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CONSTRAINED QUASICONVEXIFICATION OF THE SQUARE OF THE GRADIENT OF THE STATE IN OPTIMAL DESIGN

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Abstract. We explicitly compute the constrained quasiconvexification of the integrand associated with the square of the gradient of the state in a typical optimal design problem in which a volume constraint is enforced.

1. Introduction. We want to consider the following optimal design problem in conductivity. Given a design domain $\Omega \subset \mathbf{R}^2$, fixed amounts of two different conducting materials, the charge density on $\partial\Omega$, and the current density over Ω, decide on the best way to mix them in Ω in order to minimize the cost functional consisting in the mean square deviation of the gradient of the underlying electric potential from a given target vector field. Similar optimal design problems have been addressed in a number of papers, especially when the cost functional does not depend on derivatives of the state. Γ-convergence, G-convergence, and H-convergence ideas in the context of homogenization theory (see for example [11], [12], [13], [18], [19], [21], although an exhaustive list would have to be much longer) or variational treatments ([8]) are among the techniques utilized to analyze such problems. When cost functionals depend on derivatives of the state, more elaborate tools and ideas (always within the context of homogenization and effective properties) have also been applied ([7], [9], [20]) to the analysis and understanding of such situations. See the last section for more specific comments on these works. Recently, a purely variational format has also been proposed to examine these problems ([14]), although it had been previously indicated in [17]. In particular, when the volume constraint is not present, the fully explicit quasiconvexification of the resulting integrand when the situation is formulated as a vector variational problem has been obtained ([15]). This quasiconvex hull somehow encodes optimal microstructures or microgeometries much in the same way as is typical in non-convex vector variational problems ([5]). If we would like to keep the volume constraint, an appropriate convex hull has been defined and examined in [14]. This involves the concept of constrained quasiconvexity and constrained quasiconvexification. The aim of these pages is to explicitly

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compute in closed form the underlying constrained quasiconvexification when the optimal design problem is described as a vector variational problem depending on a vector gradient and a volume fraction. This constrained quasiconvexification enjoys a jointly convexity property in the gradient and volume fraction which is the key assumption for a relaxation theorem under constraints ([6]).

Given a regular, simply-connected domain $\Omega \subset \mathbf{R}^2$, conductivities α, β, $0 < \alpha < \beta$, a fixed volume fraction $\lambda \in (0,1)$, $f \in H^{-1}(\Omega)$, $u_0 \in H^1(\Omega)$, and $F \in L^2(\Omega)$, we want to

$$\text{Minimize } I(\chi) = \int_\Omega |\nabla u(x) - F(x)|^2 \, dx,$$

where χ is the characteristic function of a subset of Ω with mean value equal to λ, and $u \in H^1(\Omega)$ is the unique solution of

$$\begin{gathered} \operatorname{div}((\alpha\chi(x) + \beta(1-\chi(x)))\nabla u(x)) = f \text{ in } \Omega, \\ u = u_0 \text{ on } \partial\Omega. \end{gathered}$$

We know that typically the answer to the existence of optimal solution is negative ([10]), so that something must be done about the analysis and approximation of optimal (or quasi optimal) solutions. Our approach is based on a variational reformulation of the situation as a vector variational problem as in the references cited above.

Before briefly describing the reformulation of the problem, and for the sake of simplicity, we notice that

$$|\nabla u(x) - F(x)|^2 = |\nabla u(x)|^2 - 2\nabla u(x) \cdot F(x) + |F(x)|^2$$

implies that it suffices to restrict attention to $F \equiv 0$ since our final result for a non-vanishing vector field F is a direct extension. On the other hand, we can also assume that $f \equiv 0$, since the case where f is not identically zero amounts to performing a translation on the resulting integrand for the variational reformulation, and the invariance of convex hulls with respect to translations enables us to restrict attention to the case $f \equiv 0$ (see [15]). Hence we assume that F and f identically vanish.

Under the hypothesis of simple connectedness of Ω, there exists a potential $v \in H^1(\Omega)$ such that

$$\operatorname{div}((\alpha\chi(x) + \beta(1-\chi(x)))\nabla u(x)) = 0 \text{ in } \Omega$$

is equivalent to

$$(\alpha\chi(x) + \beta(1-\chi(x)))\nabla u(x) + T\nabla v(x) = 0,$$

where T is the counterclockwise $\pi/2$-rotation in the plane. If we collect both u and v in a single vector field $U = (u, v)$, it is not hard to realize that our initial optimal design problem is equivalent to

$$\text{Minimize } I(U) = \int_\Omega W(\nabla U(x)) \, dx$$

subject to

$$\begin{gathered} U \in H^1(\Omega), \quad U^{(1)} = f \text{ on } \partial\Omega, \\ \int_\Omega V(\nabla U(x)) \, dx = \lambda, \end{gathered}$$

where the densities

$$W, V\colon \mathbf{M}^{2\times 2} \to \mathbf{R}^* = \mathbf{R} \cup \{+\infty\}$$

are defined by

$$W(A) = \begin{cases} |A^{(1)}|^2, & \text{if } A \in \Lambda_\alpha \cup \Lambda_\beta, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$V(A) = \begin{cases} 1, & \text{if } A \in \Lambda_\alpha, \\ 0, & \text{if } A \in \Lambda_\beta, \\ +\infty, & \text{otherwise.} \end{cases}$$

Here, $U^{(i)}$, $i = 1, 2$, denote the i-th component of U, $A^{(i)}$, $i = 1, 2$, denote the i-th row of A, and Λ_γ designates the two-dimensional subspace of matrices defined by

$$\Lambda_\gamma = \{A \in \mathbf{M}^{2\times 2}\colon \gamma A^{(1)} + T A^{(2)} = 0\}.$$

Since it is equivalent to our original optimal design problem, this new vector variational problem does not admit optimal solutions. In such cases, however, its relaxation ([5]) usually provides all the information to understand and approximate optimal behavior. Under integral restrictions, this relaxation ought to be somewhat more elaborate. In fact, it was shown in [14] that the relevant quasiconvexification in such circumstances is

$$CQW(A, t) = \inf\left\{ \frac{1}{|D|}\int_D W(A + \nabla\varphi(x))\,dx\colon \varphi \in W_0^{1,\infty}(D), \int_D V(A + \nabla\varphi(x))\,dx = t|D| \right\},$$

where D is any regular domain. This is the appropriate integrand for a relaxation theorem. It enjoys the above-mentioned jointly-convex property

$$CQW(A, t) \le \frac{1}{|D|}\int_D CQW(A + \nabla\varphi(x), t + \theta(x))\,dx$$

whenever $\varphi \in W_0^{1,\infty}(D)$ and $\theta \in L^\infty(D)$ with vanishing mean value over D. In particular $CQW(A, t)$ must be quasiconvex for fixed t and convex for fixed A. There are a number of technical issues concerning the fact that the integrands in our specific situation, W and V, are not Carathéodory functions. We will address these issues, as well as generalizations of our computations here and those in [15], specifically in a future work ([4]), but restrict attention here to the explicit computation of the convex hull $CQW(A, t)$. CQ stands for constrained quasiconvexification.

For the statement of our main result, we need some piece of notation. All of the expressions that follow, as well as all questions concerning the correct sets where they are defined, will be discussed and clarified in the proof. Put

$$\begin{aligned} g(A) &= \alpha^2\beta^2|A^{(1)}|^4 + |A^{(2)}|^4 + (\alpha^2 + 6\alpha\beta + \beta^2)\det A^2 \\ &\quad - 2\alpha\beta|A^{(1)}|^2|A^{(2)}|^2 - 2\alpha\beta(\alpha+\beta)|A^{(1)}|^2\det A - 2(\alpha+\beta)|A^{(2)}|^2\det A, \end{aligned} \tag{1.1}$$

and

$$r_1(A) = \frac{1}{2} + \frac{1}{2(\beta-\alpha)\det A}[\alpha\beta|A^{(1)}|^2 - |A^{(2)}|^2 - \sqrt{g(A)}],$$
$$r_2(A) = \frac{1}{2} + \frac{1}{2(\beta-\alpha)\det A}[\alpha\beta|A^{(1)}|^2 - |A^{(2)}|^2 + \sqrt{g(A)}].$$

Take the matrices

$$A_{\alpha,t} = \begin{pmatrix} z_t \\ \alpha T z_t \end{pmatrix}, \quad z_t = \frac{1}{t}\frac{1}{\beta-\alpha}(\beta A^{(1)} + TA^{(2)}),$$
$$A_{\beta,i} = \begin{pmatrix} w_i \\ BTw_i \end{pmatrix}, \quad w_i = \frac{1}{1-r_i(A)}\frac{(-1)}{\beta-\alpha}(\alpha A^{(1)} + TA^{(2)}),$$
$$A_{\beta,j,t} = \begin{pmatrix} w_{j,t} \\ \beta T w_{j,t} \end{pmatrix}, \quad w_{j,t} = \frac{1}{t}\frac{r_j(A)}{1-r_j(A)}\frac{(-1)}{\beta-\alpha}(\alpha A^{(1)} + TA^{(2)}),$$

where $i \neq j$ and $A^{(k)}$ stands for the k-th row of A. Finally, put

$$s_{i,j} = \frac{(1-r_i(A))[t(1-r_j(A)) - (1-t)r_j(A)]}{t(1-r_j(A)) - (1-r_i(A))r_j(A)}.$$

The aim of this paper is to prove the following theorem.

THEOREM 1.1. The constrained quasiconvexification of W is given by

$$CQW(A,t) = \frac{1}{t\beta(\beta-\alpha)}(\beta^2|A^{(1)}|^2 + |A^{(2)}|^2 - (\alpha t + \beta(2-t))\det A)$$

if (A,t) is such that

$$\alpha\beta(\beta(1-t)+\alpha t)|A^{(1)}|^2 + (\alpha(1-t)+\beta t)|A^{(2)}|^2 \leq (t(1-t)(\beta-\alpha)^2 + 2\alpha\beta)\det A,$$

and

$$CQW(A,t) = +\infty$$

otherwise.

Moreover there are two second-order laminates supported in three matrices (except when $t = r_i(A)$ for $i = 1$ or $i = 2$ that the laminate collapses to a first-order laminate) which are optimal microstructures. Namely, bearing in mind the notation before the statement of the theorem, the two laminates

$$\nu_{i,j} = s_{i,j}\delta_{A_{\beta,i}} + (1-s_{i,j})\left(\frac{t}{1-s_{i,j}}\delta_{A_{\alpha,t}} + \frac{1-s_{i,j}-t}{1-s_{i,j}}\delta_{A_{\beta,j,t}}\right),$$

for $i \neq j$, where

$$\det(A_{\alpha,t} - A_{\beta,j,t}) = 0,$$
$$\det\left(A_{\beta,i} - \frac{t}{1-s_{i,j}}A_{\alpha,t} - \frac{1-s_{i,j}-t}{1-s_{i,j}}A_{\beta,j,t}\right) = 0,$$

are optimal, and so are any convex combination of these two.

Notice the polyconvex dependence of $CQW(A,t)$ on A and the convex dependence on t. The joint convexity property in (A,t) is, however, more than these two separate convexities. As is usual with non-convex variational problems, optimal microstructure are encoded in such relaxation results where effective energies are computed. We hope to exploit this result from the horizon of numerical simulation of optimal solutions in the near future ([1]).

The next section focuses on the lower bound for $CQW(A,t)$ while Sec. 3 is devoted to admissible optimal microstructures for which that lower bound is attained. The final section includes a comparison of some of our computations with those performed under homogenization ideas. Since this comparison is somewhat specific, we have decided to defer it.

2. The lower bound. For the set

$$\mathcal{A}(A,t) = \Big\{ \nu\colon \text{homogeneous } W^{1,\infty}\text{-Young measures with first moment } A \text{ and } \int_{\mathbf{M}} V(F)\, d\nu(F) = t \Big\},$$

we will actually focus on the following infimum ([6]):

$$CQW(A,t) = \inf\left\{ \int_{\mathbf{M}} W(F)\, d\nu(F)\colon \nu \in \mathcal{A}(A,t) \right\}. \tag{2.1}$$

Technicalities concerning whether this infimum is equal to the infimum in terms of gradients as it has been stated in the Introduction will be addressed in [4].

Step 1. By definition of W, we can restrict attention to these feasible ν's supported in the union $\Lambda_\alpha \cup \Lambda_\beta$ such that $\nu(\Lambda_\alpha) = t$. Thus we can decompose

$$\nu = t\nu_\alpha + (1-t)\nu_\beta, \quad \operatorname{supp}(\nu_\alpha) \subset \Lambda_\alpha, \operatorname{supp}(\nu_\beta) \subset \Lambda_\beta.$$

Furthermore if we set

$$A_\alpha = \int_{\Lambda_\alpha} F\, d\nu_\alpha(F), \quad A_\beta = \int_{\Lambda_\beta} F\, d\nu_\beta(F),$$

we have

$$A_\alpha \in \Lambda_\alpha, A_\beta \in \Lambda_\beta, \quad A = tA_\alpha + (1-t)A_\beta.$$

In this way, there are vectors $z, w \in \mathbf{R}^2$ such that

$$A_\alpha = \begin{pmatrix} z \\ \alpha T z \end{pmatrix}, \quad A_\beta = \begin{pmatrix} w \\ \beta T w \end{pmatrix}. \tag{2.2}$$

If we write the two vector equations enclosed in

$$A = tA_\alpha + (1-t)A_\beta$$

row-wise, it is elementary to obtain

$$\begin{aligned} z &= \frac{1}{t(\beta-\alpha)}(\beta A^{(1)} + TA^{(2)}), \\ w &= -\frac{1}{(1-t)(\beta-\alpha)}(\alpha A^{(1)} + TA^{(2)}), \end{aligned} \tag{2.3}$$

where $A^{(k)}$ designates the k-th row of A. Therefore the matrices A_α and A_β are uniquely determined by A and t, and are independent of ν itself.

Step 2. Let us exploit the weak continuity of the determinant ([5]). We know that

$$\det A = \int_{\mathbf{M}} \det F \, d\nu(F)$$
$$= t \int_{\Lambda_\alpha} \det F \, d\nu_\alpha(F) + (1-t) \int_{\Lambda_\beta} \det F \, d\nu_\beta(F).$$

But notice that

$$\det F = \alpha |F^{(1)}|^2$$

if $F \in \Lambda_\alpha$, and similarly in Λ_β. Hence

$$\det A = t\alpha \int_{\Lambda_\alpha} |F^{(1)}|^2 \, d\nu_\alpha(F) + (1-t)\beta \int_{\Lambda_\beta} |F^{(1)}|^2 \, d\nu_\beta(F). \tag{2.4}$$

On the other hand, the cost functional we would like to minimize can be written

$$\int_{\mathbf{M}} W(F) \, d\nu(F) = t \int_{\Lambda_\alpha} |F^{(1)}|^2 \, d\nu_\alpha(F) + (1-t) \int_{\Lambda_\beta} |F^{(1)}|^2 \, d\nu_\beta(F). \tag{2.5}$$

Step 3. We would like to consider a certain linear programming problem (LPP) related to our situation. Let us consider variables x_α and x_β by putting

$$x_\alpha = \int_{\Lambda_\alpha} |F^{(1)}|^2 \, d\nu_\alpha(F),$$
$$x_\beta = \int_{\Lambda_\beta} |F^{(1)}|^2 \, d\nu_\beta(F).$$

Obviously, by Jensen's inequality, we must respect the constraints

$$x_\alpha \geq |z|^2, \quad x_\beta \geq |w|^2,$$

where z and w are given in (2.3), and are independent of ν. These inequality constraints together with (2.4) and the cost functional (2.5), lead us to the LPP (see Fig. 2.1)

$$\text{Minimize } t x_\alpha + (1-t) x_\beta$$

subject to

$$x_\alpha \geq |z|^2, \quad x_\beta \geq |w|^2,$$
$$\det A = \alpha t x_\alpha + \beta(1-t) x_\beta.$$

Let $m(A,t)$ denote the optimal value of this problem. Clearly

$$m(A,t) \leq CQW(A,t).$$

Step 4. Computation of $m(A,t)$. It is very easy to find that the point

$$\left(|z|^2, \frac{\det A - \alpha t |z|^2}{(1-t)\beta} \right)$$

is the intersection of the equality constraint with the line $x_\alpha = |z|^2$. Therefore, in order to have a non-empty feasible set for our LPP, we must enforce

$$\frac{\det A - \alpha t |z|^2}{(1-t)\beta} \geq |w|^2$$

or

$$\det A \geq \alpha t |z|^2 + \beta(1-t)|w|^2. \tag{2.6}$$

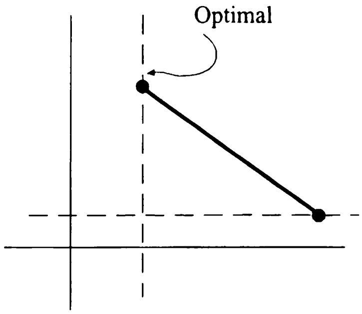

FIG. 2.1. A LPP yielding lower bound.

Otherwise $m(A,t)$ (and consequently $CQW(A,t)$) is $+\infty$.

Let us assume that A and t are such that (2.6) holds. Then the feasible region for our LPP is non-empty and the optimal value $m(A,t)$ will be attained at one of the two extreme points

$$\left(|z|^2, \frac{\det A - \alpha t|z|^2}{(1-t)\beta}\right)$$

or

$$\left(\frac{\det A - \beta(1-t)|w|^2}{t\alpha}, |w|^2\right).$$

Under (2.6) it is elementary to check that the minimum is taken on in the first of these two points and moreover

$$m(A,t) = \frac{1}{\beta}(\det A + t(\beta - \alpha)|z|^2).$$

More explicitly, using (2.3),

$$m(A,t) = \frac{1}{t\beta(\beta-\alpha)}(\beta^2|A^{(1)}|^2 + |A^{(2)}|^2 - (\alpha t + \beta(2-t))\det A).$$

Although, in principle, we have the lower bound

$$\frac{1}{t\beta(\alpha-\beta)}(\beta^2|A^{(1)}|^2 + |A^{(2)}|^2 - (\alpha t + \beta(2-t))\det A) \le CQW(A,t),$$

we claim that in fact, equality holds. This is the aim of the next section.

3. Optimal microstructures. To prove that the lower bound shown in the previous section is in fact attained, we would have to find an optimal microstructure (gradient Young measure) for which

$$x_\alpha = \int_{\Lambda_\alpha} |F^{(1)}|^2\, d\nu_\alpha(F) = |z|^2 = |A_\alpha^{(1)}|^2.$$

By the strict convexity of the square function, this is only possible if

$$\nu_\alpha = \delta_{A_\alpha}, \quad A_\alpha = \begin{pmatrix} z \\ \alpha T z \end{pmatrix},$$

and z is given by (2.3). Hence the question is if we can find a gradient Young measure ν such that

$$\nu = t\delta_{A_\alpha} + (1-t)\nu_\beta.$$

We will show that this is so by taking ν_β a certain convex combination of two Dirac masses. Indeed, our construction will deliver such a microstructure which is a second-order laminate with three mass points. To this end we must detect rank-one directions passing through a given first moment A. Notice that ν cannot be supported in two matrices unless the feasible region for our LPP in the preceding section is a single point. The analysis that follows is also part of the computations in [15].

Consider a pair (A,t) where $m(A,t)$ is finite; i.e., by (2.6) after an elementary reorganization,

$$\begin{aligned}(\beta-\alpha)^2 \det At^2 + (\alpha\beta(\alpha-\beta)|A^{(1)}|^2 + (\beta-\alpha)|A^{(2)}|^2 - (\beta-\alpha)^2\det A)t \\ + (\alpha\beta^2|A^{(1)}|^2 + \alpha|A^{(2)}|^2 - 2\alpha\beta\det A) \leq 0.\end{aligned} \tag{3.1}$$

Let $P_A(t)$ be this second-degree polynomial in t for fixed A. Notice that

$$\begin{aligned}P_A(0) &= \alpha|\beta A^{(1)} + TA^{(2)}|^2 \geq 0,\\ P_A(1) &= \beta|\alpha A^{(1)} + TA^{(2)}|^2 \geq 0.\end{aligned}$$

(3.1) forces $P_A(t)$ to be a non-degenerate, upward-parabola so that $\det A > 0$, and moreover the discriminant must be positive (non-negative), and the vertex of the parabola should lie in the interval $(0,1)$ so that the two (one) real roots are contained in the interval $(0,1)$. After some careful manipulations, the discriminant turns out to be $g(A)$ in (1.1) while the condition on the vertex together with $g(A) \geq 0$ simplifies to (see [15])

$$h(A) \geq 0, \quad h(A) = (\alpha+\beta)\det A - \alpha\beta|A^{(1)}|^2 - |A^{(2)}|^2.$$

Equality signs above correspond to admissible degenerate cases. Therefore $g(A)$ and $h(A)$ must be non-negative, and then

$$r_1(A) \leq t \leq r_2(A),$$

where $r_i(A)$ are the two roots of $P_A(t) = 0$, given also in the Introduction. These computations imply, in particular, that the set where $m(A,t)$ (or $CQW(A,t)$) is finite can also be described by saying

$$h(A) \geq 0, g(A) \geq 0, \quad t \in [r_1(A), r_2(A)].$$

Let A be such that

$$h(A) \geq 0, \quad g(A) \geq 0.$$

We will try to write

$$A = sA^\alpha + (1-s)A^\beta,$$

ere

$$s \in (0,1), \quad A^\alpha \in \Lambda_\alpha, \ A^\beta \in \Lambda_\beta, \quad \det(A^\alpha - A^\beta) = 0.$$

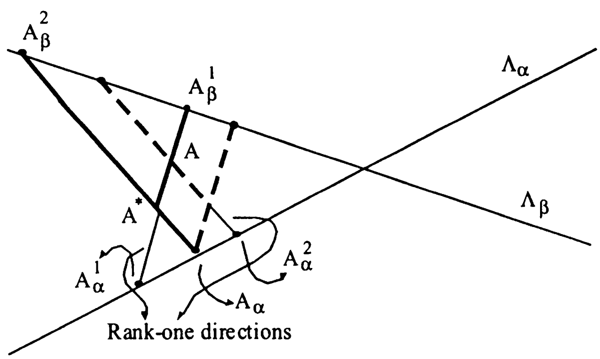

FIG. 3.1. Optimal laminates.

Taking all these conditions into account, the constraint on the vanishing determinant becomes, after some algebra,

$$\frac{\alpha}{s^2}|\beta A^{(1)}+TA^{(2)}|^2+\frac{\beta}{(1-s)^2}|\alpha A^{(1)}+TA^{(2)}|^2 \\ -\frac{\alpha+\beta}{s(1-s)}(\alpha A^{(1)}+TA^{(2)})\cdot(\beta A^{(1)}+TA^{(2)})=0.$$

Some additional algebraic manipulation shows this equation to be precisely the same

$$P_A(s)=0,$$

so that its two roots are $r_i(A)$, $i=1,2$. This means that there are two rank-one directions going through A with end-points in Λ_α and Λ_β.

We are now in a situation where we can find that optimal second-order laminate (Fig. 3.1).

Indeed we have a genuine two-dimensional framework determined by A and the two independent rank-one directions we have just found. If we set

$$A^i_\alpha=\begin{pmatrix} z_i \\ \alpha T z_i\end{pmatrix},\quad z_i=\frac{1}{r_i(A)(\beta-\alpha)}(\beta A^{(1)}+TA^{(2)}),\quad i=1,2,$$

then having in mind that $r_1(A)\le t\le r_2(A)$, it turns out that A_α is a convex combination of the two A^i_α's

$$A_\alpha=sA^1_\alpha+(1-s)A^2_\alpha,\quad \frac{1}{t}=s\frac{1}{r_1(A)}+(1-s)\frac{1}{r_2(A)}.$$

Then it is elementary to realize that

$$A^*=(1-s)A+sA^1_\alpha.$$

The second order laminate will correspond to the decomposition

$$A=\lambda A^1_\beta+(1-\lambda)A^*=\lambda A^1_\beta+(1-\lambda)(\tau A^2_\beta+(1-\tau)A_\alpha).$$

Finding the appropriate weights λ and τ in terms of t, $r_1(A)$ and $r_2(A)$ involves some straightforward but tedious algebra leading to the formulas before Theorem 1.1. Notice that in fact there is another different such laminate, associated to a parallel construction. Any convex combination of the two will also be optimal.

These two second-order laminates supported in three matrices seem the best choice in the sense that a minimal number of matrices participate. What is clear is that these two distinguished laminates are the only ones supported in three matrices. For this, keep in mind the general results on gradient Young measures supported in three matrices ([2], [16]).

The computations performed in this section also enable us to express $CQW(A,t)$ in the form

$$CQW(A,t) = \begin{cases} \frac{1}{t\beta(\beta-\alpha)}(\beta^2|A^{(1)}|^2 + |A^{(2)}|^2 - (\alpha t + \beta(2-t))\det A), & \text{if } (A,t) \in \mathcal{B}, \\ +\infty, & \text{otherwise,} \end{cases}$$

where

$$\mathcal{B} = \{(A,t) \in \mathbf{M}^{2\times 2} \times [0,1] \colon h(A) \geq 0, g(A) \geq 0, r_1(A) \leq t \leq r_2(A)\}.$$

4. Concluding remarks. We would like to comment on some of the consequences of the preceding computations and on the relationship with the previous work on these sorts of problems, mainly the references cited in the Introduction.

The pioneering work [20] addressed for the first time this optimal design problem. One of the main achievements was to conclude that for F in a dense G_δ set of L^2, optimal structures are first-order laminates with the gradient of the electric potential being parallel to the layers (see [15]), and all minimizing sequences of electric fields converging strongly in L^2.

[9] pursued further the analysis started in [20], by looking more closely into first-order rank-1 laminates. Numerical experiments conducted in this paper apparently supported the fact that the zero function belongs to the dense G_δ set mentioned above.

[7] furnishes a formula for the relaxation of the original optimal design problem in terms of effective tensors and pointwise volume fractions. When there is no volume constraint present, first-order rank-one laminates are also shown to be optimal.

Some of our conclusions are:

(1) under no volume constraint, the only optimal structures are first-order laminates ([15]);
(2) when a volume restriction must be enforced, second-order laminates are optimal in general, although there could possibly be simpler optimal microstructures.

However, since our analysis stays at the level of computing the relaxed integrand, we can say nothing about the G_δ set above. An analysis of the relaxed problem would have to be performed in order to conclude something about optimal microstructures.

It is also interesting to ask about the relationship between our approach and that based on H-convergence and homogenization. This issue has been briefly indicated in [3], although further work is probably needed to fully appreciate this connection. Notice that we have two descriptions of the same optimization problem: one in terms of designs

$$a(x) = \alpha\chi(x) + \beta(1 - \chi(x))$$

and the other one in terms of pairs of gradients

$$(\nabla u(x), \nabla v(x)).$$

Relaxation for the first formulation involves G-convergence while relaxation for the second involves weak convergence. But observe, roughly speaking, that G-convergence of designs is weak convergence of pairs of associated gradients. Therefore relaxation of both optimization problems lead to the same underlying relaxed problem. The relaxed functional in terms of gradients turns out to be doable as we have shown in this work. It is interesting to notice that, even though the original functional did not depend explicitly on the second component ∇v, its quasiconvexification does. In principle, we could recast the relaxed functional found here and in [15] in terms of effective tensors and underlying electric potentials, going back to the equilibrium law for effective tensors.

More explicitly, [7] works with the weak limits

$$\begin{aligned} \nabla \varphi_0 &= \lim_{\epsilon \to 0} \nabla \varphi_\epsilon, \\ a^* \nabla \varphi_0 &= \lim_{\epsilon \to 0} a^\epsilon \nabla \varphi_\epsilon, \\ a^* \nabla \varphi_0 \cdot \nabla \varphi_0 &= \lim_{\epsilon \to 0} a^\epsilon \nabla \varphi_\epsilon \cdot \nabla \varphi_\epsilon. \end{aligned}$$

These three limits correspond, respectively, in our framework to

$$\begin{aligned} \nabla u &= A^{(1)}, \\ -T\nabla v &= -TA^{(2)}, \\ -\nabla u \cdot T\nabla v &= \det A. \end{aligned}$$

The lower bound in [7] is

$$\lim_{\epsilon \to 0} |\nabla \varphi_\epsilon|^2 \geq \frac{|(\beta - a^*)\nabla \varphi_0|^2}{\theta\beta(\beta - \alpha)} + \frac{1}{\beta} a^* \nabla \varphi_0 \cdot \nabla \varphi_0.$$

With the identifications indicated above, this lower bound is exactly the same as our lower bound in Sec. 2. Our proof of the attainability of this lower bound differs from the one in [7] although again the underlying ideas are essentially the same: to construct a second order rank-one laminate by mixing the phases α and β in proportions t and $1-t$ in such a way that there is no oscillation in the α phase. In [7] more elaborate tools from homogenization are used to show the existence of such a composite (the attainability of the trace bound). Further analysis may reveal more profound connections between these two perspectives.

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