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THE DIRECT MEG PROBLEM IN THE PRESENCE OF AN ELLIPSOIDAL SHELL INHOMOGENEITY

By

GEORGE DASSIOS (*Division of Applied Mathematics, Department of Chemical Engineering, University of Patras, and ICEHT/FORTH*)

AND

FOTINI KARIOTOU (*Division of Applied Mathematics, Department of Chemical Engineering, University of Patras, and Hellenic Open University*)

Abstract. The forward problem of Magnetoencephalography for an ellipsoidal inhomogeneous shell-model of the brain is considered. The inhomogeneity enters through a confocal ellipsoidal shell exhibiting different conductivity than the one of the brain tissue. It is shown that, as far as the leading quadrupolic moment of the exterior magnetic field is concerned, the complicated expression associated with the field itself is the same as in the homogeneous case, while the effect of the shell is focused on the form of the generalized dipole moment. In contrast to the spherical case, where no shell inhomogeneities are "readable" outside the skull, the ellipsoidal shells establish their existence on the exterior magnetic induction field in a way that depends not only on the geometry but also on the conductivity of the shell. The degenerated spherical results are fully recovered.

1. Introduction. The mathematical theory of Electroencephalography (EEG) and Magnetoencephalography (MEG) was founded in the late 60s, mainly on the basis of the works of Geselowitz [7, 8]. Since then, many efforts have been made to produce analytic solutions for the related direct problem where the field generated by a given source is sought. We mention the works of Ilmoniemi, Hämäläinen and Knuutila [10], Sarvas [15] and Fokas, Kurylev and Marinakis [5] for the spherical brain model, the works of Cuffin and Cohen [1] and de Munck [6] for the spheroidal brain model and the work of Nolte, Fieseler and Curio [14] for perturbative models of the brain. But, as anatomy indicates, the actual geometry of the human brain is best approximated by a triaxial ellipsoid [16], a geometrical shape far more complicated than the sphere or even the spheroid. Intense efforts towards a complete analytic solution for EEG and MEG problems in ellipsoidal geometry led to results included in [3, 4, 11, 12]. The present work aims in obtaining an analytic expression of the leading quadrupolic term for the exterior magnetic field in the

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presence of an ellipsoidal shell with different conductivity from the one characterizing the brain tissue.

The result is amazingly similar to the corresponding result of the single ellipsoidal model [3]. The only effect that the inhomogeneous shell has on the form of the magnetic field, is attributed to the expression of the generalized dipole moment appearing in the quadrupole term. As we show, this generalized dipole moment is a complicated expression of the conductivity profiles in the two compartments as well as of the semiaxes of the brain and its covering shell.

Everything connected to the geometry of the problem is expressed in terms of canonical dyadics. It is important to realize that the form of the generalized dipole moment dictates the orientation of the silent sources which are drastically effected by the conductive ellipsoidal shell. It is also shown that the part of the solution associated with the observation point, where the magnetic field is measured, is not influenced by the existence of the shell.

Section 2 involves the mathematical statement of the direct MEG problem for an ellipsoidal brain which is excited by a current dipole in its interior and it is surrounded by a confocal ellipsoidal shell of different conductivity. In order to solve this problem we need to solve first the related EEG problem, a task that has been fulfilled in [11]. The main result of the present work is exposed in Section 3 where the exterior magnetic field is evaluated through a series of appropriate manipulations with elliptic integrals. In Section 4 we reduce the ellipsoidal results to the corresponding spherical ones as well as to the case of the single component homogeneous brain model.

2. Statement of the problem. First we define the geometry of the problem. The brain is modeled with the triaxial ellipsoid

$$\frac{x_1^2}{b_1^2}+\frac{x_2^2}{b_2^2}+\frac{x_3^2}{b_3^2}\leq 1 \tag{2.1}$$

which has the constant conductivity σ_b. We denote this region by V_b and let $S_b=\partial V_b$. The confocal ellipsoidal shell V_α, bounded by the ellipsoid S_b and the ellipsoid S_α given by

$$\frac{x_1^2}{\alpha_1^2}+\frac{x_2^2}{\alpha_2^2}+\frac{x_3^2}{\alpha_3^2}=1, \tag{2.2}$$

surrounds the brain and is characterized by the conductivity $\sigma_\alpha\neq\sigma_b$. The space V_α models either the cerebrospinal fluid, the bone, the skin, or an average of all of them. The semiaxes are ordered as follows:

$$\left.\begin{array}{l}\alpha_3<\alpha_2<\alpha_1,\\ b_3<b_2<b_1<\alpha_1\end{array}\right\}. \tag{2.3}$$

In terms of the ellipsoidal coordinates (ρ,μ,ν) [9], which are connected to the Cartesian coordinates (x_1, x_2, x_3) by

$$\left.\begin{aligned} h_2h_3x_1 &= \rho\mu\nu, \\ h_1h_3x_2 &= \sqrt{\rho^2-h_3^2}\sqrt{\mu^2-h_3^2}\sqrt{h_3^2-\nu^2}, \\ h_1h_2x_3 &= \sqrt{\rho^2-h_2^2}\sqrt{h_2^2-\mu^2}\sqrt{h_2^2-\nu^2}, \end{aligned}\right\} \tag{2.4}$$

where the semifocal distances h_1, h_2, h_3 are given by

$$\left.\begin{aligned} h_1^2 &= \alpha_2^2-\alpha_3^2 = b_2^2-b_3^2, \\ h_2^2 &= \alpha_1^2-\alpha_3^2 = b_1^2-b_3^2, \\ h_3^2 &= \alpha_1^2-\alpha_2^2 = b_1^2-b_2^2, \end{aligned}\right\} \tag{2.5}$$

the surface S_b corresponds to $\rho = b_1$, the surface S_α to $\rho = \alpha_1$, the core-domain V_b to $\rho \in [h_2, b_1)$, and the shell-domain V_α to $\rho \in (b_1, \alpha_1)$. The exterior domain V is then described by $\rho > \alpha_1$ and has zero conductivity. The magnetic permeability μ_0 is taken to be constant in all of $\mathbb{R}^3$.

At the point $\mathbf{r}_0 = (\rho_0, \mu_0, \nu_0)$, being within V_b, an equivalent dipole current source with dipole moment $\mathbf{Q}$ models the local electrochemical activity of the neurons.

Within the framework of quasistatic electromagnetic theory [13], which is generally accepted for MEG problems [15], this current

$$\mathbf{J}(\mathbf{r}) = \mathbf{Q}\delta(\mathbf{r}-\mathbf{r}_0) \tag{2.6}$$

gives rise to an electric potential u and to a magnetic induction field $\mathbf{B}$.

Let us denote by u_b, u_α and u the electric field in V_b, V_α and V, respectively. Then the magnetic field $\mathbf{B}$, generated in V, is a consequence of the primary current $\mathbf{J}$ and the induction currents $-\sigma_\alpha\nabla u_\alpha$ and $-\sigma_b\nabla u_b$ which are supported in V_α and V_b, respectively.

Hence, Ampere's law [13]

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\limits_G \mathbf{J}^t(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}dv(\mathbf{r}') \tag{2.7}$$

where G stands for the support of the total current $\mathbf{J}^t$, implies the representation

$$\begin{aligned} \mathbf{B}(\mathbf{r}) =& \frac{\mu_0}{4\pi}\mathbf{Q}\times\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} - \frac{\mu_0\sigma_\alpha}{4\pi}\int\limits_{V_\alpha}(\nabla_{\mathbf{r}'}u_\alpha(\mathbf{r}'))\times\left(\nabla_{\mathbf{r}'}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dv(\mathbf{r}') \\ &- \frac{\mu_0\sigma_b}{4\pi}\int\limits_{V_b}(\nabla_{\mathbf{r}'}u_b(\mathbf{r}'))\times\left(\nabla_{\mathbf{r}'}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dv(\mathbf{r}') \end{aligned} \tag{2.8}$$

or, in view of Geselowitz formula [2],

$$\begin{aligned} \mathbf{B}(\mathbf{r}) =& \frac{\mu_0}{4\pi}\mathbf{Q}\times\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} - \frac{\mu_0\sigma_\alpha}{4\pi}\int\limits_{S_\alpha}u_\alpha(\mathbf{r}')\hat{\boldsymbol{\rho}}'\times\left(\nabla_{\mathbf{r}'}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)ds(\mathbf{r}') \\ &+ \frac{\mu_0}{4\pi}\int\limits_{S_b}(\sigma_\alpha u_\alpha(\mathbf{r}')-\sigma_b u_b(\mathbf{r}'))\hat{\boldsymbol{\rho}}'\times\left(\nabla_{\mathbf{r}'}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)ds(\mathbf{r}') \end{aligned} \tag{2.9}$$

where $\hat{\boldsymbol{\rho}}'$ denotes the outward unit normal on the corresponding ellipsoid.

Representation (2.9) recasts the contribution of the conductive domains to $\mathbf{B}$ in the form of surface distributions of dipoles oriented normally to the interfaces S_α and S_b. The density of these dipole distributions is proportional to $-\sigma_\alpha u_\alpha$ on S_α and to $\sigma_\alpha u_\alpha - \sigma_b u_b$ on S_b. Therefore, our first task is to solve the following boundary value problem, for the electric potential

$$\Delta u(\mathbf{r}) = 0, \quad \mathbf{r} \in V, \tag{2.10}$$

$$\Delta u_\alpha(\mathbf{r}) = 0, \quad \mathbf{r} \in V_\alpha, \tag{2.11}$$

$$\Delta u_b(\mathbf{r}) = \frac{1}{\sigma_b} \nabla \cdot \mathbf{J}(\mathbf{r}), \quad \mathbf{r} \in V_b, \tag{2.12}$$

where $\mathbf{J}(\mathbf{r})$ is given by (2.6).

Continuity conditions demand that the fields u, u_α, u_b are connected through the surface conditions

$$u_\alpha(\mathbf{r}) = u(\mathbf{r}), \quad \mathbf{r} \in S_\alpha, \tag{2.13}$$

$$\partial_n u_\alpha(\mathbf{r}) = 0, \quad \mathbf{r} \in S_\alpha, \tag{2.14}$$

$$u_\alpha(\mathbf{r}) = u_b(\mathbf{r}), \quad \mathbf{r} \in S_b, \tag{2.15}$$

$$\sigma_\alpha \partial_n u_\alpha(\mathbf{r}) = \sigma_b \partial_n u_b(\mathbf{r}), \quad \mathbf{r} \in S_b, \tag{2.16}$$

where ∂_n stands for the outward normal differentiation on the corresponding surface.

Furthermore, the exterior electric field has to satisfy the asymptotic condition

$$u(\mathbf{r}) = O\left(\frac{1}{r}\right), \quad r \to \infty. \tag{2.17}$$

The solution of the problem (2.10)-(2.17), which has been obtained in [11], assumes the following appropriate-for-our-purpose form:

$$\begin{aligned} u(\mathbf{r}) =& d_0^1 \frac{I_0^1(\rho)}{I_0^1(\alpha_1)} + \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \frac{\mathbf{Q} \cdot \nabla \mathbb{E}_n^m(\mathbf{r}_0)}{\gamma_n^m C_n^m} \\ & \times \frac{1}{E_n^m(\alpha_1) E_n^{m\prime}(\alpha_1) \alpha_2 \alpha_3} \frac{I_n^m(\rho)}{I_n^m(\alpha_1)} \mathbb{E}_n^m(\rho, \mu, \nu) \end{aligned} \tag{2.18}$$

for $\mathbf{r} \in V$,

$$\begin{aligned} u_\alpha(\mathbf{r}) =& d_0^1 + \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \frac{\mathbf{Q} \cdot \nabla \mathbb{E}_n^m(\mathbf{r}_0)}{\gamma_n^m C_n^m} \\ & \times \left(I_n^m(\rho) - I_n^m(\alpha_1) + \frac{1}{E_n^m(\alpha_1) E_n^{m\prime}(\alpha_1) \alpha_2 \alpha_3} \right) \mathbb{E}_n^m(\rho, \mu, \nu) \end{aligned} \tag{2.19}$$

for $\mathbf{r} \in V_\alpha$, and

$$\begin{aligned} u_b(\mathbf{r}) =& d_0^1 + \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \frac{\mathbf{Q} \cdot \nabla \mathbb{E}_n^m(\mathbf{r}_0)}{\gamma_n^m C_n^m} \left[\left(\frac{C_n^m}{\sigma_b} (I_n^m(b_1) - I_n^m(\rho)) \right.\right. \\ & \left.\left. + I_n^m(b_1) - I_n^m(\alpha_1) + \frac{1}{E_n^m(\alpha_1) E_n^{m\prime}(\alpha_1) \alpha_2 \alpha_3} \right) \right] \mathbb{E}_n^m(\rho, \mu, \nu) \end{aligned} \tag{2.20}$$

for $\mathbf{r} \in V_b$, where for each $n = 1, 2, ...$ and $m = 1, 2, ..., 2n+1$,

$$\begin{aligned} C_n^m =& \sigma_\alpha + (\sigma_b - \sigma_\alpha) \\ & \times \left[(I_n^m(b_1) - I_n^m(\alpha_1)) E_n^m(b_1) E_n^{m'}(b_1) b_2 b_3 + \frac{E_n^m(b_1) E_n^{m'}(b_1) b_2 b_3}{E_n^m(\alpha_1) E_n^{m'}(\alpha_1) \alpha_2 \alpha_3} \right] \end{aligned} \tag{2.21}$$

and d_0^1 is an arbitrary constant (as any potential function owes to have).

The rest of the variables appearing in (2.18)-(2.21) are connected to the ellipsoidal harmonics

$$\mathbb{E}_n^m(\rho, \mu, \nu) = E_n^m(\rho) E_n^m(\mu) E_n^m(\nu), \tag{2.22}$$

$$\mathbb{F}_n^m(\rho, \mu, \nu) = F_n^m(\rho) E_n^m(\mu) E_n^m(\nu) = (2n+1) I_n^m(\rho) \mathbb{E}_n^m(\rho, \mu, \nu) \tag{2.23}$$

given in terms of the Lamé functions E_n^m and F_n^m and the elliptic integrals $I_n^m(\rho)$ which are defined in the Appendix.

Finally, γ_n^m denote the ellipsoidal normalization constants

$$\gamma_n^m = \int\limits_{\rho=\rho_0} [E_n^m(\mu) E_n^m(\nu)]^2 l_{\rho_0}(\mu, \nu) ds(\mu, \nu) \tag{2.24}$$

for each $n = 1, 2, ...$ and $m = 1, 2, ..., 2n+1$, where $l_{\rho_0}(\mu, \nu)$ is the ellipsoidal weighting function defined by

$$l_{\rho_0}(\mu, \nu) = [(\rho_0^2 - \mu^2)(\rho_0^2 - \nu^2)]^{-\frac{1}{2}}. \tag{2.25}$$

3. The magnetic field. Our main task in this section is to find analytic expressions for the integrals appearing in (2.9). This will be achieved via appropriate transformations, the orthogonality of the surface ellipsoidal harmonics and the evaluation of the normalization integrals. We start with the evaluation of the integral

$$\mathbf{I}_\alpha(\mathbf{r}) = \int\limits_{S_\alpha} u_\alpha(\mathbf{r}') \hat{\boldsymbol{\rho}}' \times \nabla_{\mathbf{r}'} \frac{1}{|\mathbf{r} - \mathbf{r}'|} ds(\mathbf{r}'). \tag{3.1}$$

Detailed analysis of the part $\hat{\boldsymbol{\rho}}' \times \nabla_{\mathbf{r}'} |\mathbf{r} - \mathbf{r}'|^{-1}$ leads to the expansion [3]

$$\begin{aligned} \hat{\boldsymbol{\rho}}' \times \left. \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right|_{\rho'=\alpha_1} =& l_\alpha(\mu', \nu') \left[\sum_{m=1}^{3} \boldsymbol{\beta}_m E_1^m(\mu') E_1^m(\nu') \right. \\ & \left. + \sum_{m=1}^{5} \boldsymbol{\delta}_m E_2^m(\mu') E_2^m(\nu') \right] + \mathrm{O}(el_3') \end{aligned} \tag{3.2}$$

where l_α is obtained from (2.25) by substituting $\rho_0 = \alpha_1$,

$$\boldsymbol{\beta}_m = 3\frac{\alpha_1\alpha_2\alpha_3}{h_1h_2h_3}\frac{h_m}{\alpha_m}\hat{\mathbf{x}}_m \otimes \mathbf{r} \underset{\cdot}{\times} \tilde{\mathbf{H}}_1(\rho), \quad m = 1,2,3, \tag{3.3}$$

$$\boldsymbol{\delta}_1 = -\frac{\alpha_1\alpha_2\alpha_3}{3(\Lambda_\alpha - \Lambda'_\alpha)}\tilde{\boldsymbol{\Lambda}}_\alpha \underset{\cdot}{\times} \tilde{\mathbf{F}}_\alpha(\mathbf{r}), \tag{3.4}$$

$$\boldsymbol{\delta}_2 = \frac{\alpha_1\alpha_2\alpha_3}{3(\Lambda_\alpha - \Lambda'_\alpha)}\tilde{\boldsymbol{\Lambda}}'_\alpha \underset{\cdot}{\times} \tilde{\mathbf{F}}_\alpha(\mathbf{r}), \tag{3.5}$$

$$\boldsymbol{\delta}_3 = \frac{\alpha_1\alpha_2\alpha_3}{h_1h_2h_3^2}\left[\frac{\alpha_2}{\alpha_1}\hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_2 + \frac{\alpha_1}{\alpha_2}\hat{\mathbf{x}}_2 \otimes \hat{\mathbf{x}}_1\right] \underset{\cdot}{\times} \tilde{\mathbf{F}}_\alpha(\mathbf{r}), \tag{3.6}$$

$$\boldsymbol{\delta}_4 = \frac{\alpha_1\alpha_2\alpha_3}{h_1h_2^2h_3}\left[\frac{\alpha_3}{\alpha_1}\hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_3 + \frac{\alpha_1}{\alpha_3}\hat{\mathbf{x}}_3 \otimes \hat{\mathbf{x}}_1\right] \underset{\cdot}{\times} \tilde{\mathbf{F}}_\alpha(\mathbf{r}), \tag{3.7}$$

$$\boldsymbol{\delta}_5 = \frac{\alpha_1\alpha_2\alpha_3}{h_1^2h_2h_3}\left[\frac{\alpha_3}{\alpha_2}\hat{\mathbf{x}}_2 \otimes \hat{\mathbf{x}}_3 + \frac{\alpha_2}{\alpha_3}\hat{\mathbf{x}}_3 \otimes \hat{\mathbf{x}}_2\right] \underset{\cdot}{\times} \tilde{\mathbf{F}}_\alpha(\mathbf{r}), \tag{3.8}$$

with the cross-dot product defined by

$$(\mathbf{a} \otimes \mathbf{b}) \underset{\cdot}{\times} (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \times \mathbf{c})(\mathbf{b} \cdot \mathbf{d}), \tag{3.9}$$

and the related polyadics defined as follows:

$$\tilde{\boldsymbol{\Lambda}}_\alpha = \sum_{m=1}^{3} \frac{\hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m}{\Lambda_\alpha - \alpha_m^2}, \tag{3.10}$$

$$\tilde{\boldsymbol{\Lambda}}'_\alpha = \sum_{m=1}^{3} \frac{\hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m}{\Lambda'_\alpha - \alpha_m^2}, \tag{3.11}$$

$$\tilde{\mathbf{H}}_1(\rho) = \sum_{m=1}^{3} I_1^m(\rho)\hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m, \tag{3.12}$$

$$\tilde{\tilde{\mathbf{H}}}_2(\rho) = \sum_{\substack{i,j=1\\ i\neq j}}^{3} I_2^{i+j}(\rho)\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j, \tag{3.13}$$

$$\tilde{\mathbf{F}}_\alpha(\mathbf{r}) = -\frac{\mathbb{F}_2^1(\mathbf{r})}{\Lambda_\alpha - \Lambda'_\alpha}\tilde{\boldsymbol{\Lambda}}_\alpha + \frac{\mathbb{F}_2^2(\mathbf{r})}{\Lambda_\alpha - \Lambda'_\alpha}\tilde{\boldsymbol{\Lambda}}'_\alpha + 15\mathbf{r} \otimes \mathbf{r} : \tilde{\tilde{\mathbf{H}}}_2(\rho). \tag{3.14}$$

The constants $\Lambda_\alpha, \Lambda'_\alpha$ are given in (A.7) and the notation $\mathrm{O}(el'_3)$ indicates ellipsoidal terms in $\mathbf{r}'$ that are of the octapole or higher multipole type.

A similar expansion for the potential u_α on S_α leads to

$$u_\alpha(\alpha_1, \mu, \nu) = d_0^1 + \sum_{m=1}^{3} \zeta_1^m E_1^m(\mu)E_1^m(\nu) + \sum_{m=1}^{5} \zeta_2^m E_2^m(\mu)E_2^m(\nu) + \mathrm{O}(el'_3) \tag{3.15}$$

where for $n = 1, 2$ and $m = 1, 2, ..., 2n + 1$,

$$\zeta_n^m = \frac{\mathbf{Q} \cdot \nabla \mathbb{E}_n^m(\mathbf{r}_0)}{\gamma_n^m C_n^m} \frac{1}{\alpha_2 \alpha_3 E_n^{m\prime}(\alpha_1)}. \tag{3.16}$$

Using the expression (2.21) as well as (A.3)-(A.19) we can rewrite the ζ_n^m for $n = 1, 2$ and $m = 1, 2, ..., 2n + 1$ as follows:

$$\zeta_1^m = \frac{3h_m \alpha_m}{4\pi h_1 h_2 h_3 \alpha_1 \alpha_2 \alpha_3} \cdot \frac{\mathbf{Q} \cdot \hat{\mathbf{x}}_m}{\sigma_\alpha + (\sigma_b - \sigma_\alpha)\left((I_n^m(b_1) - I_1^m(\alpha_1))b_1 b_2 b_3 + \frac{b_1 b_2 b_3}{\alpha_1 \alpha_2 \alpha_3}\right)}; \tag{3.17}$$

for $m = 1, 2, 3$,

$$\begin{aligned}
\zeta_2^1 =& -\frac{5}{8\pi(\Lambda_\alpha - \Lambda'_\alpha)\alpha_1 \alpha_2 \alpha_3} \\
&\times \frac{\mathbf{Q} \otimes \mathbf{r}_0 : \tilde{\mathbf{\Lambda}}_\alpha}{\sigma_\alpha + (\sigma_b - \sigma_\alpha)\left((I_2^1(b_1) - I_2^1(\alpha_1))2\Lambda_b b_1 b_2 b_3 + \frac{\Lambda_b b_1 b_2 b_3}{\Lambda_\alpha \alpha_1 \alpha_2 \alpha_3}\right)}, \qquad (3.18)\\
\zeta_2^2 =& \frac{5}{8\pi(\Lambda_\alpha - \Lambda'_\alpha)\alpha_1 \alpha_2 \alpha_3} \\
&\times \frac{\mathbf{Q} \otimes \mathbf{r}_0 : \tilde{\mathbf{\Lambda}}'_\alpha}{\sigma_\alpha + (\sigma_b - \sigma_\alpha)\left((I_2^2(b_1) - I_2^2(\alpha_1))2\Lambda'_b b_1 b_2 b_3 + \frac{\Lambda'_b b_1 b_2 b_3}{\Lambda'_\alpha \alpha_1 \alpha_2 \alpha_3}\right)}, \qquad (3.19)\\
\zeta_2^{i+j} =& -\frac{15\alpha_i \alpha_j h_i h_j}{4\pi(h_1 h_2 h_3)^2(\alpha_i^2 + \alpha_j^2)\alpha_1 \alpha_2 \alpha_3} \\
&\times \frac{\mathbf{Q} \cdot (x_{0i}\hat{\mathbf{x}}_j + x_{0j}\hat{\mathbf{x}}_i)}{\sigma_\alpha + (\sigma_b - \sigma_\alpha)\left((I_2^{i+j}(b_1) - I_2^{i+j}(\alpha_1))(b_i^2 + b_j^2)b_1 b_2 b_3 + \frac{(b_i^2 + b_j^2)b_1 b_2 b_3}{(\alpha_i^2 + \alpha_j^2)\alpha_1 \alpha_2 \alpha_3}\right)} \qquad (3.20)
\end{aligned}$$

with $i, j \in \{1, 2, 3\}$ and $i \neq j$.

Inserting (3.2) and (3.15) in (3.1) and using orthogonality we arrive at

$$\mathbf{I}_\alpha(\mathbf{r}) = \sum_{m=1}^{5} \zeta_1^m \gamma_1^m \boldsymbol{\beta}_m(\mathbf{r}) + \sum_{m=1}^{5} \zeta_2^m \gamma_2^m \boldsymbol{\delta}_m(\mathbf{r}) + \mathrm{O}(el_3) \tag{3.21}$$

or, in view of (3.3)-(3.14), (3.17)-(3.20) and some long calculations, we can rewrite

(3.21) as

$$
\begin{aligned}
\mathbf{I}_\alpha(\mathbf{r}) = \sum_{m=1}^{3} & \frac{3\mathbf{Q}\cdot\hat{\mathbf{x}}_m\otimes\hat{\mathbf{x}}_m\otimes\mathbf{r}\underset{\cdot}{\times}\tilde{\mathbf{H}}_1(\rho)}{\sigma_\alpha+(\sigma_b-\sigma_\alpha)\left((I_1^m(b_1)-I_1^m(\alpha_1))b_1b_2b_3+\frac{b_1b_2b_3}{\alpha_1\alpha_2\alpha_3}\right)} \\
& - \frac{(\Lambda_\alpha-\alpha_1^2)(\Lambda_\alpha-\alpha_2^2)(\Lambda_\alpha-\alpha_3^2)\mathbf{Q}\otimes\mathbf{r}_0:\tilde{\mathbf{\Lambda}}_\alpha\otimes\tilde{\mathbf{\Lambda}}_\alpha\underset{\cdot}{\times}\tilde{\mathbf{F}}_\alpha(\mathbf{r})}{3(\Lambda_\alpha-\Lambda'_\alpha)\left[\sigma_\alpha+(\sigma_b-\sigma_\alpha)\left((I_2^1(b_1)-I_2^1(\alpha_1))b_1b_2b_3+\frac{\Lambda_b b_1b_2b_3}{\Lambda_\alpha\alpha_1\alpha_2\alpha_3}\right)\right]} \\
& + \frac{(\Lambda'_\alpha-\alpha_1^2)(\Lambda'_\alpha-\alpha_2^2)(\Lambda'_\alpha-\alpha_3^2)\mathbf{Q}\otimes\mathbf{r}_0:\tilde{\mathbf{\Lambda}}'_\alpha\otimes\tilde{\mathbf{\Lambda}}'_\alpha\underset{\cdot}{\times}\tilde{\mathbf{F}}_\alpha(\mathbf{r})}{3(\Lambda_\alpha-\Lambda'_\alpha)\left[\sigma_\alpha+(\sigma_b-\sigma_\alpha)\left((I_2^2(b_1)-I_2^2(\alpha_1))b_1b_2b_3+\frac{\Lambda'_b b_1b_2b_3}{\Lambda'_\alpha\alpha_1\alpha_2\alpha_3}\right)\right]} \\
& + \mathbf{Q}\otimes\mathbf{r}_0:\left[\sum_{\substack{i,j=1\\ i\neq j}}^{3}\frac{(\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j+\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i)\otimes(\alpha_i^2\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i+\alpha_j^2\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j)}{(\alpha_i^2+\alpha_j^2)}\right. \\
& \times\left.\frac{1}{\sigma_\alpha+(\sigma_b-\sigma_\alpha)\left((I_2^{i+j}(b_1)-I_2^{i+j}(\alpha_1))(b_i^2+b_j^2)b_1b_2b_3+\frac{(b_i^2+b_j^2)b_1b_2b_3}{(\alpha_i^2+\alpha_j^2)\alpha_1\alpha_2\alpha_3}\right)}\right] \\
& \underset{\cdot}{\times}\tilde{\mathbf{F}}_\alpha(\mathbf{r})+\mathrm{O}(el_3) \qquad (3.22)
\end{aligned}
$$

where the actual dependence on the position $\mathbf{r}_0$ and moment $\mathbf{Q}$ of the source dipole, on the observation point $\mathbf{r}$, on the geometrical parameters $\alpha_i, b_i,\ i=1,2,3$, and on the conductivities σ_α and σ_b, is explicitly shown.

Next we move to the integral

$$
\mathbf{I}_b(\mathbf{r}) = \int_{S_b} u_b(\mathbf{r}')\hat{\boldsymbol{\rho}}'\times\nabla_{\mathbf{r}'}\frac{1}{|\mathbf{r}-\mathbf{r}'|}ds(\mathbf{r}'). \qquad (3.23)
$$

Following the same track of calculations as with the integral $\mathbf{I}_\alpha(\mathbf{r})$ we obtain

$$
\begin{aligned}
\hat{\boldsymbol{\rho}}'\times\left.\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right|_{\rho'=b_1} = & l_b(\mu',\nu')\left[\sum_{m=1}^{3}\boldsymbol{\beta}'_m E_1^m(\mu')E_1^m(\nu')\right. \\
& \left.+\sum_{m=1}^{5}\boldsymbol{\delta}'_m E_2^m(\mu')E_2^m(\nu')\right]+\mathrm{O}(el'_3), \qquad (3.24)
\end{aligned}
$$

$$
\boldsymbol{\beta}'_m = 3\frac{b_1b_2b_3}{h_1h_2h_3}\frac{h_m}{b_m}\hat{\mathbf{x}}_m\otimes\mathbf{r}\underset{\cdot}{\times}\tilde{\mathbf{H}}_1(\rho), \quad m=1,2,3, \qquad (3.25)
$$

$$
\boldsymbol{\delta}'_1 = -\frac{b_1b_2b_3}{3(\Lambda_b-\Lambda'_b)}\tilde{\mathbf{\Lambda}}_b\underset{\cdot}{\times}\tilde{\mathbf{F}}_b(\mathbf{r}), \qquad (3.26)
$$

$$
\boldsymbol{\delta}'_2 = \frac{b_1b_2b_3}{3(\Lambda_b-\Lambda'_b)}\tilde{\mathbf{\Lambda}}'_b\underset{\cdot}{\times}\tilde{\mathbf{F}}_b(\mathbf{r}), \qquad (3.27)
$$

$$
\boldsymbol{\delta}'_3 = \frac{b_1b_2b_3}{h_1h_2h_3^2}\left[\frac{b_2}{b_1}\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_2+\frac{b_1}{b_2}\hat{\mathbf{x}}_2\otimes\hat{\mathbf{x}}_1\right]\underset{\cdot}{\times}\tilde{\mathbf{F}}_b(\mathbf{r}), \qquad (3.28)
$$

$$\boldsymbol{\delta}_4' = \frac{b_1 b_2 b_3}{h_1 h_2^2 h_3}\left[\frac{b_3}{b_1}\hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_3 + \frac{b_1}{b_3}\hat{\mathbf{x}}_3 \otimes \hat{\mathbf{x}}_1\right] \underset{\cdot}{\times} \tilde{\mathbf{F}}_b(\mathbf{r}), \tag{3.29}$$

$$\boldsymbol{\delta}_5' = \frac{b_1 b_2 b_3}{h_1^2 h_2 h_3}\left[\frac{b_3}{b_2}\hat{\mathbf{x}}_2 \otimes \hat{\mathbf{x}}_3 + \frac{b_2}{b_3}\hat{\mathbf{x}}_3 \otimes \hat{\mathbf{x}}_2\right] \underset{\cdot}{\times} \tilde{\mathbf{F}}_b(\mathbf{r}), \tag{3.30}$$

where Λ_b, Λ_b' and $l_b(\mu', \nu')$ are given by (A.7) and (A.14) respectively, with α_i replaced by $b_i,\ i = 1, 2, 3,$

$$\left.\begin{aligned} \tilde{\boldsymbol{\Lambda}}_b &= \sum_{m=1}^{3} \frac{\hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m}{\Lambda_b - b_m^2} \\ \tilde{\boldsymbol{\Lambda}}_b' &= \sum_{m=1}^{3} \frac{\hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m}{\Lambda_b' - b_m^2} \end{aligned}\right\} \tag{3.31}$$

and

$$\tilde{\mathbf{F}}_b(\mathbf{r}) = -\frac{\mathbb{F}_2^1(\mathbf{r})}{\Lambda_b - \Lambda_b'}\tilde{\boldsymbol{\Lambda}}_b + \frac{\mathbb{F}_2^2(\mathbf{r})}{\Lambda_b - \Lambda_b'}\tilde{\boldsymbol{\Lambda}}_b' + 15\mathbf{r} \otimes \mathbf{r} : \tilde{\tilde{\mathbf{H}}}_2(\rho). \tag{3.32}$$

The dyadic function $\tilde{\mathbf{H}}_1(\rho)$ and the tetradic function $\tilde{\tilde{\mathbf{H}}}_2(\rho)$ are defined by (3.12) and (3.13) respectively.

It is straightforward to show that as a consequence of confocality

$$\Lambda_\alpha - \Lambda_\alpha' = \Lambda_b - \Lambda_b', \tag{3.33}$$

$$\left.\begin{aligned} \Lambda_\alpha - \alpha_i^2 &= \Lambda_b - b_i^2, \ \ i = 1, 2, 3 \\ \Lambda_\alpha' - \alpha_i^2 &= \Lambda_b' - b_i^2, \ \ i = 1, 2, 3 \end{aligned}\right\} \tag{3.34}$$

and

$$\left.\begin{aligned} \tilde{\boldsymbol{\Lambda}}_\alpha &= \tilde{\boldsymbol{\Lambda}}_b = \tilde{\boldsymbol{\Lambda}} \\ \tilde{\boldsymbol{\Lambda}}_\alpha' &= \tilde{\boldsymbol{\Lambda}}_b' = \tilde{\boldsymbol{\Lambda}}' \end{aligned}\right\}. \tag{3.35}$$

Hence

$$\tilde{\mathbf{F}}_\alpha(\mathbf{r}) = \tilde{\mathbf{F}}_b(\mathbf{r}) = \tilde{\mathbf{F}}(\mathbf{r}). \tag{3.36}$$

The potential (2.20) provides

$$u_b(b_1, \mu, \nu) = d_0^1 + \sum_{m=1}^{3} \theta_1^m E_1^m(\mu) E_1^m(\nu) + \sum_{m=1}^{5} \theta_2^m E_2^m(\mu) E_2^m(\nu) + \mathrm{O}(el_3) \tag{3.37}$$

where for $n = 1, 2$ and $m = 1, 2, ..., 2n + 1,$

$$\theta_n^m = \frac{\mathbf{Q} \cdot \nabla \mathbb{E}_n^m(\mathbf{r}_0) E_n^m(b_1)}{\gamma_n^m C_n^m}\left(I_n^m(b_1) - I_n^m(\alpha_1) + \frac{1}{E_n^m(\alpha_1) E_n^{m\prime}(\alpha_1)\alpha_2\alpha_3}\right). \tag{3.38}$$

Further manipulations imply that

$$\theta_1^m = \frac{3h_m b_m}{4\pi h_1 h_2 h_3} \cdot \frac{\mathbf{Q} \cdot \hat{\mathbf{x}}_m \left(I_1^m(b_1) - I_1^m(\alpha_1) + \frac{1}{\alpha_1\alpha_2\alpha_3}\right)}{\sigma_\alpha + (\sigma_b - \sigma_\alpha)\left((I_1^m(b_1) - I_1^m(\alpha_1))b_1 b_2 b_3 + \frac{b_1 b_2 b_3}{\alpha_1\alpha_2\alpha_3}\right)} \tag{3.39}$$

for $m = 1, 2, 3$,

$$\theta_2^1 = -\frac{5\mathbf{Q}\otimes\mathbf{r}_0 : \tilde{\mathbf{\Lambda}}}{8\pi(\Lambda_b - \Lambda_b')} \cdot \frac{(I_2^1(b_1) - I_2^1(\alpha_1))2\Lambda_b + \frac{\Lambda_b}{\Lambda_\alpha\alpha_1\alpha_2\alpha_3}}{\sigma_\alpha + (\sigma_b - \sigma_\alpha)\left((I_2^1(b_1) - I_2^1(\alpha_1))2\Lambda_b b_1 b_2 b_3 + \frac{\Lambda_b b_1 b_2 b_3}{\Lambda_\alpha\alpha_1\alpha_2\alpha_3}\right)}, \tag{3.40}$$

$$\theta_2^2 = \frac{5\mathbf{Q}\otimes\mathbf{r}_0 : \tilde{\mathbf{\Lambda}}'}{8\pi(\Lambda_b - \Lambda_b')} \cdot \frac{(I_2^2(b_1) - I_2^2(\alpha_1))2\Lambda_b' + \frac{\Lambda_b'}{\Lambda_\alpha'\alpha_1\alpha_2\alpha_3}}{\sigma_\alpha + (\sigma_b - \sigma_\alpha)\left((I_2^2(b_1) - I_2^2(\alpha_1))2\Lambda_b' b_1 b_2 b_3 + \frac{\Lambda_b' b_1 b_2 b_3}{\Lambda_\alpha'\alpha_1\alpha_2\alpha_3}\right)}, \tag{3.41}$$

$$\begin{aligned}\theta_2^{i+j} &= \frac{15\mathbf{Q}\cdot(x_{0i}\hat{\mathbf{x}}_j + x_{0j}\hat{\mathbf{x}}_i)h_i h_j b_i b_j}{4\pi h_1^2 h_2^2 h_3^2} \\ &\times \frac{I_2^{i+j}(b_1) - I_2^{i+j}(\alpha_1) + \frac{1}{(\alpha_i^2+\alpha_j^2)\alpha_1\alpha_2\alpha_3}}{\sigma_\alpha + (\sigma_b - \sigma_\alpha)\left((I_2^{i+j}(b_1) - I_2^{i+j}(\alpha_1))(b_i^2 + b_j^2)b_1 b_2 b_3 + \frac{(b_i^2+b_j^2)b_1 b_2 b_3}{(\alpha_i^2+\alpha_j^2)\alpha_1\alpha_2\alpha_3}\right)}\end{aligned} \tag{3.42}$$

with $i, j = \{1, 2, 3\}$ and $i \neq j$.

Using orthogonality, the integral (3.23), with the help of the multipole expansions (3.24) and (3.37), provides

$$\mathbf{I}_b(\mathbf{r}) = \sum_{m=1}^{3} \theta_1^m \gamma_1^m \boldsymbol{\beta}_m'(\mathbf{r}) + \sum_{m=1}^{5} \theta_2^m \gamma_2^m \boldsymbol{\delta}_m'(\mathbf{r}) + \mathrm{O}(el_3). \tag{3.43}$$

Using (3.25)-(3.30) and (3.39)-(3.42), as well as a long series of calculations, it is possible to rewrite (3.43) as

$$\begin{aligned}\mathbf{I}_b(\mathbf{r}) = &\sum_{m=1}^{3} \frac{(I_1^m(b_1) - I_1^m(\alpha_1))b_1 b_2 b_3 + \frac{b_1 b_2 b_3}{\alpha_1\alpha_2\alpha_3}}{\sigma_\alpha + (\sigma_b - \sigma_\alpha)\left((I_1^m(b_1) - I_1^m(\alpha_1))b_1 b_2 b_3 + \frac{b_1 b_2 b_3}{\alpha_1\alpha_2\alpha_3}\right)} \\ &\times 3\mathbf{Q}\cdot\hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m \otimes \mathbf{r} \underset{\cdot}{\times} \tilde{\mathbf{H}}_1(\rho) \\ &- \frac{(\Lambda_\alpha - \alpha_1^2)(\Lambda_\alpha - \alpha_2^2)(\Lambda_\alpha - \alpha_3^2)\left((I_2^1(b_1) - I_2^1(\alpha_1))2\Lambda_b b_1 b_2 b_3 + \frac{\Lambda_b b_1 b_2 b_3}{\Lambda_\alpha\alpha_1\alpha_2\alpha_3}\right)}{3(\Lambda_\alpha - \Lambda_\alpha')\left[\sigma_\alpha + (\sigma_b - \sigma_\alpha)\left((I_2^1(b_1) - I_2^1(\alpha_1))2\Lambda_b b_1 b_2 b_3 + \frac{\Lambda_b b_1 b_2 b_3}{\Lambda_\alpha\alpha_1\alpha_2\alpha_3}\right)\right]} \\ &\times \mathbf{Q}\otimes\mathbf{r}_0 : \tilde{\mathbf{\Lambda}} \otimes \tilde{\mathbf{\Lambda}} \underset{\cdot}{\times} \mathbf{F}(\mathbf{r}) \\ &+ \frac{(\Lambda_\alpha' - \alpha_1^2)(\Lambda_\alpha' - \alpha_2^2)(\Lambda_\alpha' - \alpha_3^2)\left((I_2^2(b_1) - I_2^2(\alpha_1))2\Lambda_b' b_1 b_2 b_3 + \frac{\Lambda_b' b_1 b_2 b_3}{\Lambda_\alpha'\alpha_1\alpha_2\alpha_3}\right)}{3(\Lambda_\alpha - \Lambda_\alpha')\left[\sigma_\alpha + (\sigma_b - \sigma_\alpha)\left((I_2^2(b_1) - I_2^2(\alpha_1))2\Lambda_b' b_1 b_2 b_3 + \frac{\Lambda_b' b_1 b_2 b_3}{\Lambda_\alpha'\alpha_1\alpha_2\alpha_3}\right)\right]} \\ &\times \mathbf{Q}\otimes\mathbf{r}_0 : \tilde{\mathbf{\Lambda}}' \otimes \tilde{\mathbf{\Lambda}}' \underset{\cdot}{\times} \mathbf{F}(\mathbf{r}) \\ &+ \sum_{\substack{i,j=1\\ i\neq j}}^{3} \frac{(I_2^{i+j}(b_1) - I_2^{i+j}(\alpha_1))(b_i^2 + b_j^2)b_1 b_2 b_3 + \frac{(b_i^2+b_j^2)b_1 b_2 b_3}{(\alpha_i^2+\alpha_j^2)\alpha_1\alpha_2\alpha_3}}{\sigma_\alpha + (\sigma_b - \sigma_\alpha)\left((I_2^{i+j}(b_1) - I_2^{i+j}(\alpha_1))(b_i^2 + b_j^2)b_1 b_2 b_3 + \frac{(b_i^2+b_j^2)b_1 b_2 b_3}{(\alpha_i^2+\alpha_j^2)\alpha_1\alpha_2\alpha_3}\right)} \\ &\times \frac{\mathbf{Q}\otimes\mathbf{r}_0\cdot(\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j + \hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i)\otimes(b_i^2\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i + b_j^2\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j) \underset{\cdot}{\times} \tilde{\mathbf{F}}(\mathbf{r})}{b_i^2 + b_j^2}.\end{aligned} \tag{3.44}$$

The multipole expansion of the first term on the right-hand side of (2.9) implies that [3]

$$\mathbf{Q} \times \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} = 3\mathbf{Q} \times \tilde{\mathbf{H}}_1(\rho) \cdot \mathbf{r} + \mathbf{Q} \times \mathbf{r}_0 \cdot \tilde{\mathbf{F}}(\mathbf{r}) + O(el_3). \tag{3.45}$$

At this stage we have the multipole expansions of all terms appearing in (2.9), which by virtue of the continuity condition (2.15) is written as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\mathbf{Q} \times \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} - \sigma_\alpha \mathbf{I}_\alpha(\mathbf{r}) - (\sigma_b - \sigma_\alpha)\mathbf{I}_b(\mathbf{r})\right]. \tag{3.46}$$

First we observe that the dipole term of the expression involving $\mathbf{I}_\alpha(\mathbf{r})$ and $\mathbf{I}_b(\mathbf{r})$ gives

$$\begin{aligned}
&[\sigma_\alpha \mathbf{I}_\alpha(\mathbf{r}) + (\sigma_b - \sigma_\alpha)\mathbf{I}_b(\mathbf{r})]_{n=1} = \sigma_\alpha \sum_{m=1}^{3} \zeta_1^m \boldsymbol{\beta}_m(\mathbf{r})\gamma_1^m + (\sigma_b - \sigma_\alpha)\sum_{m=1}^{3} \theta_1^m \boldsymbol{\beta}'_m(\mathbf{r})\gamma_1^m \\
&= \sum_{m=1}^{3} \frac{3\sigma_\alpha \mathbf{Q}\cdot\hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m \otimes \mathbf{r} \underset{\cdot}{\times} \tilde{\mathbf{H}}_1(\rho)}{\sigma_\alpha + (\sigma_b - \sigma_\alpha)\left((I_1^m(b_1) - I_1^m(\alpha_1))b_1b_2b_3 + \frac{b_1b_2b_3}{\alpha_1\alpha_2\alpha_3}\right)} \\
&\quad + \frac{(\sigma_b - \sigma_\alpha)\left((I_1^m(b_1) - I_1^m(\alpha_1))b_1b_2b_3 + \frac{b_1b_2b_3}{\alpha_1\alpha_2\alpha_3}\right)}{\sigma_\alpha + (\sigma_b - \sigma_\alpha)\left((I_1^m(b_1) - I_1^m(\alpha_1))b_1b_2b_3 + \frac{b_1b_2b_3}{\alpha_1\alpha_2\alpha_3}\right)} 3\mathbf{Q}\cdot\hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m \otimes \mathbf{r} \underset{\cdot}{\times} \tilde{\mathbf{H}}_1(\rho) \\
&= 3\mathbf{Q}\cdot\sum_{m=1}^{3} \hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m \otimes \mathbf{r} \underset{\cdot}{\times} \tilde{\mathbf{H}}_1(\rho) = 3\mathbf{Q}\cdot\tilde{\mathbf{I}} \otimes \mathbf{r} \underset{\cdot}{\times} \tilde{\mathbf{H}}_1(\rho) = 3\mathbf{Q} \otimes \mathbf{r} \underset{\cdot}{\times} \tilde{\mathbf{H}}_1(\rho)
\end{aligned} \tag{3.47}$$

where $\tilde{\mathbf{I}}$ stands for the identity dyadic. Combining this result with (3.45) we see that the dipole contribution to $\mathbf{B}$ vanishes. This result is in accord with the physical fact that no magnetic monopole exists in nature. Consequently, the leading multipole of $\mathbf{B}$ is of quadrupolic character.

Analyzing each one of the five types of quadrupoles appearing in the part of $\mathbf{B}$ that is due to the conductive current we obtain

$$\begin{aligned}
&[\sigma_\alpha \mathbf{I}_\alpha(\mathbf{r}) + (\sigma_b - \sigma_\alpha)\mathbf{I}_b(\mathbf{r})]_2^1 \\
&\qquad = -\frac{(\Lambda_\alpha - \alpha_1^2)(\Lambda_\alpha - \alpha_2^2)(\Lambda_\alpha - \alpha_3^2)}{3(\Lambda_\alpha - \Lambda'_\alpha)} \mathbf{Q} \otimes \mathbf{r}_0 : \tilde{\boldsymbol{\Lambda}} \otimes \tilde{\boldsymbol{\Lambda}} \underset{\cdot}{\times} \tilde{\mathbf{F}}(\mathbf{r}),
\end{aligned} \tag{3.48}$$

$$\begin{aligned}
&[\sigma_\alpha \mathbf{I}_\alpha(\mathbf{r}) + (\sigma_b - \sigma_\alpha)\mathbf{I}_b(\mathbf{r})]_2^2 \\
&\qquad = \frac{(\Lambda'_\alpha - \alpha_1^2)(\Lambda'_\alpha - \alpha_2^2)(\Lambda'_\alpha - \alpha_3^2)}{3(\Lambda_\alpha - \Lambda'_\alpha)} \mathbf{Q} \otimes \mathbf{r}_0 : \tilde{\boldsymbol{\Lambda}}' \otimes \tilde{\boldsymbol{\Lambda}}' \underset{\cdot}{\times} \tilde{\mathbf{F}}(\mathbf{r}),
\end{aligned} \tag{3.49}$$

and for $i,j=\{1,2,3\}$ with $i\neq j$,

$$
\begin{aligned}
&[\sigma_\alpha \mathbf{I}_\alpha(\mathbf{r})+(\sigma_b-\sigma_\alpha)\mathbf{I}_b(\mathbf{r})]_2^{i+j}\\
&=\mathbf{Q}\otimes\mathbf{r}_0:\frac{(\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j+\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i)\otimes(b_i^2\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i+b_j^2\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j)}{b_i^2+b_j^2}\times\tilde{\mathbf{F}}(\mathbf{r})\\
&+\frac{\sigma_\alpha\mathbf{Q}\otimes\mathbf{r}_0:(\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j+\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i)}{\sigma_\alpha+(\sigma_b-\sigma_\alpha)\left((I_2^{i+j}(b_1)-I_2^{i+j}(\alpha_1))(b_i^2+b_j^2)+\frac{(b_i^2+b_j^2)b_1b_2b_3}{(\alpha_i^2+\alpha_j^2)\alpha_1\alpha_2\alpha_3}\right)}\\
&\otimes\left(\frac{\alpha_i^2\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i+\alpha_j^2\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j}{\alpha_i^2+\alpha_j^2}-\frac{b_i^2\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i+b_j^2\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j}{b_i^2+b_j^2}\right)\times\tilde{\mathbf{F}}(\mathbf{r}).
\end{aligned}
\tag{3.50}
$$

Substituting (3.48)-(3.50) in (3.46) and using the identities

$$
\begin{aligned}
&\frac{(\Lambda_\alpha-\alpha_1^2)(\Lambda_\alpha-\alpha_2^2)(\Lambda_\alpha-\alpha_3^2)}{3(\Lambda_\alpha-\Lambda'_\alpha)}\tilde{\mathbf{\Lambda}}\otimes\tilde{\mathbf{\Lambda}}\\
&\quad-\frac{(\Lambda'_\alpha-\alpha_1^2)(\Lambda'_\alpha-\alpha_2^2)(\Lambda'_\alpha-\alpha_3^2)}{3(\Lambda_\alpha-\Lambda'_\alpha)}\tilde{\mathbf{\Lambda}}'\otimes\tilde{\mathbf{\Lambda}}'\\
&=\frac{1}{3}\tilde{\mathbf{I}}\otimes\tilde{\mathbf{I}}-\sum_{i=1}^{3}\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_i,
\end{aligned}
\tag{3.51}
$$

$$
\begin{aligned}
&\frac{(\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j+\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i)\otimes(b_j^2\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j+b_i^2\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i)}{b_i^2+b_j^2}\\
&=\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j\\
&\quad+\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i+\frac{(b_i^2\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j-b_j^2\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i)\otimes(\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i-\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j)}{b_i^2+b_j^2},\quad i\neq j,
\end{aligned}
\tag{3.52}
$$

$$
\left.\begin{aligned}
\hat{\mathbf{x}}_1\times\tilde{\mathbf{I}}&=\hat{\mathbf{x}}_3\otimes\hat{\mathbf{x}}_2-\hat{\mathbf{x}}_2\otimes\hat{\mathbf{x}}_3\\
\hat{\mathbf{x}}_2\times\tilde{\mathbf{I}}&=\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_3-\hat{\mathbf{x}}_3\otimes\hat{\mathbf{x}}_1\\
\hat{\mathbf{x}}_3\times\tilde{\mathbf{I}}&=\hat{\mathbf{x}}_2\otimes\hat{\mathbf{x}}_1-\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_2
\end{aligned}\right\},
\tag{3.53}
$$

$$
\begin{aligned}
&\sum_{\substack{(i,j)=\{(1,2)\\(1,3)\\(2,3)\}}}^{3}\frac{(\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j+\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i)\otimes(b_j^2\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j+b_i^2\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i)}{b_i^2+b_j^2}\\
&=\sum_{\substack{i,j=1\\i\neq j}}^{3}\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j\\
&\quad+\left\{\frac{(b_1^2\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_2-b_2^2\hat{\mathbf{x}}_2\otimes\hat{\mathbf{x}}_1)\otimes\hat{\mathbf{x}}_3}{b_1^2+b_2^2}+\frac{(-b_1^2\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_3+b_3^2\hat{\mathbf{x}}_3\otimes\hat{\mathbf{x}}_1)\otimes\hat{\mathbf{x}}_2}{b_1^2+b_3^2}\right.\\
&\quad\left.+\frac{(b_2^2\hat{\mathbf{x}}_2\otimes\hat{\mathbf{x}}_3-b_3^2\hat{\mathbf{x}}_3\otimes\hat{\mathbf{x}}_2)\otimes\hat{\mathbf{x}}_1}{b_2^2+b_3^2}\right\}\times\tilde{\mathbf{I}},
\end{aligned}
\tag{3.54}
$$

$$\mathbf{Q}\otimes\mathbf{r}_0 : \left(\sum_{i=1}^{3}\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_i + \sum_{\substack{i,j=1\\ i\neq j}}^{3}\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j\right)\underset{\cdot}{\times}\tilde{\mathbf{F}}(\mathbf{r}) \tag{3.55}$$
$$= \mathbf{Q}\otimes\mathbf{r}_0 : \tilde{\tilde{\mathbf{I}}}\underset{\cdot}{\times}\tilde{\mathbf{F}}(\mathbf{r}) = \mathbf{Q}\otimes\mathbf{r}_0\underset{\cdot}{\times}\tilde{\mathbf{F}}(\mathbf{r}) = \mathbf{Q}\times\mathbf{r}_0\cdot\tilde{\mathbf{F}}(\mathbf{r})$$

and

$$\tilde{\mathbf{I}}\underset{\cdot}{\times}\tilde{\mathbf{F}}(\mathbf{r}) = \tilde{\mathbf{O}} \tag{3.56}$$

where $\tilde{\tilde{\mathbf{I}}}$ is the identity tetradic defined by

$$\tilde{\tilde{\mathbf{I}}} = \sum_{i,j=1}^{3}\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j\otimes\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j \tag{3.57}$$

we arrive at the compact expression

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}(\mathbf{d}-\mathbf{d}_b+\mathbf{d}_\alpha)\cdot \left(\frac{\mathbb{F}_2^1(\mathbf{r})}{\Lambda_\alpha-\Lambda'_\alpha}\tilde{\mathbf{\Lambda}} - \frac{\mathbb{F}_2^2(\mathbf{r})}{\Lambda_\alpha-\Lambda'_\alpha}\tilde{\mathbf{\Lambda}}' - 15\sum_{\substack{i,j=1\\ i\neq j}}^{3} x_i x_j I_2^{i+j}(\rho)\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_j\right) + \mathrm{O}(el_3) \tag{3.58}$$

where

$$\mathbf{d} = (\mathbf{Q}\cdot\tilde{\mathbf{M}}(b_1)\times\mathbf{r}_0)\cdot\tilde{\mathbf{N}}(b_1), \tag{3.59}$$
$$\mathbf{d}_b = (\mathbf{Q}\cdot\tilde{\mathbf{M}}(b_1)\times\mathbf{r}_0)\cdot\tilde{\mathbf{N}}_c(b_1), \tag{3.60}$$
$$\mathbf{d}_\alpha = (\mathbf{Q}\cdot\tilde{\mathbf{M}}(\alpha_1)\times\mathbf{r}_0)\cdot\tilde{\mathbf{N}}_c(\alpha_1), \tag{3.61}$$
$$\tilde{\mathbf{M}}(b_1) = \sum_{i=1}^{3} b_i^2\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_i, \tag{3.62}$$
$$\tilde{\mathbf{M}}(\alpha_1) = \sum_{i=1}^{3} \alpha_i^2\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_i, \tag{3.63}$$
$$\tilde{\mathbf{N}}(b_1) = \sum_{i=1}^{3}\frac{\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_i}{b_1^2+b_2^2+b_3^2-b_i^2}, \tag{3.64}$$
$$\tilde{\mathbf{N}}_c(b_1) = \sum_{i=1}^{3} C^{6-i}\frac{\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_i}{b_1^2+b_2^2+b_3^2-b_i^2}, \tag{3.65}$$
$$\tilde{\mathbf{N}}_c(\alpha_1) = \sum_{i=1}^{3} C^{6-i}\frac{\hat{\mathbf{x}}_i\otimes\hat{\mathbf{x}}_i}{\alpha_1^2+\alpha_2^2+\alpha_3^2-\alpha_i^2}, \tag{3.66}$$

and

$$C^{6-i} = \frac{\sigma_\alpha}{\sigma_\alpha + (\sigma_b - \sigma_\alpha)\left((I_2^{6-i}(b_1) - I_2^{6-i}(\alpha_1))(b_1^2 + b_2^2 + b_3^2 - b_i^2) + \frac{(b_1^2+b_2^2+b_3^2-b_i^2)b_1b_2b_3}{(\alpha_1^2+\alpha_2^2+\alpha_3^2-\alpha_i^2)\alpha_1\alpha_2\alpha_3}\right)} = \frac{\sigma_\alpha}{C_2^{6-i}} \tag{3.67}$$

with C_2^{6-i} defined in (2.21).

Obviously, the coincidence of the conductivities, which geometrically is reflected upon the absorption of the shell into the conductive core, i.e.,

$$\sigma_\alpha \to \sigma_b \Leftrightarrow \alpha_i \to b_i, \quad i = 1, 2, 3 \tag{3.68}$$

implies that

$$C^{6-i} = 1, \quad i = 1, 2, 3, \tag{3.69}$$

$$\tilde{\mathbf{N}}_c(b_i) \to \tilde{\mathbf{N}}_c(\alpha_i) \to \tilde{\mathbf{N}}(b_i) \tag{3.70}$$

and finally,

$$\mathbf{d}_b \to \mathbf{d}_\alpha \to \mathbf{d}. \tag{3.71}$$

Then (3.58) is reduced to the corresponding expression for the magnetic induction field for the single homogeneous model [3]

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{d}\cdot\left(\frac{\mathbb{F}_2^1(\mathbf{r})}{\Lambda_b - \Lambda_b'}\tilde{\mathbf{\Lambda}} - \frac{\mathbb{F}_2^2(\mathbf{r})}{\Lambda_b - \Lambda_b'}\tilde{\mathbf{\Lambda}}' - 15\sum_{\substack{i,j=1\\ i\neq j}}^{3} x_i x_j I_2^{i+j}(\rho)\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j\right) + O(el_3). \tag{3.72}$$

4. Reduction to spherical geometry. It is hard, at a first glance, to accept that results on the ellipsoidal system do not reduce trivially to the corresponding results for the spherical system. Nevertheless, a more careful consideration helps to realize that this is an intrinsic difficulty which is due to the way the two systems are constructed. In particular, while the spherical system reduces down to a point, i.e. the center, the ellipsoidal system reduces to a two-dimensional manifold; that of the focal ellipse. As a consequence, the reduction from the ellipsoidal expressions to the spherical ones leads to complicated indeterminate forms which are not easy to handle. An appropriate, but not always obvious, grouping of terms is necessary to overcome these difficulties. This is a cumbersome process which will not be explained further here. As an example, we give the limit [3]

$$\lim_{e\to s}\left(\frac{\mathbb{F}_2^1(\mathbf{r})}{\Lambda_\alpha - \Lambda_\alpha'}\tilde{\mathbf{\Lambda}} - \frac{\mathbb{F}_2^2(\mathbf{r})}{\Lambda_\alpha - \Lambda_\alpha'}\tilde{\mathbf{\Lambda}}' - 15\sum_{\substack{i,j=1\\ i\neq j}}^{3} x_i x_j I_2^{i+j}(\rho)\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j\right) = \frac{\tilde{\mathbf{I}}}{r^3} - \frac{3}{r^5}\sum_{m=1}^{3} x_m^2 \hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m - 15\left(\frac{\mathbf{r}\otimes\mathbf{r}}{5r^5} - \frac{1}{5r^5}\sum_{m=1}^{3} x_m^2 \hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m\right) = \frac{\tilde{\mathbf{I}} - 3\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}}{r^3} \tag{4.1}$$

where the notation $e \to s$ denotes the reduction of the ellipsoidal to the spherical system. More limits of this kind can be found in [3].

If we denote by α and b the radii of the exterior and the interior reduced spheres respectively, then it is not hard to show that

$$\lim_{e\to s} \Lambda_\alpha = \lim_{e\to s} \Lambda'_\alpha = \alpha^2, \tag{4.2}$$

$$\lim_{e\to s} h_i = 0, \quad i = 1, 2, 3, \tag{4.3}$$

$$\lim_{e\to s} \mu = \lim_{e\to s} \nu = 0, \tag{4.4}$$

$$\lim_{e\to s} \rho = r, \tag{4.5}$$

and finally that

$$\lim_{e\to s} I_2^m(\rho) = \frac{1}{5r^5}, \quad m = 1, 2, 3, 4, 5. \tag{4.6}$$

Furthermore, the limits

$$\lim_{e\to s} \tilde{\mathbf{M}}(b_1) = b^2 \sum_{i=1}^{3} \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_i = b^2 \tilde{\mathbf{I}}, \tag{4.7}$$

$$\lim_{e\to s} \tilde{\mathbf{M}}(\alpha_1) = \alpha^2 \tilde{\mathbf{I}}, \tag{4.8}$$

$$\lim_{e\to s} \tilde{\mathbf{N}}(b_1) = \frac{1}{2b^2} \tilde{\mathbf{I}}, \tag{4.9}$$

$$\lim_{e\to s} C^{6-i} = \frac{1}{1 + \left(\frac{\sigma_b}{\sigma_\alpha} - 1\right)\left[\frac{3}{5}\left(\frac{b}{\alpha}\right)^5 + \frac{2}{5}\right]} = C_s, \tag{4.10}$$

$$\lim_{e\to s} \tilde{\mathbf{N}}_c(b_1) = \frac{C_s}{2b^2} \tilde{\mathbf{I}}, \tag{4.11}$$

$$\lim_{e\to s} \tilde{\mathbf{N}}_c(\alpha_1) = \frac{C_s}{2\alpha^2} \tilde{\mathbf{I}} \tag{4.12}$$

imply that

$$\lim_{e\to s} \mathbf{d} = (\mathbf{Q} \cdot \tilde{\mathbf{I}} \times \mathbf{r}_0) \cdot \frac{1}{2} \tilde{\mathbf{I}} = \frac{\mathbf{Q} \times \mathbf{r}_0}{2}, \tag{4.13}$$

$$\lim_{e\to s} \mathbf{d}_b = (\mathbf{Q} \cdot \tilde{\mathbf{I}} \times \mathbf{r}_0) \cdot \frac{C_s}{2} \tilde{\mathbf{I}} = C_s \frac{\mathbf{Q} \times \mathbf{r}_0}{2}, \tag{4.14}$$

$$\lim_{e\to s} \mathbf{d}_\alpha = (\mathbf{Q} \cdot \tilde{\mathbf{I}} \times \mathbf{r}_0) \cdot \frac{C_s}{2} \tilde{\mathbf{I}} = C_s \frac{\mathbf{Q} \times \mathbf{r}_0}{2}, \tag{4.15}$$

which they recover finally the known [3] result for the sphere

$$\lim_{e\to s} \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{8\pi} \mathbf{Q} \times \mathbf{r}_0 \cdot \frac{\tilde{\mathbf{I}} - 3\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}}{r^3} + O\left(\frac{1}{r^4}\right). \tag{4.16}$$

Note that the spherical quadrupolic term on the right-hand side of (4.16) coincides with the corresponding result for a single homogeneous sphere and this is in accord with common knowledge that shells are not "readable" by MEG recordings. We remark here that this property of spherical geometry is due to the non-dependence of the radial

component of the magnetic field on the electric potential [15]. Our results though show that this is not true for the more realistic ellipsoidal model of the brain.

Appendix A. Ellipsoidal Harmonics. The reader can find the basic theory of ellipsoidal harmonics in the monumental work of Hobson [9]. Here, we provide only the expressions that will make the present work self-readable.

The interior Lamé functions $E_n^m(x)$ and the exterior Lamé functions $F_n^m(x)$, where $n = 1, 2, ...$ denotes the degree and $m = 1, 2, ..., 2n+1$ denotes the order of the corresponding function, solve the Lamé equation,

$$(x^2 - h_2^2)(x^2 - h_3^2)E''(x) + x(2x^2 - h_2^2 - h_3^2)E'(x) + (Ax^2 + B)E(x) = 0. \tag{A.1}$$

The constants A and B are appropriately associated with the degree n and the order m. We only use the Lamé functions of degree less or equal to 2, and these are

$$E_0^1(x) = 1, \tag{A.2}$$

$$E_1^m(x) = |x^2 - \alpha_1^2 + \alpha_m^2|^{\frac{1}{2}}, \quad m = 1, 2, 3, \tag{A.3}$$

$$E_2^1(x) = x^2 - \alpha_1^2 + \Lambda, \tag{A.4}$$

$$E_2^2(x) = x^2 - \alpha_1^2 + \Lambda' \tag{A.5}$$

$$E_2^{6-m}(x) = \frac{E_1^1(x)E_1^2(x)E_1^3(x)}{E_1^m(x)}, \quad m = 1, 2, 3, \tag{A.6}$$

where the constants

$$\left.\begin{matrix}\Lambda \\ \Lambda'\end{matrix}\right\} = \frac{1}{3}\sum_{n=1}^{3}\alpha_n^2 \pm \frac{2}{3}\left[\sum_{n=1}^{3}\left(\alpha_n^4 - \frac{\alpha_1^2\alpha_2^2\alpha_3^2}{\alpha_n^2}\right)\right]^{\frac{1}{2}} \tag{A.7}$$

are the roots of

$$\sum_{n=1}^{3}\frac{1}{\Lambda - \alpha_n^2} = 0. \tag{A.8}$$

As Lamé showed, the harmonic eigensolutions in ellipsoidal form are given by the Lamé products

$$\mathbb{E}_n^m(\rho, \mu, \nu) = E_n^m(\rho)E_n^m(\mu)E_n^m(\nu) \tag{A.9}$$

which are regular at the origin, and the Lamé products

$$\mathbb{F}_n^m(\rho, \mu, \nu) = F_n^m(\rho)E_n^m(\mu)E_n^m(\nu) \tag{A.10}$$

which are regular at infinity.

The Lamé functions $E_n^m(\rho)$ and $F_n^m(\rho)$ are connected via the formula

$$F_n^m(\rho) = (2n+1)E_n^m(\rho)\mathrm{I}_n^m(\rho) \tag{A.11}$$

where

$$I_n^m(\rho) = \int_{\rho}^{+\infty}\frac{dt}{[E_n^m(t)]^2\sqrt{t^2 - h_2^2}\sqrt{t^2 - h_3^2}}. \tag{A.12}$$

Note that the only difference of the factors E_n^m in (A.9) is attributed to the interval of variation of the ellipsoidal coordinates (ρ,μ,ν) which vary in the successive intervals

$$-h_3 \leq \nu \leq h_3 \leq \mu \leq h_2 \leq \rho < +\infty. \tag{A.13}$$

The surface ellipsoidal harmonics $E_n^m(\mu)E_n^m(\nu)$ are orthogonal with respect to the weighting function

$$l_\rho(\mu,\nu) = \frac{1}{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}} \tag{A.14}$$

over the surface of any ellipsoid specified by a fixed $\rho > h_2$. The corresponding normalization constants, defined by (2.24) assume the values

$$\gamma_0^1 = 4\pi, \tag{A.15}$$

$$\gamma_1^m = \frac{4\pi}{3}\frac{h_1^2h_2^2h_3^2}{h_m^2}, \quad m = 1,2,3, \tag{A.16}$$

$$\gamma_2^1 = -\frac{8\pi}{5}(\Lambda-\Lambda')(\Lambda-\alpha_1^2)(\Lambda-\alpha_2^2)(\Lambda-\alpha_3^2), \tag{A.17}$$

$$\gamma_2^2 = \frac{8\pi}{5}(\Lambda-\Lambda')(\Lambda'-\alpha_1^2)(\Lambda'-\alpha_2^2)(\Lambda'-\alpha_3^2), \tag{A.18}$$

and

$$\gamma_2^{6-m} = \frac{4\pi}{15}h_1^2h_2^2h_3^2h_m^2, \quad m = 1,2,3. \tag{A.19}$$

Usefull expressions for the gradients of the ellipsoidal harmonics, connection formulae between ellipsoidal harmonics and their Cartesian forms, as well as some basic relations among the elliptic integrals can be found in the appendices in [3].

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