QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXII, NUMBER 3
SEPTEMBER 2004, PAGES 477–492

A SMALL DOMAIN LIMIT FOR NONLINEAR DIELECTRIC MEDIA

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1. Introduction. The paper is concerned with a nonlinear system consisting of Maxwell's equations

$$\varepsilon\partial_t\mathbf{E} = \text{ curl } \mathbf{H} - \partial_t\tilde{\mathbf{P}} - \mathbf{j}, \quad \mu\partial_t\mathbf{H} = -\text{ curl } \mathbf{E}, \tag{1.1}$$

on $(0,\infty)\times\Omega$ coupled with the equation

$$\partial_t^2\mathbf{P} + \beta\partial_t\mathbf{P} + \nabla_P V(x,\mathbf{P}) = \gamma\mathbf{E} \tag{1.2}$$

on $(0,\infty)\times G$. Here $\Omega\subset\mathbb{R}^3$ denotes the spatial domain and $G\subset\Omega$ is an open subset.

The unknown functions are the electric and magnetic fields $\mathbf{E},\mathbf{H}$, which depend on the time $t\geq 0$ and the space-variable $x\in\Omega$, and the dielectric polarization $\mathbf{P}$ defined on $\mathbb{R}^+\times G$. In 1.1, the function $\tilde{\mathbf{P}}$ is the extension of $\mathbf{P}$ on $\mathbb{R}^+\times\Omega$ defined by zero on the set $\mathbb{R}^+\times(\Omega\setminus G)$. This system, which describes the propagation of electromagnetic waves in a dielectric medium occupying the set G, is supplemented by the initial boundary conditions

$$\vec{n}\wedge\mathbf{E} = 0 \text{ on } (0,\infty)\times\Gamma_1 \text{ and } \vec{n}\wedge\mathbf{H} = 0 \text{ on } (0,\infty)\times\Gamma_2 \tag{1.3}$$

$$\mathbf{E}(0,x) = \mathbf{E}_0(x), \mathbf{H}(0,x) = \mathbf{H}_0(x) \tag{1.4}$$

and

$$\mathbf{P}(0,x) = \mathbf{P}_0(x), \quad \partial_t\mathbf{P}(0,x) = \mathbf{P}_1(x) \text{ on } G. \tag{1.5}$$

Here $\Gamma_1\subset\partial\Omega$ is the perfectly conducting part of the boundary and $\Gamma_2 \stackrel{\text{def}}{=} \partial\Omega\setminus\Gamma_1$. Also the whole space case $\Omega=\mathbb{R}^3$ without boundary condition 1.3 is considered.

The potential energy function $V: G\times\mathbb{R}^3\to[0,\infty)$ provides a generally nonlinear restoring force in the ordinary differential equation governing $\mathbf{P}$. As in [6], the coefficients $\beta>0$ and $\gamma>0$ may depend on the space variable $x\in G$. They take into account the possibly variable mass, dipole moment, and density of the oscillating charged particles. Furthermore, $\varepsilon,\mu\in L^\infty(\mathbb{R}^3)$ denote the dielectric and magnetic susceptibilities respectively, which are assumed to be uniformly positive matrix-valued functions. An

Received December 19, 2002.
2000 *Mathematics Subject Classification.* Primary 35Q60, 35L25; Secondary 78A40.
Key words and phrases. Nonlinear differential equations in electromagnetism, singular limit, asymptotic behavior.
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external current $\mathbf{j} \in L^1((0,\infty), L^2(\Omega))$ is included also. The physical background of this system is described, for example, in [1], [11], and [13]. This system is also closely related to the Debye polarization model for microwaves involving the first order differential equation $\partial_t \mathbf{P} + \alpha(T)\mathbf{P} = \beta(T)\mathbf{E}$ instead of 1.2, [10].

The main topic of this paper is a quasistationary limit, which is of interest if the size of the domain is very small compared with the wave length.

For this purpose, consider a family of domains $\Omega_\alpha = \{\alpha y : y \in \Omega\}$ and $G_\alpha = \{\alpha y : y \in G\} \subset \Omega_\alpha$, $\mathbf{j}_\alpha(t,x) \stackrel{\text{def}}{=} \mathbf{j}(t, \alpha^{-1}x)$ and coefficients $\varepsilon_\alpha(x) \stackrel{\text{def}}{=} \varepsilon(\alpha^{-1}x)$, $\mu_\alpha(x) \stackrel{\text{def}}{=} \mu(\alpha^{-1}x)$ for $x \in \Omega_\alpha$. Now, let $(\mathbf{E}_\alpha, \mathbf{H}_\alpha)$, defined on $[0,\infty)\times\Omega_\alpha$, and $\mathbf{P}_\alpha$ defined on $[0,\infty)\times G_\alpha$, satisfy 1.1 with $\mathbf{j}$, ε, and μ replaced by $\mathbf{j}_\alpha$, ε_α, and μ_α, respectively. Then $\mathbf{e}_\alpha(t,y) \stackrel{\text{def}}{=} \mathbf{E}_\alpha(t,\alpha y)$, $\mathbf{h}_\alpha(t,y) \stackrel{\text{def}}{=} \mathbf{H}_\alpha(t,\alpha y)$, defined on $[0,\infty) \times \Omega$, and $\mathbf{p}_\alpha(t,y) \stackrel{\text{def}}{=} \mathbf{P}_\alpha(t,\alpha y)$, defined on $[0,\infty) \times G$, satisfy

$$\varepsilon\partial_t \mathbf{e}_\alpha = \alpha^{-1} \operatorname{curl} \mathbf{h}_\alpha - \mathbf{j} - \partial_t \tilde{\mathbf{p}}_\alpha, \quad \mu\partial_t \mathbf{h}_\alpha = -\alpha^{-1} \operatorname{curl} \mathbf{e}_\alpha \text{ on } [0,\infty) \times \Omega. \tag{1.6}$$

The main objective of this paper is the investigation of the limit $\alpha \to 0$, which corresponds to the case of a very small partial domain. Since 1.2 does not contain derivatives with respect to the space variables, this equation is invariant under the above scaling. It will be shown in Sec. 4 that $(\mathbf{e}_\alpha, \mathbf{h}_\alpha, \mathbf{p}_\alpha)$ converge for $\alpha \to 0$ to functions $(\mathbf{E}, \mathbf{H}, \mathbf{P})$ satisfying the equations from electrostatics, i.e.,

$$\operatorname{curl} \mathbf{E} = 0 \text{ and } \operatorname{div} [\varepsilon\mathbf{E}(t) + \tilde{\mathbf{P}}(t)] = \rho(t) \stackrel{\text{def}}{=} \operatorname{div} \left[\varepsilon\mathbf{E}_0 + \tilde{\mathbf{P}}_0 - \int_0^t \mathbf{j}(s)ds\right]$$

on $(0,\infty) \times \Omega$, where the charge density ρ is determined by the prescribed initial data $\mathbf{E}_0, \mathbf{P}_0$ and the external current $\mathbf{j}$. Furthermore, a rate for this convergence will be given for the case where Ω is an exterior domain. Finally, in Sec. 5, the asymptotic behavior is investigated. It is shown that the solution to the limit problem decays for $t \to \infty$ in the case of vanishing space charge.

In [7], a similar limit is carried out for the Landau-Lifschitz equations coupled with Maxwell's equations

$$\varepsilon\partial_t \mathbf{E} = \operatorname{curl} \mathbf{H} - \mathbf{j}, \quad \mu\partial_t \mathbf{H} = -\operatorname{curl} \mathbf{E} - \mu\partial_t \tilde{\mathbf{M}}, \tag{1.7}$$

on $\mathbb{R}^+ \times \mathbb{R}^3$ coupled with the equation

$$\partial_t \mathbf{M} = F(x, \mathbf{M}) \cdot \mathbf{H} + \mathbf{a}(x, \mathbf{M}) \tag{1.8}$$

on $\mathbb{R}^+ \times G$. Here $G \subset \mathbb{R}^3$ is an open set. In 1.7, the function $\tilde{\mathbf{M}}$ denotes again the extension of $\mathbf{M}$ on $\mathbb{R}^+ \times \mathbb{R}^3$ defined by zero on the set $\mathbb{R}^+ \times (\mathbb{R}^3 \setminus G)$. This system, which describes the propagation of electromagnetic waves in a ferromagnetic medium occupying the set G, is supplemented by the initial-boundary conditions 1.4 and $\mathbf{M}(0,x) = \mathbf{M}_0(x)$ on G. Here the initial state for the magnetic induction $\mathbf{B}_0 \stackrel{\text{def}}{=} \mu(\mathbf{H}_0 + \tilde{\mathbf{M}}_0)$ is assumed to be divergence free.

The term $\mathbf{a}(x, \mathbf{M})$ takes into account a possible anisotropy of the medium. A physically relevant example for F is $F(x, \mathbf{m})\mathbf{h} = \gamma\mathbf{m} \wedge \mathbf{h} + \alpha\mathbf{m} \wedge (\mathbf{m} \wedge \mathbf{h})$ with $\alpha \geq 0$ including a damping term $\alpha\mathbf{m} \wedge (\mathbf{m} \wedge \mathbf{h})$. Equation 1.8 provides a uniform L^∞ bound for the magnetization $\mathbf{M}$, which does not hold for the polarization field governed by 1.2.

In the quasistationary limit for 1.7, 1.2 one obtains, if the external current $\mathbf{j}$ is divergence free, the equation

$$\text{curl } \mathbf{H} = \mathbf{j} \text{ and } \text{ div } (\mu[\mathbf{H} + \tilde{\mathbf{M}}]) = 0 \text{ on } \mathbb{R}^+ \times \mathbb{R}^3$$

for the magnetostatic field.

2. Basic definitions, assumptions, and statement of main results. Let $\Omega \subset \mathbb{R}^3$ be an arbitrary spatial domain, $\Gamma_1 \subset \partial\Omega$, $\Gamma_2 \stackrel{\text{def}}{=} \partial\Omega \setminus \Gamma_1$, and $G \subset \Omega$ a non-empty open subset. Throughout this paper the following assumptions are imposed on the potential energy function $V : G \times \mathbb{R}^3 \to [0, \infty)$. First $V(\cdot, y) \in L^\infty(G)$ for all $y \in \mathbb{R}^3$, $V(x, \cdot) \in C^2(\mathbb{R}^3, \mathbb{R})$,

$$V(x, 0) = 0 \text{ and } (\nabla_P V)(x, 0) = 0 \text{ for all } x \in G. \tag{2.1}$$

It is assumed that $(\nabla_P V)$ is Lipschitz-continuous with respect to y, i.e., there exists some $L_0 \in (0, \infty)$, such that

$$|(\nabla_P V)(x, y) - (\nabla_P V)(x, z)| \leq L_0 |y - z| \text{ for all } x \in G, y, z \in \mathbb{R}^3. \tag{2.2}$$

The dielectric and magnetic susceptibilities $\varepsilon, \mu \in L^\infty(\mathbb{R}^3)$ are assumed to be uniformly positive functions; that means

$$\varepsilon(x), \mu(x) \geq a_0 \text{ on } \mathbb{R}^3 \text{ with some } a_0 > 0. \tag{2.3}$$

Let $\beta \in L^\infty(G)$ be a uniformly positive and $\gamma \in L^\infty(G)$ be a positive, but not necessarily uniformly positive, function on G. Let

$$\mathcal{G} \stackrel{\text{def}}{=} \{\mathbf{f} : G \to \mathbb{R}^3 \text{ with } \gamma^{-1/2}\mathbf{f} \in L^2(G)\}$$

be the weighted $L^2(G)$-space for the polarization field endowed with the scalar product

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{G}} \stackrel{\text{def}}{=} \int_G \gamma^{-1}(x) \mathbf{f}(x) \mathbf{g}(x) dx.$$

Note that $\gamma^{-1} V(x, \mathbf{f}) \in L^1(G)$ and $(\nabla_y V)(x, \mathbf{f}) \in \mathcal{G}$ for all $\mathbf{f} \in \mathcal{G}$ by 2.1 and 2.2.

The first three and the last three components of a vector $\mathbf{u} \in \mathbb{C}^6$ are denoted by $\underline{\mathbf{u}}_1 \in \mathbb{C}^3$ and $\underline{\mathbf{u}}_2 \in \mathbb{C}^3$, respectively. Next, some function spaces are introduced. For an arbitrary open set $K \subset \mathbb{R}^3$, the space of all infinitely differentiable functions with compact support contained in K is denoted by $C_0^\infty(K)$. Let $H_{\text{curl}}(K)$ be defined as the space of all $\mathbf{E} \in L^2(K, \mathbb{C}^3)$ with curl $\mathbf{E} \in L^2(K)$. Let W_H denote the closure of $C_0^\infty(\mathbb{R}^3 \setminus \overline{\Gamma_2}, \mathbb{C}^3)$ in $H_{\text{curl}}(\Omega)$, where $H_{\text{curl}}(\Omega)$ is the space of all $\mathbf{E} \in L^2(\Omega, \mathbb{C}^3)$ with curl $\mathbf{E} \in L^2(\Omega)$ in the sense of distributions.
Next, W_E is defined as the set of all $\mathbf{E} \in H_{\text{curl}}(\Omega)$, such that

$$\int_\Omega (\mathbf{E} \text{ curl } \mathbf{F} - \mathbf{F} \text{ curl } \mathbf{E}) dx = 0 \text{ for all } \mathbf{F} \in W_H,$$

which includes a weak formulation of the boundary condition $\vec{n} \wedge \mathbf{E} = 0$ on Γ_1; see [5], [6].

Now, let $D(B) \stackrel{\text{def}}{=} W_E \times W_H$ and

$$B(\mathbf{E}, \mathbf{H}) \stackrel{\text{def}}{=} (\varepsilon^{-1} \text{curl } \mathbf{H}, -\mu^{-1} \text{ curl } \mathbf{E}) \text{ for } (\mathbf{E}, \mathbf{H}) \in D(B).$$

Then B is a densely defined skew adjoint operator in the Hilbert-space $X \stackrel{\text{def}}{=} L^2(\Omega, \mathbb{C}^6)$ endowed with the weighted scalar-product

$$\langle (\mathbf{E}, \mathbf{H}), (\mathbf{F}, \mathbf{G}) \rangle_X \stackrel{\text{def}}{=} \int_\Omega \left(\varepsilon \mathbf{E} \cdot \overline{\mathbf{F}} + \mu \mathbf{H} \cdot \overline{\mathbf{G}} \right) dx.$$

In what follows, $W_{E,0}$ and $W_{H,0}$ denote the spaces of all $\mathbf{f} \in W_E$ and $\mathbf{g} \in W_H$ with $\operatorname{curl} \mathbf{f} = \operatorname{curl} \mathbf{g} = 0$, respectively. Let Y_E denote the spaces of all $\mathbf{A} \in L^2(\Omega, \mathbb{C}^3)$ with

$$\int_\Omega \varepsilon \mathbf{A} \cdot \mathbf{f} dx = 0 \text{ for all } \mathbf{f} \in W_{E,0}, \tag{2.4}$$

in particular $\operatorname{div}(\varepsilon \mathbf{A}) = 0$ on Ω, since $\nabla \varphi \in W_{E,0}$ for all $\varphi \in C_0^\infty(\Omega)$.

Let X_0 be the kernel of B, i.e.,

$$X_0 \stackrel{\text{def}}{=} \{(\mathbf{f}, \mathbf{g}) \in D(B) : B(\mathbf{f}, \mathbf{g}) = 0\} = W_{E,0} \times W_{H,0}. \tag{2.5}$$

Let Q denote the orthogonal projector on $X_0^\perp = (\ker B)^\perp = \overline{\operatorname{ran} B}$. Note that $1 - Q$ is of the form

$$(1 - Q)(\mathbf{f}, \mathbf{g}) = ((1 - Q_E)\mathbf{f}, (1 - Q_H)\mathbf{g}) \text{ for } (\mathbf{f}, \mathbf{g}) \in X, \tag{2.6}$$

where $1 - Q_E$ and $1 - Q_H$ are the orthogonal projectors on $W_{E,0}$ and $W_{H,0}$ with respect to the weighted scalar products

$$\langle \mathbf{E}, \mathbf{F} \rangle_\varepsilon \stackrel{\text{def}}{=} \int_\Omega \varepsilon \mathbf{E} \cdot \overline{\mathbf{F}} dx \text{ and } \langle \mathbf{G}, \mathbf{H} \rangle_\mu \stackrel{\text{def}}{=} \int_\Omega \mu \mathbf{G} \cdot \overline{\mathbf{H}} dx,$$

respectively. In particular

$$\operatorname{ran}\,(1 - Q_E) = W_{E,0} \text{ and } \operatorname{ran}\, Q_E = Y_E. \tag{2.7}$$

This means that Q_E is the orthogonal projector on Y_E, which is the orthogonal complement of $W_{E,0}$ with respect to $\langle \cdot, \cdot \rangle_\varepsilon$.

Let $W^{k,p}_{\text{loc}}([0, \infty), Y)$ denote for a Banach-space Y the space of all functions defined on $[0, \infty)$ with values in Y, whose derivatives up to order k belong to $L^p((0, T), Y)$ for all $T > 0$.

The assumptions on the initial data $\mathbf{E}_0, \mathbf{H}_0, \mathbf{P}_0, \mathbf{P}_1$, and $\mathbf{j}$ are

$$\mathbf{j} \in L^1((0, \infty), L^2(\Omega)), \quad (\mathbf{E}_0, \mathbf{H}_0) \in X, \quad \mathbf{P}_0 \in \mathcal{G}, \text{ and } \mathbf{P}_1 \in \mathcal{G}. \tag{2.8}$$

The main result of this paper is:

THEOREM 2.1. Assume 2.1–2.3 and 2.8. Let $(\mathbf{E}_\alpha, \mathbf{H}_\alpha, \mathbf{P}_\alpha)$ be the solution to the equations

$$\varepsilon \partial_t \mathbf{E}_\alpha = \alpha^{-1} \operatorname{curl} \mathbf{H}_\alpha - \mathbf{j} - \partial_t \tilde{\mathbf{P}}_\alpha, \quad \mu \partial_t \mathbf{H}_\alpha = -\alpha^{-1} \operatorname{curl} \mathbf{E}_\alpha, \tag{2.9}$$

on $\mathbb{R}^+ \times \Omega$ coupled with the equation

$$\partial_t^2 \mathbf{P}_\alpha + \beta \partial_t \mathbf{P}_\alpha + (\nabla_P V)(x, \mathbf{P}_\alpha) = \gamma \mathbf{E}_\alpha \tag{2.10}$$

on $\mathbb{R}^+ \times G$ supplemented by the initial-boundary conditions 1.3, 1.4, and 1.5. Then

$$\mathbf{E}_\alpha \stackrel{\alpha \to 0}{\longrightarrow} \mathbf{E} \text{ in } L^\infty((0, T), L^2(\Omega)) \text{ weak } * \tag{2.11}$$

and

$$\|\mathbf{P}_\alpha - \mathbf{P}\|_{W^{1,\infty}((0,T), L^2(K \cap G))} \stackrel{\alpha \to 0}{\longrightarrow} 0 \text{ for all compact } K \subset \Omega. \tag{2.12}$$

Here $\mathbf{P} \in C^2([0,\infty),\mathcal{G})$ and $\mathbf{E} \in C([0,\infty),W_{E,0})$ obey

$$\mathbf{E}(t) = (1-Q_E)\left[\mathbf{E}_0 + \varepsilon^{-1}\tilde{\mathbf{P}}_0 - \int_0^t \varepsilon^{-1}\mathbf{j}(s)ds - \varepsilon^{-1}\tilde{\mathbf{P}}(t)\right], \tag{2.13}$$

i.e., $\mathbf{E}(t) \in W_{E,0}$, and

$$\int_\Omega [\varepsilon\mathbf{E}(t) + \tilde{\mathbf{P}}(t)]\mathbf{f}dx = \int_\Omega [\varepsilon\mathbf{E}_0 + \tilde{\mathbf{P}}_0 - \int_0^t \mathbf{j}(s)ds]\mathbf{f}dx \text{ for all } \mathbf{f} \in W_{E,0} \text{ and } t \in (0,\infty).$$

(Here the function $\tilde{\mathbf{P}}(t)$ is the extension of $\mathbf{P}(t)$ on Ω defined by zero on the set $\Omega \setminus G$.) Furthermore

$$\partial_t^2\mathbf{P} + \beta\partial_t\mathbf{P} + (\nabla_P V)(x,\mathbf{P}) = \gamma\mathbf{E} \text{ on } (0,\infty)\times G \tag{2.14}$$

with initial condition

$$\mathbf{P}(0) = \mathbf{P}_0 \text{ and } \partial_t\mathbf{P}(0) = \mathbf{P}_1. \tag{2.15}$$

The solution to problem 2.13–2.15 is unique.

The initial-boundary value problem 2.9, 2.10, 1.3, 1.4, and 1.5 is satisfied in the sense described in [5], [6]; see also the beginning of Sec. 3.

REMARK 2.2. By 2.13, one has in particular

$$\text{curl } \mathbf{E} = 0 \text{ and } \text{ div } [\varepsilon\mathbf{E}(t) + \tilde{\mathbf{P}}(t)] = \rho(t) \stackrel{\text{def}}{=} \text{ div } [\varepsilon\mathbf{E}_0 + \tilde{\mathbf{P}}_0 - \int_0^t \mathbf{j}(s)ds] \tag{2.16}$$

on $(0,\infty)\times\Omega$. If, in addition, Ω is simply connected, then $\mathbf{E}(t) = -\nabla\varphi(t)$ with some potential function $\varphi(t) \in H^1(\Omega)$, which solves Poisson's equation

$$\text{div } [\varepsilon\nabla\varphi(t) - \tilde{\mathbf{P}}(t)] = -\rho(t).$$

2.11 implies convergence in time mean, i.e.,

$$\int_0^t \mathbf{E}_\alpha(s)ds \stackrel{\alpha\to 0}{\longrightarrow} \int_0^t \mathbf{E}(s)ds \text{ in } L^2(\Omega) \text{ weakly } \text{ for all } t \in [0,\infty),$$

but in general not pointwise with respect to time. In particular $\mathbf{E}_\alpha(0) = \mathbf{E}_0 \neq \mathbf{E}(0)$ in general, where $\mathbf{E}$ is as in 2.13 2.15. This follows from the possible rapid oscillations with respect to time coming from the large factor α^{-1} in 2.9.

3. Some auxiliary results. In what follows, the assumptions 2.1–2.3 and 2.8 will be satisfied. Next some auxiliary results concerning problem (1.1)–(1.5) will be given. First, it follows from the contraction mapping principle as in [6] that Problem (1.1)–(1.5) has a unique weak solution $(\mathbf{E},\mathbf{H},\mathbf{P})$ with the properties $(\mathbf{E},\mathbf{H}) \in C([0,\infty),X)$ and $\mathbf{P} \in C^2([0,\infty),\mathcal{G})$. In particular, 1.1 is satisfied in the sense that

$$(\mathbf{E}(t),\mathbf{H}(t)) = \exp(tB)(\mathbf{E}_0,\mathbf{H}_0) \tag{3.1}$$
$$-\int_0^t \exp((t-s)B)\left[\mathcal{R}\partial_t\mathbf{P}(s) + (\varepsilon^{-1}\mathbf{j}(s),0)\right]ds,$$

in particular

$$\frac{d}{dt}\langle(\mathbf{E}(t),\mathbf{H}(t)),\mathbf{a}\rangle_X = -\langle(\mathbf{E}(t),\mathbf{H}(t)),B\mathbf{a}\rangle_X - \langle\mathcal{R}\partial_t\mathbf{P}(s) + (\varepsilon^{-1}\mathbf{j}(s),0),\mathbf{a}\rangle_X \tag{3.2}$$

for all $\mathbf{a} \in D(B)$; see [2], [12].

In what follows, let $H^{(q)}_{\mathrm{curl}}(\Omega)$ denote for $q \in [1,2]$ the space of all $\mathbf{h} \in L^2(\Omega)$ with curl $\mathbf{h} \in L^q(\Omega) + L^2(\Omega)$ in the sense of distributions, i.e., curl $\mathbf{h}$ admits a decomposition curl $\mathbf{h} = \mathbf{g}_1 + \mathbf{g}_2$ with

$$\mathbf{g}_1 \in L^q(\Omega) \text{ and } \mathbf{g}_2 \in L^2(\Omega).$$

Next, let $W^{(q)}_{\mathrm{div}}$ be the space of all $\mathbf{f} \in L^2(\Omega)$, such that div $(\varepsilon \mathbf{f}) \in L^q(\Omega) + L^2(\Omega)$.

PROPOSITION 3.1. i) Let $q \in (6/5, 2]$. Then $H^{(q)}_{\mathrm{curl}}(\Omega) \cap W^{(q)}_{\mathrm{div}}(\Omega)$ is compactly embedded in $L^2(K)$ for all compact $K \subset \Omega$.

In particular, $H^{(q)}_{\mathrm{curl}}(\Omega) \cap Y_E$ and $W_E \cap Y_E$ are compactly embedded in $L^2(K)$ for all compact $K \subset \Omega$.

ii) For all $\chi \in W^{1,\infty}(\mathbb{R}^3)$, the commutator $[\chi, Q_E] \stackrel{\mathrm{def}}{=} \chi \cdot Q_E - Q_E \cdot \chi$ is compact as an operator from $L^2(\Omega)$ to $L^2(K)$ for all compact $K \subset \Omega$.

Proof. Note that $W_E \subset H^{(q)}_{\mathrm{curl}}(\Omega)$ and, by 2.4, $Y_E \subset W^{(q)}_{\mathrm{div}}(\Omega)$. Let $K \subset \Omega$ be compact and choose $\chi_0 \in C_0^\infty(\Omega)$ with $\chi_0(x) = 1$ on K. Suppose that $\{\mathbf{f}_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^{(q)}_{\mathrm{curl}}(\Omega) \cap W^{(q)}_{\mathrm{div}}(\Omega)$, which means that $\{\mathbf{f}_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^3)$, whereas $\{\text{curl } \mathbf{f}_n\}_{n\in\mathbb{N}}$ and $\{\text{div } [\varepsilon \mathbf{f}_n]\}_{n\in\mathbb{N}}$ are bounded in $L^q(\Omega) + L^2(\Omega)$. Let $\mathbf{u}_n \stackrel{\mathrm{def}}{=} \chi_0 \mathbf{f}_n \in L^2(\mathbb{R}^3)$ (defined as zero outside supp χ_0). Then, since $q \le 2$ and supp χ_0 is bounded, $\{\text{curl } \mathbf{u}_n\}_{n\in\mathbb{N}}$ and $\{\text{div } [\varepsilon \mathbf{u}_n]\}_{n\in\mathbb{N}}$ are bounded in $L^q(\mathbb{R}^3)$ and supp $\mathbf{u}_n \subset$ supp χ_0. Now, it follows from Lemma 3.1 in [7] that $(\mathbf{u}_n)_{n\in\mathbb{N}}$ is pre-compact in $L^2(\mathbb{R}^3)$, whence $(\mathbf{f}_n)_{n\in\mathbb{N}}$ is pre-compact in $L^2(K)$, since $\chi_0(x) = 1$ on K. This completes the proof of the first assertion.

To prove ii), suppose $\mathbf{e} \in L^2(\Omega)$. Then $\mathbf{f} \stackrel{\mathrm{def}}{=} \chi \cdot Q_E \mathbf{e} - Q_E(\chi \mathbf{e})$ obeys by 2.4 and 2.7,

$$\text{div } (\varepsilon \mathbf{f}) = \text{ div } (\chi \varepsilon Q_E \mathbf{e}) = \varepsilon\, (Q_E \mathbf{e})\, \nabla \chi \in L^2(\Omega). \tag{3.3}$$

Since $\mathbf{f} = \chi \cdot (Q_E - 1)\mathbf{e} - (Q_E - 1)(\chi \mathbf{e})$, and $1 - Q_E$ is the orthogonal projector on $W_{E,0} \subset H_{\mathrm{curl}}(\Omega)$ by 2.7, one obtains $\mathbf{f} \in H_{\mathrm{curl}}(\Omega)$ and

$$\text{curl } \mathbf{f} = ((1 - Q_E)\mathbf{e}) \wedge \nabla \chi \in L^2(\Omega).$$

By 3.3 and part (i), this completes the proof of (ii). □

Assertion (i) is well known for $q = 2$; see [14] and [15].

COROLLARY 3.2. For all $\chi \in C_0^\infty(\Omega)$, the commutator $[\chi, Q_E]$ is a compact operator in $L^2(\Omega)$.

Proof. Choose $\chi_0 \in C_0^\infty(\Omega)$ with $\chi_0(x) = 1$ on supp χ. Note that, since Q and χ are symmetric,

$$[\chi, Q_E] = [\chi_0 \chi, Q_E] = \chi_0 [\chi, Q_E] + [\chi_0, Q_E]\chi = \chi_0 [\chi, Q_E] - (\chi [\chi_0, Q_E])^*,$$

where $(\chi[\chi_0, Q_E])^*$ denotes the adjoint operator with respect to the weighted scalar product $\langle \cdot, \cdot \rangle_\varepsilon$. By Proposition 3.1 (ii), the operators $\chi[\chi_0, Q_E]$ and $\chi_0[\chi, Q_E]$ are compact in $L^2(\Omega)$. Hence $(\chi[\chi_0, Q_E])^*$ is also compact. □

LEMMA 3.3. Suppose that $\{\mathbf{E}_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty((0,T), L^2(\Omega))$, and $\{\mathbf{B}_n\}_{n\in\mathbb{N}}$ is bounded in $L^2((0,T), L^q(\Omega) + L^2(\Omega))$ with some $q \in (6/5, 2]$, such that

$$\mathbf{E}_n \stackrel{n\to\infty}{\longrightarrow} \mathbf{E} \text{ in } L^\infty((0,T), L^2(\Omega)) \text{ weak } * \tag{3.4}$$

with some $\mathbf{E} \in L^\infty((0,T), L^2(\Omega))$ and

$$\text{curl } \mathbf{E}_n = -\partial_t \mathbf{B}_n \tag{3.5}$$

in the sense that

$$\int_\Omega \mathbf{E}_n(t) \cdot \text{ curl } \mathbf{f} dx = -\frac{d}{dt} \int_\Omega \mathbf{B}_n(t) \cdot \mathbf{f} dx \text{ for all } \mathbf{f} \in C_0^\infty(\Omega).$$

Then $\mathbf{A}_n(t) \stackrel{\text{def}}{=} \int_0^t Q_E \mathbf{E}_n(s) ds$ and $\mathbf{A}(t) \stackrel{\text{def}}{=} \int_0^t Q_E \mathbf{E}(s) ds$ satisfy

$$\|\mathbf{A}_n - \mathbf{A}\|_{L^\infty((0,T),L^2(K))} \stackrel{n\to\infty}{\longrightarrow} 0 \text{ and all compact } K \subset \Omega.$$

Proof. Let $K \subset \Omega$ be compact. First, it follows from 3.4 that

$$\mathbf{A}_n \stackrel{n\to\infty}{\longrightarrow} \mathbf{A} \text{ in } L^\infty((0,T), L^2(\Omega)) \text{ weak } *. \tag{3.6}$$

By 2.7, one has

$$\int_\Omega (1 - Q_E)\mathbf{E}_n(s) \text{ curl } \mathbf{f} dx = 0$$

and therefore

$$\int_\Omega \mathbf{A}_n(t) \text{ curl } \mathbf{f} dx = \int_0^t \int_\Omega Q_E \mathbf{E}_n(s) \text{ curl } \mathbf{f} dx ds$$

$$= \int_0^t \int_\Omega \mathbf{E}_n(s) \text{ curl } \mathbf{f} dx ds = -\int_0^t \frac{d}{dt} \int_\Omega \mathbf{B}_n(s) \cdot \mathbf{f} dx ds = \int_\Omega [\mathbf{B}_n(0) - \mathbf{B}_n(t)] \cdot \mathbf{f} dx$$

for all $\mathbf{f} \in C_0^\infty(\Omega)$. Hence

$$\text{curl } \mathbf{A}_n(t) = \mathbf{B}_n(0) - \mathbf{B}_n(t) \in L^q(\mathbb{R}^3) + L^2(\mathbb{R}^3),$$

which implies by Proposition 3.1 (i) and the boundedness of $\{\mathbf{B}_n\}_{n\in\mathbb{N}}$ in $L^\infty((0,T), L^q(\Omega) + L^2(\Omega))$ that

$$\{\mathbf{A}_n(t)\}_{n\in\mathbb{N}} \text{ is precompact in } L^2(K) \text{ for fixed } t \in (0,T). \tag{3.7}$$

Since $\partial_t \mathbf{A}_n(t) = Q_E \mathbf{E}_n(t)$, it follows from the boundedness of $\{Q_E \mathbf{E}_n\}_{n\in\mathbb{N}}$ that $\{\mathbf{A}_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,\infty}((0,T), L^2(K))$. Hence it follows from 3.7 and Arzela's theorem that this sequence is precompact in $C([0,T], L^2(K))$. Thus, 3.6 yields

$$\|\mathbf{A}_n(t) - \mathbf{A}(t)\|_{L^2(K)} \stackrel{n\to\infty}{\longrightarrow} 0 \text{ uniformly on } (0,T).$$

□

LEMMA 3.4. Suppose that $\{\mathbf{Q}_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $W^{1,\infty}((0,T), L^2(\Omega))$ with

$$\mathbf{Q}_n \stackrel{n\to\infty}{\longrightarrow} 0 \text{ in } L^\infty((0,T), L^2(\Omega)) \text{ weak } *. \tag{3.8}$$

Furthermore, let $\rho \in C_0^\infty(\Omega)$. Then

$$\|[\rho, Q_E]\mathbf{Q}_n\|_{L^\infty((0,T),L^2(\Omega))} \stackrel{n\to\infty}{\longrightarrow} 0.$$

Proof. It follows from the boundedness of the sequence $\{\mathbf{Q}_n\}_{n\in\mathbb{N}}$ in $L^\infty((0,T), L^2(\mathbb{R}^3))$ and Proposition 3.1 (ii) that

$$\{[\rho, Q_E]\mathbf{Q}_n(t)\}_{n\in\mathbb{N}} \text{ is precompact in } L^2(\Omega) \text{ for fixed } t \in (0,T). \tag{3.9}$$

By the boundedness of $\{\mathbf{Q}_n\}_{n\in\mathbb{N}}$ in $W^{1,2}((0,T), L^2(\Omega))$, the sequence

$$\{[\rho, Q_E]\mathbf{Q}_n(\cdot)\}_{n\in\mathbb{N}}$$

is bounded in $W^{1,2}((0,T),L^2(\Omega))$. Hence it follows from 3.9 and Arzela's theorem that this sequence is precompact in $C([0,T],L^2(\Omega))$. Thus, 3.8 yields

$$\|[\rho,Q_E]\mathbf{Q}_n(t)\|_{L^2(\Omega)} \overset{n\to\infty}{\longrightarrow} 0 \text{ uniformly on } (0,T), \tag{3.10}$$

which completes the proof. □

LEMMA 3.5. Suppose that $\{\mathbf{E}_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty((0,T),L^2(\Omega))$, and let $\{\mathbf{P}_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $W^{2,\infty}((0,T),L^2(G))$, such that

$$\mathbf{E}_n \overset{n\to\infty}{\longrightarrow} \mathbf{E} \text{ in } L^\infty((0,T),L^2(\Omega)) \text{ weak } *, \tag{3.11}$$

$$\mathbf{P}_n \overset{n\to\infty}{\longrightarrow} \mathbf{P} \text{ in } W^{1,\infty}((0,T),L^2(G)) \text{ weak } *, \tag{3.12}$$

$$\mathbf{P}_n(0) \overset{n\to\infty}{\longrightarrow} \mathbf{P}_0 \text{ and } \partial_t\mathbf{P}_n(0) \overset{n\to\infty}{\longrightarrow} \mathbf{P}_1 \text{ in } L^2(G) \text{ strongly}. \tag{3.13}$$

Furthermore, assume that

$$(1-Q_E)[\mathbf{E}_n(t)+\varepsilon^{-1}\tilde{\mathbf{P}}_n(t)] \overset{n\to\infty}{\longrightarrow} \mathbf{F} \text{ in } L^1((0,T),L^2(K)) \text{ strongly} \tag{3.14}$$

for all compact $K\subset\Omega$ with some $\mathbf{F}\in L^1((0,T),L^2(\Omega))$.

$$\partial_t^2\mathbf{P}_n+\beta\partial_t\mathbf{P}_n+(\nabla_P V)(x,\mathbf{P}_n)=\gamma\mathbf{E}_n \text{ on } \mathbb{R}^+\times G \tag{3.15}$$

and

$$\operatorname{curl}\mathbf{E}_n=-\partial_t\mathbf{B}_n, \tag{3.16}$$

where $\{\mathbf{B}_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^2((0,T),L^q(\Omega)+L^2(\Omega))$ with some $q\in(6/5,2]$. Then

$$\|\mathbf{P}_n-\mathbf{P}\|_{W^{1,\infty}((0,T),L^2(K\cap G))} \overset{n\to\infty}{\longrightarrow} 0 \text{ for all compact } K\subset\Omega, \tag{3.17}$$

$$\partial_t^2\mathbf{P}+\beta\partial_t\mathbf{P}+(\nabla_P V)(x,\mathbf{P})=\gamma\mathbf{E} \text{ on } \mathbb{R}^+\times G, \tag{3.18}$$

$$\mathbf{P}(0)=\mathbf{P}_0 \text{ and } \partial_t\mathbf{P}(0)=\mathbf{P}_1. \tag{3.19}$$

Proof. Let

$$\mathbf{F}_n(t)\overset{\text{def}}{=}(1-Q_E)[\mathbf{E}_n(t)+\varepsilon^{-1}\tilde{\mathbf{P}}_n(t)], \text{ and } \mathbf{A}_n(t)\overset{\text{def}}{=}\int_0^t Q_E\mathbf{E}_n(s)ds.$$

Then 3.15 yields

$$\begin{aligned}
&\partial_t\mathbf{P}_n(t)-\partial_t\mathbf{P}_m(t)+\beta[\mathbf{P}_n(t)-\mathbf{P}_m(t)] \\
&\quad+\int_0^t[(\nabla_P V)(x,\mathbf{P}_n(s))-(\nabla_P V)(x,\mathbf{P}_m(s))]ds \\
&=\partial_t\mathbf{P}_n(0)-\partial_t\mathbf{P}_m(0)+\beta[\mathbf{P}_n(0)-\mathbf{P}_m(0)]+\gamma\int_0^t(\mathbf{E}_n(s)-\mathbf{E}_m(s))ds \\
&=\partial_t\mathbf{P}_n(0)-\partial_t\mathbf{P}_m(0)+\beta[\mathbf{P}_n(0)-\mathbf{P}_m(0)]+\gamma\mathbf{A}_n(t)-\gamma\mathbf{A}_m(t) \\
&+\gamma\int_0^t\Big\{\mathbf{F}_n(s)-\mathbf{F}_m(s)-(1-Q_E)(\varepsilon^{-1}[\tilde{\mathbf{P}}_n(s)-\tilde{\mathbf{P}}_m(s)])\Big\}ds \text{ on } \mathbb{R}^+\times G.
\end{aligned} \tag{3.20}$$

Now, suppose that $K\subset\Omega$ is compact, choose $\chi\in C_0^\infty(\Omega)$ with $\chi=1$ on K, and let

$$\mathbf{p}_{n,m}(t)\overset{\text{def}}{=}\chi[\mathbf{P}_n(t)-\mathbf{P}_m(t)].$$

Then

$$\partial_t\mathbf{p}_{n,m}(t)+\beta\mathbf{p}_{n,m}(t)+\int_0^t\chi[(\nabla_P V)(x,\mathbf{P}_n(s))-(\nabla_P V)(x,\mathbf{P}_m(s))]ds \tag{3.21}$$

$$+\gamma\int_0^t (1-Q_E)(\varepsilon^{-1}\tilde{\mathbf{p}}_{n,m}(s))ds$$
$$= \chi[\partial_t\mathbf{P}_n(0)-\partial_t\mathbf{P}_m(0)]+\chi\beta[\mathbf{P}_n(0)-\mathbf{P}_m(0)]+\chi\gamma\mathbf{A}_n(t)-\chi\gamma\mathbf{A}_m(t)$$
$$+\gamma\int_0^t \left\{\chi\mathbf{F}_n(s)-\chi\mathbf{F}_m(s)+[\chi,Q_E](\varepsilon^{-1}[\tilde{\mathbf{P}}_n(s)-\tilde{\mathbf{P}}_m(s)])\right\}ds.$$

By assumption 2.2, one has

$$\|\chi[(\nabla_P V)(x,\mathbf{P}_n(s))-(\nabla_P V)(x,\mathbf{P}_m(s))]\|_{L^2(G)} \le K\|\mathbf{p}_{n,m}(s)\|_{L^2(G)}. \tag{3.22}$$

Furthermore,

$$\|\gamma(1-Q_E)(\varepsilon^{-1}\tilde{\mathbf{p}}_{n,m}(s))\|_{L^2(G)} \le C_1\|\varepsilon^{1/2}(1-Q_E)(\varepsilon^{-1}\tilde{\mathbf{p}}_{n,m}(s))\|_{L^2(\Omega)} \tag{3.23}$$
$$\le C_1\|\varepsilon^{-1/2}\tilde{\mathbf{p}}_{n,m}(s)\|_{L^2(\Omega)} \le C_2\|\mathbf{p}_{n,m}(s)\|_{L^2(G)}.$$

By a standard Gronwall-type estimate, it follows from 3.21–3.23 that

$$\|\mathbf{p}_{n,m}\|_{W^{1,\infty}((0,T),L^2(G))} \le C_3(\|\partial_t\mathbf{P}_n(0)-\partial_t\mathbf{P}_m(0)\|_{L^2(G)} \tag{3.24}$$
$$+\|\mathbf{P}_n(0)-\mathbf{P}_m(0)\|_{L^2(G)}+\|\chi\mathbf{A}_n-\chi\mathbf{A}_m\|_{L^\infty((0,T),L^2(G))}$$
$$+\|\chi\mathbf{F}_n-\chi\mathbf{F}_m\|_{L^1((0,T),L^2(G))}+\|[\chi,Q_E](\varepsilon^{-1}[\tilde{\mathbf{P}}_n(\cdot)-\tilde{\mathbf{P}}_m(\cdot)])\|_{L^1((0,T),L^2(G))}).$$

Next, 3.11, 3.12 and Lemmas 3.3 and 3.4 yield

$$\|[\chi,Q_E](\varepsilon^{-1}[\tilde{\mathbf{P}}_n(\cdot)-\tilde{\mathbf{P}}_m(\cdot)])\|_{L^1((0,T),L^2(G))} \tag{3.25}$$
$$+\|\chi\mathbf{A}_n-\chi\mathbf{A}_m\|_{L^\infty((0,T),L^2(G))} \overset{n,m\to\infty}{\longrightarrow} 0.$$

By the definition of $\mathbf{p}_{n,m}$, it follows from 3.13, 3.14, 3.24, and 3.25 that the convergence in 3.12 is locally strong, i.e.,

$$\|\mathbf{P}_n-\mathbf{P}\|_{W^{1,\infty}((0,T),L^2(K\cap G))} \le \|\chi[\mathbf{P}_n-\mathbf{P}]\|_{W^{1,\infty}((0,T),L^2(G))} \overset{n\to\infty}{\longrightarrow} 0,$$

whence 3.17 and, thus,

$$\|(\nabla_P V)(x,\mathbf{P}_n(\cdot))-(\nabla_P V)(x,\mathbf{P}(\cdot))\|_{L^\infty((0,T),L^2(K\cap G))} \overset{n\to\infty}{\longrightarrow} 0. \tag{3.26}$$

Finally, it follows from 3.11 and 3.13 again, 3.15, and 3.26 that $\mathbf{E}$ and $\mathbf{P}$ obey 3.18 and 3.19. □

4. The quasistationary limit. In this section, Theorem 2.1 is proved. Setting $\mathbf{U}_\alpha \overset{\text{def}}{=} (\mathbf{E}_\alpha,\mathbf{H}_\alpha)$, 2.9–2.10 reads as

$$\partial_t\mathbf{U}_\alpha = \alpha^{-1}B\mathbf{U}_\alpha - \mathcal{R}\partial_t\mathbf{P}_\alpha(t)-(\varepsilon^{-1}\mathbf{j}(t),0) \tag{4.1}$$
$$\text{and } \mathbf{U}_\alpha(0)=\mathbf{U}_0 \overset{\text{def}}{=} (\mathbf{E}_0,\mathbf{H}_0).$$

Recall that $\mathbf{U}_\alpha \in C([0,\infty),X)$ obeys the variation of constant formula

$$\mathbf{U}_\alpha(t) = \exp(\alpha^{-1}tB)\mathbf{U}_0 \tag{4.2}$$
$$-\int_0^t \exp(\alpha^{-1}(t-s)B)\left[\mathcal{R}\partial_t\mathbf{P}_\alpha(s)+(\varepsilon^{-1}\mathbf{j}(s),0)\right]ds,$$

where $\exp(tB), t\in\mathbb{R}$ is the unitary group generated by B.
In particular, one obtains the energy balance

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{U}_\alpha(t)\|_X^2 = -\langle\mathcal{R}\partial_t\mathbf{P}_\alpha(t)+(\varepsilon^{-1}\mathbf{j}(t),0),\mathbf{U}_\alpha(t)\rangle_X \tag{4.3}$$

$$\leq \|\mathbf{U}_\alpha(t)\|_X^2 + \|\mathcal{R}\partial_t \mathbf{P}_\alpha(t)\|_X^2 + \|(\varepsilon^{-1}\mathbf{j}(t),0)\|_X^2$$
$$\leq \|\mathbf{U}_\alpha(t)\|_X^2 + C_1\|\partial_t \mathbf{P}_\alpha(t)\|_{L^2(G)}^2 + C_1\|\mathbf{j}(t)\|_{L^2(\Omega)}^2,$$

since $\exp(tB)$ is unitary in X. The standard energy estimate for 1.2 gives

$$\|\partial_t \mathbf{P}_\alpha(t)\|_{L^2(G)}^2 \leq C_2 \int_0^t \|\mathbf{E}_\alpha(s)\|_{L^2(G)}^2 ds + C_2 \leq C_3 \int_0^t \|\mathbf{U}_\alpha(s)\|_X^2 ds + C_3. \tag{4.4}$$

By 4.3, 4.4, and Gronwall's lemma, one obtains

$$\|\mathbf{U}_\alpha\|_{L^\infty((0,T),X)} + \|\mathbf{P}_\alpha\|_{W^{1,\infty}((0,T),L^2(G))} \leq C_4 \tag{4.5}$$

with some constant C_4 independent of n. Thus, there exist $\mathbf{P} \in W^{1,\infty}((0,T),L^2(G))$, $\mathbf{E} \in L^\infty((0,T),L^2(\Omega))$, and a subsequence, $(\mathbf{E}_{(\alpha_m)}, \mathbf{H}_{(\alpha_m)}, \mathbf{P}_{(\alpha_m)})$ with $\alpha_m \overset{m\to\infty}{\longrightarrow} 0$, such that

$$\mathbf{P}_{(\alpha_m)} \overset{m\to\infty}{\longrightarrow} \mathbf{P} \text{ in } L^\infty((0,T),L^2(G)) \text{ weak } * \tag{4.6}$$
$$\text{and } \mathbf{E}_{(\alpha_m)} \overset{m\to\infty}{\longrightarrow} \mathbf{E} \text{ in } L^\infty((0,T),L^2(\Omega)) \text{ weak } *. \tag{4.7}$$

The aim of the following consideration is to show that ($\mathbf{E}_{(\alpha_m)}$ and $\mathbf{P}_{(\alpha_m)}$) satisfy the conditions of Lemma 3.5. It follows from 4.2 that

$$(1-Q)\mathbf{U}_\alpha(t) = (1-Q)\left(\mathbf{U}_0 - \int_0^t \left[\mathcal{R}\partial_t \mathbf{P}_\alpha(s) + (\varepsilon^{-1}\mathbf{j}(s),0)\right] ds\right) \tag{4.8}$$
$$= (1-Q)\left(\mathbf{U}_0 + \mathcal{R}\mathbf{P}_0 - \mathcal{R}\mathbf{P}_\alpha(s) - \int_0^t (\varepsilon^{-1}\mathbf{j}(s),0)ds\right).$$

This gives

$$(1-Q_E)\mathbf{E}_\alpha(t) = (1-Q_E)\left(\mathbf{E}_0 + \varepsilon^{-1}\tilde{\mathbf{P}}_0 - \int_0^t \varepsilon^{-1}\mathbf{j}(s)ds - \varepsilon^{-1}\tilde{\mathbf{P}}_\alpha(t)\right), \tag{4.9}$$

whence 3.14 with $\mathbf{F}(t) \overset{\text{def}}{=} (1-Q_E)\left(\mathbf{E}_0 + \varepsilon^{-1}\tilde{\mathbf{P}}_0 - \int_0^t \varepsilon^{-1}\mathbf{j}(s)ds\right)$. Let $\mathbf{g} \in W_H$. Since $(0,\mathbf{g}) \in D(B)$, it follows from 4.1 that

$$\int_\Omega \mathbf{E}_\alpha(t)\cdot \text{ curl } \mathbf{g}dx = \langle \mathbf{U}_\alpha(t), B(0,\mathbf{g})\rangle_X \tag{4.10}$$
$$= -\alpha\left\{\frac{d}{dt}\langle \mathbf{U}_\alpha(t),(0,\mathbf{g})\rangle_X + \langle \mathcal{R}\partial_t \mathbf{P}_\alpha(t) - (\varepsilon^{-1}\mathbf{j}(t),0),(0,\mathbf{g})\rangle_X\right\}$$
$$= -\alpha\frac{d}{dt}\langle \mathbf{U}_\alpha(t),(0,\mathbf{g})\rangle_X = -\frac{d}{dt}\int_\Omega \mathbf{B}_\alpha(t)\cdot \mathbf{g}dx$$

with $\mathbf{B}_\alpha \overset{\text{def}}{=} \alpha\mu\mathbf{H}_\alpha$. Since $C_0^\infty(\Omega) \subset W_H$ and $(\mathbf{B}_\alpha)_{\alpha\in\mathbb{N}}$ is bounded in $L^\infty((0,T),L^2(\Omega))$ by 4.5, condition 3.16 is also satisfied by 4.10. Now, it follows from Lemma 3.5 that

$$\|\mathbf{P}_{\alpha_m} - \mathbf{P}\|_{W^{1,\infty}((0,T),L^2(K\cap G))} \overset{m\to\infty}{\longrightarrow} 0 \text{ for all compact } K \subset \Omega \tag{4.11}$$

and that $\mathbf{E}$ and $\mathbf{P}$ satisfy 2.14 and 2.15. Next, 4.7 and 4.10 yield

$$\int_0^t \int_\Omega \mathbf{E}(s)\cdot \text{ curl } \mathbf{g}dxds = \lim_{m\to\infty}\int_0^t \int_\Omega \mathbf{E}_{\alpha_m}(s)\cdot \text{ curl } \mathbf{g}dxds$$
$$= \lim_{m\to\infty} \alpha_m\langle \mathbf{U}_0 - \mathbf{U}_{\alpha_m}(t),(0,\mathbf{g})\rangle_X = 0 \text{ for all } \mathbf{g} \in W_H,$$

which implies that

$$\mathbf{E}(t) \in W_{E,0} \text{ for all } t \in (0,T). \tag{4.12}$$

Furthermore, it follows from 4.6, 4.7, and 4.9 that

$$\int_0^\tau (1-Q_E)\left(\mathbf{E}(t)+\varepsilon^{-1}\tilde{\mathbf{P}}(t)\right)dt = \lim_{m\to\infty}\int_0^\tau (1-Q_E)\left(\mathbf{E}_{\alpha_m}(t)+\varepsilon^{-1}\tilde{\mathbf{P}}_{\alpha_m}(t)\right)dt$$
$$= \int_0^\tau (1-Q_E)\left(\mathbf{E}_0+\varepsilon^{-1}\tilde{\mathbf{P}}_0-\int_0^t \varepsilon^{-1}\mathbf{j}(s)ds\right)dt.$$

By 4.12 this yields 2.13. The uniqueness of the solution to problem 2.13–2.15 follows easily from the estimates

$$\|(\nabla_P V)(x,\mathbf{p})-(\nabla_P V)(x,\mathbf{q})\|_{L^2(G)} \le K\|\mathbf{p}-\mathbf{q}\|_{L^2(G)} \tag{4.13}$$

by assumption 2.2, and

$$\|\gamma(1-Q_E)(\varepsilon^{-1}\tilde{\mathbf{p}})\|_{L^2(G)} \le C_1\|\varepsilon^{1/2}(1-Q_E)(\varepsilon^{-1}\tilde{\mathbf{p}})\|_{L^2(\Omega)} \tag{4.14}$$
$$\le C_1\|\varepsilon^{-1/2}\tilde{\mathbf{p}}\|_{L^2(\Omega)} \le C_2\|\mathbf{p}\|_{L^2(G)} \text{ for all } \mathbf{p}\in L^2(G).$$

In particular $\mathbf{E}$ and $\mathbf{P}$ defined in 4.7 are the only possible accumulation points for $\alpha\to 0$. This completes the proof of Theorem 2.1.

Next, a rate for the convergence in Theorem 2.1 will be given for the case where Ω is an exterior domain. Here a Poincaré type inequality for divergence free vector fields is used which has been proved in [6], Lemma 4.3.

LEMMA 4.1. Assume that $\Omega\subset\mathbb{R}^3$ be a domain with bounded complement, such that $\mathbb{R}^3\setminus\overline{\Omega}$ is a Lipschitz-domain and $G\subset\Omega$ an open subset. Suppose that $B_R\cap\Omega$ (for all $R>0$ with $\mathbb{R}^3\setminus\Omega\subset B_R$) and the decomposition $\partial\Omega=\Gamma_1\cup\Gamma_2$ of the boundary satisfy the assumptions in [6]. Furthermore, assume that $\varepsilon(x)=1$ on all of Ω.

Then there exists a constant $K_3\in(0,\infty)$, such that

$$\|\mathbf{A}\|_{L^2(\Omega\cap B_{(R_0)})}+\|\mathbf{A}\|_{L^6(\mathbb{R}^3\setminus B_{(R_0)})} \le K_3\|\text{curl }\mathbf{A}\|_{L^2(\Omega)}$$

for all $\mathbf{A}\in W_E\cap Y_E$ (with Y_E as in 2.4).

THEOREM 4.2. Suppose that the conditions of Theorem 2.1 and Lemma 4.1 are satisfied and assume that the coefficient γ occurring in 1.2 satisfies

$$\gamma\in L^r(G) \text{ for some } r\in[3,\infty). \tag{4.15}$$

Let $(\mathbf{E}_\alpha,\mathbf{H}_\alpha,\mathbf{P}_\alpha)$ be as in Theorem 2.1 and $T>0$.
Then

$$\sup_{t\in[0,T]}\left\{\left\|\int_0^t Q_E\mathbf{E}_\alpha(s)ds\right\|_{L^2(\Omega\cap B_{(R_0)})}\right. \tag{4.16}$$
$$\left.+\left\|\int_0^t Q_E\mathbf{E}_\alpha(s)ds\right\|_{L^6(\mathbb{R}^3\setminus B_{(R_0)})}\right\} \le K_1\alpha$$

and

$$\|(1-Q_E)[\mathbf{E}_\alpha(\cdot)-\mathbf{E}(\cdot)]\|_{W^{1,\infty}((0,T),L^2(\Omega))} \tag{4.17}$$
$$+\|\mathbf{P}_\alpha-\mathbf{P}\|_{W^{1,\infty}((0,T),L^2(G))} \le K_1\alpha^{3/r}$$

for all $\alpha\in[0,1]$ with some constant $K_1>0$ independent of α,t. Here $\mathbf{P}\in C^2([0,\infty),\mathcal{G})$ and $\mathbf{E}\in C([0,\infty),W_{E,0})$ obey 2.13–2.15.

Proof. As in Lemma 3.5, let

$$\mathbf{A}_\alpha(t) \stackrel{\text{def}}{=} \int_0^t Q_E \mathbf{E}_\alpha(s) ds \in Y_E.$$

Then it follows from 4.10 that, for all $\mathbf{g} \in W_H$,

$$\int_\Omega \mathbf{A}_\alpha(t) \cdot \text{ curl } \mathbf{g} dx = -\alpha \int_0^t \frac{d}{dt} \langle \mathbf{U}_\alpha(s), (0, \mathbf{g}) \rangle_X ds$$

$$= -\alpha \langle \mathbf{U}_\alpha(t) - \mathbf{U}_\alpha(0), (0, \mathbf{g}) \rangle_X = -\alpha \int_\Omega \mu[\mathbf{H}_\alpha(t) - \mathbf{H}_\alpha(0)] \cdot \mathbf{g} dx,$$

which implies that $\mathbf{A}_\alpha \in W_E$ with

$$\| \text{ curl } \mathbf{A}_\alpha(t) \|_{L^2(\Omega)} \leq \alpha \| \mu[\mathbf{H}_\alpha(t) - \mathbf{H}_\alpha(0)] \|_{L^2(\Omega)} \leq C_1 \alpha \tag{4.18}$$

for all $t \in (0, T)$. Now, Lemma 4.1 and 4.18 give

$$\| \mathbf{A}_\alpha(t) \|_{L^2(\Omega \cap B_{(R_0)})} + \| \mathbf{A}_\alpha(t) \|_{L^6(\mathbb{R}^3 \setminus B_{(R_0)})} \leq K_3 C_1 \alpha, \tag{4.19}$$

whence 4.16. Let

$$p \stackrel{\text{def}}{=} \left(\frac{1}{2} - \frac{1}{r} \right)^{-1} \in (2, 6]$$

with r as in assumption 4.15. Then it follows from 4.5 and 4.19 that

$$\begin{aligned} &\| \mathbf{A}_\alpha(t) \|_{L^2(\Omega \cap B_{(R_0)})} + \| \mathbf{A}_\alpha(t) \|_{L^p(\mathbb{R}^3 \setminus B_{(R_0)})} \\ &\leq \| \mathbf{A}_\alpha(t) \|_{L^2(\Omega \cap B_{(R_0)})} + \| \mathbf{A}_\alpha(t) \|^{1-3/r}_{L^2(\mathbb{R}^3 \setminus B_{(R_0)})} \| \mathbf{A}_\alpha(t) \|^{3/r}_{L^6(\mathbb{R}^3 \setminus B_{(R_0)})} \\ &\leq K_4 C_1 \alpha^{3/r}. \end{aligned} \tag{4.20}$$

Next, it follows from 4.20, assumption 4.15, and Hölder's inequality that

$$\begin{aligned} &\| \gamma \mathbf{A}_\alpha(t) \|_{L^2(G)} \\ &\leq \| \gamma \|_{L^\infty(G)} \| \mathbf{A}_\alpha(t) \|_{L^2(G \cap B_{(R_0)})} + \| \gamma \|_{L^r(G)} \| \mathbf{A}_\alpha(t) \|_{L^p(G \setminus B_{(R_0)})} \\ &\leq K_5 C_1 \alpha^{3/r} \text{ for all } \alpha \geq 0, t \in [0, T]. \end{aligned} \tag{4.21}$$

As in 3.20, one obtains

$$\begin{aligned} &\partial_t \mathbf{P}_\alpha(t) - \partial_t \mathbf{P}_\beta(t) + \beta[\mathbf{P}_\alpha(t) - \mathbf{P}_\beta(t)] \\ &+ \int_0^t [(\nabla_P V)(x, \mathbf{P}_\alpha(s)) - (\nabla_P V)(x, \mathbf{P}_\beta(s))] ds \\ &= \gamma \mathbf{A}_\alpha(t) - \gamma \mathbf{A}_\beta(t) - \gamma \int_0^t (1 - Q_E)(\varepsilon^{-1}[\tilde{\mathbf{P}}_\alpha(s) - \tilde{\mathbf{P}}_\beta(s)]) ds \text{ on } \mathbb{R}^+ \times G. \end{aligned} \tag{4.22}$$

By assumption 2.2, one has

$$\| (\nabla_P V)(x, \mathbf{P}_\alpha(s)) - (\nabla_P V)(x, \mathbf{P}_\beta(s)) \|_{L^2(G)} \leq K \| \mathbf{P}_\alpha(s) - \mathbf{P}_\beta(s) \|_{L^2(G)}. \tag{4.23}$$

Furthermore,

$$\begin{aligned} &\| \gamma(1 - Q_E)(\varepsilon^{-1}[\tilde{\mathbf{P}}_\alpha(t) - \tilde{\mathbf{P}}_\beta(t)]) \|_{L^2(G)} \leq C_2 \| \varepsilon^{1/2}(1 - Q_E)(\varepsilon^{-1}[\tilde{\mathbf{P}}_\alpha(t) - \tilde{\mathbf{P}}_\beta(t)]) \|_{L^2(\Omega)} \\ &\leq C_3 \| \tilde{\mathbf{P}}_\alpha(t) - \tilde{\mathbf{P}}_\beta(t) \|_{L^2(\Omega)} = C_3 \| \mathbf{P}_\alpha(t) - \mathbf{P}_\beta(t) \|_{L^2(G)}. \end{aligned} \tag{4.24}$$

As in 3.24, it follows from 4.21–4.24 that

$$\|\mathbf{P}_\alpha - \mathbf{P}_\beta\|_{W^{1,\infty}((0,T),L^2(G))} \leq C_4 \|\gamma \mathbf{A}_\alpha - \gamma \mathbf{A}_\beta\|_{L^\infty((0,T),L^2(G))}. \tag{4.25}$$

Finally, the assertion 4.17 follows from 4.21 and 4.25. □

REMARK 4.3. Condition 4.15 is satisfied if the subset G, on which the polarization $\mathbf{P}$ is located, is bounded.

5. Asymptotic behavior. In this section, the asymptotic behavior of the solution $(\mathbf{E}, \mathbf{P})$ to problem 2.13–2.15 is studied. The main result of this section is:

THEOREM 5.1. Suppose that the conditions of Theorem 2.1 are satisfied and assume that the potential V satisfies the attraction condition

$$0 \leq V(x,y) \leq Ky(\nabla_P V)(x,y) \text{ for all } x \in G, y \in \mathbb{R}^3 \tag{5.1}$$

with some constant $K > 0$.
Furthermore, suppose that the initial data $(\mathbf{E}_0, \mathbf{P}_0)$ for $(\mathbf{E}_\alpha, \mathbf{P}_\alpha)$ satisfy

$$(1 - Q_E)[\mathbf{E}_0 + \varepsilon^{-1}\tilde{\mathbf{P}}_0 - \int_0^\infty \varepsilon^{-1}\mathbf{j}(s)ds] = 0. \tag{5.2}$$

Then the solution $(\mathbf{E}, \mathbf{P})$ to problem 2.13–2.15 satisfies

$$\|\mathbf{E}(t)\|_{L^2(\Omega)} \overset{t\to\infty}{\longrightarrow} 0.$$

If, in addition, $\mathbf{j}(t) = 0$ for all $t > T_1$ with some $T_1 > 0$, then there exists some constant $K > 0$ such that

$$\|\mathbf{E}(t)\|_{L^2(\Omega)} + \|\partial_t \mathbf{P}(t)\|_{\mathcal{G}}^2 + \int_G \gamma^{-1} V(x, \mathbf{P}(t))dx \leq Kt^{-1}.$$

Assumption 5.1 allows that $|(\nabla_P V)(x,y)|$ tends to zero for $|y| \to \infty$ as in [8]. The linear case $(\nabla_P V)(x,y) = ay$ with some $a > 0$ is included also.

Condition 5.2 includes

$$\operatorname{div}\, [\varepsilon \mathbf{E}_0 + \tilde{\mathbf{P}}_0 - \int_0^\infty \mathbf{j}(s)ds] = 0.$$

Physically this means that the space charge determined by the initial-state $(\mathbf{E}_0, \mathbf{P}_0)$ and the prescribed current $\mathbf{j}$ vanishes as $t \to \infty$. In analogy to Lemma 2 in [6], one obtains

LEMMA 5.2. Suppose that the conditions of Theorem 2.1 are satisfied. Then the solution to problem 2.13–2.15 has the properties

i) $\mathbf{E} \in L^\infty((0,\infty), W_{E,0})$ and $\partial_t \mathbf{P} \in L^\infty((0,\infty), \mathcal{G}) \cap L^2((0,\infty), \mathcal{G})$.

ii) The energy

$$\mathcal{E}(t) \overset{\text{def}}{=} \|\mathbf{E}(t)\|_\varepsilon^2 + \|\partial_t \mathbf{P}(t)\|_{\mathcal{G}}^2 + 2\int_G \gamma^{-1} V(x, \mathbf{P})dx \tag{5.3}$$

obeys

$$\frac{d}{dt}\mathcal{E}(t) \leq K_2 \|\mathbf{j}(t)\|_{L^2(\Omega)} - 2\|\beta^{1/2}\partial_t \mathbf{P}(t)\|_{\mathcal{G}}^2 \tag{5.4}$$

with some constant $K_2 > 0$ independent of t.

Proof. By 2.14 one has

$$\frac{d}{dt}\left(1/2\|\partial_t \mathbf{P}(t)\|_{\mathcal{G}}^2 + \int_G \gamma^{-1} V(x,\mathbf{P})dx\right) \tag{5.5}$$
$$= -\int_G \gamma^{-1}\beta|\partial_t \mathbf{P}|^2 dx + \int_G \mathbf{E}\partial_t \mathbf{P} dx$$

whereas 2.13 yields

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{E}(t)\|_\varepsilon^2 = \langle \mathbf{E}(t), (1-Q_E)\partial_t[\mathbf{E}_0 + \varepsilon^{-1}\tilde{\mathbf{P}}_0 - \int_0^t \varepsilon^{-1}\mathbf{j}(s)ds - \varepsilon^{-1}\tilde{\mathbf{P}}(t)]\rangle_\varepsilon \tag{5.6}$$
$$= -\int_G \mathbf{E}\partial_t \mathbf{P} dx - \int_\Omega \mathbf{E}\mathbf{j} dx.$$

By 5.5 and 5.6 one obtains the energy estimate

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}(t) = -\int_G \gamma^{-1}\beta|\partial_t \mathbf{P}|^2 dx - \int_\Omega \mathbf{E}\mathbf{j} dx \tag{5.7}$$
$$\leq C_1 \mathcal{E}(t)^{1/2}\|\mathbf{j}(t)\|_{L^2(\Omega)} - \|\beta^{1/2}\partial_t \mathbf{P}(t)\|_{\mathcal{G}}^2.$$

Since $\|\mathbf{j}(\cdot)\|_{L^2(\Omega)} \in L^1(0,\infty)$, the first assertion (i) follows from this inequality. Finally, 5.3, 5.7, (i), and $\|\mathbf{j}(\cdot)\|_{L^2(\Omega)} \in L^1(0,\infty)$ yield (ii). □

LEMMA 5.3. Under assumption 5.2 and the conditions of Theorem 2.1, it follows that

$$t^{-1}\int_0^t \left(\|\mathbf{E}(s)\|_\varepsilon^2 + \int_G \gamma^{-1}\mathbf{P}(s)\cdot(\nabla_P V)(x,\mathbf{P}(s))dx\right)ds \leq K_3\left[t^{-1} + t^{-1}\int_0^t R(s)ds\right]$$

with some constant $K_3 > 0$ independent of t. Here $R(t) \stackrel{\text{def}}{=} \int_t^\infty \|\mathbf{j}(s)\|_{L^2(\Omega)}ds$.

Proof. It follows from 2.13, Condition 5.2, and Lemma 5.2 that

$$\|\mathbf{E}(t)\|_\varepsilon^2 = \langle \mathbf{E}(t), (1-Q_E)[\mathbf{E}_0 + \varepsilon^{-1}\tilde{\mathbf{P}}_0 - \int_0^t \varepsilon^{-1}\mathbf{j}(s)ds - \varepsilon^{-1}\tilde{\mathbf{P}}(t)]\rangle_\varepsilon$$
$$= -\langle \mathbf{E}(t), \varepsilon^{-1}\tilde{\mathbf{P}}(t) + \mathbf{r}(t)\rangle_\varepsilon \text{ with } \mathbf{r}(t) \stackrel{\text{def}}{=} (1-Q_E)\int_t^\infty \varepsilon^{-1}\mathbf{j}(s)ds.$$

Thus,

$$\|\mathbf{E}(t)\|_\varepsilon^2 \leq -\int_G \mathbf{E}(t)\mathbf{P}(t)dt + C_1\|\mathbf{r}(t)\|_{L^2(\Omega)}$$
$$= C_1\|\mathbf{r}(t)\|_{L^2(\Omega)} - \int_G \gamma^{-1}\left[\partial_t^2\mathbf{P}(t) + \beta\partial_t\mathbf{P}(t) + (\nabla_y V)(x,\mathbf{P}(t))\right]\mathbf{P}dx.$$

Now,

$$t^{-1}\int_0^t \left(\|\mathbf{E}(s)\|_\varepsilon^2 + \int_G \gamma^{-1}\mathbf{P}(s)\cdot(\nabla_P V)(x,\mathbf{P}(s))dx\right)ds \tag{5.8}$$
$$\leq C_1 t^{-1}\int_0^t \|\mathbf{r}(s)\|_{L^2(\Omega)}ds + C_2/t - 1/2t^{-1}\int_G \gamma^{-1}\beta|\mathbf{P}(t)|^2 dx$$
$$-t^{-1}\int_G \gamma^{-1}\partial_t\mathbf{P}(t)\mathbf{P}(t)dx + t^{-1}\int_0^t\int_G \gamma^{-1}|\partial_t\mathbf{P}(s)|^2 dxds$$
$$\leq C_1 t^{-1}\int_0^t \|\mathbf{r}(s)\|_{L^2(\Omega)}ds + C_3/t + C_3 t^{-1}\|\partial_t\mathbf{P}(t)\|_{\mathcal{G}}^2 + t^{-1}\|\partial_t\mathbf{P}\|_{L^2((0,\infty),\mathcal{G})}^2$$

$$\leq C_1 t^{-1} \int_0^t \|\mathbf{r}(s)\|_{L^2(\Omega)} ds + C_4/t$$

by Lemma 5.2 i) again. In the previous estimates C_j are constants independent of t, which completes the proof. □

COMPLETION OF THE PROOF OF THEOREM 5.1: First, it follows from assumption 5.1 and Lemma 5.3 that

$$t^{-1} \int_0^t \left(\|\mathbf{E}(s)\|_\varepsilon^2 + \int_G \gamma^{-1} V(x, \mathbf{P}(s)) dx \right) ds \leq C_1 \left[t^{-1} + t^{-1} \int_0^t R(s) ds \right]. \tag{5.9}$$

Furthermore, $t^{-1} \int_0^t \|\partial_t \mathbf{P}(s)\|_{\mathcal{G}}^2 ds \leq t^{-1} \|\partial_t \mathbf{P}\|_{L^2((0,\infty),\mathcal{G})}^2 \leq C_2 t^{-1}$ by Lemma 5.2 i) and therefore

$$t^{-1} \int_0^t \mathcal{E}(s) ds \leq C_3 \left[t^{-1} + t^{-1} \int_0^t R(s) ds \right] \tag{5.10}$$

by 5.3 and 5.9. Furthermore, one has by 5.4

$$\mathcal{E}(t) \leq \mathcal{E}(s) + K_2 \int_s^t \|\mathbf{j}(r)\|_{L^2(\Omega)} dr \leq \mathcal{E}(s) + K_2 R(s),$$

for $0 \leq s \leq t$ and, thus,

$$\mathcal{E}(t) \leq t^{-1} \int_0^t (\mathcal{E}(s) + K_2 R(s)) ds. \tag{5.11}$$

Now, 5.10 and 5.11 yield

$$\mathcal{E}(t) \leq C_4 \left[t^{-1} + t^{-1} \int_0^t R(s) ds \right] \stackrel{t \to \infty}{\longrightarrow} 0. \tag{5.12}$$

This completes the proof of the decay of $\|\mathbf{E}(t)\|_\varepsilon$ for $t \to \infty$.

Now, suppose that $\mathbf{j}(t) = 0$ for all $t > T_1$ with some $T_1 > 0$. Then $r(s) = 0$ for all $s > T_1$. Hence it follows from 5.12 that for all $t > T_1$,

$$\mathcal{E}(t) \leq C_4 \left[t^{-1} + t^{-1} \int_0^{T_1} R(s) ds \right] \leq C_5 t^{-1},$$

which completes the proof.

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