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CONVERGENCE TO EQUILIBRIUM RAREFACTION WAVES FOR DISCONTINUOUS SOLUTIONS OF SHALLOW WATER WAVE EQUATIONS WITH RELAXATION

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Abstract. The purpose of this paper is to study the discontinuous solutions to a shallow water wave equation with relaxation. The typical initial value problem of discontinuous solutions is the Riemann problem. Unlike the homogeneous hyperbolic conservation laws, due to the inhomogeneity of the system studied here, the solutions of the Riemann problem do not have a self-similar structure anymore. This problem can be formulated as a free boundary problem. We show that the Riemann solutions still have a piecewise smooth structure globally and converge to the rarefaction waves of the equilibrium equation as time tends to infinity.

1. Introduction. The propagation of surface shallow water waves in a river with constant inclined bottom topography can be described in Eulerian coordinates by the following system of hyperbolic conservation laws with a relaxation term:

$$\begin{cases} h_\tau + (hu)_\xi = 0, \\ (hu)_\tau + (hu^2 + \frac{1}{2}g'h^2)_\xi = g'hS - C_f u^2. \end{cases} \tag{1.1}$$

The assumptions of hydrostatic pressure distribution, gravitational effect, and frictional bottom topography are used in deriving the shallow water river model (cf. [42]). In this case, $g' = g\cos\alpha$, $S = \tan\alpha$, with $0 < \alpha < \pi/2$, g is the gravitational acceleration, α is a constant representing the inclination angle of the river, $C_f > 0$ is the constant frictional coefficient, $h > 0$ and $u > 0$ are the depth and velocity of the water respectively, and τ and ξ are the time and space variables, respectively.

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Since it is more convenient to consider system (1.1) in Lagrangian coordinates, we use the usual transformation $x = \int_{\xi(\tau)}^{\xi} h(y,\tau)dy$ and $t = \tau$, where $\xi(\tau)$ is an arbitrary particle path satisfying $\dot{\xi}(\tau) = u(\xi(\tau),\tau)$. Under this transformation, system (1.1) becomes

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = g'S - C_f u^2 v, \end{cases} \tag{1.2}$$

where $v = 1/h$, and $p(v) = \frac{1}{2}g'v^{-2}$. This system is strictly hyperbolic when $0 < v < \infty$ with two distinct characteristic speeds $\lambda_1(v) = -\sqrt{-p'(v)} = -\sqrt{g'}v^{-3/2}$, $\lambda_2(v) = \sqrt{-p'(v)} = \sqrt{g'}v^{-3/2}$, and two Riemann invariants

$$w(u,v) = u + m(v), \qquad z(u,v) = u - m(v), \tag{1.3}$$

with $m(v) = -2\sqrt{g'/v}$ satisfying $m'(v) = \lambda_2(v)$. When inertia is neglected, the relaxation term $g'S - C_f u^2 v$ vanishes, the system is in "equilibrium" and the "equilibrium" equation corresponding to (1.2) is given by

$$v_t - f(v)_x = 0, \tag{1.4}$$

where $f(v) = \pm\sqrt{\frac{g'S}{C_f v}}$ satisfies $g'S - C_f(f(v))^2 v = 0$. In the following, we consider the case when (v,u) is in a small neighborhood of a point on the "equilibrium" curve $u = \sqrt{\frac{g'S}{C_f v}}$, i.e., $f(v) = \sqrt{\frac{g'S}{C_f v}}$. It is expected that system (1.2), as $t \to \infty$, is well approximated by the "equilibrium" equation (1.4), provided the subcharacteristic condition $|f'(v)| < \sqrt{-p'(v)}$ holds. This subcharacteristic condition serves as the stability condition (see [42] and [22]), and it turns out to be very simple in the present situation, i.e.,

$$\tan\alpha = S < 4C_f, \tag{1.5}$$

which means the inclination angle is less than a critical value.

Previous works on nonlinear hyperbolic systems with relaxation are mainly on smooth solutions with small derivatives (cf. [22] , [50], [10], [54] and [17]). In general, if the derivatives of initial data are not small, discontinuities will develop in finite time. Therefore, it is quite natural to study discontinuous solutions. For water waves, discontinuities satisfying the Rankine-Hugoniot condition represent turbulent bores, or "hydraulic jumps" in water wave theory (cf. [42]).

The simplest problem with discontinuous initial data is the Riemann problem, i.e., the initial value problem of (1.2) with the following initial data:

$$(v(x,0),u(x,0)) = \begin{cases} (v_l,u_l), & -\infty < x < 0, \\ (v_r,u_r) & 0 < x < +\infty. \end{cases} \tag{1.6}$$

The Riemann solutions play an important role in the study of quasi-linear hyperbolic systems. It is very important and interesting to understand the structure of Riemann solutions in the study of hyperbolic conservation laws, since the Riemann solutions serve as building blocks of general discontinuous solutions and the standard test model of mathematical theories and numerical methods for solving nonlinear hyperbolic systems. Also, the Riemann solutions may determine the large-time behavior of general discontinuous solutions. Moreover, for system (1.2), the current techniques are inadequate to give a

good understanding of the structure and behavior of general discontinuous solutions of system (1.2). Of course the Riemann solutions of homogeneous quasilinear hyperbolic systems of the form $U_t + F(U)_x = 0$ are well understood (cf. [35]). For a homogeneous system, since the system and the Riemann initial data are invariant under the uniform stretching of coordinates $(x,t) \to (ax, at)$, the Riemann solutions admit self-similar solutions of the form $U(x/t)$. Therefore, for homogeneous systems, the Riemann solutions can be given by explicit formulas whenever they exist. However, for inhomogeneous systems, the Riemann solutions lose their self-similar structure, and have much more complicated and richer structures.

In this paper, we assume that the two states (v_l, u_l) and (v_r, u_r), satisfying $v_r > 0,\ v_l > 0,\ u_r > 0,\ u_l > 0$, are connected by S_1S_2 in the (v,u) phase plane (see [35]). That is, there exists an intermediate state (v_m, u_m) such that (v_m, u_m) is connected to (v_l, u_l) by a 1-shock wave, i.e.

$$u_m - u_l = -\sqrt{-(p(v_m) - p(v_l))(v_m - v_l)}, \qquad v_l > v_m, \tag{1.7}$$

and (v_r, u_r) is connected to (v_m, u_m) by a 2-shock wave,

$$u_r - u_m = \sqrt{-(p(v_m) - p(v_r))(v_m - v_r)}, \qquad v_m < v_r. \tag{1.8}$$

When (v_r, u_r) and (v_l, u_l) are close enough, it is easy to check that $v_m > 0$ and $u_m > 0$.

In the case $v_l < v_r$, the global structure and large time behavior of solutions to the above Riemann problem are studied in ([24]). It is shown in ([24]) that solutions to the above Riemann problem converge to traveling wave solutions when $v_l < v_r$. When $v_l > v_r$, the large time asymptotic behavior will be quite different. We expect that the long time behavior of solutions to the relaxation system (1.2) can be described by that of the "equilibrium" equation (1.4). The large time behavior of solutions for the "equilibrium" equation (1.4) (cf. [20]) is determined by the end states v_l and v_r as $x \to \pm\infty$. The long time behavior of solutions to (1.4) is a shock wave when $v_l < v_r$ and a rarefaction wave when $v_l > v_r$ (cf. [20]). Therefore, we expect that the solution of the Riemann problem (1.2) and (1.6) will converge to the rarefaction wave of the "equilibrium" equation (1.4) when $v_l > v_r$ as $t \to \infty$. One of the purposes in this paper is to confirm this expectation. The proof of the convergence to rarefaction waves in this paper is quite different from that of the convergence to the traveling wave as in [24]. This is because the traveling wave solves (1.2), but the rarefaction wave solution of (1.4) does not solve equation (1.2). This causes much of the difficulty in our analysis.

Here we highlight the new ingredients in this paper, compared with related previous works as follows:

i) A semi-linear model proposed by Jin and Xin ([16]) has been extensively studied (cf. [2], [16], [13], [17], [18], [26], [27], [29], [33], [39], [40], [43], [46], [47], [48] and [49]). The feature of the model in [16] is that the left-hand side of the system is linear and the coefficients are constant, while the left-hand side of system (1.2) is nonlinear.

ii) For some 2×2 hyperbolic systems with relaxation, with the left hand side being nonlinear, the behavior of solutions are studied mainly for **smooth** solutions. Here we mention two results of this type. First, in [21], convergence to "equilibrium" rarefaction waves and shock profiles is proved respectively for *smooth* solutions of general 2×2

relaxation systems, with the assumption that the end states at $x = \pm\infty$ are on the "equilibrium" curve. In the present paper, the two end states satisfying (1.7) and (1.8) are not on the "equilibrium" curve $u = f(v)$. For the case when the end states are not on the "equilibrium" curve, the large time behavior of *smooth* solutions of the p-system with relaxation of the form $f(v) - u$ is considered in [54]. The relaxation form is linear in u, and u and v are separated, thus simplifying the analysis. For system (1.2), the relaxation form $S - C_f u^2 v$ is more complicated and more physically reasonable, i.e., it is nonlinear in u, and u and v are not separated. This causes considerable difficulty in obtaining desired estimates. Therefore, one can expect that the analysis of (1.2) will be more difficult than the p-system studied in [54], even for smooth solutions.

iii) There has been some progress in Riemann solutions for some inhomogeneous hyperbolic systems. In [20], the local in time existence and structure of Riemann solutions for general inhomogeneous hyperbolic systems are discussed. In [12] and [11], the global existence of Riemann solutions and convergence to diffusion waves for Riemann solutions containing a single shock for systems with damping are studied, respectively, where the source term is very simple (the right-hand side of the second equation is replaced by $-\alpha u$). The source term in (1.2) is much more complicated, and the solutions of the Riemann problem exhibit completely different behavior. In [9], the Riemann problem of a system arising from viscoelasticity is discussed. This system has the form of $u_t + \sigma_x = 0$, $(\sigma - f(u))_t + (\sigma - \mu f(u)) = 0$ with the second equation being an ordinary differential equation for $\sigma - f(u)$ which can be introduced as a new variable v to simplify the arguments. In ([24]), as mentioned above, convergence to traveling waves for solutions of the Riemann problem of (1.2) and (1.6) is proved for the case $v_l < v_r$. However, as mentioned above, the proof of convergence to rarefaction waves is quite different from that for traveling waves.

REMARK 1. Only the Riemann problem is studied in this paper. The ideas and techniques employed here can also be used to handle the perturbed Riemann problem with the data

$$(v(x,0), u(x,0)) = \begin{cases} (v_l(x), u_l(x)), & -\infty < x < 0, \\ (v_r(x), u_r(x)) & 0 < x < +\infty, \end{cases} \tag{1.9}$$

where the functions $(v_l(x), u_l(x))$ and $(v_r(x), u_r(x))$ are small and smooth perturbations of the constant states (v_l, u_l) and (v_r, u_r). Moreover, a combination of the ideas and techniques used here and those used in [23] can be used to handle the interaction of shocks.

Relaxation problems have attracted considerable attention in the recent years ([3], [4], [5], [8], [14], [21], [25], [28], [31], [32], [34], [36], [37], [50], [51], [52], [53]). The general setting can be found in [4]. Convergence to planar traveling waves for smooth solutions for the 2-D shallow water wave equation with relaxation is shown in [10]. The fluid dynamic limit (cf. [45] and [44]) and the boundary layer problem (cf. [19]) of the 3×3 nonlinear Broadwell model of Boltzmann equations were studied. The Riemann problem of a modified Broadwell model with self-similar structure was investigated in [7] and [38]. The existence and stability of shock profiles (traveling waves) for a general nonlinear $n \times n$ system with relaxation were studied in [52] and [31], respectively. For the initial

boundary value problem, one can refer to [34], [41], [46], [47] and [48]. A quasilinear model of gas dynamics equations was investigated in [53]. A modified Glimm's scheme ([25]) and a wave front tracking method ([1]) are used respectively to prove the existence of global BV solutions of p-system with relaxation for $p(v) = kv^{-1}$.

Closely related to this paper is a result obtained in [15] for system (1.2) with viscosity in the case that the subcharacteristic condition (1.5) is violated, the weakly nonlinear limit is verified, and the underlying relaxation system reduces to the Burgers equation with a source term (cf [15]). It would be important and interesting to study the behavior of discontinuous solutions of the inviscid system (1.2) when the subcharacteristic condition (1.5) is violated. As pointed out in [42], the resulting flow in this case is not necessarily completely chaotic or without structure. In favorable circumstances, it takes the form of "roll waves" with a periodic structure of discontinuous bores separated by smooth profiles.

The rest of the paper is organized as follows. In Section 2, we state the main result of this paper. In Section 3, we gather some decay estimates of the discontinuities, which play an important role in our analysis. In Section 4, some properties of the "equilibrium" rarefaction wave is presented. The problem is reformulated in Section 5. In Section 6, the energy method is used to study the problem formulated in Section 5.

2. Statement of the result. We now present the main results in the paper. Before that, let us define the shock waves for (1.2) as follows.

A discontinuity along $x = x_1(t)$ is called a 1-shock if the Rankine- Hugoniot condition

$$\begin{cases} \frac{dx_1(t)}{dt} = -\sqrt{-\frac{p(v_+)-p(v_-)}{v_+-v_-}}, \\ u_+ - u_- = -\dot{x}_1(t)(v_+ - v_-), \\ p(v_+) - p(v_-) = \dot{x}_1(t)(u_+ - u_-), \end{cases} \tag{2.1}$$

and the entropy condition

$$v_+(t) < v_-(t)$$

hold, where

$$(v_-(t), u_-(t)) = (v, u)(x_1(t) - 0, t), (v_+(t), u_+(t)) = (v, u)(x_1(t) + 0, t).$$

The 2-shock can be defined similarly as a discontinuity $x = x_2(t)$ satisfying the Rankine-Hugoniot condition

$$\begin{cases} \frac{dx_2(t)}{dt} = \sqrt{-\frac{p(v_+)-p(v_-)}{v_+-v_-}}, \\ u_+ - u_- = -\dot{x}_2(t)(v_+ - v_-), \\ p(v_+) - p(v_-) = \dot{x}_2(t)(u_+ - u_-), \end{cases} \tag{2.2}$$

and the entropy condition

$$v_+(t) > v_-(t),$$

where

$$(v_-(t), u_-(t)) = (v, u)(x_2(t) - 0, t), (v_+(t), u_+(t)) = (v, u)(x_2(t) + 0, t).$$

The Riemann problem of (1.2) and (1.6) for $0 < t \leq T$ can be formulated as the following free boundary problem.

FBP: 1-shock discontinuity $x = x_1(t)$ issuing from (0,0), satisfying the Rankine-Hugoniot condition, the entropy condition

$$v(x_1(t)-,t) > v(x_1(t)+,t)$$

and the initial condition

$$\lim_{t\to 0}(v,u)(x_1(t)-,t) = (v_l,u_l),\ \lim_{t\to 0}(v,u)(x_1(t)+,t) = (v_m,u_m);$$

while a 2-shock $x = x_2(t)$ issuing from $(0,0)$, satisfying the Rankine-Hugoniot condition, the entropy condition

$$v(x_2(t)-,t) < v(x_2(t)+,t)$$

and the initial condition

$$\lim_{t\to 0}(v,u)(x_2(t)-,t) = (v_m,u_m), \quad \lim_{t\to 0}(v,u)(x_2(t)+,t) = (v_r,u_r).$$

The solution (v,u) is smooth in the region

$$S(T) = \{(x,t)|0 < t \le T, x_1(t) \le x \le x_2(t)\}.$$

In the outer region $O_1(T) = \{(x,t)|0 \le t \le T, -\infty < x < x_1(t)\}$, the solution is completely determined by the initial left state (v_l,u_l) because of the entropy condition. Similarly, the solution in $O_2(T) = \{(x,t)|0 \le t \le T, x_2(t) < x < \infty\}$ is completely determined by (v_r,u_r). In the following, for simplicity, we set $g' = 1$. Therefore,

$$f(v) = \sqrt{\frac{S}{C_f v}}.$$

It is easy to check that the solution in $O_1(T)$ is given by

$$(v,u)(x,t) = (v_l,u^l(x,t)) = (v_l,\sqrt{\frac{S}{C_f v_l}}\frac{1+y^l}{1-y^l}),\ x < x_1(t), \tag{2.3}$$

where

$$y^l = \frac{u_l\sqrt{C_f v_l} - \sqrt{S}}{u_l\sqrt{C_f v_l} + \sqrt{S}}\exp(-2\sqrt{SC_f v_l}t). \tag{2.4}$$

The solution (v,u) in $O_1(T)$ can be obtained by solving the following initial value problem of the system of ODEs:

$$v_t = 0, u_t = \frac{1}{\epsilon}(S - C_f u^2 v),$$
$$(v,u)|_{t=0} = (v_l,u_l).$$

Similarly, the solution in $O_2(T)$ is given by

$$(v,u)(x,t) = (v_r,u^r(x,t)) = (v_r,\sqrt{\frac{S}{C_f v_l}}\frac{1+y^r}{1-y^r}),\ x > x_2(t), \tag{2.5}$$

where

$$y^r = \frac{u_r\sqrt{C_f v_r} - \sqrt{S}}{u_r\sqrt{C_f v_r} + \sqrt{S}}\exp(-2\sqrt{SC_f v_r}t). \tag{2.6}$$

It follows from (2.3) and (2.5) that

$$|u^l(x,t) - f(v_l)| \le O(1)|u_l - f(v_l)|\exp(-\sqrt{SC_f v_l}t),\ x < x_1(t) \tag{2.7}$$

and

$$|u^r(x,t) - f(v_r)| \le O(1)|u_r - f(v_r)| \exp(-\sqrt{SC_f v_r}t),\ x > x_2(t). \tag{2.8}$$

Here and in the following, we use $O(1)$ to denote a generic positive bounded quantity independent of t. (2.7) and (2.8) indicate that, in the outer region $O_i(T)$, $i = 1, 2$, the solution (v, u) approaches to the equilibrium state $v = f(u)$ exponentially fast.

The local existence of solutions to the above free boundary problem is a simple corollary of Li and Yu's general theorem on quasilinear hyperbolic systems ([20]). In order to extend the local solution for all time, we need to establish a uniform C^1-estimate in the region $S(T)$ defined above for $T > 0$. This will be carried out in Sections 3 and 4 by the observation that the subcharacteristic condition forces the discontinuity of the solution and its derivatives to decay exponentially with respect to time. Thus, as $t \to \infty$, the solution of the free boundary problem will approach to a continuous function. Note that the large time asymptotic state depends on the relationship between v_l and v_r. As stated in Section 1, when $v_l < v_r$, the large time asymptotic state of the solution of the Riemann problem (1.2) and (1.6) is a traveling wave, as shown in [24]. When $v_l > v_r$, the Riemann solution of equilibrium equation (1.4) with the Riemann data (v_l, v_r) is a rarefaction wave, which is expected to be the asymptotic state of the solution to the Riemann problem (1.2) and (1.6).

The rarefaction wave V^R of equilibrium equation (1.4) is the solution to (1.4) with the Riemann initial data

$$V^R(x,0) = \begin{cases} v_l, & -\infty < x < 0, \\ v_r, & 0 < x < +\infty. \end{cases} \tag{2.9}$$

The solution V^R is given by

$$V^R(x,t) = \begin{cases} v_l, & -\infty < x < -f'(v_l)t, \\ (-f')^{-1}(x/t), & -f'(v_l)t \le x \le -f'(v_r)t, \\ v_r, & -f'(v_r)t < x < +\infty; \end{cases} \tag{2.10}$$

here and in the following $(-f')^{-1}$ denotes the inverse of $-f'$.

Now we can state our main theorem. In the following $[\ell](x(t))$ denotes the jump of the function ℓ along a curve $x = x(t)$, i.e., $[\ell](x(t) = \ell(x(t)+, t) - \ell(x(t)-, t)$.

THEOREM 2.1. [Structure and asymptotic behavior of solutions to the Riemann problem] Suppose $v_l > v_r$. If $|v_r - v_l| + |u_r - f(v_r)|$ is small enough, and the subcharacteristic condition (1.5) holds, then there exists a global smooth solution to the above (FBP) for any $T > 0$ in $S(T)$. Moreover, we have the following estimates:

Along the shocks $x = x_i(t), i = 1, 2$,

$$|[u](x_i(t))| + |[v](x_i(t))| + |[u_x](x_i(t))| + |[v_x](x_i(t))| \le O(1)|v_r - v_l| \exp(-\alpha t), \tag{2.11}$$

for some $\alpha > 0$. Furthermore,

$$\lim_{t\to\infty} \sup_{x_1(t)\le x\le x_2(t)} (|v(x,t) - V^R(x,t)| + |u(x,t) - f(V^R)(x,t)|) = 0. \tag{2.12}$$

3. Decay estimate of discontinuities. In this section, we gather the decay estimates of the discontinuities of the solutions $(v,\ u)$ and their derivatives along the shock curves $x = x_1(t)$ and $x = x_2(t)$, and other estimates. Those estimates are important for the proof of Theorem 2.1. In the following, we always use w and z to denote the Riemann invariants defined in (1.3). The free boundary problem has the boundaries 1-shock $x = x_1(t)$ and 2-shock $x = x_2(t)$. In the following we denote

$$S(T) = \{(x,t) : 0 \le t \le T, x_1(t) \le x \le x_2(t)\}. \tag{3.1}$$

LEMMA 3.1. Let (v,u) be a smooth solution of the free boundary problem (FBP) as stated in section 2 in $S(T)$. Assume the subcharacteristic condition (1.5) is satisfied. Suppose there exist two positive constants v_1 and v_2 independent of t such that $v_1 \le v(x,t) \le v_2$ as $(x,t) \in S(T)$. Then we have, for some $\gamma > 0$,

$$|[u](x_i(t))| + |[v](x_i(t))| + |[u_x](x_i(t))| + |[v_x](x_i(t))| \le O(1)|v_r - v_l| \exp(-\gamma t), \tag{3.2}$$

where $[\ ](x_i(t))$ denote the usual jump along the curve $x = x_i(t)$ $(i = 1,\ 2)$

$$\begin{aligned}&(|w_x| + |z_x|)(x_1(t)+, t) + (|w_x| + |z_x|)(x_2(t)-, t)\\ &\le O(1)(|v_r - v_l| + |u_r - f(v_r)|)e^{-\gamma t},\end{aligned} \tag{3.3}$$

$$\begin{aligned}&\sup_{(x,t)\in S(T)} (|w_x| + |z_x| + |w_{xx}|(x,t) + |z_{xx}| + |u - f(v)|)(x,t)\\ &\le O(1)|v_r - v_l| + |u_r - f(v_r)|,\end{aligned} \tag{3.4}$$

provided $|v_r - v_l| + |u_r - f(v_r)|$ is small enough.

The proof of this lemma can be found in [24], where the characteristic method is employed to derive the above estimates. Those estimates are true no matter whether $v_l < v_r$ or $v_l > v_r$ (see [24] for details).

4. The equilibrium rarefaction waves. The rarefaction wave V^R of equilibrium equation (1.4) given by (2.10) can be approximated as $t \to \infty$ by the smooth solution $V(x,t)$ of the following Cauchy problem:

$$V_t - f(V)_x = 0, \tag{4.1}$$

$$V(x,0) = (v^r + v^l)/2 + m(x)(v^r - v^l)/2, \tag{4.2}$$

where $m(x)$ is a smooth function satisfying $m(x) = -1$ for $x \le -k_0$, $m(x) = 1$ for $x \ge k_0$ and $m'(x) \ge 0$. Here k_0 is a fixed constant. Since $V_0'(x) \le 0$, then $\frac{d(-f'(V_0(x))}{dx} \ge 0$, so the solution $V(x,t)$ of (4.1) and (4.2) is an expansion wave and thus smooth. The following lemma gives some estimates on V.

LEMMA 4.1. Let V be the solution to the Cauchy problem (4.1) and (4.2). Then we have the following estimates:

a)

$$\lim_{t\to\infty} \sup_{-\infty<x<+\infty} |V - V^R|(x,\ t) = 0, \tag{4.3}$$

where V^R is the rarefaction wave V^R of equilibrium equation (1.4) given by (2.10).

b)

$$V_x(x,t) \le 0, \tag{4.4}$$

for $x \in \mathbf{R^1}$ and $t \ge 0$,

$$|D^\alpha V|(x,t) = \begin{cases} 0, & \text{for } |x| \ge k_0 + mt, \\ O(1)|v_l - v_r|(t+1)^{-|\alpha|}, & \text{for } |x| \le k_0 + mt, \end{cases} \tag{4.5}$$

where

$$m = \max\{|f'(v)| : v_r \le v \le v_l\} = |f'(v_r)|, \tag{4.6}$$

and for any partial derivative $D^\alpha V$ of V with respect to x and t

$$V(x,t) = \begin{cases} v_l, & \text{for } x < -k_0 - mt, \\ v_r, & \text{for } x > k_0 + mt. \end{cases} \tag{4.7}$$

Proof. a) Any solutions of (4.1) with the same end states are time asymptotically equivalent (i.e., they converge to each other in L^∞-norm as t tends to infinity [22]).

b) Since the wave is expansive, i.e. $\frac{\partial(-f'(V))(x,t)}{\partial x} \ge 0$, and $f'(V) < 0$ so $V_x \le 0$. This proves (4.4). (4.5) and (4.7) can be found in ([21] p. 158). □

5. Reformulation of the problem. In order to prove (2.12), it is sufficient to show that

$$\lim_{t\to\infty} \sup_{-\infty<x<+\infty} |v - V|(x,\ t) = 0, \tag{5.1}$$

in view of (4.3). For this purpose, we let

$$\phi = v - V,\ \psi = u - f(V).$$

Then ϕ and ψ satisfy the following system of equations for $x_1(t) < x < x_2(t)$:

$$\phi_t - \psi_x = 0, \tag{5.2}$$

$$\psi_t + p(V+\phi)_x + (f'(V))^2 V_x = R(\phi,\ \psi,\ V), \tag{5.3}$$

where

$$R(\phi,\ \psi,\ V) = S - C_f(\psi + f(V))^2(\phi + V). \tag{5.4}$$

We write $p(V+\phi)$ as

$$p(V+\phi) = P(V) + p'(V)\phi + Q(V,\phi). \tag{5.5}$$

Then $Q(V,\phi)$ satisfies

$$Q(V,\phi) = O(1)\phi^2. \tag{5.6}$$

In this setting, (5.3) can be written as

$$\psi_t + (p'(V)\phi)_x = R(\phi,\ \psi,\ V) - Q(V,\phi) - F(V)_x, \tag{5.7}$$

where $F(V)$ is a function such that $F'(V) = p'(V) + (f'(V))^2$, which can be taken as

$$F(V) = \int_{v_-}^{V} (p'(z) + (f'(z))^2)dz. \tag{5.8}$$

Differentiating (5.7) with respect to x, using (5.2), we obtain,

$$\phi_{tt} + (p'(V)\phi)_{xx} = R_x - Q_x - F(V)_{xx}. \tag{5.9}$$

We will work on equation (5.9) by using the energy method in $S(T) = \{(x,t)|0 < t \leq T, x_1(t) \leq x \leq x_2(t)\}$. For this purpose, we establish the following estimates of ϕ and its derivatives on the boundaries of $S(T)$.

LEMMA 5.1. Suppose there exist two positive constants v_1 and v_2 independent of t such that $v_1 \leq v(x,t) \leq v_2$ as $(x,t) \in S(T)$. Then

$$(|\phi| + |\psi|)(x_1(t)+,t) + (|\phi| + |\psi|)(x_2(t)-,t) \leq O(1)(|v_r - v_l| + |u_r - f(v_r))|e^{-\gamma t}, \tag{5.10}$$

$$\begin{aligned}&(|\phi_x| + |\psi_x|)(x_1(t)+,t) + (|\phi_x| + |\psi_x|)(x_2(t)-,t)\\ &\leq O(1)(|v_r - v_l| + |u_r - f(v_r)|)e^{-\gamma t},\end{aligned} \tag{5.11}$$

$$\begin{aligned}&(|D^2\phi| + |D^2\psi|)(x_1(t)+,t) + (|D^2\phi| + |D^2\psi|)(x_2(t)-,t)\\ &\leq O(1)(|v_r - v_l| + |u_r - f(v_r)|),\end{aligned} \tag{5.12}$$

where $\gamma > 0$ is a positive constant, provided $|v_r - v_l| + |u_r - f(v_r)|$ is small and the subcharacteristic condition (1.5) holds..

Proof. First we note that

$$|\phi(x_1(t)+,t)| = |v - V|(x_1(t)+,t) \leq |v(x_1(t)+,t) - v_l| + |V(x_1(t),t) - v_l|. \tag{5.13}$$

By (3.2) and the fact $v(x_1(t)+,t) = v_l$, we have

$$|v(x_1(t)+,t) - v_l| \leq O(1)(|v_r - v_l| + |u_r - f(v_r))|e^{-\gamma t}, \tag{5.14}$$

for some $\gamma > 0$. On the other hand, due to the subcharacteristic condition (1.5), there exists a positive constant $a > 0$ such that

$$\dot{x}_1(t) + m < -a,$$

where $m = -|f'(v_r)|$. So $x_1(t) < -mt - at$ for $t > 0$. Therefore, there exists $T_0 > 0$ such that $x_1(t) < -k_0 - mt$ for $t < T_0$, where k_0 is the constant in (5.1). So, by (5.1),

$$V(x_1(t),t) = v_l \tag{5.15}$$

for $t > T_0$. For $t < T_0$, it is obvious that

$$|V(x_1(t),t) - v_l| \leq |v_r - v_l|, \tag{5.16}$$

due to (4.4). This, together with (5.13) and (5.14), implies

$$|\phi(x_1(t)+,t)| \leq O(1)(|v_r - v_l| + |u_r - f(v_r))|e^{-\gamma t}. \tag{5.17}$$

Similarly,

$$|\psi(x_1(t)+,t)| = |u - f(V)|(x_1(t)+,t) \leq |u(x_1(t)+,t) - f(v_l)| + |f(V)(x_1(t),t) - f(v_l)|. \tag{5.18}$$

The term $|f(V)(x_1(t),t) - f(v_l)|$ can be handled by (5.15) and (5.16).

On the other hand,

$$\begin{aligned}&|u(x_1(t)+,t) - f(v_l)|\\ &\leq |u(x_1(t)+,t) - u(x_1(t)-,t)| + |u(x_1(t)-,t) - f(v_l)|\\ &\leq O(1)(|v_r - v_l| + |u_r - f(v_r))|e^{-\gamma t}.\end{aligned} \tag{5.19}$$

So

$$|\psi(x_1(t)+,t)| \le O(1)(|v_r - v_l| + |u_r - f(v_r))|e^{-\gamma t}. \tag{5.20}$$

Similarly, one can show,

$$|\phi(x_2(t)-,t)| + |\psi(x_2(t)-,t)| \le O(1)(|v_r - v_l| + |u_r - f(v_r))|e^{-\gamma t}, \tag{5.21}$$

and (5.10) follows. Inequality (5.11) can be proved similarly by using $(4.5)_1$ and (3.3), and (5.19) can be proved similarly by using $(4.5)_1$ and (3.4). □

6. Energy estimate. In this section, we prove the following *a priori* estimate.

LEMMA 6.1. Let $t_0 > 0$ be a small time such that the local solution of (1.2) and (1.6) exists in $(0, t_0)$. For any $T > t_0$, suppose there exist two positive constants v_1 and v_2 independent of t such that $v_1 \le v(x,t) \le v_2$ as $(x,t) \in S(T)$. Then we have the following *a priori* estimate:

$$\begin{aligned}
&\int_{x_1(t)}^{x_2(t)} \sum_{j=0}^{3} (|D^j\phi|^2 + |D^j\psi|^2)(x,t)dx \\
&\quad + \int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \{\sum_{j=1}^{3}(|D^j\phi|^2 + |D^j\psi|^2) + |V_x|(\phi^2+\psi^2)\}(x,s)dxds \\
&\le O(1)\int_{x_1(t_0)}^{x_2(t_0)} \sum_{j=0}^{3}(|D^j\phi|^2 + |D^j\psi|^2)(x,t_0)dx + O(1)(|v_r - v_l| + |u_r - f(v_r)|),
\end{aligned} \tag{6.1}$$

for $t_0 \le t \le T$ provided $\max_{t_0\le t\le T}\int_{x_1(t)}^{x_2(t)}(\phi^2+\psi^2)(x,t)dx + \delta$ and $|v_r - v_l| + |u_r - f(v_r)|$ are small, where D^j denote the partial derivatives with respect to x and t. Here and in the following,

$$\delta = \max_{t_0\le t\le T, x_1(t)\le x\le x_2(t)} (|\phi| + |\psi|)(x,t). \tag{6.2}$$

Theorem 2.1 follows from Lemma 6.1 immediately by the local existence result of [20] and standard continuation argument (cf. [21]).

REMARK 2. If $|v_r - v_l| + |u_r - f(v_r)|$ is small, then it follows from (3.4), (3.4) and (4.5) that

$$\begin{aligned}
&\max_{t_0\le t\le T, x_1(t)\le x\le x_2(t)} \left(\sum_{j=1}^{2}(|D^j\phi| + |D^j\psi|)\right) \\
&\le O(1)|v_r - v_l| + |u_r - f(v_r)|.
\end{aligned} \tag{6.3}$$

In fact, (3.4) gives the estimates of the derivatives of x. By (3.4) and equation (1.2), one can easily get the estimates of the derivatives of t. Estimate (6.3) will be used for the energy estimates.

Proof of Lemma 6.1. Multiplying (5.9) by ϕ, we have

$$(\phi_{tt} + (p'(V)\phi)_{xx})\phi = (R_x - Q_x - F(V)_{xx})\phi, \tag{6.4}$$

for $x_1(t) < x < x_2(t)$. We estimate each term as follows. For $x_1(t) < x < x_2(t)$,

$$(\phi_{tt}\phi = (\phi_t\phi)_t - \phi_t^2, \tag{6.5}$$

$$((p'(V)\phi)_{xx})\phi = ((p'(V)\phi)_x\phi)_x - p''(V)V_x\phi\phi_x - p'(V)\phi_x^2. \tag{6.6}$$

The term $-Q_x\phi$ can be estimated as follows. It follows from (5.5) that

$$Q = \left(\int_0^1 \int_0^\theta p''(V + \beta\phi)d\beta d\theta\right)\phi^2 =: D(V,\phi)\phi^2, \tag{6.7}$$

so we have

$$\begin{aligned} Q_x\phi &= D_x\phi^3 + D(\phi^2)_x\phi \\ &= D_x\phi^3 + \frac{2}{3}D(\phi^3)_x \\ &= D_x\phi^3 + \frac{2}{3}(D\phi^3)_x - \frac{2}{3}D_x\phi^3 = \frac{2}{3}(D\phi^3)_x + \frac{1}{3}D_x\phi^3 \\ &= \frac{2}{3}(D\phi^3)_x + \frac{1}{3}(D_V V_x + D_\phi\phi_x)\phi^3. \end{aligned} \tag{6.8}$$

Moreover,

$$-F_{xx}\phi \le O(1)(V_x^2|\phi| + |V_{xx}\phi|). \tag{6.9}$$

Therefore, as $x_1(t) < x < x_2(t)$,

$$\begin{aligned} &(\phi_t\phi)_t + (-p'(V)\phi_x^2) - \phi_t^2 \\ &\le \left((p'(V)\phi)_x\phi - \frac{2}{3}D\phi^3\right)_x + R_x\phi \\ &+ O(1)(|V_x\phi\phi_x| + |V_x\phi^3| + |\phi^3\phi_x| + V_x^2|\phi| + |V_{xx}\phi|). \end{aligned} \tag{6.10}$$

We estimate $R_x\phi$ as follows. At first, it is easy to check, since $2f'(V)V + f(V) = 0$ and $\phi_t = \psi_x$, that

$$\begin{aligned} R_x &= R_V V_x + R_\psi\psi_x + R_\phi\phi_t \\ &= -C_f(f(V) + \psi)\,(2f'(V)\phi + \psi)\,V_x - 2C_f(f(V) + \psi)(\phi + V)\phi_t \\ &- C_f(f(V) + \psi)^2\phi_x. \end{aligned} \tag{6.11}$$

After some calculations, we arrive at (in the following formula, β_i $(i = 1, 2)$ are quantities satisfying $|\beta_i| \le O(1)\delta$ with δ given by (6.2))

$$\begin{aligned}
R_x\phi &\le SV^{-2}V_x\phi^2 - C_f f(V)V_x\phi\psi - \frac{1}{2}C_f(f(V)+\psi)^2(\phi^2)_x \\
&\quad - C_f(f(V)+\psi)(V+\phi)(\phi^2)_t + O(1)|V_x|(\phi^2+\psi^2) \\
&= SV^{-2}V_x\phi^2 - C_f f(V)V_x\phi\psi - \frac{1}{2}C_f\{(f(V)+\psi)^2(\phi^2)\}_x \\
&\quad - C_f\{(f(V)+\psi)(V+\phi)(\phi^2)\}_t + C_f\phi^2(f(V)+\psi)(f'(V)V_x+\psi_x) \\
&\quad + C_f\phi^2\{(f'(V)V_t+\psi_t)(V+\phi) + (f(V)+\psi)(V_t+\phi_t)\} \\
&\quad + O(1)|V_x|(\phi^2+\psi^2) \\
&\le \{SV^{-2} + C_f f(V)f'(V)\}V_x\phi^2 - C_f f(V)V_x\phi\psi - \frac{1}{2}C_f\{((f(V))^2+\beta_1)\phi^2\}_x \\
&\quad - C_f\{(f(V)V+\beta_2)(\phi^2)\}_t + C_f(f(V)+\psi)\psi_x\phi^2 \\
&\quad + C_f(f'(V)V+f(V))V_t\phi^2 + C_f(\phi^3\psi_t + f(V)\phi^2\phi_t) \\
&\quad + O(1)\delta|V_x|(\phi^2+\psi^2).
\end{aligned} \tag{6.12}$$

By virtue of the fact that $f(V) = \sqrt{\frac{S}{C_f V}}$ and $V_t = f'(V)V_x$, we obtain

$$C_f f(V)f'(V) = -\frac{1}{2}SV^{-2}, C_f(f'(V)V+f(V))V_t\phi^2 = -\frac{1}{4}SV^{-2}V_x\phi^2. \tag{6.13}$$

Also, since $\phi_t = \psi_x$, we have

$$\begin{aligned}
&C_f(\phi^3\psi_t + f(V)\phi^2\phi_t) + C_f(f(V)+\psi)\psi_x\phi^2 \\
&= C_f\phi^3\psi_t + \frac{C_f}{3}f(V)(\phi^3)_t + \frac{C_f}{3}(f(V)+\psi)(\phi^3)_t \\
&\le \frac{2C_f}{3}\phi^3\psi_t + \frac{2C_f}{3}(2f(V)\phi^3 + \psi\phi^3)_t + O(1)\delta|V_x|\phi^2,
\end{aligned} \tag{6.14}$$

where δ is given by (6.2). (6.12)–(6.14) imply

$$\begin{aligned}
R_x\phi &\le \frac{1}{2}SV^{-2}V_x\phi^2 - C_f f(V)V_x\phi\psi + \frac{2C_f}{3}\phi^3\psi_t \\
&\quad - C_f\{(f(V)V+\beta_3)\phi^2\}_t - C_f\{(\frac{(f(V))^2}{2}+\beta_4)\phi^2\}_x \\
&\quad + O(1)\delta|V_x|(\phi^2+\psi^2),
\end{aligned} \tag{6.15}$$

where β_3 and β_4 are quantities satisfying

$$|\beta_3| + |\beta_4| \le O(1)\delta. \tag{6.16}$$

Inequalities (6.10) and (6.15) imply, in view of the fact $|V_x| < 0$, that

$$\begin{aligned}
&A_t + (-p'(V))\phi_x^2 + \frac{1}{2}C_f V^{-2}|V_x|\phi^2 - \phi_t^2 \\
&\le B_x - C_f f(V)V_x\phi\psi + \frac{2C_f}{3}\phi^3\psi_t \\
&\quad + O(1)(|V_x\phi\phi_x| + \phi^3|\phi_x| + V_x^2|\phi| + |V_{xx}\phi|) + O(1)\delta|V_x|(\phi^2+\psi^2),
\end{aligned} \tag{6.17}$$

as $x_1(t) \le x \le x_2(t)$, where

$$A = \phi_t\phi + (C_f f(V)V + \beta)\phi^2,\ |\beta| < O(1)\delta, \tag{6.18}$$

$$\begin{cases} B = (p'(V)\phi)_x\phi - \frac{2}{3}D\phi^3 - C_f(\frac{(f(V))^2}{2} + \beta_4)\phi^2), \\ |B| \le O(1)(|\phi| + |\phi_x|). \end{cases} \tag{6.19}$$

Multiplying (5.9) by ϕ_t, we have

$$(\frac{1}{2}\phi_t^2)_t + (p'(V)\phi)_{xx}\phi_t = (R_x - Q_x - F(V)_{xx})\phi_t. \tag{6.20}$$

Each term can be handled as follows:

$$(p'(V)\phi)_{xx}\phi_t = ((p'(V)\phi)_x\phi_t)_x - (p'(V)\phi)_x\phi_{xt}. \tag{6.21}$$

On the other hand,

$$\begin{aligned} (p'(V)\phi)_x\phi_{xt} &= p''(V)V_x\phi\phi_{xt} + \frac{1}{2}p'(V)(\phi_x^2)_t \\ &= p''(V)V_x\phi\phi_{xt} + \frac{1}{2}(p'(V)\phi_x^2)_t - \frac{1}{2}(p''(V)V_t\phi_x^2). \end{aligned} \tag{6.22}$$

(6.21) and (6.22) imply

$$(p'(V)\phi)_{xx}\phi_t = ((p'(V)\phi)_x\phi_t)_x - p''(V)V_x\phi\phi_{xt} - \frac{1}{2}(p'(V)\phi_x^2)_t + \frac{1}{2}p''(V)V_t\phi_x^2. \tag{6.23}$$

By (5.5), we have

$$Q_x = (p'(V+\phi) - p'(V))(\phi_x + V_x) - p''(V)V_x\phi. \tag{6.24}$$

This implies

$$|Q_x\phi_t| \le O(1)(|V_x||\phi||\phi_t| + |\phi||\phi_x||\phi_t|). \tag{6.25}$$

Obviously,

$$|F_{xx}\phi_t| \le O(1)(V_x^2 + |V_{xx}|)|\phi_t|. \tag{6.26}$$

So, we have from (6.20), (6.23)–(6.26) that

$$\begin{aligned} &(\frac{1}{2}\phi_t^2 + \frac{1}{2}(-p'(V))\phi_x^2)_t + \frac{1}{2}(p''(V)V_t\phi_x^2) \\ &\le -((p'(V)\phi)_x\phi_t)_x + R_x\phi_t \\ &+ O(1)(|V_x\phi\phi_{xt}| + |V_x||\phi||\phi_t| + |\phi||\phi_x||\phi_t|) + O(1)(V_x^2 + |V_{xx}|)|\phi_t|. \end{aligned} \tag{6.27}$$

We estimate $R_x\phi_t$ as follows. Since $\phi_t = \psi_x$, we have

$$\begin{aligned} R_x\phi_t &= (R_\phi\phi_x + R_\psi\psi_x + R_V V_x)\phi_t \\ &= R_\phi\phi_x\phi_t + R_\psi\phi_t^2 + R_V V_x\phi_t, \end{aligned} \tag{6.28}$$

where $R_\phi = \partial_\phi R(V,\phi,\psi), R_V = \partial_V R(V,\phi,\psi)$ and $R_\psi = \partial_\psi R(V,\phi,\psi)$. Since $R = S - C_f(f(V)+\psi)^2(V+\phi)$, it is easy to check that

$$|R_\phi\phi_x\phi_t| \le |C_f\,(f(V))^2\,\phi_x\phi_t| + O(1)|\psi||\phi_x||\phi_t| \tag{6.29}$$

and

$$|R_V V_x\phi_t| \le O(1)(|\phi| + |\psi|)|V_x||\phi_t|. \tag{6.30}$$

It follows from (6.28)–(6.30) that

$$R_x\phi_t \le R_\psi\phi_t^2 + |C_f\left(f(V)\right)^2\phi_x\phi_t| + O(1)|\psi||\phi_x||\phi_t| + O(1)(|\phi|+|\psi|)|V_x||\phi_t|. \tag{6.31}$$

This, together with (6.26), implies

$$\begin{aligned}
&\frac{1}{2}\left(\phi_t^2 + (-p'(V))\phi_x^2\right)_t + (-R_\psi)\phi_t^2 + \frac{1}{2}p''(V)V_t\phi_x^2\\
&\le -\left((p'(V)\phi)_x\,\phi_t\right)_x + |C_f\left(f(V)\right)^2\phi_x\phi_t|\\
&+ O(1)(|V_x\phi\phi_{xt}| + |V_x||\phi||\phi_t| + |\phi||\phi_x||\phi_t|) + O(1)(V_x^2 + |V_{xx}|)|\phi_t|\\
&+ O(1)|\psi||\phi_x||\phi_t| + O(1)|\psi||V_x||\phi_t|.
\end{aligned} \tag{6.32}$$

(6.17)+ $k\times$(6.32), for some positive constant k to be determined later, yields, after some rearrangements,

$$\begin{aligned}
&\left(A + \frac{k}{2}\phi_t^2 + \frac{k}{2}(-p'(V))\phi_x^2\right)_t + H_x + (-p'(V) + \frac{k}{2}p''(V)V_t)\phi_x^2\\
&+ ((-kR_\psi) - 1)\phi_t^2 - |kC_f\left(f(V)\right)^2\phi_x\phi_t| + \frac{1}{2}C_fV^{-2}|V_x|\phi^2\\
&\le R.H.S.,
\end{aligned} \tag{6.33}$$

where

$$H = (p'(V)\phi)_x\phi_t - B,$$

with B given by (6.19). $R.H.S.$ is given by

$$\begin{aligned}
R.H.S. &= -C_f f(V)V_x\phi\psi\\
&+ \frac{2C_f}{3}\phi^3\psi_t + O(1)(|V_x\phi\phi_{xt}| + |V_x||\phi||\phi_t| + |\phi||\phi_x||\phi_t|) + O(1)(V_x^2 + |V_{xx}|)|\phi_t|\\
&+ O(1)|\psi||\phi_x||\phi_t| + O(1)|\psi||V_x||\phi_t|\\
&+ O(1)(|V_x\phi\phi_x| + \phi^3|\phi_x| + V_x^2|\phi| + |V_{xx}\phi|) + O(1)\delta|V_x|(\phi^2+\psi^2).
\end{aligned} \tag{6.34}$$

We have, from (6.19),

$$|H| \le O(1)(|\phi| + |\phi_x| + |\phi_t|), \tag{6.35}$$

and, as given by (6.18),

$$A = \phi_t\phi + (C_f f(V)V + \beta)\phi^2,\ |\beta| < O(1)\delta,$$

and $\delta = \sup(|\phi| + |\psi|)$. We next choose a positive constant k such that

$$\phi_t\phi + C_f f(V)V\phi^2 + \frac{k}{2}\phi_t^2 > 0$$

and

$$((-kR_\psi) - 1)\phi_t^2 - |kC_f\left(f(V)\right)^2\phi_x\phi_t| + (-p'(V))\phi_x^2 > 0.$$

At first, since $v_l > V > v_r$, we have

$$\phi_t\phi + C_f f(V)V\phi^2 + \frac{k}{2}\phi_t^2 \ge C_f v_r f(v_r)\phi^2 + \phi_t\phi + \frac{k}{2}\phi_t^2. \tag{6.36}$$

Since $k > 0$ and $C_f v_r f(v_r) > 0$, in order to guarantee the quadratic form $C_f v_r f(v_r)\phi^2 + \phi_t\phi + \frac{k}{2}\phi_t^2$ to be positive definite, we only require $1 - 2kC_f v_r f(v_r) < 0$, i.e.,

$$k > \frac{1}{2C_f v_r f(v_r)} = \frac{1}{2\sqrt{C_f S v_r}}. \tag{6.37}$$

If (6.37) is satisfied, then there exists a positive constant c_1 such that

$$\phi_t\phi + C_f f(V)V\phi^2 + \frac{k}{2}\phi_t^2 \geq c_1(\phi^2 + \phi_t^2). \tag{6.38}$$

On the other hand, from the formula for R, we have

$$-R_\psi = 2C_f(f(V) + \psi)(\phi + V) \geq 2C_f v_r f(v_r) - O(1)\delta, \tag{6.39}$$

in view of the fact that $v_l > V > v_r$. By virtue of (6.39) and the fact that $p(v) = \frac{1}{2}v^{-2}$ and $f(v) = \sqrt{\frac{S}{C_f v}}$, we have

$$\begin{aligned}
&((-kR_\psi) - 1)\phi_t^2 - |kC_f\left(f(V)\right)^2\phi_x\phi_t| + (-p'(V))\phi_x^2 \\
&\geq (2kC_f v_r f(v_r) - 1)\phi_t^2 - kSV^{-1}|\phi_x\phi_t| + V^{-3}\phi_x^2 - O(1)\delta\phi_t^2 \\
&\geq (2kC_f v_r f(v_r) - 1)\phi_t^2 - kSv_r^{-1}|\phi_x\phi_t| + v_r^{-3}\phi_x^2 - O(1)\delta\phi_t^2 - O(1)|v_l - v_r|(\phi_x^2 + \phi_t^2).
\end{aligned} \tag{6.40}$$

In order to guarantee the quadratic form $(2kC_f v_r f(v_r) - 1)\phi_t^2 - kSv_r^{-1}|\phi_x\phi_t| + v_r^{-3}\phi_x^2$ to be positively definite, we require that

$$2kC_f v_r f(v_r) - 1 > 0 \tag{6.41}$$

and

$$(kSv_r^{-1})^2 - 4(2kC_f v_r f(v_r) - 1)v_r^{-3} < 0, \tag{6.42}$$

and note that (6.41) is the same as (6.37). We obtain from (3.51) and the fact $f(v) = \sqrt{\frac{S}{C_f v}}$ that

$$\frac{4\sqrt{SC_f} - 2\sqrt{S(4C_f - S)}}{S^2\sqrt{v_r}} < k < \frac{4\sqrt{SC_f} + 2\sqrt{S(4C_f - S)}}{S^2\sqrt{v_r}}. \tag{6.43}$$

Here the subcharacteristic condition $S < 4C_f$ is used. In view of (6.37) and (6.42), we choose k satisfying

$$\max\{\frac{4\sqrt{SC_f} - 2\sqrt{S(4C_f - S)}}{S^2\sqrt{v_r}}, \frac{1}{2\sqrt{C_f S v_r}}\} < k < \frac{4\sqrt{SC_f} + 2\sqrt{S(4C_f - S)}}{S^2\sqrt{v_r}}. \tag{6.44}$$

Due to the subcharacteristic condition $S < 4C_f$, we have $\frac{1}{2\sqrt{C_f S v_r}} < \frac{4\sqrt{SC_f}+2\sqrt{S(4C_f-S)}}{S^2\sqrt{v_r}}$, so the positive number satisfying (6.44) can actually be chosen. By this choice of k, we have

$$(2kC_f v_r f(v_r) - 1)\phi_t^2 - kSv_r^{-1}|\phi_x\phi_t| + v_r^{-3}\phi_x^2 \geq c_3(\phi_t^2 + \phi_x^2), \tag{6.45}$$

for some positive constant c_3. This, together with (6.40), implies

$$((-kR_\psi) - 1)\phi_t^2 - |kC_f\left(f(V)\right)^2\phi_x\phi_t| + (-p'(V))\phi_x^2 \geq c_4(\phi_t^2 + \phi_x^2), \tag{6.46}$$

for some positive constant c_4, if $\delta = \sup(|\phi| + |\psi|)$ and $|v_l - v_r|$ are small. Since $V_t = f'(V)V_x$, $f'(V) < 0$ and $V_x < 0$, we have $V_t > 0$. So the third term in (6.33) is positive. Moreover, since $f'(V) < 0$ and $V_x < 0$, we have $\frac{k}{2}C_f f(V)f'(V)V_x\phi^2 \geq c_5|V_x|\phi^2$ for some positive constant c_5. This, together with (6.46) and (6.33), yields

$$\begin{aligned}&\left(A + \frac{k}{2}\phi_t^2 + \frac{k}{2}(-p'(V))\phi_x^2\right)_t + H_x + c_6(\phi_t^2 + \phi_x^2) + c_7|V_x|\phi^2\\&\leq R.H.S.,\end{aligned} \tag{6.47}$$

for some positive constants c_6 and c_7, if $\delta = \sup(|\phi| + |\psi|)$ and $|v_l - v_r|$ are small, where $R.H.S.$ is given by (6.34). For $R.H.S.$, using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&R.H.S.\\&\leq \frac{1}{2}\left(c_6(\phi_t^2 + \phi_x^2) + c_7|V_x|\phi^2\right) + O(1)\left(|V_x|\psi^2 + \phi^6 + V_x^4 + V_{xx}^2 + \delta\phi_{xt}^2\right)\\&+ O(1)\delta(\phi_t^2 + \phi_x^2) + O(1)(V_x^2 + |V_{xx}|)|\phi| + O(1)|\phi^3\psi_t|.\end{aligned} \tag{6.48}$$

So, (6.47) and (6.48) imply

$$\begin{aligned}&\left(A + \frac{k}{2}\phi_t^2 + \frac{k}{2}(-p'(V))\phi_x^2\right)_t + H_x + (\phi_t^2 + \phi_x^2) + |V_x|\phi^2\\&\leq O(1)\left(|V_x|\psi^2 + \phi^6 + V_x^4 + V_{xx}^2 + \delta\phi_{xt}^2\right) + O(1)(V_x^2 + |V_{xx}|)|\phi|.\end{aligned} \tag{6.49}$$

Integrating (6.49) over the region $\{(x,s)|x_1(s) \leq x \leq x_2(s), t_0 \leq s \leq t\}$ and using Green's formula and (6.46), we obtain

$$\begin{aligned}&\int_{x_1(t)}^{x_2(t)} E(x,t)dx + \int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \left\{(\phi_t^2 + \phi_x^2) + |V_x|\phi^2\right\}(x,s)dxds\\&\leq \int_{x_1(t_0)}^{x_2(t_0)} E(x,t_0)dx + O(1)\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \left(|V_x|\psi^2 + \phi^6 + V_x^4 + V_{xx}^2 + \delta\phi_{xt}^2 + |\phi^3\psi_t|\right)dxds\\&+ O(1)\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} (V_x^2 + |V_{xx}|)|\phi|dxds\\&+ \int_{t_0}^{t}\left\{E(x_2(s)-,s)\dot{x}_2(s) - E(x_1(s)+,s)\dot{x}_1(s) + H(x_1(s)+,s) - H(x_2(s)-,s)\right\}ds,\end{aligned} \tag{6.50}$$

where

$$E = A + \frac{k}{2}\phi_t^2 + \frac{k}{2}(-p'(V))\phi_x^2. \tag{6.51}$$

It follows from (6.38) that

$$c_8(\phi^2+\phi_t^2+\phi_x^2)\le E\le c_9(\phi^2+\phi_t^2+\phi_x^2), \tag{6.52}$$

for some positive constants c_8 and c_9. Moreover, by Lemma 5.1 and (6.35), we have

$$\begin{aligned}&\int_{t_0}^t\{E(x_2(s)-,s)\dot x_2(s)-E(x_1(s)+,s)\dot x_1(s)+H(x_1(s)+,s)-H(x_2(s)-,s)\}\,ds\\&\le O(1)(|v_r-v_l|+|v_r-f(v_r)|).\end{aligned} \tag{6.53}$$

By Lemma 4.1, we have

$$\int_{t_0}^t\int_{x_1(s)}^{x_2(s)}(V_x^4+V_{xx}^2)dxds\le O(1)|v_r-v_l|. \tag{6.54}$$

We estimate $\int_{t_0}^t\int_{x_1(s)}^{x_2(s)}\phi^6(x,s)dxds$ as follows. At first, for $x_1(t)<x<x_2(t)$, we have

$$\phi^2(x,\ t)=\phi^2(x_1(t)-,\ t)+\int_{x_1(t)}^x 2\phi\phi_x(y,\ t)dy. \tag{6.55}$$

Hence

$$\begin{aligned}\phi^4(x,\ t)&=O(1)\phi^4(x_1(t)+,\ t)+\left(\int_{x_1(t)}^{x_2(t)}|\phi\phi_x|(x,t)dx\right)^2\\&\le O(1)\phi^4(x_1(t)+,\ t)+\int_{x_1(t)}^{x_2(t)}\phi^2(x,t)dx\int_{x_1(t)}^{x_2(t)}\phi_x^2(x,t)dx.\end{aligned}$$

Thus, by (5.10) and the fact that $x_2(s)-x_1(s)=O(1)s$, we have

$$\begin{aligned}&\int_{t_0}^t\int_{x_1(s)}^{x_2(s)}\phi^6dxds\\&\le O(1)\int_{t_0}^t\phi^4(x_1(s)+,\ s)(x_2(s)-x_1(s))ds\\&+O(1)\max_{t_0\le s\le t}\left(\int_{x_1(s)}^{x_2(s)}\phi^2(x,s)ds\right)^2\int_{t_0}^t\int_{x_1(s)}^{x_2(s)}\phi_x^2dxds\\&\le O(1)(|v_r-v_l|+|u_r-f(v_r)|)+O(1)\max_{t_0\le s\le t}\left(\int_{x_1(s)}^{x_2(s)}\phi^2(x,s)ds\right)^2\int_{t_0}^t\int_{x_1(s)}^{x_2(s)}\phi_x^2dxds.\end{aligned} \tag{6.56}$$

Next, we estimate the third term on the right-hand side of (6.50). At first, by (6.55), we have

$$\max_{x_1(t)\le x\le x_2(t)}|\phi|\le|\phi(x_1(t)-,\ t)|+(\int_{x_1(t)}^{x_2(t)}\phi^2dx)^{1/4}(\int_{x_1(t)}^{x_2(t)}\phi_x^2dx)^{1/4}. \tag{6.57}$$

Let I_3 be the third term on the right-hand side of (6.50). It follows from (5.1), (5.10) and (6.57) that

$$\begin{aligned}
I_3 &\le O(1)\int_{t_0}^{t}\left(\max_{x_1(s)\le x\le x_2(s)}|\phi|(x,s)\int_{x_1(s)}^{x_2(s)}(V_x^2+|V_{xx}|)dx\right)ds\\
&\le O(1)\int_{t_0}^{t}|\phi(x_1(s)+,s)|ds\\
&+\int_{t_0}^{t}\left(\int_{x_1(s)}^{x_2(s)}\phi^2dx)^{1/4}(\int_{x_1(s)}^{x_2(s)}\phi_x^2dx)^{1/4}\int_{x_1(s)}^{x_2(s)}(V_x^2+|V_{xx}|)dx\right)ds\\
&\le O(1)(|v_r-v_l|+|u_r-f(v_r)|)+\frac{1}{2}\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)}\phi_x^2dxds\\
&+O(1)\left(\max_{t_0\le s\le t}\int_{x_1(s)}^{x_2(s)}\phi^2(x,s)dx\right)^{1/3}\int_{t_0}^{t}\left(\int_{x_1(s)}^{x_2(s)}(V_x^2+|V_{xx}|)dx\right)^{4/3}ds\\
&\le \frac{1}{2}\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)}\phi_x^2dxds+O(1)(|v_r-v_l|+|u_r-f(v_r)|). \qquad (6.58)
\end{aligned}$$

So, it follows from (6.50)–(6.58) that

$$\begin{aligned}
&\int_{x_1(t)}^{x_2(t)}(\phi^2+\phi_t^2+\phi_x^2)(x,t)dx+\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)}\left\{(\phi_t^2+\phi_x^2)+|V_x|\phi^2\right\}(x,s)dxds\\
&\le O(1)\int_{x_1(t_0)}^{x_2(t_0)}(\phi^2+\phi_t^2+\phi_x^2)(x,t_0)dx+O(1)\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)}(|V_x|\psi^2+\delta\phi_{xt}^2+|\phi^3\psi_t|)dxds\\
&+O(1)(|v_r-v_l|+|u_r-f(v_r)|), \qquad (6.59)
\end{aligned}$$

if $|v_r-v_l|+|u_r-f(v_r)|$ and $\max_{t_0\le s\le T}\int_{x_1(s)}^{x_2(s)}\phi^2(x,s)dx$ are small. We estimate

$$\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)}|V_x|\psi^2dxds$$

as follows. Multiplying (5.9) by $-V_x\psi$, we get

$$\begin{aligned}
&(-\frac{1}{2}V_x\psi^2)_t+\frac{1}{2}V_{xt}\psi^2-p''(V)V_x^2\phi\psi\\
&=-RV_x\psi+QV_x\psi+F(V)_xV_x\psi. \qquad (6.60)
\end{aligned}$$

Since $C_f(f(V))^2V=S$, we have

$$R=-C_f(2f(V)\psi+\psi^2)(\phi+V). \qquad (6.61)$$

So, we see that

$$-RV_x\phi\le 2C_ff(V)VV_x\psi^2+O(1)|V_x|\psi^2|\phi|. \qquad (6.62)$$

From (6.7), we have

$$|QV_x\psi|\le O(1)|V_x||\psi|\phi^2. \qquad (6.63)$$

Obviously, it then follows that

$$|F(V)_xV_x\psi|\le O(1)|V_x|^2|\psi|. \qquad (6.64)$$

Since $V_x \le 0$, (6.60)–(6.64) and Cauchy-Schwarz inequality imply that

$$\begin{aligned}&(\frac{1}{2}|V_x\psi^2|)_t + |V_x|\psi^2\\&\le O(1)|v_r - v_l|\phi_x^2 + O(1)\delta|V_x|\phi^2 + O(1)(|V_x|^2 + |V_{xt}|)|\psi|.\end{aligned} \tag{6.65}$$

Here, we use the fact that $|V_x| \le |v_r - v_l|$ and $\delta = \sup(|\phi| + |\psi|)$. Integrating (6.65) over the region $(x,t) : x_1(s) \le x \le x_2(s), t_0 \le s \le t$, and using the (5.10), we obtain

$$\begin{aligned}&\int_{x_1(t)}^{x_2(t)} \frac{1}{2}|V_x\psi^2|(x,t)dx + \int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} |V_x|\psi^2 dxds\\&\le O(1)\int_{x_1(t_0)}^{x_2(t_0)} \frac{1}{2}|V_x\psi^2|(x,t_0)dx + O(1)\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \left(|v_r - v_l|\phi_x^2 + O(1)\delta|V_x|\phi^2\right) dxds\\&+ O(1)\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} (|V_x|^2 + |V_{xt}|)|\psi|dxds + O(1)(|v_r - v_l| + |u_r - f(v_r)|).\end{aligned} \tag{6.66}$$

Furthermore, (6.59) and (6.66) imply that

$$\begin{aligned}&\int_{x_1(t)}^{x_2(t)} (\phi^2 + \phi_t^2 + \phi_x^2)(x,t + |V_x|\psi^2)dx\\&+ \int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \left\{\phi_t^2 + \phi_x^2 + |V_x|(\phi^2 + \psi^2)\right\}(x,s)dxds\\&\le O(1)\int_{x_1(t_0)}^{x_2(t_0)} (\phi^2 + \phi_t^2 + \phi_x^2 + |V_x|\psi^2)(x,t_0)dx + O(1)\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} (\delta\phi_{xt}^2 + |\phi^3\psi_t|)dxds\\&+ O(1)(|v_r - v_l| + |u_r - f(v_r)|) + O(1)\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} (|V_x|^2 + |V_{xt}|)|\psi|dxds,\end{aligned} \tag{6.67}$$

if $|v_r - v_l| + |u_r - f(v_r)|$ and $\max_{t_0 \le s \le T}\int_{x_1(s)}^{x_2(s)} \phi^2(x,s)dx$ are small. We use I_4 to denote the last term in (6.66). Similar to (6.58), we have

$$\begin{aligned}I_4 &\le \frac{1}{2}\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \psi_x^2 dxds + O(1)(|v_r - v_l| + |u_r - f(v_r)|)\\&= \frac{1}{2}\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \phi_t^2 dxds + O(1)(|v_r - v_l| + |u_r - f(v_r)|), \text{ since } \phi_t = \psi_x,\end{aligned} \tag{6.68}$$

if $\max_{t_0 \le s \le T}\int_{x_1(s)}^{x_2(s)} \psi^2(x,s)dx$ is small. So, by (6.67) and (6.68),

$$\begin{aligned}&\int_{x_1(t)}^{x_2(t)} (\phi^2 + \phi_t^2 + \phi_x^2 + |V_x|\psi^2)(x,t)dx\\&+ \int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \left\{\phi_t^2 + \phi_x^2 + |V_x|(\phi^2 + \psi^2)\right\}(x,s)dxds\\&\le O(1)\int_{x_1(t_0)}^{x_2(t_0)} (\phi^2 + \phi_t^2 + \phi_x^2 + |V_x|\psi^2)(x,t_0)dx + O(1)\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} (\delta\phi_{xt}^2 + |\phi^3\psi_t|)dxds\\&+ O(1)(|v_r - v_l| + |u_r - f(v_r)|),\end{aligned} \tag{6.69}$$

if $|v_r - v_l| + |u_r - f(v_r)|$ and $\max_{t_0 \le s \le T} \int_{x_1(s)}^{x_2(s)} (\phi^2 + \psi^2)(x,s)dx + \delta$ are small. We turn to the estimate of $\int_{t_0}^t \int_{x_1(s)}^{x_2(s)} |\phi^3 \psi_t| dxds$. Differentiate (5.7) with respect to t and multiply by ψ_t to get

$$\frac{1}{2}(\psi_t^2)_t = R_t \psi_t - ((p'(V)\phi)_{xt} + Q_t + F(V)_{xt})\,\psi_t. \tag{6.70}$$

Similar to (6.11), we have

$$\begin{aligned} R_t = &-C_f(f(V)+\psi)\,(2f'(V)\phi + \psi)\,V_t - 2C_f(f(V)+\psi)(\phi + V)\psi_t \\ &- C_f(f(V)+\psi)^2 \phi_t. \end{aligned} \tag{6.71}$$

Since $V > v_r$, thus $f(V)V > f(v_r)v_r > 0$. Moreover, $|V_t| = O(1)|V_x|$. Therefore, (6.71) implies

$$R_t\psi_t \le -2C_f f(v_r)v_r\psi_t^2 + O(1)(\delta\psi_t^2 + |\phi_t||\psi_t|) + O(1)|V_x|(|\phi| + |\psi|)|\psi_t|. \tag{6.72}$$

Moreover, it is easy to verify, with the help of (4.1), (5.8), (6.7) and the Cauchy-Schwarz inequality, that

$$\begin{aligned} &|\,((p'(V)\phi)_{xt} + Q_t + F(V)_{xt})\,\psi_t| \\ &\le O(1)(V_x^4 + V_{xx}^2) + O(1)(\phi_t^2 + \phi_{xt}^2) + C_f f(v_r)v_r\psi_t^2, \end{aligned} \tag{6.73}$$

if $|\phi| + |\psi|$ is small. Therefore, (6.70), (6.72) and (6.73) imply, by using the Cauchy-Schwarz inequality,

$$\frac{1}{2}(\psi_t^2)_t + \psi_t^2 \le O(1)(V_x^4 + V_{xx}^2 + \phi_t^2 + \phi_{xt}^2) + O(1)|V_x|(|\phi|^2 + |\psi|^2), \tag{6.74}$$

if δ is small. Integrating (6.74) over the region $\{(x,s)|x_1(s) \le x \le x_2(s),\ t_0 \le s \le t\}$, using (6.54), we obtain

$$\begin{aligned} &\int_{x_1(t)}^{x_2(t)} \psi_t^2(x,t)dx + \int_{t_0}^t \int_{x_1(s)}^{x_2(s)} \psi_t^2(x,s)dxds \\ &\le O(1)\int_{x_1(t_0)}^{x_2(t_0)} (\psi_t^2)(x,t_0)dx + O(1)\int_{t_0}^t \int_{x_1(s)}^{x_2(s)} (\phi_t^2 + \phi_{xt}^2)dxds \\ &+ \int_{t_0}^t \int_{x_1(s)}^{x_2(s)} |V_x|(|\phi|^2 + |\psi|^2)dxds + O(1)(|v_r - v_l|). \end{aligned} \tag{6.75}$$

For the term $\int_{t_0}^t \int_{x_1(s)}^{x_2(s)} |\phi^3\psi_t| dxds$ in (6.69), we can use the Cauchy-Schwarz inequality to obtain

$$\int_{t_0}^t \int_{x_1(s)}^{x_2(s)} |\phi^3\psi_t| dxds \le O(1)\frac{1}{\epsilon}\int_{t_0}^t \int_{x_1(s)}^{x_2(s)} \phi^6 dxds + \epsilon \int_{t_0}^t \int_{x_1(s)}^{x_2(s)} \psi_t^2, \tag{6.76}$$

for any $\epsilon > 0$. $\int_{t_0}^t \int_{x_1(s)}^{x_2(s)} \phi^6 dxds$ can be estimated by (6.56).

Next, we turn to the higher order estimates. This is similar to the above estimates and the method used in [21], the difference being that we will need the boundary estimates (5.10)–(5.12). So we only sketch the proof as follows.

Here and in the following, we set

$$\delta_2 = \max_{t_0\le t\le T, x_1(t)\le x\le x_2(t)} \left(\sum_{j=1}^{2}(|D^j\phi| + |D^j\psi|)\right), \tag{6.77}$$

and by (6.3), we have

$$\delta_2 \le O(1)(|v_r - v_l| + |u_r - f(v_r)|). \tag{6.78}$$

At first, we differentiate (5.9) with respect to t and multiply by ϕ_{tx}. Then we integrate the resulting equation to obtain

$$\begin{aligned}
&\int_{x_1(t)}^{x_2(t)} \phi_{xt}^2(x,t)dx + \int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \phi_{xt}^2(x,s)dxds \\
&\le O(1)\int_{x_1(t_0)}^{x_2(t_0)} (\phi_{xt}^2 + \phi_{xx}^2)(x,t_0)dx \\
&+ O(1)\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \delta_2\left(\phi_{xxt}^2 + \phi_{xtt}^2 + |\nabla\phi|^2 + |\nabla\psi|^2 + |D^2(\phi,\psi)|^2 + |V_x|(\phi^2+\psi^2)\right)dxds \\
&+ O(1)(|v_r - v_l| + |u_r - f(v_r)|).
\end{aligned} \tag{6.79}$$

Therefore, by (6.56), (6.69), (6.76) and (6.79), we can choose ϵ suitably in (6.76) such that

$$\begin{aligned}
&\int_{x_1(t)}^{x_2(t)} (\phi^2 + \phi_t^2 + \phi_x^2) + |V_x|\psi^2 + \phi_{xt}^2(x,t)dx \\
&+ \int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \left\{\phi_t^2 + \phi_x^2 + |V_x|(\phi^2+\psi^2) + \phi_{xt}^2\right\}(x,s)dxds \\
&\le O(1)\int_{x_1(t_0)}^{x_2(t_0)} (\phi^2 + \phi_t^2 + \phi_x^2 + |V_x|\psi^2 + \psi_t^2 + \phi_{xt}^2 + \phi_{xx}^2)(x,t_0)dx \\
&+ O(1)\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \delta_2\left(\phi_{xxt}^2 + \phi_{xtt}^2 + |D^2(\phi,\psi)|^2\right)dxds \\
&+ O(1)(|v_r - v_l| + |u_r - f(v_r)|),
\end{aligned} \tag{6.80}$$

if $|v_r - v_l| + |u_r - f(v_r)|$ and $\max_{t_0\le s\le T}\int_{x_1(s)}^{x_2(s)}(\phi^2+\psi^2)(x,s)dx + \delta$ are small. We differentiate (5.9) with respect to x and multiply by ϕ_x, differentiate (5.9) with respect to t and multiply by ϕ_{tt}, and differentiate (5.9) with respect to x twice and multiply by respectively ϕ_{xx} and ϕ_{xxt}. We integrate these resulting equations respectively to get

$$\begin{aligned}
&\int_{x_1(t)}^{x_2(t)} \phi_x^2(x,t)dx + \int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \phi_{xx}^2(x,s)dxds \\
&\le O(1)\int_{x_1(t_0)}^{x_2(t_0)} \phi_x^2(x,t_0)dx \\
&+ O(1)\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} (\phi_{xt}^2 + \delta_2(|\nabla\phi|^2 + |\nabla\psi|^2) + |V_x|(\phi^2+\psi^2) + |V_x|^3)dxds \\
&+ O(1)(|v_r - v_l| + |u_r - f(v_r)|),
\end{aligned} \tag{6.81}$$

$$\begin{aligned}
&\int_{x_1(t)}^{x_2(t)} \phi_{tt}^2(x,t)dx + \int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \phi_{tt}^2(x,s)dxds \\
&\le O(1)\int_{x_1(t_0)}^{x_2(t_0)} (\phi_{tt}^2+\phi_{xt}^2)(x,t_0)dx + O(1)\int_{x_1(t)}^{x_2(t)} \phi_{xt}^2(x,t)dx \\
&+ O(1)\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \delta_2(\phi_{xtt}^2+\phi_{ttt}^2) + |\nabla\phi|^2 + |\nabla\psi|^2 + |D^2(\phi,\psi)|^2 + |V_x|(\phi^2+\psi^2)dxds \\
&+ O(1)(|v_r - v_l| + |u_r - f(v_r)|),
\end{aligned} \tag{6.82}$$

$$\begin{aligned}
&\int_{x_1(t)}^{x_2(t)} \phi_{xx}^2(x,t)dx + \int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \phi_{xxx}^2(x,s)dxds \\
&\le O(1)\int_{x_1(t_0)}^{x_2(t_0)} (\phi_{xx}^2+\phi_{xxt}^2)(x,t_0)dx + O(1)\int_{x_1(t)}^{x_2(t)} \phi_{xxt}^2(x,t)dx \\
&+ O(1)\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \delta_2\left(\psi_{xxx}^2 + |\nabla\phi|^2 + |\nabla\psi|^2 + |D^2(\phi,\psi)|^2 + |V_x|(\phi^2+\psi^2) + |V_x|^3\right)dxds \\
&+ O(1)(|v_r - v_l| + |u_r - f(v_r)|).
\end{aligned} \tag{6.83}$$

Differentiating (5.9) with respect to t and then with respect to x, multiplying by ϕ_{ttx}, and integrating the resulting equation, we obtain

$$\begin{aligned}
&\int_{x_1(t)}^{x_2(t)} (\phi_{xxx}^2+\phi_{xxt}^2)(x,t)dx + \int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \phi_{xxt}^2(x,s)dxds \\
&\le O(1)\int_{x_1(t_0)}^{x_2(t_0)} \phi_{xxx}^2(x,t_0)dx + O(1)\int_{x_1(t)}^{x_2(t)} \phi_{xxt}^2(x,t)dx \\
&+ O(1)\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \delta_2\left(|\nabla\phi|^2 + |\nabla\psi|^2 + |D^2(\phi,\psi)|^2 + |D^3(\phi,\psi)|^2\right. \\
&\left.+|V_x|(\phi^2+\psi^2) + |V_x|^3\right)dxds \\
&+ O(1)(|v_r - v_l| + |u_r - f(v_r)|).
\end{aligned} \tag{6.84}$$

Differentiating (5.9) with respect to t twice, multiplying by ϕ_{ttt}, and integrating the resulting equation, we get

$$\begin{aligned}
&\int_{x_1(t)}^{x_2(t)} (\phi_{xtt}^2+\phi_{xxt}^2)(x,t)dx + \int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \phi_{xtt}^2(x,s)dxds \\
&\le O(1)\int_{x_1(t_0)}^{x_2(t_0)} \phi_{xtt}^2(x,t_0)dx + O(1)\int_{x_1(t)}^{x_2(t)} \phi_{xxt}^2(x,t)dx \\
&+ O(1)\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \delta_2\left(|\nabla\phi|^2 + |\nabla\psi|^2 + |D^2(\phi,\psi)|^2 + |D^3(\phi,\psi)|^2\right. \\
&\left.+|V_x|(\phi^2+\psi^2) + |V_x|^3\right)dxds \\
&+ O(1)(|v_r - v_l| + |u_r - f(v_r)|).
\end{aligned} \tag{6.85}$$

When we get these two estimates, the boundary terms are bounded by

$$O(1)\int_{t_0}^{t}(\sum_{j=0}^{1}|D^j(\phi,\psi)||D^2(\phi,\psi)(x_1(s)+,s)+\sum_{j=0}^{1}|D^j(\phi,\psi)||D^2(\phi,\psi)(x_2(s)-,s))ds.$$

By (5.10)–(5.12), this is bounded by $O(1)(|v_r-v_l|+|u_r-f(v_r)|)$. Following a similar argument as in [21], using Lemma 5.1, one can get the estimate on ψ,

$$\begin{aligned}
&\int_{x_1(t)}^{x_2(t)}\sum_{j=0}^{3}|D^j\psi|^2(x,t)dx+\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)}\sum_{j=0}^{3}|D^j\psi|^2(x,s)dxds\\
&\le O(1)\int_{x_1(t_0)}^{x_2(t_0)}\sum_{j=0}^{3}|D^j\psi|^2(x,t_0)dx+O(1)\int_{x_1(t)}^{x_2(t)}\sum_{j=1}^{3}|D^j\phi|^2(x,t)dx\\
&+O(1)\int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)}\left(\sum_{j=1}^{3}|D^j\phi|^2+|V_x|\phi^2\right)dxds\\
&+O(1)(|v_r-v_l|+|u_r-f(v_r)|). \qquad (6.86)
\end{aligned}$$

Again, using a similar argument as in [21], and (6.80)–(6.86), Lemma 6.1 follows. Theorem 2.1 follows from Lemma 6.1 immediately by the local existence result of [20] and standard continuation argument (cf. [21]).

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