\sigma}, \bar{m}) \in \mathcal{N}\ ,$$

where

$$\mathcal{N} = \{(\sigma, m) \in \mathcal{H} |\ \sigma_\tau = 0,\ t^\nu(m) = 0,\ |m_\nu| \leq -\sigma_\nu \text{ on } \Gamma_c\}\ ,$$

$$\mathcal{H} = \{(\sigma, m) |\ \sigma = \{\sigma_{ij}\}, m = \{m_{ij}\};\ \sigma, \mathrm{div}\sigma \in L^2(\Omega),$$
$$m, \nabla\nabla m \in L^2(\Omega)\}\ .$$

Here m_ν are the bending moments and $t^\nu(m)$ are transverse forces.

THEOREM 1.2. There exists a unique solution to the problem (11)–(14).

The proof of Theorem 1.2 is given in Sec. 3.

Note that the case of the cracks which come out at $\Gamma = \partial\Omega$ is also treated by the smooth domain formulation. This means that the method is applied to the case when $\bar{\Gamma}_c$ crosses the external boundary Γ (see Remarks 2.2, 3.2).

2. Two-dimensional elasticity. In this section, the detailed proof of Theorem 1.1 is given. We start with the variational inequality for frictionless contact on crack faces in the two-dimensional elasticity.

2.1. *Variational formulation.* The equilibrium problem for a linear elastic body occupying the domain Ω_c with the interior crack Γ_c can be formulated as follows [8]. We have to find functions $u = (u_1, u_2), \sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\mathrm{div}\sigma = f \quad \text{in} \quad \Omega_c\ , \tag{15}$$

$$C\sigma - \varepsilon(u) = 0 \quad \text{in} \quad \Omega_c\ , \tag{16}$$

$$u = 0 \quad \text{on} \quad \Gamma\ , \tag{17}$$

$$[u]\nu \geq 0, \quad [\sigma_\nu] = 0, \quad \sigma_\nu \cdot [u]\nu = 0 \quad \text{on} \quad \Gamma_c\ , \tag{18}$$

$$\sigma_\nu \leq 0, \quad \sigma_\tau = 0 \quad \text{on} \quad \Gamma_c^\pm\ . \tag{19}$$

Here $[u] = u^+ - u^-$ is the jump of the displacement field across Γ_c, the signs $\pm$ indicate the positive and negative directions of the normal ν, $f = (f_1, f_2) \in L^2(\Omega)$ is a given

external force acting on the body, and the following notations are used:

$$\sigma_\nu = \sigma_{ij}\nu_j\nu_i, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \cdot \nu, \quad \sigma_\tau = \{\sigma_\tau^i\}_{i=1}^2, \quad \sigma\nu = \{\sigma_{ij}\nu_j\}_{i=1}^2 ,$$
$$\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i,j = 1,2, \quad \varepsilon(u) = \{\varepsilon_{ij}(u)\}_{i,j=1}^2,$$
$$\{C\sigma\}_{ij} = c_{ijkl}\sigma_{kl}, \quad c_{ijkl} = c_{jikl} = c_{klij}, \quad c_{ijkl} \in L^\infty(\Omega) .$$

Tensor C satisfies the ellipticity condition

$$c_{ijkl}\xi_{ji}\xi_{kl} \geq c_0|\xi|^2, \quad \forall \xi_{ji} = \xi_{ij},\ c_0 > 0 . \tag{20}$$

We use the summation convention over repeated indices $i, j, k, l = 1, 2$.

Equations and inequalities (18)–(19) describe the mutual nonpenetration between crack faces without friction. Relation (15) is the equilibrium equations, the equation (16) is the Hooke constitutive law, and the condition (17) corresponds to the fixed displacements on the boundary Γ.

In order to introduce the variational formulation of the problem (15)–(19), we need the following Sobolev space

$$H^{1,0}(\Omega_c) = \{v = (v_1, v_2) | v_i \in H^1(\Omega_c), v_i = 0 \quad \text{on} \quad \Gamma, \quad i = 1,2\}$$

and a closed convex set of admissible displacements

$$K = \{v \in H^{1,0}(\Omega_c) \mid [v]\nu \geq 0 \quad \text{a.e. on} \quad \Gamma_c\} . \tag{21}$$

In this case we can consider the minimisation problem

$$\min_{v \in K} \left\{ \frac{1}{2}(\sigma(v), \varepsilon(v))_{\Omega_c} - (f, v)_{\Omega_c} \right\}, \tag{22}$$

which admits the unique solution $u \in K$ satisfying the variational inequality

$$(\sigma(u), \varepsilon(v-u))_{\Omega_c} \geq (f, v-u)_{\Omega_c} \quad \forall v \in K . \tag{23}$$

Here $(\cdot,\cdot)_{\Omega_c}$ is the scalar product in $L^2(\Omega_c)$ and the stress tensor $\sigma(u) = \sigma$ is found from the Hooke law (16). From (23) it follows that equilibrium equation (15) is satisfied in the sense of distributions. To verify this, it suffices to substitute $v = u \pm \varphi$, $\varphi \in C_0^\infty(\Omega_c)$, in the variational inequality (23). It can be shown [8] that for the solution to the variational inequality (23), all of the boundary conditions (18)–(19) are satisfied. In the next section we specify the meaning of these conditions.

2.2. *Mixed formulation.* Consider the space of stresses

$$H(\text{div}) = \{\sigma = \{\sigma_{ij}\} | \ \sigma \in L^2(\Omega_c), \text{div}\sigma \in L^2(\Omega_c)\}$$

equipped with the norm

$$\|\sigma\|^2_{H(div)} = \|\sigma\|^2_{L^2(\Omega_c)} + \|\text{div}\sigma\|^2_{L^2(\Omega_c)}$$

and define the set of admissible stresses

$$H(\text{div}; \Gamma_c) = \{\sigma \in H(\text{div}) | \ [\sigma\nu] = 0 \quad \text{on} \quad \Gamma_c; \ \ \sigma_\nu \leq 0, \sigma_\tau = 0 \quad \text{on} \quad \Gamma_c^\pm\} .$$

For the sake of simplicity, the same notation $L^2(\Omega_c)$ is used for the space of scalar functions and the space $[L^2(\Omega_c)]^2 = L^2(\Omega_c; \mathbb{R}^2)$ of vector functions, as well as for the space $[L^2(\Omega_c)]^4$ of tensor valued functions.

Introduce the space $H^{\frac{1}{2}}(\Sigma)$ with the norm

$$\|\varphi\|^2_{H^{\frac{1}{2}}(\Sigma)} = \|\varphi\|^2_{L^2(\Sigma)} + \int_\Sigma \int_\Sigma \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^2} dxdy$$

and denote by $H^{-\frac{1}{2}}(\Sigma)$ the dual space of $H^{\frac{1}{2}}(\Sigma)$. Note that for $\sigma \in H(\mathrm{div})$, the traces $(\sigma\nu)^\pm$ can be defined as elements of $H^{-\frac{1}{2}}(\Sigma)$ (see [8], [21]) and the trace operators are continuous from $H(\mathrm{div})$ to $H^{-\frac{1}{2}}(\Sigma)$. Also, it is possible to define $\sigma^\pm_\nu, (\sigma^i_\tau)^\pm \in H^{-\frac{1}{2}}(\Sigma), i = 1,2$, such that the Green formula holds,

$$(\mathrm{div}\sigma, \psi)_{\Omega_1} = -(\sigma, \varepsilon(\psi))_{\Omega_1} + \langle \sigma^-_\nu, \psi_\nu\rangle_{\frac{1}{2}} + \langle \sigma^-_\tau, \psi_\tau\rangle_{\frac{1}{2}},$$
$$\forall \psi = (\psi_1, \psi_2) \in H^1(\Omega_1)\ ,$$

where ν is assumed to be the external normal vector to the boundary $\partial\Omega_1 = \Sigma$ and $\langle\cdot,\cdot\rangle_{\frac{1}{2}}$ is the duality pairing between $H^{-\frac{1}{2}}(\Sigma)$ and $H^{\frac{1}{2}}(\Sigma)$. A similar formula takes place for the domain Ω_2 with the external normal vector $-\nu$ to the part Σ of its boundary $\Gamma \cup \Sigma$. Zero jump condition for $\sigma\nu$ in the definition of $H(\mathrm{div};\Gamma_c)$ means

$$\langle (\sigma\nu)^+ - (\sigma\nu)^-, \varphi\rangle_{\frac{1}{2}} = 0 \quad \forall \varphi = (\varphi_1, \varphi_2) \in H^{\frac{1}{2}}(\Sigma)\ .$$

Since $(\sigma\nu)^+$ and $(\sigma\nu)^-$ coincide, it follows that $\sigma^+_\nu = \sigma^-_\nu$, $(\sigma^i_\tau)^+ = (\sigma^i_\tau)^-, i = 1,2$. Let $\mathrm{supp}\varphi$ denote the support of the function φ. The second and the third conditions in the definition of $H(\mathrm{div};\Gamma_c)$ can be written as

$$\langle \sigma^\pm_\nu, \varphi\rangle_{\frac{1}{2}} \le 0 \quad \forall \varphi \in H^{\frac{1}{2}}(\Sigma), \quad \varphi \ge 0 \quad \text{a.e. on } \Gamma_c\ , \mathrm{supp}\ \varphi \subset \Gamma_c$$

and

$$\langle \sigma^\pm_\tau, \varphi\rangle_{\frac{1}{2}} = 0 \quad \forall \varphi = (\varphi_1, \varphi_2) \in H^{\frac{1}{2}}(\Sigma), \quad \varphi_i\nu_i = 0 \text{ a.e. on } \Gamma_c\ , \mathrm{supp}\ \varphi \subset \Gamma_c\ ,$$

respectively. Therefore, the convex cone $H(\mathrm{div};\Gamma_c)$ is closed in the space $H(\mathrm{div})$. Hence $H(\mathrm{div};\Gamma_c)$ is weakly closed in $H(\mathrm{div})$.

The above arguments allow us to define functional spaces on Γ_c. Recall the definition of the weighted Sobolev space on Γ_c (see, e.g., [7] for details),

$$H^{\frac{1}{2}}_{00}(\Gamma_c) = \{\varphi \in H^{\frac{1}{2}}(\Gamma_c) | \frac{\varphi}{\sqrt{\rho}} \in L^2(\Gamma_c)\}$$

equipped with the norm

$$\|\varphi\|^2_{\frac{1}{2},00} = \|\varphi\|^2_{\frac{1}{2}} + \int_{\Gamma_c} \rho^{-1}\varphi^2\ ,$$

where $\rho(x) = \mathrm{dist}(x, \partial\Gamma_c)$ and $\|\cdot\|_{\frac{1}{2}}$ is the norm in the space $H^{\frac{1}{2}}(\Gamma_c)$. It is well known [14] that functions from the space $H^{\frac{1}{2}}_{00}(\Gamma_c)$ can be extended to Σ by zero, and such an extension is an element of the space $H^{\frac{1}{2}}(\Sigma)$. The extension of φ is denoted by $\bar\varphi$, i.e.,

$$\bar\varphi(x) = \begin{cases} \varphi(x), & x \in \Gamma_c \\ 0, & x \in \Sigma \setminus \Gamma_c \end{cases}$$

and we have $\varphi \in H^{\frac{1}{2}}_{00}(\Gamma_c)$ if and only if $\bar\varphi$ belongs to $H^{\frac{1}{2}}(\Sigma)$.

Let us observe that by the above formulae, the elements $\sigma_\nu \in (H^{\frac{1}{2}}_{00}(\Gamma_c))^*$ and $\sigma^i_\tau \in (H^{\frac{1}{2}}_{00}(\Gamma_c))^*$, $i = 1, 2$, can be defined [8]. The inequalities on Γ_c are understood in the sense of duality; i.e., $[\sigma_\nu] = 0, \sigma_\nu \leq 0$ in the definition of $H(\mathrm{div}; \Gamma_c)$ mean

$$\langle \sigma_\nu, \varphi \rangle_{\frac{1}{2},00} \leq 0 \quad \forall \varphi \in H^{\frac{1}{2}}_{00}(\Gamma_c) \quad \text{such that } \varphi \geq 0 \quad \text{a.e. on } \Gamma_c \ ,$$

and furthermore, the condition $\sigma_\tau = 0$ on $\Gamma_c^\pm$ in the definition of the cone $H(\mathrm{div}; \Gamma_c)$ takes the form

$$\langle \sigma_\tau, \varphi \rangle_{\frac{1}{2},00} = 0 \quad \forall \varphi = (\varphi_1, \varphi_2) \in H^{\frac{1}{2}}_{00}(\Gamma_c) \quad \text{such that } \varphi_i \nu_i = 0 \text{ a.e. on } \Gamma_c \ .$$

Here $\langle \cdot, \cdot \rangle_{\frac{1}{2},00}$ is the duality pairing between $(H^{\frac{1}{2}}_{00}(\Gamma_c))^*$ and $H^{\frac{1}{2}}_{00}(\Gamma_c)$.

It is important that in the above formulae the curve Σ is assumed to be arbitrary, but it should be sufficiently smooth. This means that the formulae mentioned are valid for the closed curves Σ which are smooth enough. All boundary conditions for σ included in the definition of $H(\mathrm{div}; \Gamma_c)$ are precisely the same as the boundary conditions for the solution $\sigma(u) = \sigma$ of the variational inequality (23). Let us note that a solution dependence on domain variations for classical boundary value problems is analysed in [20]. For domain variations in the free boundary crack problems, we refer the reader to [8], [10] (see also [13]).

Now, we are in a position to give the mixed formulation for the problem (15)–(19). We have to find functions $u = (u_1, u_2)$, $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$u \in L^2(\Omega_c), \sigma \in H(\mathrm{div}; \Gamma_c) \ , \tag{24}$$

$$-\mathrm{div}\sigma = f \quad \text{in } \Omega_c \ , \tag{25}$$

$$(C\sigma, \bar{\sigma} - \sigma)_{\Omega_c} + (u, \mathrm{div}\bar{\sigma} - \mathrm{div}\sigma)_{\Omega_c} \geq 0 \quad \forall \bar{\sigma} \in H(\mathrm{div}; \Gamma_c) \ . \tag{26}$$

Boundary value problem (15)–(19) is formally equivalent to (24)–(26). Indeed, assuming that the solutions to (24)–(26) are sufficiently regular, we can derive from (26) the Hooke law by taking test functions of the form $\bar{\sigma} = \pm\tilde{\sigma} + \sigma$, where $\tilde{\sigma}$ are smooth functions with compact supports in Ω_c,

$$C\sigma - \varepsilon(u) = 0 \quad \text{in } \Omega_c \ .$$

The boundary conditions

$$[u]\nu \geq 0, \quad \sigma_\nu \cdot [u]\nu = 0 \quad \text{on } \Gamma_c \tag{27}$$

follow from (26) by an application of the Green formula. Thus, all boundary conditions (18)–(19) are fulfilled. On the other hand, by multiplication of (16) by $\bar{\sigma} - \sigma$, $\bar{\sigma} \in H(\mathrm{div}; \Gamma_c)$, it can be shown that the inequality (26) follows from (15)–(19).

Note that the set $H(\mathrm{div}; \Gamma_c)$ includes only the restriction imposed on the stress tensor components. As for the relations (27), they are included in the problem (24)–(26). On the other hand, the set K in the variational inequality (23) includes only the restriction imposed on the displacement u, and the equations and inequalities (18), (19) can be derived from (23).

We aim toward investigation of the problem (24)–(26). First, we prove the existence of a solution.

THEOREM 2.1. There exists a solution to the system (24)–(26).

Proof. In order to establish a priori estimates for solutions, we introduce a function $\sigma^0 \in H(\mathrm{div};\Gamma_c)$ which solves the equilibrium equations

$$-\mathrm{div}\sigma^0 = f \quad \text{in } \Omega_c \ . \tag{28}$$

Such a function can be found by solving the variational inequality (23) with an arbitrary Hooke's law satisfying the condition (20). Let us point out that the existence of a solution to the system (24)–(26) can be in fact established directly by solving (23); however, we provide the proof by a different argument without any requirement on the solvability of (23). The reason to proceed in this way is that in the sequel we can use exactly the same arguments in order to analyse the smooth domain formulation for the problem under consideration.

To prove the existence of solutions to (24)–(26) we introduce the regularized boundary value problem depending on a parameter $\delta > 0$. Then the existence of a solution for the regularized problem is shown and a priori estimates are obtained. The proof is completed by the passage to the limit $\delta \to 0$.

Let us fix $0 < \delta < \delta_0$. The regularized problem takes the form

$$u^\delta \in L^2(\Omega_c), \sigma^\delta \in H(\mathrm{div};\Gamma_c) \ , \tag{29}$$

$$\delta u^\delta - \mathrm{div}\sigma^\delta = f \quad \text{in } \Omega_c \ , \tag{30}$$

$$(C\sigma^\delta, \bar\sigma - \sigma^\delta)_{\Omega_c} + (u^\delta, \mathrm{div}\bar\sigma - \mathrm{div}\sigma^\delta)_{\Omega_c} \geq 0 \quad \forall \bar\sigma \in H(\mathrm{div};\Gamma_c) \ . \tag{31}$$

From (30), (31) it follows that

$$\delta(u^\delta, u^\delta)_{\Omega_c} - (\mathrm{div}\sigma^\delta, u^\delta)_{\Omega_c} = (f, u^\delta)_{\Omega_c} \ ,$$
$$(C\sigma^\delta, \sigma^0 - \sigma^\delta)_{\Omega_c} + (u^\delta, \mathrm{div}\sigma^0 - \mathrm{div}\sigma^\delta)_{\Omega_c} \geq 0 \ ,$$

and the following estimate is obtained:

$$\delta\|u^\delta\|^2_{L^2(\Omega_c)} + \|\sigma^\delta\|^2_{L^2(\Omega_c)} \leq c \tag{32}$$

with the constant c uniform with respect to δ. Moreover, (30) implies that

$$\mathrm{div}\sigma^\delta = \delta u^\delta - f \quad \text{in } \Omega_c \ .$$

Thus, in view of (32), the following uniform estimate is obtained:

$$\|\mathrm{div}\sigma^\delta\|^2_{L^2(\Omega_c)} \leq c \ . \tag{33}$$

Let us show that, for a given δ, there exists a solution to the problem (29)–(31). Indeed, from (30) it follows $u^\delta = \frac{1}{\delta}(f + \mathrm{div}\sigma^\delta)$. Substituting this value of u^δ in (31) we derive the variational inequality

$$(C\sigma^\delta, \bar\sigma - \sigma^\delta)_{\Omega_c} + \frac{1}{\delta}(f + \mathrm{div}\sigma^\delta, \mathrm{div}\bar\sigma - \mathrm{div}\sigma^\delta)_{\Omega_c} \geq 0 \quad \forall \bar\sigma \in H(\mathrm{div};\Gamma_c) \ .$$

It is clear that solving this variational inequality is equivalent to minimisation of the functional

$$G(\sigma) = \frac{1}{2}(C\sigma, \sigma)_{\Omega_c} + \frac{1}{2\delta}(\mathrm{div}\sigma, \mathrm{div}\sigma)_{\Omega_c} + \frac{1}{\delta}(f, \mathrm{div}\sigma)_{\Omega_c}$$

over the weakly closed convex set $H(\mathrm{div};\Gamma_c)$. The functional G is coercive and weakly lower semicontinuous on the space $H(\mathrm{div})$, hence the minimisation problem has a (unique) solution $\sigma = \sigma^\delta$. Having found σ^δ we define u^δ from (30). The solution u^δ, σ^δ satisfies the relations (29)–(31). Now we perform the passage to the limit in (29)–(31) as $\delta \to 0$. From (31) it follows that

$$C\sigma^\delta - \varepsilon(u^\delta) = 0 \quad \text{in } \Omega_c$$

in the sense of distributions, i.e., in particular $\varepsilon(u^\delta) \in L^2(\Omega_c)$. Since $u^\delta \in L^2(\Omega_c)$, by an application of the second Korn inequality which holds in the domain Ω_c, it follows that $u^\delta \in H^1(\Omega_c)$. On the other hand,

$$u^\delta = 0 \quad \text{on } \Gamma \ ,$$

which can be deduced from (31) taking into account that the vector function $\bar{\sigma}n$ is free on Γ. Hence $u^\delta \in H^{1,0}(\Omega_c)$, and, by the first Korn inequality, the uniform estimate with respect to δ is obtained,

$$\|u^\delta\|_{H^{1,0}(\Omega_c)} \le c \ . \tag{34}$$

Taking into account (32), (33), we have the uniform estimate with respect to δ,

$$\|\sigma^\delta\|_{L^2(\Omega_c)} + \|\mathrm{div}\sigma^\delta\|_{L^2(\Omega_c)} \le c \ . \tag{35}$$

Therefore, there exist elements u, σ such that for $\delta \to 0$ we have the following convergences for subsequences:

$$u^\delta \to u \ \text{ weakly in } H^{1,0}(\Omega_c) \ \text{ and strongly in } \ L^2(\Omega_c) \ ,$$
$$\sigma^\delta \to \sigma \ \text{ weakly in } \ L^2(\Omega_c) \ ,$$
$$\mathrm{div}\sigma^\delta \to \mathrm{div}\sigma \text{ weakly in } \ L^2(\Omega_c) \ .$$

Finally, for $\delta \to 0$ we pass to the limit in (30), (31), and (24)–(26) follows, which completes the proof of Theorem 2.1. □

Note that the solution to (24)–(26) is unique. Indeed, if there are two solutions $(u^1, \sigma^1), (u^2, \sigma^2)$, from (26), it follows that $\sigma^1 = \sigma^2$. Since $C\sigma^i = \varepsilon(u^i)$, $i = 1, 2$, we have $\varepsilon(u^1 - u^2) = 0$, hence $u^1 = u^2$.

REMARK 2.1. The mixed formulation of the problem (15)–(19) can be applied to the case when $\bar{\Gamma}_c$ crosses the external boundary Γ and also to the case when Γ_c is only of $C^{0,1}-$ regularity. The $C^{1,1}-$ regularity of the curve Σ was needed to define σ_ν, σ_τ. It is possible to avoid the interpretation of the boundary conditions included in the set $H(\mathrm{div};\Gamma_c)$. Indeed, consider a crack Γ_c of $C^{0,1}-$regularity such that $\bar{\Gamma}_c$ crosses the boundary Γ (see Fig. 4). Assume that the angle between Γ and $\bar{\Gamma}_c$ at the common point x_c is nonzero.

Introduce the set of admissible stresses in the following equivalent form

$$H(\mathrm{div};\Gamma_c) = \{\sigma \in H(\mathrm{div}) \mid \int_{\Omega_c} (\sigma\varepsilon(\bar{u}) + \bar{u}\mathrm{div}\sigma) \ge 0 \ \ \forall\, \bar{u} \in K\} \ ,$$

where the set K is defined in (21). For such a definition of $H(\mathrm{div};\Gamma_c)$, we can verify that all boundary conditions for stresses are fulfilled provided that the function σ is sufficiently regular. Note that if Γ_c divides Ω into two separate domains Ω^1 and Ω^2, we obtain a

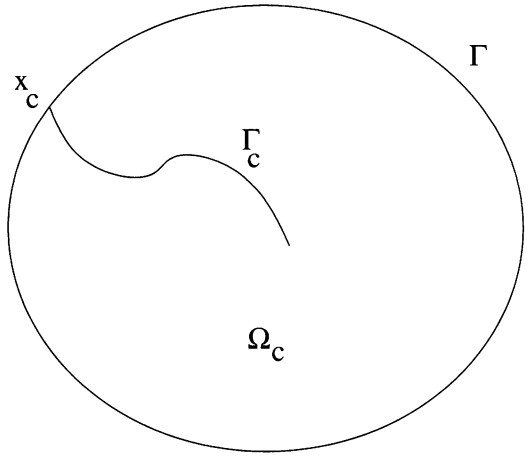

FIG. 4. Boundary crack

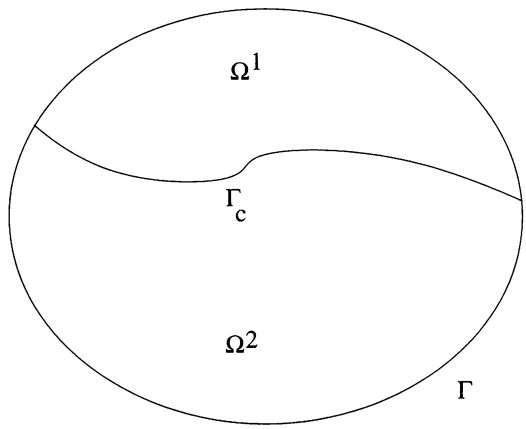

FIG. 5. Contact problem

contact problem for two elastic bodies occupying the domains Ω^1, Ω^2 with inequality type boundary conditions (18)–(19) imposed on the common boundary Γ_c (see Fig. 5).

2.3. *Smooth domain method.* In this section the smooth domain method to the crack problem for two-dimensional elasticity is formulated. The main feature of such a formulation is that the constraints on the stress tensor are imposed on subsets of the smooth domain Ω and the unknown functions u, σ are defined in the smooth domain Ω. We extend unknown functions u, σ from the nonsmooth domain Ω_c to the smooth domain Ω (cf. [1]). Such an extension reduces in fact to a definition of two fields u, σ on the curve Γ_c. We shall use the same notation u, σ for the extended functions defined on Ω and write the problem (15)–(19) in the domain Ω. The problem takes the following form.

We have to find functions $u = (u_1, u_2)$, $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\mathrm{div}\sigma = f \quad \text{in } \Omega \ , \tag{36}$$

$$C\sigma - \varepsilon(u) + p(u)\delta_{\Gamma_c} = 0 \quad \text{in } \Omega \ , \tag{37}$$

$$u = 0 \quad \text{on } \Gamma \ , \tag{38}$$

$$[u]\nu \geq 0 \ , \quad \sigma_\nu \leq 0 \ , \quad \sigma_\tau = 0 \ , \quad \sigma_\nu \cdot [u]\nu = 0 \quad \text{on } \Gamma_c \ , \tag{39}$$

where we denote $p(u)_{ij} = \frac{1}{2}([u_i]\nu_j + [u_j]\nu_i)$, and δ_{Γ_c} is the single layer distribution on Γ_c defined by the following formula:

$$\langle \delta_{\Gamma_c}, \varphi \rangle = \int_{\Gamma_c} \varphi \ , \quad \forall \varphi \in C_0^\infty(\Omega) \ .$$

We denoted by $\langle T, \varphi \rangle$ the value of a distribution T on the function $\varphi \in C_0^\infty(\Omega)$. Let us point out that solutions to the system (15)–(19) determined from the variational inequality (23) satisfy the jump condition

$$[\sigma\nu] = 0 \quad \text{on } \Gamma_c \tag{40}$$

and, therefore, Eq. (36) is of the same form as Eq. (15). Let us verify this statement. It follows from (23) that $\sigma = \sigma(u)$ satisfies

$$\sigma \in L^2(\Omega_c), \ \mathrm{div}\sigma \in L^2(\Omega_c), \ \ -\mathrm{div}\sigma = f \ \text{ in } \ \Omega_c \ . \tag{41}$$

Then in view of (40), (41) it follows that for any $\varphi \in C_0^\infty(\Omega)$,

$$\langle \sigma_{ij,j} + f_i, \varphi \rangle = -(\sigma_{ij}, \varphi_{,j})_{\Omega_1} - (\sigma_{ij}, \varphi_{,j})_{\Omega_2} + (f_i, \varphi)_\Omega$$

$$= \langle [\sigma_{ij}\nu_j], \varphi \rangle_{\frac{1}{2}} + (\sigma_{ij,j} + f_i, \varphi)_{\Omega_1} + (\sigma_{ij,j} + f_i, \varphi)_{\Omega_2} = 0 \ , i = 1, 2 \ ,$$

which proves that Eq. (36) holds in the sense of distributions.
The difference between the system (15)–(19) and the system (36)–(39) is that now the conditions (39) are considered to be internal constraints for the solutions which are imposed on the curve Γ_c located in the interior of the smooth domain Ω. Let us note that the equivalence of the systems (15)–(19) and (36)–(39) is straightforward for smooth solutions. We show that it is also the case for the weak solution. We need the following notation for the space of stresses and the convex cone of admissible stresses in the smooth domain Ω,

$$\mathcal{H}(\mathrm{div}) = \{\sigma = \{\sigma_{ij}\} | \sigma, \mathrm{div}\sigma \in L^2(\Omega)\} \ ,$$

$$\mathcal{H}(\mathrm{div}; \Gamma_c) = \{\sigma \in \mathcal{H}(\mathrm{div}) | \sigma_\tau = 0 \ , \quad \sigma_\nu \leq 0 \quad \text{on } \Gamma_c\} \ .$$

The norm in the space $\mathcal{H}(\mathrm{div})$ is defined by the formula

$$\|\sigma\|^2_{\mathcal{H}(div)} = \|\sigma\|^2_{L^2(\Omega)} + \|\mathrm{div}\sigma\|^2_{L^2(\Omega)} \ .$$

As was indicated before for the cone $H(\mathrm{div}; \Gamma_c)$, the convex cone $\mathcal{H}(\mathrm{div}; \Gamma_c)$ is also closed in the space $\mathcal{H}(\mathrm{div})$ since the conditions $\sigma_\tau = 0$, $\sigma_\nu \leq 0$ on Γ_c are well defined for any element $\sigma \in \mathcal{H}(\mathrm{div})$. Indeed, for any curve Σ satisfying the prescribed conditions, the

functionals σ_ν, $\sigma_\tau^i, i = 1, 2$, are uniquely defined in the space $H^{-\frac{1}{2}}(\Sigma)$. The conditions $\sigma_\tau = 0, \sigma_\nu \leq 0$ on Γ_c in the definition of $\mathcal{H}(\mathrm{div}; \Gamma_c)$ are understood in the sense that

$$\langle \sigma_\tau, \varphi \rangle_{\frac{1}{2}} = 0 \quad \forall \varphi = (\varphi_1, \varphi_2) \in H^{\frac{1}{2}}(\Sigma), \quad \varphi_i \nu_i = 0 \text{ a.e. on } \Gamma_c \text{ , supp } \varphi \subset \Gamma_c \text{ ,}$$

and

$$\langle \sigma_\nu, \varphi \rangle_{\frac{1}{2}} \leq 0 \quad \forall \varphi \in H^{\frac{1}{2}}(\Sigma), \quad \varphi \geq 0 \quad \text{a.e. on } \Gamma_c \text{ , supp } \varphi \subset \Gamma_c \text{ ,}$$

respectively.

The weak formulation of the system (36)–(39) takes the form of the following problem in Ω.

Find $u = (u_1, u_2)$, $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$u \in L^2(\Omega), \quad \sigma \in \mathcal{H}(\mathrm{div}; \Gamma_c) \text{ ,} \tag{42}$$

$$-\mathrm{div}\sigma = f \quad \text{in } \Omega \text{ ,} \tag{43}$$

$$(C\sigma, \bar{\sigma} - \sigma)_\Omega + (u, \mathrm{div}\bar{\sigma} - \mathrm{div}\sigma)_\Omega \geq 0 \quad \forall \bar{\sigma} \in \mathcal{H}(\mathrm{div}; \Gamma_c) \text{ .} \tag{44}$$

Note that (44) follows directly from (26), since we can change the integration over Ω_c by the integration over Ω.

THEOREM 2.2. There exists a solution to the problem (42)–(44).

Proof. The general scheme of the proof remains the same as in the proof of Theorem 2.1. First of all, the function σ^0 defined in the proof of Theorem 2.1 can be extended to the domain Ω, the extended function is also denoted by σ^0, $\sigma^0 \in \mathcal{H}(\mathrm{div}; \Gamma_c)$, and furthemore the equilibrium equations are satisfied,

$$-\mathrm{div}\sigma^0 = f \quad \text{in } \Omega \text{ .}$$

Now, for a positive parameter δ, consider the regularized problem

$$u^\delta \in L^2(\Omega), \quad \sigma^\delta \in \mathcal{H}(\mathrm{div}; \Gamma_c) \text{ ,} \tag{45}$$

$$\delta u^\delta - \mathrm{div}\sigma^\delta = f \quad \text{in } \Omega \text{ ,} \tag{46}$$

$$(C\sigma^\delta, \bar{\sigma} - \sigma^\delta)_\Omega + (u^\delta, \mathrm{div}\bar{\sigma} - \mathrm{div}\sigma^\delta)_\Omega \geq 0 \quad \forall \bar{\sigma} \in \mathcal{H}(\mathrm{div}; \Gamma_c) \text{ .} \tag{47}$$

From (45)–(47) we can obtain the uniform with respect to δ estimate

$$\delta \|u\|^2_{L^2(\Omega)} + \|\sigma\|^2_{L^2(\Omega)} + \|\mathrm{div}\sigma\|^2_{L^2(\Omega)} \leq c \text{ .} \tag{48}$$

In the same way as in the proof of Theorem 2.1, from (46)–(47) the following estimate is obtained, uniform with respect to δ,

$$\|u^\delta\|_{H^{1,0}(\Omega_c)} \leq c \text{ .} \tag{49}$$

By estimates (48), (49), we have, as $\delta \to 0$, the following convergences for subsequences:

$$u^\delta \to u \quad \text{strongly in } L^2(\Omega) \text{ ,}$$

$$\sigma^\delta \to \sigma \quad \text{weakly in } L^2(\Omega) \text{ ,}$$

$$\mathrm{div}\sigma^\delta \to \mathrm{div}\sigma \quad \text{weakly in } L^2(\Omega) \text{ .}$$

Consequently, we can pass to the limit as $\delta \to 0$ in (45)–(47) and obtain (42)–(44), which completes the proof of Theorem 2.2. □

The solution to (42)–(44) is unique.

Formulation of the free boundary crack problem in the form (42)–(44) is attractive from a numerical point of view since the domain Ω does not contain nonsmooth components of the boundary. Moreover, the restrictions imposed on the stress tensor components are given on subsets of Ω. So the problem formulation (42)–(44) resembles that of classical contact problems with restrictions imposed on subsets of the domain. A wide class of contact problems with restrictions imposed on subsets of domains can be found in [9].

REMARK 2.2. Similar to the mixed problem formulation (see Remark 2.1), we can consider an equivalent definition of the admissible stresses,

$$\mathcal{H}(\mathrm{div};\Gamma_c) = \{\sigma \in \mathcal{H}(\mathrm{div}) \mid \int_{\Omega_c} (\sigma\varepsilon(\bar{u}) + \bar{u}\mathrm{div}\sigma) \geq 0 \ \ \forall\, \bar{u} \in K\} \ .$$

The set K is defined in (21). The above definition of $\mathcal{H}(\mathrm{div};\Gamma_c)$ can be applied for both the interior and boundary cracks (see Figs. 4, 5).

REMARK 2.3. Now observe that the classical approach to the two-dimensional crack problem is characterized by the equality type boundary conditions (cf. (18), (19))

$$\sigma_\nu = \sigma_\tau = 0 \ \text{ on } \ \Gamma_c^{\pm} \ . \tag{50}$$

In this case the smooth domain method can be successfully applied to the problem (15)–(17), (50). Indeed, the set of admissible stresses is defined as follows:

$$\mathcal{H}(\mathrm{div};\Gamma_c) = \{\sigma \in \mathcal{H}(\mathrm{div}) |\ \sigma_\nu = 0, \ \ \sigma_\tau = 0 \quad \text{on } \Gamma_c\} \ . \tag{51}$$

Instead of (44), we obtain the identity

$$(C\sigma, \bar{\sigma})_\Omega + (u, \mathrm{div}\bar{\sigma})_\Omega = 0 \quad \forall \bar{\sigma} \in \mathcal{H}(\mathrm{div};\Gamma_c) \ . \tag{52}$$

Hence, the smooth domain method for the classical boundary value crack problem can be formulated in the form (42), (43), (52), where $\mathcal{H}(\mathrm{div};\Gamma_c)$ is defined in (51).

3. Kirchhoff plate with a crack. In this section we show that the smooth domain method can be applied to equilibrium problems for the Kirchhoff plates having cracks with inequality type boundary conditions given at the crack faces. As in the two-dimensional elasticity, these boundary conditions describe a mutual nonpenetration between the crack faces. The problem formulation is as follows [8].

In the domain Ω_c, we have to find the functions $u = (u_1, u_2), w, \sigma = \{\sigma_{ij}\}, m = \{m_{ij}\}$, $i, j = 1, 2$, such that

$$-\mathrm{div}\sigma = f \quad \text{in} \quad \Omega_c \ , \tag{53}$$

$$-\nabla\nabla m = F \quad \text{in} \quad \Omega_c \ , \tag{54}$$

$$C\sigma - \varepsilon(u) = 0 \quad \text{in} \quad \Omega_c \ , \tag{55}$$

$$Dm + \nabla\nabla w = 0 \quad \text{in} \quad \Omega_c \ , \tag{56}$$

$$u = 0, \quad w = \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \Gamma \ , \tag{57}$$

$$[u]\nu \geq \left| \left[\frac{\partial w}{\partial \nu} \right] \right|, \quad [\sigma_\nu] = 0, \quad [m_\nu] = 0 \quad \text{on} \quad \Gamma_c \ , \tag{58}$$

$$|m_\nu| \leq -\sigma_\nu, \quad \sigma_\nu \cdot [u]\nu - m_\nu \left[\frac{\partial w}{\partial \nu} \right] = 0 \quad \text{on} \quad \Gamma_c \ , \tag{59}$$

$$\sigma_\tau = 0, \quad t^\nu(m) = 0 \quad \text{on} \quad \Gamma_c^\pm \ . \tag{60}$$

Here $f = (f_1, f_2)$; $F, f_i \in L^2(\Omega)$ are given functions, $i = 1, 2$,

$$\nabla\nabla m = m_{ij,ij} \ , \quad m_\nu = m_{ij}\nu_j\nu_i, \quad \nabla\nabla w = \{w_{,ij}\}_{i,j=1}^2 \ ,$$
$$t^\nu(m) = m_{ij,j}\nu_j + m_{ij,k}\tau_k\tau_j\nu_i \ , \ (\tau_1, \tau_2) = (-\nu_2, \nu_1) \ .$$

We use the same notations as in the previous sections. Tensor C is symmetric and satisfies the condition (20). Similar conditions are imposed on the tensor D,

$$\{Dm\}_{ij} = d_{ijkl}m_{kl}, \ i, j = 1, 2 \ .$$

Note that (53), (54) are equilibrium equations; relations (55), (56) provide the constitutive law. Boundary conditions (57) mean that the plate is clamped along the external boundary Γ. Equations and inequalities (58)–(60) describe a mutual nonpenetration between the crack faces $\Gamma_c^\pm$. The functions u, w are horizontal and vertical displacements of the mid-surface points of the plate; σ, m are stress tensor and moment tensor, respectively.

For a variational formulation of the problem (53)–(60), we need the Sobolev space

$$H^{2,0}(\Omega_c) = \{v \in H^2(\Omega_c) |\ v = \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \Gamma\}.$$

Consider the convex set of admissible displacements,

$$K_c = \{(u, w) \in [H^{1,0}(\Omega_c)]^2 \times H^{2,0}(\Omega_c) |\ \ [u]\nu \geq \left| \left[\frac{\partial w}{\partial \nu} \right] \right| \ \text{ a.e. on } \Gamma_c\}. \tag{61}$$

There exists a solution to the following minimisation problem

$$\min_{(u,w)\in K_c} \left\{ \frac{1}{2}(\sigma(u), \varepsilon(u))_{\Omega_c} - \frac{1}{2}(m(w), \nabla\nabla w)_{\Omega_c} - (f, u)_{\Omega_c} - (F, w)_{\Omega_c} \right\},$$

which is equivalent to the variational inequality

$$(u, w) \in K_c \ , \ \ (\sigma(u), \varepsilon(\bar{u} - u))_{\Omega_c} - (m(w), \nabla\nabla\bar{w} - \nabla\nabla w)_{\Omega_c} \tag{62}$$
$$\geq (f, \bar{u} - u)_{\Omega_c} + (F, \bar{w} - w)_{\Omega_c} \quad \forall(\bar{u}, \bar{w}) \in K_c \ .$$

The set K_c is weakly closed and the minimized functional is coercive and weakly lower semicontinuous in the space $[H^{1,0}(\Omega_c)]^2 \times H^{2,0}(\Omega_c)$. Hence, the problem (62) is solvable. Furthermore, the solution is unique.

Now, we introduce a mixed formulation of the problem (53)–(60). Consider the space

$$H(\Omega_c) = \{(\sigma, m) |\, \sigma = \{\sigma_{ij}\}, m = \{m_{ij}\};\ \sigma, \mathrm{div}\sigma \in L^2(\Omega_c), \\ m, \nabla\nabla m \in L^2(\Omega_c)\}$$

equipped with the norm

$$\|(\sigma, m)\|^2_{H(\Omega_c)} = \|\sigma\|^2_{L^2(\Omega_c)} + \|\mathrm{div}\sigma\|^2_{L^2(\Omega_c)} + \|m\|^2_{L^2(\Omega_c)} \\ + \|\nabla\nabla m\|^2_{L^2(\Omega_c)} .$$

Introduce the set of admissible stresses and moments,

$$K(\Omega_c) = \{(\sigma, m) \in H(\Omega_c) |\ [\sigma\nu] = [m_\nu] = [t^\nu(m)] = 0 \ \text{ on } \ \Gamma_c; \\ |m_\nu| \le -\sigma_\nu\ ,\ \sigma_\tau = 0,\ t^\nu(m) = 0 \text{ on } \ \Gamma_c^\pm\} .$$

Also, consider the space $H^{\frac{3}{2}}(\Sigma)$ with the norm

$$\|\varphi\|^2_{H^{\frac{3}{2}}(\Sigma)} = \|\varphi\|^2_{H^1(\Sigma)} + \int_\Sigma \int_\Sigma \frac{|\nabla\varphi(x) - \nabla\varphi(y)|^2}{|x-y|^2} dxdy$$

and its dual $H^{-\frac{3}{2}}(\Sigma)$. In the domain Ω_1, we can define traces on the boundary Σ, in particular, $m_\nu^- \in H^{-\frac{1}{2}}(\Sigma), t^\nu(m)^- \in H^{-\frac{3}{2}}(\Sigma)$, and the following Green formula holds [8], [21]:

$$(w, \nabla\nabla m)_{\Omega_1} = (\nabla\nabla w, m)_{\Omega_1} + \langle t^\nu(m)^-, w\rangle_{\frac{3}{2}} - \langle m_\nu^-, \frac{\partial w}{\partial \nu}\rangle_{\frac{1}{2}} , \\ \forall w \in H^2(\Omega_1) , \tag{63}$$

where $\langle\cdot,\cdot\rangle_{\frac{3}{2}}$ stands for the duality pairing between $H^{-\frac{3}{2}}(\Sigma)$ and $H^{\frac{3}{2}}(\Sigma)$.

For the domain Ω_2, we can write the Green formula similar to (63). In this case the boundary of Ω_2 contains two parts, Σ and Γ.

In addition to the two-dimensional elasticity, we should explain in what sense boundary conditions are fulfilled in the definition of $K(\Omega_c)$. Zero jump condition for $t^\nu(m)$ means

$$\langle t^\nu(m)^+ - t^\nu(m)^-, \varphi\rangle_{\frac{3}{2}} = 0 \quad \forall\varphi \in H^{\frac{3}{2}}(\Sigma) . \tag{64}$$

The condition $t^\nu(m) = 0$ on $\Gamma_c^\pm$ reads

$$\langle t^\nu(m)^\pm, \varphi\rangle_{\frac{3}{2}} = 0 \quad \forall\varphi \in H^{\frac{3}{2}}(\Sigma), \quad \mathrm{supp}\ \varphi \subset \Gamma_c . \tag{65}$$

It is seen that the set $K(\Omega_c)$ is convex. By the continuity of the trace operators, the set $K(\Omega_c)$ is closed. Hence $K(\Omega_c)$ is weakly closed.

Like in the two-dimensional elasticity, in (65) we can choose test functions $\bar\varphi$, where $\bar\varphi$ is an extension of φ to Σ by zero, $\varphi \in H^{\frac{3}{2}}_{00}(\Gamma_c)$. The norm in the space $H^{\frac{3}{2}}_{00}(\Gamma_c)$ is defined by the formula

$$\|\phi\|^2_{H^{\frac{3}{2}}_{00}(\Gamma_c)} = \|\phi\|^2_{H^{\frac{3}{2}}(\Gamma_c)} + \int_{\Gamma_c} \rho^{-1}|\nabla\phi|^2.$$

It is known that $\varphi \in H^{\frac{3}{2}}_{00}(\Gamma_c)$ if and only if $\bar\varphi \in H^{\frac{3}{2}}(\Sigma)$ [14].

Now we are in position to provide the mixed formulation for the analysed problem (53)–(60).

We have to find the functions $u = (u_1, u_2), w, \sigma = \{\sigma_{ij}\}, m = \{m_{ij}\}$, such that

$$u = (u_1, u_2) \in L^2(\Omega_c), w \in L^2(\Omega_c), (\sigma, m) \in K(\Omega_c) \ , \tag{66}$$

$$-\mathrm{div}\sigma = f \quad \text{on } \Omega_c \ , \tag{67}$$

$$-\nabla\nabla m = F \quad \text{on } \Omega_c \ , \tag{68}$$

$$(u, \mathrm{div}\bar{\sigma} - \mathrm{div}\sigma)_{\Omega_c} + (w, \nabla\nabla\bar{m} - \nabla\nabla m)_{\Omega_c} \tag{69}$$

$$+(C\sigma, \bar{\sigma} - \sigma)_{\Omega_c} + (Dm, \bar{m} - m)_{\Omega_c} \geq 0 \quad \forall(\bar{\sigma}, \bar{m}) \in K(\Omega_c) \ .$$

Inequality (69) follows from (55)–(56). It suffices to multiply these equations by $\bar{\sigma} - \sigma, \bar{m} - m$, respectively, with $(\bar{\sigma}, \bar{m}) \in K(\Omega_c)$. On the other hand, the equations (55), (56) follow from (69). To prove this, it suffices to take in (69) the test functions $(\bar{\sigma}, \bar{m}) = (\sigma, m) + (\tilde{\sigma}, \tilde{m})$, $(\tilde{\sigma}, \tilde{m}) \in C_0^\infty(\Omega_c)$. Moreover, the relations (66)–(69) contain all boundary conditions (57)–(60).

The existence of a solution to (66)–(69) can be shown by a procedure that is used in the proof of Theorem 3.1 below. The solution is unique.

REMARK 3.1. We can observe that, as in the two-dimensional elasticity, it is possible to avoid the explicit formulation of the boundary conditions for stresses and moments included in the set $K(\Omega_c)$. Namely, it suffices to introduce the set of admissible stresses and moments by using the "dual" formula

$$K(\Omega_c) = \{(\sigma, m) \in H(\Omega_c) \mid \int_{\Omega_c} (\sigma\varepsilon(\bar{u}) + \bar{u}\mathrm{div}\sigma)$$

$$+ \int_{\Omega_c} (\bar{w}\nabla\nabla m - m\nabla\nabla\bar{w}) \geq 0 \quad \forall\ (\bar{u}, \bar{w}) \in K_c\} \ ,$$

where the set K_c is defined in (61). This equivalent definition of the set $K(\Omega_c)$ is suitable also in the case when $\bar{\Gamma}_c$ crosses the external boundary Γ (see Figs. 4, 5). In particular, if Γ_c divides Ω into two separate domains Ω^1, Ω^2, we obtain the contact problem for two elastic plates occupying the domains Ω^1, Ω^2 with contact conditions (58)–(60) on the common boundary Γ_c.

Now we can formulate the smooth domain method for the problem (53)–(60). In this case the solution is defined in the smooth domain Ω. In fact, we extend the unknown functions from the domain Ω_c to the domain Ω. To simplify formulae below we use the same notations for the extended functions. The formulation of the problem is as follows.

In the domain Ω, we have to find functions $u = (u_1, u_2), w, \sigma = \{\sigma_{ij}\}, m = \{m_{ij}\}$, $i, j = 1, 2$, such that

$$-\mathrm{div}\sigma = f \quad \text{in} \quad \Omega\ , \tag{70}$$

$$-\nabla\nabla m = F \quad \text{in} \quad \Omega\ , \tag{71}$$

$$C\sigma - \varepsilon(u) + p(u)\delta_{\Gamma_c} = 0 \quad \text{in} \quad \Omega\ , \tag{72}$$

$$Dm + \nabla\nabla w + P(w) = 0 \quad \text{in} \quad \Omega\ , \tag{73}$$

$$u = 0, \quad w = \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \Gamma\ , \tag{74}$$

$$[u]\nu \geq \left|\left[\frac{\partial w}{\partial \nu}\right]\right|,\ \sigma_\tau = 0, \quad t^\nu(m) = 0 \quad \text{on} \quad \Gamma_c\ , \tag{75}$$

$$|m_\nu| \leq -\sigma_\nu, \quad \sigma_\nu \cdot [u]\nu - m_\nu \left[\frac{\partial w}{\partial \nu}\right] = 0 \quad \text{on} \quad \Gamma_c\ . \tag{76}$$

Here

$$P(w)_{ij} = -([w]\nu_i\delta_{\Gamma_c})_{,j} - [w_{,i}]\nu_j\delta_{\Gamma_c}\ .$$

It is very important that the solution of the problem (53)–(60) determined from the variational inequality (62) possesses the properties

$$[\sigma\nu] = 0, \quad [m_\nu] = 0, \quad [t^\nu(m)] = 0 \quad \text{on} \quad \Gamma_c\ . \tag{77}$$

It allows us to write equilibrium equations (53), (54) in the domain Ω in the same form. Let us verify this statement. The validity of the equation (70) in the domain Ω is already shown (see Sec. 2). So we just check (71). From (62) it follows that

$$m \in L^2(\Omega_c)\ ,\ \nabla\nabla m \in L^2(\Omega_c), \quad -\nabla\nabla m = F \quad \text{in} \quad \Omega_c\ . \tag{78}$$

Let m denote the extended function, defined in Ω. Then, by (77)–(78), we have for any $\varphi \in C_0^\infty(\Omega)$,

$$\langle \nabla\nabla m + F, \varphi\rangle = (m, \nabla\nabla\varphi)_{\Omega_1} + (m, \nabla\nabla\varphi)_{\Omega_2} + (F, \varphi)_\Omega$$

$$= (\nabla\nabla m + F, \varphi)_{\Omega_1} + (\nabla\nabla m + F, \varphi)_{\Omega_2}$$

$$+\langle [t^\nu(m)], \varphi\rangle_{\frac{3}{2}} - \langle [m_\nu], \frac{\partial\varphi}{\partial\nu}\rangle_{\frac{1}{2}} = 0\ .$$

Hence the equilibrium equation (71) holds in Ω in the sense of distributions. To give the weak formulation of (70)–(76), we need additional notations. Consider the space

$$\mathcal{H}(\Omega) = \{(\sigma, m) |\ \sigma = \{\sigma_{ij}\}, m = \{m_{ij}\};\ \sigma, \mathrm{div}\sigma \in L^2(\Omega),$$
$$m, \nabla\nabla m \in L^2(\Omega)\}$$

equipped with norm

$$\|(\sigma, m)\|^2_{\mathcal{H}(\Omega)} = \|\sigma\|^2_{L^2(\Omega)} + \|\mathrm{div}\sigma\|^2_{L^2(\Omega)} + \|m\|^2_{L^2(\Omega)}$$
$$+\|\nabla\nabla m\|^2_{L^2(\Omega)}\ .$$

Introduce the admissible set of stresses and moments

$$\mathcal{K}(\Omega) = \{(\sigma, m) \in \mathcal{H}(\Omega) |\ \sigma_\tau = 0,\ t^\nu(m) = 0,\ |m_\nu| \le -\sigma_\nu \text{ on } \Gamma_c\}\ .$$

Interpretation of the conditions imposed on σ, m in the definition of $\mathcal{K}(\Omega)$ is simpler compared to the case of the nonsmooth domain Ω_c, since the jumps on Σ of the functions $\sigma\nu, m_\nu, t^\nu(m)$ are equal to zero by definition. Hence the equalities and inequality are fulfilled in the following sense:

$$\langle \sigma_\nu \pm m_\nu, \varphi\rangle_{\frac{1}{2}} \le 0 \ \ \forall\varphi \in H^{\frac{1}{2}}(\Sigma), \quad \varphi \ge 0 \quad \text{a.e. on } \Gamma_c\ ,$$
$$\text{supp}\ \varphi \subset \Gamma_c\ ,$$

$$\langle \sigma_\tau, \varphi\rangle_{\frac{1}{2}} = 0 \ \ \forall\varphi = (\varphi_1, \varphi_2) \in H^{\frac{1}{2}}(\Sigma), \ \ \varphi_i\nu_i = 0 \text{ a.e. on } \Gamma_c\ ,$$
$$\text{supp}\ \varphi \subset \Gamma_c\ ,$$

$$\langle t^\nu(m), \varphi\rangle_{\frac{3}{2}} = 0 \quad \forall\varphi \in H^{\frac{3}{2}}(\Sigma), \quad \text{supp}\ \varphi \subset \Gamma_c\ .$$

In the weak formulation of the problem (70)–(76), unknown functions u, w, σ, m are such that

$$u = (u_1, u_2) \in L^2(\Omega),\ w \in L^2(\Omega),\ (\sigma, m) \in \mathcal{K}(\Omega)\ , \tag{79}$$
$$-\text{div}\sigma = f \quad \text{in } \Omega\ , \tag{80}$$
$$-\nabla\nabla m = F \quad \text{in } \Omega\ , \tag{81}$$
$$(u, \text{div}\bar\sigma - \text{div}\sigma)_\Omega + (w, \nabla\nabla\bar m - \nabla\nabla m)_\Omega \tag{82}$$
$$+(C\sigma, \bar\sigma - \sigma)_\Omega + (Dm, \bar m - m)_\Omega \ge 0 \quad \forall(\bar\sigma, \bar m) \in \mathcal{K}(\Omega)\ .$$

We can prove the following statement.

THEOREM 3.1. There exists a unique solution to the problem (79)–(82).

Proof. The general scheme of the proof is the same as in Theorem 2.2. We introduce functions $(\sigma^0, m^0) \in \mathcal{K}(\Omega)$ satisfying the equations

$$-\text{div}\sigma^0 = f\ \ ,\ -\nabla\nabla m^0 = F \quad \text{in } \Omega\ .$$

The functions (σ^0, m^0) can be obtained by solving variational inequality (62) with arbitrary constitutive laws (55)–(56) for any given tensors C, D. Of course the tensors C, D should satisfy the required conditions. To prove the existence of a solution, a similar regularization procedure is used. For a positive parameter δ, the regularized problem is considered:

$$u^\delta = (u_1^\delta, u_2^\delta) \in L^2(\Omega), w^\delta \in L^2(\Omega), (\sigma^\delta, m^\delta) \in \mathcal{K}(\Omega)\ , \tag{83}$$
$$\delta u^\delta - \text{div}\sigma^\delta = f \quad \text{in } \Omega\ , \tag{84}$$
$$\delta w^\delta - \nabla\nabla m^\delta = F \quad \text{in } \Omega\ , \tag{85}$$
$$(C\sigma^\delta, \bar\sigma - \sigma^\delta)_\Omega + (Dm^\delta, \bar m - m^\delta)_\Omega + (u^\delta, \text{div}\bar\sigma - \text{div}\sigma^\delta)_\Omega \tag{86}$$
$$+(w^\delta, \nabla\nabla\bar m - \nabla\nabla m^\delta)_\Omega \ge 0 \quad \forall(\bar\sigma, \bar m) \in \mathcal{K}(\Omega)\ .$$

Taking $(\bar{\sigma}, \bar{m}) = (\sigma^0, m^0)$ in (86) and multiplying (84), (85) by u^δ, w^δ, respectively, we derive the a priori estimate

$$\delta\|u^\delta\|^2_{L^2(\Omega)} + \delta\|w^\delta\|^2_{L^2(\Omega)} + \|\sigma^\delta\|^2_{L^2(\Omega)} + \|m^\delta\|^2_{L^2(\Omega)} \le c, \tag{87}$$

where the constant c is uniform with respect to δ. By (87), from (84), (85), we have uniformly in δ

$$\|\mathrm{div}\sigma^\delta\|^2_{L^2(\Omega)} + \|\nabla\nabla m^\delta\|^2_{L^2(\Omega)} \le c\ . \tag{88}$$

Solvability of the problem (83)–(86) can be obtained by the variational approach. To this end, it suffices to substitute the values u^δ, w^δ, taken from (84), (85), into (86). In such a way we obtain the variational inequality for $(\sigma^\delta, m^\delta)$ which admits a solution. Let us perform the passage to the limit in (84)–(86) as $\delta \to 0$. From (86) it follows that

$$C\sigma^\delta - \varepsilon(u^\delta) = 0, \quad Dm^\delta + \nabla\nabla w^\delta = 0 \quad \text{in} \quad \Omega_c\ , \tag{89}$$

hence $\varepsilon(u^\delta) \in L^2(\Omega_c)$. By the second Korn inequality in Ω_c, since $u^\delta \in L^2(\Omega_c)$, we obtain $u^\delta \in H^1(\Omega_c)$. On the other hand,

$$u^\delta = 0 \quad \text{on} \quad \Gamma\,,$$

and consequently $u^\delta = (u^\delta_1, u^\delta_2) \in H^{1,0}(\Omega_c)$. We use the first Korn inequality,

$$\|u^\delta_1\|_{H^{1,0}(\Omega_c)} + \|u^\delta_2\|_{H^{1,0}(\Omega_c)} \le c\|\varepsilon(u^\delta)\|_{L^2(\Omega_c)}\ ,$$

where the constant c depends only on Ω_c. Since the deformations $\varepsilon(u^\delta)$ are bounded in $L^2(\Omega_c)$ uniformly in δ, the following estimate holds:

$$\|u^\delta_i\|_{H^{1,0}(\Omega_c)} \le c,\ i = 1, 2\ . \tag{90}$$

Next, the second equation of (89) implies $\nabla\nabla w^\delta \in L^2(\Omega_c)$. Consequently, $w^\delta \in H^2(\Omega_c)$. Taking into account the boundary conditions

$$w^\delta = \frac{\partial w^\delta}{\partial n} = 0 \quad \text{on} \quad \Gamma,$$

it follows that $w^\delta \in H^{2,0}(\Omega_c)$. We can use the inequality

$$\|w^\delta\|_{H^{2,0}(\Omega_c)} \le c\|\nabla\nabla w^\delta\|_{L^2(\Omega_c)}$$

with the constant c independent of δ, which leads to the uniform estimate with respect to δ,

$$\|w^\delta\|_{H^{2,0}(\Omega_c)} \le c\ . \tag{91}$$

Hence, by (87), (88), (90), (91), we can assume that as $\delta \to 0$

$$u^\delta_i \to u_i \quad \text{strongly in } L^2(\Omega)\ , i = 1, 2,$$
$$w^\delta \to w \quad \text{strongly in } L^2(\Omega)\ ,$$
$$(\sigma^\delta, m^\delta) \to (\sigma, m) \quad \text{weakly in } \mathcal{H}(\Omega)\ .$$

These convergences allow us to pass to the limit in (83)–(86) as $\delta \to 0$, which implies (79)–(82).

The solution is unique. Indeed, assume that we have two solutions $(u^1, w^1, \sigma^1, m^1)$, $(u^2, w^2, \sigma^2, m^2)$. From (82) it follows that $\sigma^1 = \sigma^2, m^1 = m^2$. Since

$$C\sigma^i - \varepsilon(u^i) = 0, \quad Dm^i + \nabla\nabla w^i = 0 \quad \text{in} \quad \Omega_c, \ i = 1, 2 \ ,$$

we obtain $\varepsilon(u^1 - u^2) = 0$, $\nabla\nabla(w^1 - w^2) = 0$. Consequently, $u^1 = u^2, w^1 = w^2$. □

REMARK 3.2. Similar to the two-dimensional elasticity, we can use the definition of the admissible stresses and moments which is suitable for both interior and boundary cracks, namely,

$$\mathcal{K}(\Omega) = \{(\sigma, m) \in \mathcal{H}(\Omega) \mid \int_{\Omega_c} (\sigma\varepsilon(\bar{u}) + \bar{u}\mathrm{div}\sigma)$$
$$+ \int_{\Omega_c} (\bar{w}\nabla\nabla m - m\nabla\nabla\bar{w}) \geq 0 \quad \forall\, (\bar{u}, \bar{w}) \in K_c\} \ .$$

In particular, this definition is useful for the contact problems (see Fig. 5).

REMARK 3.3. Consider the classical crack problem for the Kirchhoff plate. In this case, instead of (58)–(60), we have the linear boundary conditions

$$m_\nu = t^\nu(m) = \sigma_\nu = \sigma_\tau = 0 \quad \text{on} \ \Gamma_c^\pm \ . \tag{92}$$

The smooth domain method proposed in the paper can be applied to the problem (53)–(57), (92). Admissible set of stresses and moments in this linear case is introduced as follows:

$$\mathcal{K}(\Omega) = \{(\sigma, m) \in \mathcal{H}(\Omega) |\ m_\nu = t^\nu(m) = \sigma_\nu = \sigma_\tau = 0 \text{ on } \ \Gamma_c\} \ . \tag{93}$$

The inequality (82) should be changed by the identity

$$(u, \mathrm{div}\bar{\sigma})_\Omega + (w, \nabla\nabla\bar{m})_\Omega \tag{94}$$
$$+ (C\sigma, \bar{\sigma})_\Omega + (Dm, \bar{m})_\Omega = 0 \quad \forall (\bar{\sigma}, \bar{m}) \in \mathcal{K}(\Omega) \ .$$

Hence the smooth domain method in the classical crack problem for plates can be formulated in the form (79)–(81), (94), where the set $\mathcal{K}(\Omega)$ is defined in (93).

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