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EXISTENCE OF OPTIMAL SHAPE FOR A SYSTEM OF CONSERVATION LAWS IN A FREE AIR-POROUS DOMAIN

By

ARIAN NOVRUZI

Department of Mathematics and Statistics, University of Ottawa, Canada

Abstract. We consider a shape optimization problem related to a nonlinear system of PDE describing the gas dynamics in a free air-porous domain, including gas concentrations, temperature, velocity and pressure. The velocity and pressure are described by the Stokes and Darcy laws, while concentrations and temperature are given by mass and heat conservation laws. The system represents a simplified dry model of gas dynamics in the channel and graphite diffusive layers of hydrogen fuel cells. The model is coupled with the other part of the domain through some mixed boundary conditions, involving nonlinearities, and pressure boundary conditions. Under some assumptions we prove that the system has a solution and that there exists a channel domain in the class of Lipschitz domains minimizing a certain functional measuring the membrane temperature distribution, total current, water vapor transport and channel inlet/outlet pressure drop.

1. Introduction. Position of the problem. In this paper we consider a two-dimensional nonlinear PDE system which comprises Stokes and Darcy's laws coupled with a system of mass and heat conservation lows in a free air-porous domain. The equations describe the fluid dynamics in the cathode channel and graphite diffusive layers in hydrogen fuel cells (HFC).

HFC are useful devices, producing electricity by reacting the oxygen and hydrogen as shown in Figure 1. Namely, this is realized by pumping fresh air (O_2) in the cathode channel and hydrogen in the anode channel. The oxygen in the cathode channels diffuses through the cathode graphite diffusive layer (GDL), a porous domain, while hydrogen diffuses through the anode GDL. The hydrogen, at the anode catalyst layer (CL) contact, dissociates into ions. The electrons, through an external circuit, travel towards the

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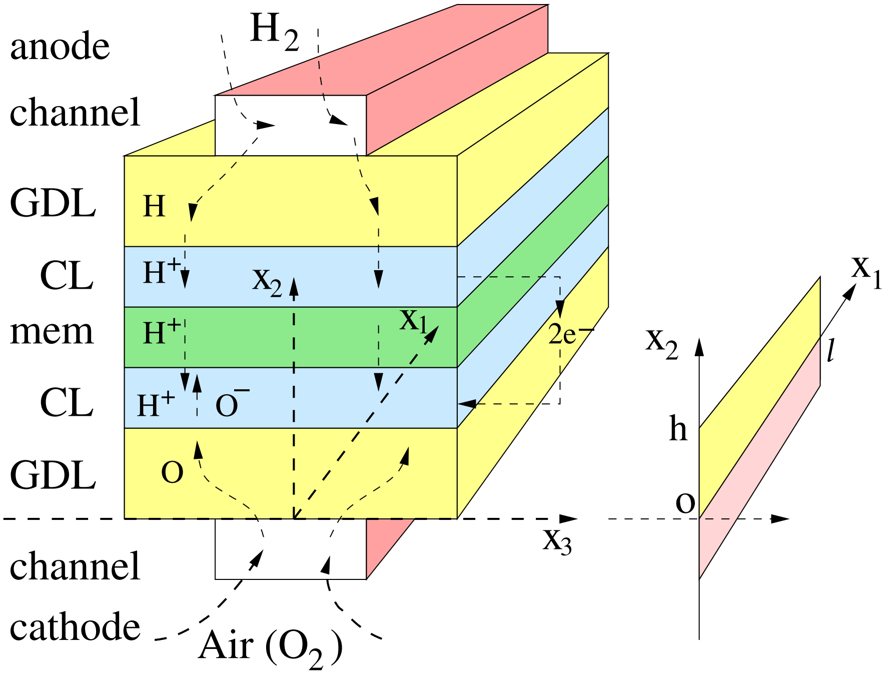
Current address: Department of Mathematics and Statistics, University of Ottawa, 585 King Edward Ave., Ottawa, ON, K1N 6N5, Canada.
E-mail address: `novruzi@uottawa.ca`

cathode GDL, producing useful electric current, and ionize the oxygen molecules at the cathode CL. Anode hydrogen ions diffuse through the membrane, and upon contact with oxygen ions at the cathode CL and membrane, enter into reaction and produce heat and water. It has been observed experimentally that the reaction is located at the CL layer, on the cathode side, mainly close to the inlet (close to $x_1 = 0$; see Figure 1), which exposes this part of the HFC to high temperatures and thus reduces its lifetime. Thus, it is required to operate the fuel cell at a uniform temperature. Meanwhile, it is required to increase the total current produced, which is very closely related to water transport to the cathode outlet (at $x_1 = l$; see Figure 1). Also, for reducing the cost of current production, it is required to reduce the cathode channel drop pressure to between $x_1 = 0$ and $x_1 = l$.

In this paper we deal with a two dimensional dry model in a cathode channel and GDL layers, and we consider the optimal channel shape optimizing a shape functional, motivated by the constraints mentioned in the previous paragraph.



a) 3D figure b) 2D cross−section

FIG. 1. a) The air (O_2, H_2O vapor) flows through the cathode channel and diffuses in the cathode GDL layer. The hydrogen flows through the anode channel and diffuses through the anode GDL layer. The reaction takes place in the cathode catalyst layer and the membrane. b) A 2D (x_1, x_2) cross section

Moreover, we will consider the two-dimensional case in the (x_1, x_2) cross section, as indicated in Figure 1b. We assume the gas contains oxygen and water vapor with mass concentration respectively $\hat{c}^o$ and $\hat{c}^v$. As $\hat{c}^o + \hat{c}^v = 1$ we can eliminate one of them, let's say $\hat{c}^v$, from the analysis. Thus the unknowns are $(\hat{c}, \hat{\tau}, \hat{\mathbf{u}}, \hat{p})$, where $\hat{c} = \hat{c}^o$, $\hat{\tau}$ is the temperature, $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2)$ is the gas mixture velocity and $\hat{p}$ is the pressure. The variable $\hat{c}$ obeys the mass conservation law, while $\hat{\tau}$ obeys the heat conservation law, both in the channel and the GDL. Usually, the gas velocity $\hat{\mathbf{u}}$ in the channel obeys the Stokes equation to a good approximation, and the gas is considered incompressible; thus the density $\hat{\rho} = \rho_0$, ρ_0 constant. In the GDL the velocity obeys the Darcy law. We will assume that even in the GDL the gas is also incompressible.

The assumption for $\hat{\rho}$ ensures the existence of a solution for our system of equations. Otherwise, the resulting PDE system is not trivially with elliptic principal part. The main difficulty is to establish an appropriate L^∞ estimation for $\hat{p}$, sufficient for making the system unconditionally elliptic.

To couple the velocities in the channel and the GDL, we will impose $u_1(0^-) = 0$ on the interface Σ at $x_2 = 0$ separating the channel and the GDL, which physically states the no-slip condition on the air-porous domain interface. Other boundary conditions are used in [2], [10], and [11], where a slip condition is considered. This condition is reported to better represent the underlying physics, though often it leads to several analytic difficulties, mainly due to the control of tangential stress $\partial_2 u_1$ on Σ. We consider a no-slip condition and continuity of normal velocity and pressure, which is a common choice in the engineering literature and leads to a more attractive mathematical analysis.

Let $k > 0$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $\boldsymbol{\beta} = (\beta_1, \beta_2)$ be given satisfying $\alpha_0 \le \alpha_1 < 0$, $\beta_0 \le \beta_1 < 0$, and consider $\mathcal{O}$, the set of uniform Lipschitz functions, as follows:

$$\begin{aligned} \mathcal{O} \quad = \quad & \{\gamma : [0,l] \mapsto (-\infty, 0],\ |\gamma(x_1) - \gamma(y_1)| \le k|x_1 - y_1|, \\ & \alpha_0 \le \gamma(0) \le \alpha_1,\ \beta_0 \le \gamma(l) \le \beta_1\}. \end{aligned} \tag{1.1}$$

For $\gamma \in \mathcal{O}$, set

$$\begin{array}{lll} A^\gamma = \{(x_1, x_2),\ x_1 \in (0,l),\ \gamma(x_1) < x_2 < 0)\}, & & G = (0,l) \times (0,h), \\ \Gamma_\gamma = \{(x_1, \gamma(x_1)),\ x_1 \in (0,l)\}, & \Gamma_i = \{0\} \times (\gamma(0), 0), & \Gamma_o = \{l\} \times (\gamma(l), 0), \\ \Sigma = (0,l) \times \{0\}, & M = (0,l) \times \{h\}, & \Gamma_w = (\{0\} \cup \{l\}) \times (0,h), \\ \Omega_\gamma = A^\gamma \cup \Sigma \cup G. & & \end{array}$$

Also, we define $\mathbf{n}^\gamma$, resp. $\mathbf{n}$, and ν_γ (or simply ν when there is no confusion) to be the exterior unit normal vector to A^γ, resp. G, Ω_γ. From the mass and heat conservation laws for $\hat{c}$, $\hat{\tau}$ and the Stokes and Darcy laws for $\hat{\mathbf{u}}$, with the assumption that the density $\hat{\rho}$ is constant, it follows that $(\hat{c}, \hat{\tau}, \hat{\mathbf{u}})$ satisfies

$$\nabla \cdot (-D\nabla \hat{c} + \hat{c}\hat{\mathbf{u}}) = 0 \quad \text{in } \Omega_\gamma, \tag{1.2}$$

$$\nabla(-\kappa\nabla\hat{\tau} + \hat{\tau}\hat{\mathbf{u}}) = 0 \quad \text{in } \Omega_\gamma, \tag{1.3}$$

$$(-\mu\Delta\hat{\mathbf{u}} + \nabla\hat{p})\mathbb{1}_{A^\gamma} + (\frac{\mu}{K}\hat{\mathbf{u}} + \nabla\hat{p})\mathbb{1}_G = 0 \quad \text{in } A^\gamma \cup G, \tag{1.4}$$

$$\nabla \cdot \hat{\mathbf{u}} = 0 \quad \text{in } A^\gamma \cup G. \tag{1.5}$$

These equations are equipped with the following boundary conditions, which are common in the HFC engineering literature:

$$\left\{\begin{array}{lclclclclcll}
\hat c - c_i & = & \hat\tau - \tau_i & = & \phi - \int_{\Gamma_i} \hat u_1 & = & \hat u_2 & = & \hat p - p_i & = & 0, & \Gamma_i, \\
\partial_\nu \hat c & = & \partial_\nu \hat\tau & = & & & \hat u_2 & = & \hat p - p_o & = & 0, & \Gamma_o, \\
\partial_\nu \hat c & = & \partial_\nu \hat\tau + (\hat\tau - \tau_w) & = & \hat u_1 & = & \hat u_2 & & & = & 0, & \Gamma_\gamma, \\
[\hat c] = [\partial_2 \hat c] & = & [\hat\tau] = [\partial_2 \hat\tau] & = & \hat u_1(0^-) & = & [\hat u_2] & = & [\hat p] & = & 0, & \Sigma, \\
\partial_{\mathbf n} \hat c & = & \partial_{\mathbf n} \hat\tau + (\hat\tau - \tau_w) & = & \hat u_1 & & & & & = & 0, & \Gamma_w \\
\partial_{\mathbf n} \hat c + \hat c & = & \partial_{\mathbf n} \hat\tau - \hat c & & & = & \hat u_2 + g(\hat c) & & & = & 0, & M,
\end{array}\right. \tag{1.6}$$

with $g(\hat c) = \frac{\hat c}{1+\hat c}$, D, κ, ϕ, p_o given constants and p_i an unknown constant. As the pressure is defined up to a constant we will take $p_o = 0$.

Here and throughout this paper, for a function φ defined in $A_\gamma \cup G$, $[\varphi] : \Sigma \mapsto \mathbb{R}$ denotes the so-called slope function on Σ. If φ is smooth on each side of Σ, say $\varphi \in C^0(A_\gamma \cup \Sigma)$, $\varphi \in C^0(G \cup \Sigma)$, then

$$[\varphi](x_1) = \lim_{x_2 \to 0, x_2 > 0} \varphi(x_1, x_2) - \lim_{x_2 \to 0, x_2 > 0} \varphi(x_1, -x_2), \quad \forall x_1 \in (0, l).$$

If φ is less regular, then the trace of φ on Σ, in any appropriate sense, will be considered. For example, if $\varphi \in H^1(A_\gamma \cup G)$, then $[\varphi] \in H^{1/2}(\Sigma)$ is defined as the difference of the trace on Σ of $\varphi \in H^1(G)$ with the trace on Σ of $\varphi \in H^1(A_\gamma)$, and the above formula for $[\varphi](x)$ holds for almost all $x_1 \in (0, l)$.

In real applications, the O_2 concentration $\hat c$ on M has a large variation, which leads to a non-uniform current production. This implies a non-uniform temperature distribution with a maximum value near $(0, h)$, which decreases the HFC lifetime. So, it is required to control the $\hat c$ concentration by making it as constant as possible, while making the $\hat c$ total membrane mass ($L^1(M)$ norm) as high as possible. Also, it is required to optimize the water (vapor) transport through the outlet Γ_o (in order to maintain a stable reaction) and to reduce the amount of pressure drop between Γ_i and Γ_o (in order to reduce current production cost).

The only control we consider is γ. For given γ set $\hat c(\gamma) = \hat c$, $\hat c^v(\gamma) := 1 - \hat c(\gamma)$, $\hat\tau(\gamma)$, $\hat{\mathbf u}(\gamma)$, $\hat p(\gamma)$, let be the solution of (1.2)–(1.6) corresponding to the domain A^γ. The discussion in the previous paragraph motivates the introduction of the following shape functional

$$E(\gamma) = \|\hat c(\gamma) - -\!\!\!\!\!\!\int_M \hat c(\gamma)\|^2_{L^2(M)} - \lambda \int_M \hat c(\gamma) - \delta \int_{\Gamma_o} \hat c^v(\gamma) + \sigma(p_i - p_o), \tag{1.7}$$

where λ, δ, σ are positive parameters. We look for a γ_* solution of

$$E(\gamma_*) = \min\{E(\gamma), \quad \gamma \in \mathcal{O}\}. \tag{1.8}$$

Let us point out that assuming (1.2)–(1.6) has a smooth solution, one can easily obtain

$$\mu \int_{A^\gamma} |\nabla \hat{\mathbf u}(\gamma)|^2 + \frac{\mu}{K} \int_G \hat{\mathbf u}^2 = \int_{\Gamma_i} \hat p(\gamma) \hat u_1(\gamma) - \int_{\Gamma_o} \hat p(\gamma) \hat u_1(\gamma) - \int_M \hat p(\gamma) \hat u_2(\gamma).$$

As $\hat p = p_i$ on Γ_i, from (1.6) it follows that $\int_{\Gamma_i} \hat p(\gamma) \hat u_1 = p_i \phi$, which gives

$$p_i = \frac{1}{\phi} \left(\mu \int_{A^\gamma} |\nabla \hat{\mathbf u}(\gamma)|^2 + \frac{\mu}{K} \int_G \hat{\mathbf u}^2 + p_o \int_{\Gamma_o} \hat u_1(\gamma) + \int_M \hat p(\gamma) \hat u_2(\gamma) \right). \tag{1.9}$$

Then, the functional $E(\gamma)$ takes the form

$$\begin{aligned} E(\gamma) &= \|\hat{c}(\gamma) - \fint_M \hat{c}(\gamma)\|^2_{L^2(M)} - \lambda \int_M \hat{c}(\gamma) - \delta \int_{\Gamma_o} \hat{c}^v(\gamma) \\ &+ \frac{\sigma}{\phi}\left(\mu \int_{A^\gamma} |\nabla \hat{\mathbf{u}}(\gamma)|^2 + \frac{\mu}{K}\int_G \hat{\mathbf{u}}^2 + p_o \int_{\Gamma_o} \hat{u}_1(\gamma) + \int_M \hat{p}(\gamma)\hat{u}_2(\gamma)\right). \qquad (1.10) \end{aligned}$$

2. Variational formulation.

Assuming (1.2)–(1.6) has a smooth solution, we multiply (1.2)–(1.5) by smooth test functions φ, θ, $\mathbf{v}$, with $\nabla \cdot \mathbf{v} = 0$, and integrate second order derivative terms by parts. We get

$$\begin{aligned} \int_{\Omega_\gamma} D(\nabla \hat{c} \cdot \nabla \varphi) + (\hat{\mathbf{u}} \cdot \nabla \hat{c})\varphi &= \int_{\Gamma_i} D\varphi \partial_\nu \hat{c} + \int_{\Gamma_\gamma \cup \Gamma_o \cup \Gamma_w} D\varphi \partial_\nu \hat{c} + \int_M D\varphi \partial_\nu \hat{c} \\ &= \int_{\Gamma_i} D\varphi \partial_\nu \hat{c} - \int_M D\hat{c}\varphi, \qquad (2.1) \\ \int_{\Omega_\gamma} \kappa(\nabla \hat{\tau} \cdot \nabla \theta) + (\hat{\mathbf{u}} \cdot \nabla \hat{\tau})\theta &= \int_{\Gamma_i} \kappa\ \theta \partial_\nu \hat{\tau} + \int_{\Gamma_\gamma \cup \Gamma_o \cup \Gamma_w} \kappa\ \theta \partial_\nu \hat{\tau} + \int_M \kappa\ \theta \partial_\nu \hat{\tau} \\ &= \int_{\Gamma_i} \kappa\ \theta \partial_\nu \hat{\tau} + \int_{\Gamma_\gamma \cup \Gamma_w} \kappa\ \theta(\tau_w - \hat{\tau}) + \int_M \kappa\ \hat{c}\theta, \qquad (2.2) \\ \int_{A^\gamma} \mu(\nabla \hat{\mathbf{u}} \cdot \nabla \mathbf{v}) + \int_G \frac{\mu}{K}(\hat{\mathbf{u}} \cdot \mathbf{v}) &= \int_{\partial A^\gamma} \mu(\mathbf{v} \cdot \partial_{\mathbf{n}^\gamma} \hat{\mathbf{u}}) - \int_{\partial A^\gamma} \hat{p}(\mathbf{v} \cdot \mathbf{n}^\gamma) - \int_{\partial G} \hat{p}(\mathbf{v} \cdot \mathbf{n}) \\ &= \int_{\Gamma_i} \mu(-v_1 \partial_1 \hat{u}_1 - v_2 \partial_1 \hat{u}_2) + \int_{\Gamma_o} \mu(v_1 \partial_1 \hat{u}_1 + v_2 \partial_1 \hat{u}_2) \\ &\quad + \int_{\Gamma_\gamma} \mu(\mathbf{v} \cdot \partial_{\mathbf{n}^\gamma} \hat{\mathbf{u}}) + \int_\Sigma \mu(v_1 \partial_2 \hat{u}_1 + v_2 \partial_2 \hat{u}_2) \\ &\quad + \int_{\Gamma_i} p_i v_1 - \int_{\Gamma_o} p_o v_1 - \int_{\Gamma_\gamma \cup \Sigma} \hat{p}(\mathbf{v} \cdot \mathbf{n}^\gamma) - \int_{\Sigma \cup \Gamma_w \cup M} \hat{p}(\mathbf{v} \cdot \mathbf{n}) \\ &= -\int_{\Gamma_i} \mu v_2 \partial_1 \hat{u}_2 + \int_{\Gamma_o} \mu v_2 \partial_1 \hat{u}_2 + \int_{\Gamma_\gamma} \mu(\mathbf{v} \cdot \partial_{\mathbf{n}^\gamma} \hat{\mathbf{u}}) \\ &\quad + \int_\Sigma \mu v_1(\cdot, 0^-)\partial_2 \hat{u}_1(\cdot, 0^-) + \int_{\Gamma_i} p_i v_1 - \int_{\Gamma_\gamma \cup \Gamma_w \cup M} \hat{p}(\mathbf{v} \cdot \nu_\gamma). \qquad (2.3) \end{aligned}$$

In the last equality we have used the boundary conditions (1.6), in particular the condition concerning the slope $[\cdot]$, so that the terms on Σ originating from both the A_γ and G domain equations disappear. Also, we use the divergence free condition $\nabla \cdot \mathbf{u} = 0$, assumed to be true on the closures $\overline{A^\gamma}$ and $\overline{G}$, which implies $\partial_1 \hat{u}_1 = -\partial_2 \hat{u}_2 = 0$ on $\Gamma_i \cup \Gamma_o$ because $\hat{u}_2 = 0$, and $\partial_2 \hat{u}_2 = -\partial_1 \hat{u}_1 = 0$ on Σ because $\hat{u}_1 = 0$ (let us note that $\hat{\mathbf{u}}$ and $\mathbf{v}$ are assumed smooth enough, say C^2 functions). The boundary conditions (1.6) suggest the choice of spaces associated to $\hat{c}$, $\hat{\tau}$ and $\hat{\mathbf{u}}$. Namely, let us introduce the following spaces:

$$\begin{aligned} \hat{\mathcal{C}}(A^\gamma) &= \{\hat{c} \in \mathcal{D}(\mathbb{R}^2),\ \ \hat{c} = c_i \text{ on } \overline{\Gamma}_i\}, \qquad \mathcal{C}(A^\gamma) = \hat{\mathcal{C}}(A^\gamma) - c_i, \\ \hat{\boldsymbol{\mathcal{C}}}(A^\gamma) &= \overline{\hat{\mathcal{C}}(A^\gamma)}^{\|\cdot\|_{H^1(\Omega_\gamma)}}, \qquad \boldsymbol{\mathcal{C}}(A^\gamma) = \overline{\mathcal{C}(A^\gamma)}^{\|\cdot\|_{H^1(\Omega_\gamma)}}, \qquad (2.4) \end{aligned}$$

where the overline sign denotes the closure with respect to the corresponding norm. Similarly, let us introduce the spaces for $\hat{\tau}$ and $\hat{\mathbf{u}}$.

$$\begin{aligned}
&\hat{\mathcal{T}}(A^\gamma) = \{\hat{\tau} \in \mathcal{D}(\mathbb{R}^2), \ \ \hat{\tau} = \tau_i \text{ on } \overline{\Gamma}_i\}, \mathcal{T}(A^\gamma) = \hat{\mathcal{T}}(A^\gamma) - \tau_i, \\
&\qquad \hat{\boldsymbol{\mathcal{T}}}(A^\gamma) = \overline{\hat{\mathcal{T}}(A^\gamma)}^{\|\cdot\|_{H^1(\Omega_\gamma)}}, \boldsymbol{\mathcal{T}}(A^\gamma) = \overline{\mathcal{T}(A^\gamma)}^{\|\cdot\|_{H^1(\Omega_\gamma)}} (= \boldsymbol{\mathcal{C}}(A^\gamma)), \qquad (2.5) \\
&\hat{\mathcal{U}}(A^\gamma) = \{\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2) \in \mathcal{D}(\mathbb{R}^2; \mathbb{R}^2), \ \nabla \cdot \hat{\mathbf{u}} = 0, \\
&\qquad \int_{\Gamma_i} \hat{u}_1 = \phi, \quad \hat{u}_1|_{\overline{\Gamma} \cup \overline{\Sigma} \cap \overline{\Gamma}_w} = 0, \quad \hat{u}_2|_{\overline{\Gamma} \cup \overline{\Gamma}_i \cap \overline{\Gamma}_o} = 0\}, \\
&\mathcal{U}(A^\gamma) = \{\mathbf{v} = (v_1, v_2) = \hat{\mathbf{u}} - \hat{\mathbf{u}}^0, \quad \hat{\mathbf{u}}, \ \hat{\mathbf{u}}^0 \in \hat{\mathcal{U}}(A^\gamma), \ \hat{\mathbf{u}}^0 \text{ fixed}, \ \ v_2|_{\overline{M}} = 0\}, \\
&\qquad \hat{\boldsymbol{\mathcal{U}}}(A^\gamma) = \overline{\hat{\mathcal{U}}(A^\gamma)}^{\|\cdot\|_{H^1(A^\gamma)} + \|\cdot\|_{L^2(G)}}, \ \ \boldsymbol{\mathcal{U}}(A^\gamma) = \overline{\mathcal{U}(A^\gamma)}^{\|\cdot\|_{H^1(A^\gamma)} + \|\cdot\|_{L^2(G)}}. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.6)
\end{aligned}$$

We point out that the “hat”(ˆ) sets are affine spaces and the corresponding “non-hat” sets are linear spaces.

Remark 2.1. From the construction of the space $\hat{\boldsymbol{\mathcal{U}}}(A^\gamma)$, it follows that for all $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2) \in \hat{\boldsymbol{\mathcal{U}}}(A^\gamma)$, the trace $\hat{u}_2(\cdot, 0)$ on Σ is well defined and $\hat{u}_2(\cdot, 0) \in H^{1/2}(\Sigma)$. Indeed, let $\hat{\mathbf{u}}^n = (\hat{u}_1^n, \hat{u}_2^n) \in \hat{\mathcal{U}}(A^\gamma)$ with $\hat{\mathbf{u}}^n \to \hat{\mathbf{u}}$ in $\hat{\boldsymbol{\mathcal{U}}}(A^\gamma)$. It follows that $\hat{\mathbf{u}}^n \to \hat{\mathbf{u}}$ in $H^1(A^\gamma)$, which implies $\hat{u}_2^n(\cdot, 0^-) \to \hat{u}_2(\cdot, 0^-)$ in $H^{1/2}(\Sigma)$. On the other hand, we have $\hat{\mathbf{u}}^n \to \hat{\mathbf{u}}$ in $L^2(G)$. This implies that $\hat{u}_2(\cdot, 0^+)$, the trace on Σ of $\hat{u}_2 \in L^2(G)$, is well defined in $H^{-1/2}(\Sigma)$ because $\int_\Sigma \hat{u}_2 v_2 = \int_G \hat{\mathbf{u}} \cdot \nabla v_2$, $\mathbf{v} = (v_1, v_2) \in \boldsymbol{\mathcal{U}}(A^\gamma)$; see [18]. This gives $\hat{u}_2^n(\cdot, 0^+) \to \hat{u}_2(\cdot, 0^+)$ in $H^{-1/2}(\Sigma)$. But $\hat{u}_2^n(\cdot, 0^-) = \hat{u}_2^n(\cdot, 0^+)$, which implies $\hat{u}_2(\cdot, 0^-) = \hat{u}_2(\cdot, 0^+) \in H^{1/2}(\Sigma)$.

For the $\hat{u}_1$ component of $\hat{\mathbf{u}}$ in general, unlike for $\hat{u}_2$, we do not have “continuity” on Σ. In fact, we have

$$\hat{u}_1(\cdot, 0^-) = 0 \ \text{ in } H^{1/2}(\Sigma), \quad \partial_1 \hat{u}_1(\cdot, 0^-) = \partial_2 \hat{u}_2(\cdot, 0^-) = 0 \ \text{ in } H^{-1/2}(\Sigma). \qquad (2.7)$$

Indeed, the first equality comes from the continuity of the embedding $H^1(A^\gamma) \subset H^{1/2}(\Sigma)$. For the second equality of (2.7), from $\nabla \cdot \hat{\mathbf{u}} = 0$ and $\hat{\mathbf{u}} \cdot \mathbf{n}^\gamma \in H^{1/2}(\partial A^\gamma)$, it follows that $\hat{\mathbf{u}} \in C^\infty(A^\gamma)$. Therefore, we have $\partial_2 \hat{\mathbf{u}} \in L^2(A^\gamma) \times L^2(A^\gamma)$ and $\nabla \cdot \partial_2 \hat{\mathbf{u}} = \partial_2(\nabla \cdot \hat{\mathbf{u}}) = 0$. It follows that the map $\partial_2 \hat{\mathbf{u}} \in L^2(A^\gamma) \times L^2(A^\gamma) \mapsto (\partial_2 \hat{\mathbf{u}} \cdot \mathbf{n}^\gamma) \in H^{-1/2}(\partial A^\gamma)$ is continuous (see [18]), which implies the continuity of the map $\partial_2 \hat{\mathbf{u}} \in L^2(A^\gamma) \times L^2(A^\gamma) \mapsto \partial_2 \hat{u}_2 \in H^{-1/2}(\Sigma)$. As $\partial_1 \hat{u}_1 = -\partial_2 \hat{u}_2$ in $L^2(A^\gamma)$ it follows that $\partial_2 \hat{\mathbf{u}} = (\partial_2 \hat{u}_1, -\partial_1 \hat{u}_1) \in L^2(A^\gamma) \times L^2(A^\gamma) \mapsto -\partial_1 \hat{u}_1 \in H^{-1/2}(\Sigma)$ is also continuous, and therefore $\partial_1 \hat{u}_1 \in H^{-1/2}(\Sigma)$ and $\partial_1 \hat{u}_1 = -\partial_2 \hat{u}_2$ in $H^{-1/2}(\Sigma)$. Finally, we get $\partial_1 \hat{u}_1 = \partial_2 \hat{u}_2 = 0$ in $H^{-1/2}(\Sigma)$ because $\hat{\boldsymbol{\mathcal{U}}}$ is the $H^1(A^\gamma) \times H^1(A^\gamma)$ closure of free divergence elements $\hat{\mathbf{u}}$ with $0 = \hat{u}_1 = \partial_1 \hat{u}_1 = -\partial_2 \hat{u}_2$ on Σ, and therefore the continuity of $\partial_2 \hat{\mathbf{u}} \in L^2(A^\gamma) \times L^2(A^\gamma) \mapsto \partial_2 \hat{u}_2 = -\partial_1 \hat{u}_1 \in H^{-1/2}(\Sigma)$ proves (2.7). □

Using (1.6), (2.1)–(2.3) and the spaces (2.4)–(2.6), we get the following weak formulation. Find $(\hat{c}, \hat{\tau}, \hat{\mathbf{u}}) \in \hat{\boldsymbol{\mathcal{C}}}(A^\gamma) \times \hat{\boldsymbol{\mathcal{T}}}(A^\gamma) \times \hat{\boldsymbol{\mathcal{U}}}(A^\gamma)$ with $\hat{u}_2 = -g(\hat{c})$ on M, satisfying

$$\begin{aligned}
\int_{\Omega_\gamma} D(\nabla\hat{c}\cdot\nabla\varphi) + (\hat{\mathbf{u}}\cdot\nabla\hat{c})\varphi + \int_M D\hat{c}\varphi &= 0, \ \forall\varphi\in\boldsymbol{\mathcal{C}}(A^\gamma), && (2.8)\\
\int_{\Omega_\gamma} \kappa(\nabla\hat{\tau}\cdot\nabla\theta) + (\hat{\mathbf{u}}\cdot\nabla\hat{\tau})\theta + \int_{\Gamma_\gamma\cup\Gamma_w} \kappa\ \hat{\tau}\theta - \int_M \kappa\ \hat{c}\theta &= \int_{\Gamma_\gamma\cup\Gamma_w} \kappa\ \tau_w\theta, \ \forall\theta\in\boldsymbol{\mathcal{T}}(A^\gamma), && (2.9)\\
\int_{A^\gamma} \mu(\nabla\hat{\mathbf{u}}\cdot\nabla\mathbf{v}) + \int_G \frac{\mu}{K}(\hat{\mathbf{u}}\cdot\mathbf{v}) &= 0, \ \forall\mathbf{v}\in\boldsymbol{\mathcal{U}}(A^\gamma). && (2.10)
\end{aligned}$$

The boundary terms of (2.3) disappear because of the choice of the space $\boldsymbol{\mathcal{U}}$. Finally, we can look for the solution of (2.8)–(2.10) in the form

$$\begin{aligned}
(c,\tau,\mathbf{u}) \in \boldsymbol{\mathcal{C}}(A^\gamma)\times\boldsymbol{\mathcal{T}}(A^\gamma)\times\boldsymbol{\mathcal{U}}(A^\gamma), \quad & \hat{c} = c + c_i, \ \ \hat{\tau} = \tau + \tau_i, \ \ \hat{\mathbf{u}} = \mathbf{u} + \mathbf{u}^c,\\
& \mathbf{u}^c = (u_1^c, u_2^c) \in \hat{\boldsymbol{\mathcal{U}}}(A^\gamma), \ \ u_2^c = -g(\hat{c}) \text{ on } M,
\end{aligned} \tag{2.11}$$

satisfying

$$\begin{aligned}
\int_{\Omega_\gamma} D(\nabla c\cdot\nabla\varphi) + (\hat{\mathbf{u}}\cdot\nabla c)\varphi + \int_M Dc\varphi &= -\int_M Dc_i\varphi, \quad \forall\varphi\in\boldsymbol{\mathcal{C}}(A^\gamma), && (2.12)\\
\int_{\Omega_\gamma} \kappa(\nabla\tau\cdot\nabla\theta) + (\hat{\mathbf{u}}\cdot\nabla\tau)\theta + \int_{\Gamma_\gamma\cup\Gamma_w} \kappa\ \tau\theta - \int_M \kappa\ c\theta &= \int_{\Gamma_\gamma\cup\Gamma_w} \kappa\ (\tau_w - \tau_i)\theta\\
&\quad + \int_M \kappa c_i\theta, \quad \forall\theta\in\boldsymbol{\mathcal{T}}(A^\gamma), && (2.13)\\
\int_{A^\gamma} \mu(\nabla\mathbf{u}\cdot\nabla\mathbf{v}) + \int_G \frac{\mu}{K}(\mathbf{u}\cdot\mathbf{v}) &= \int_{A^\gamma} \mu(-\nabla\mathbf{u}^c\cdot\nabla\mathbf{v}) + \int_G \frac{\mu}{K}(-\mathbf{u}^c\cdot\mathbf{v}),\\
&\quad \forall\mathbf{v}\in\boldsymbol{\mathcal{U}}(A^\gamma). && (2.14)
\end{aligned}$$

Let us emphasize that the choice of the space $\boldsymbol{\mathcal{U}}(A^\gamma)$ and the decomposition $\hat{\mathbf{u}} = \mathbf{u} + \mathbf{u}^c$ are appropriate for proving the existence of the solution, as they eliminate the pressure term (the integral on M) from the $\hat{\mathbf{u}}$ equation (2.3).

Finally, problems (2.8)–(2.10) and (2.12)–(2.14) are equivalent in the following sense. A solution $(c, \tau, \mathbf{u})$ of (2.12)–(2.14) gives a solution $(\hat{c}, \hat{\tau}, \hat{\mathbf{u}}) = (c + c_i, \tau + \tau_i, \mathbf{u} + \mathbf{u}^c)$ of (2.8)–(2.10). On the other hand, a solution $(\hat{c}, \hat{\tau}, \hat{\mathbf{u}})$ of (2.8)–(2.10) in general may give many solutions $(c, \tau, \mathbf{u})$ of (2.12)–(2.14), depending on the decomposition $\hat{\mathbf{u}} = \mathbf{u} + \mathbf{u}^c$. In what follows, for a given $(\hat{c}, \hat{\tau}, \hat{\mathbf{u}})$, we will consider a unique decomposition $\hat{\mathbf{u}} = \mathbf{u} + \mathbf{u}^c$, with $\mathbf{u}^c$ given by Proposition 3.5.

3. Existence of the state solution and of the optimal shape.

In this section we consider the system of PDEs (2.11), (2.12)–(2.14), and the shape optimization problem (1.8). We will prove that the system has a solution using a compactness argument. Namely, we will first show that for a given $\hat{\mathbf{u}} \in L^q(\Omega_\gamma)$, $q > 2$, there exists a unique $(\hat{c}, \hat{\tau}) \in \hat{\boldsymbol{\mathcal{C}}}(A^\gamma) \times \hat{\boldsymbol{\mathcal{T}}}(A^\gamma)$ solution of (2.8), (2.9), uniformly bounded in $H^1(\Omega_\gamma)^2$. Then, (2.10) has a unique solution $\hat{\mathbf{u}}$ uniformly bounded in $H^1(\Omega_\gamma)$. A compactness argument gives an existence result for (2.8)–(2.10).

There is a large amount of literature for elliptic nonlinear PDE systems, for example (certainly a non-exhaustive list) [3], [4], [12], [15]. The particularity of the system (2.8)–(2.10) is that the principal part of the third equation does not involve the second derivatives in the whole domain and so, in general, the terms $(\hat{\mathbf{u}} \cdot \nabla \hat{c})\varphi$ and $(\hat{\mathbf{u}} \cdot \nabla \hat{\tau})\theta$ are not well defined. Also, the set of boundary conditions requires particular attention as they involve nonlinearities and the pressure boundary conditions. For these reasons this system of equations needs a particular treatment.

Proposition 3.1. Let $\hat{\mathbf{u}} \in L^q(A^\gamma \cup G)$, $q > 2$, $(c, \tau) \in \boldsymbol{\mathcal{C}}(A^\gamma) \times \boldsymbol{\mathcal{T}}(A^\gamma)$ satisfying (2.8), (2.9). Then $(c, \tau) \in C^\alpha(\overline{\Omega}_\gamma)$, $0 < \alpha < 1$.

Proof. The function c satisfies, in the weak sense, $-D\Delta c = f$ in Ω_γ, $f = -\hat{\mathbf{u}} \cdot \nabla c$, with mixed Dirichlet and Neumann boundary conditions on $\partial\Omega_\gamma$. Moreover, $f \in (W^{1,p}(\Omega_\gamma))'$, $p = (2q)/(q-2) > 2$ because for $\varphi \in W^{1,p}(\Omega_\gamma)$ we have

$$\left| \int_{\Omega_\gamma} f\varphi \right| \leq \|\nabla c\|_{L^2(\Omega_\gamma)} \|\hat{\mathbf{u}}\|_{L^q(\Omega_\gamma)} \|\varphi\|_{L^p(\Omega_\gamma)}.$$

From [5] (Theorem 4), it follows that $c \in W^{1,\overline{p}}(\Omega_\gamma)$, for a $\overline{p} > 2$, and from Morrey's theorem it follows that $c \in C^\alpha(\overline{\Omega}_\gamma)$. The proof for τ is exactly as for c. □

Remark 3.2. (i) The continuity and C^α-regularity of c, τ may be proven in different ways, for example using the techniques in [12] estimating osc (Sec. 4, Chapter 2).
(ii) The previous proposition provides C^0 bounds for $\hat{c}$, $\hat{\tau}$. In order to obtain a more explicit dependence of all the constants involved on these bounds, we will prove directly the C^0 boundedness of $\hat{c}$, $\hat{\tau}$.

Proposition 3.3. Assume $\hat{\mathbf{u}} \in L^q(A^\gamma \cup G)$, $q > 2$, $(c, \tau) \in \boldsymbol{\mathcal{C}}(A^\gamma) \times \boldsymbol{\mathcal{T}}(A^\gamma)$ satisfying (2.12), (2.13). Then $\hat{c} = c + c_i$, $\hat{\tau} = \tau + \tau_i$ satisfy

$$0 \leq \quad \hat{c} \quad \leq c_i, \quad x \in \Omega_\gamma, 2 \tag{3.1}$$

$$\hat{\tau}_m := \min\{\tau_i, \inf \tau_w\} \leq \quad \hat{\tau} \quad \leq \hat{\tau}_M, \quad x \in \Omega_\gamma, \tag{3.2}$$

where $\hat{\tau}_M$ depends only on $(c_i, \tau_i, \tau_w, k, \boldsymbol{\alpha}, \boldsymbol{\beta}, G)$.

Proof. The proof follows the techniques used in [6], [12].
a) $\hat{c} \geq 0$. We can apply the technique used for proving the weak maximum principle in [6]. Indeed, let $m = \inf\{\hat{c}(x),\ x \in \Omega_\gamma\}$. Assume for a moment that $m < 0$. For $m < k < 0$, set $c_k = \min\{\hat{c} - k, 0\}$. Then $c_k \in \boldsymbol{\mathcal{C}}(A^\gamma)$, $\nabla c_k = \nabla \hat{c} = \nabla c$ in $Z_k := \{\hat{c} < k\}$, and of course $c_k = c - k$ in Z_k and $c_k = 0$ in $\Omega_\gamma \backslash Z_k$. Taking $\varphi = c_k$ in (2.12), we get

$$\begin{aligned}
0 \leq \int_{Z_k} D|\nabla c_k|^2 \quad &= \quad -\int_M D\hat{c}c_k - \int_{Z_k} (\hat{\mathbf{u}} \cdot \nabla c)c_k \\
(\text{as } \hat{c}c_k \geq 0) \quad &\leq \quad \int_{Z_k} |(\hat{\mathbf{u}} \cdot \nabla c_k)c_k| \\
\left(p = \frac{2q}{q-2}\right) \quad &\leq \quad \|\hat{\mathbf{u}}\|_{L^q(Z_k)} \|1\|_{L^{2p}(Z_k)} \|c_k\|_{L^{2p}(Z_k)} \|\nabla c_k\|_{L^2(Z_k)} \\
(\text{from Sob. ineq.}) \quad &\leq \quad C(\Omega_\gamma) \|\hat{\mathbf{u}}\|_{L^q(Z_k)} |Z_k|^{\frac{1}{2p}} \|\nabla c_k\|^2_{L^2(Z_k)},
\end{aligned}$$

which implies $D \leq C(\Omega_\gamma)\|\hat{\mathbf{u}}\|_{L^q(Z_k)}|Z_k|^{1/(2p)}$ with $C(\Omega_\gamma)$ not depending on Z_k. As $k \neq m$ it follows that $\|\nabla c_k\|_{L^2(Z_k)} \neq 0$. This implies that $1 \leq K|Z_k|^{1/(2p)}$, which is impossible if we let $k \to m$ because $|Z_k|^{1/(2p)} \to 0$. This proves $m \geq 0$.

b) $\hat{c} \leq c_i$. Let $m = \sup\{\hat{c}(x),\ x \in \Omega_\gamma\}$. Assume for a moment that $m > c_i$. Then for $c_i < k < m$, as in part a), set $c_k = \max\{\hat{c} - k, 0\}$, $Z_k = \{\hat{c} > k\}$. Again, $c_k \in \boldsymbol{\mathcal{C}}(A^\gamma)$ and $cc_k \geq 0$. As in case a), we find that

$$0 \leq \int_{Z_k} D|\nabla c_k|^2 \leq \|\hat{\mathbf{u}}\|_{L^q(Z_k)} \|1\|_{L^{2p}(Z_k)} \|c_k\|_{L^{2p}(Z_k)} \|\nabla c_k\|_{L^2(Z_k)}.$$

We proceed exactly as in part a) and we find that $m > c_i$ leads to a contradiction, which implies $\hat{c} \leq c_i$.

c) $\hat{\tau} \geq \tau_m$. The proof very closely follows the proof for part a). Indeed, let $m = \inf\{\hat{\tau}(x),\ x \in \Omega_\gamma\}$ and assume for a moment that $m < \hat{\tau}_m$. For $m < k < \hat{\tau}_m$, set $\tau_k = \min\{\hat{\tau} - k, 0\}$. Then $\tau_k \in \boldsymbol{\mathcal{T}}(A^\gamma)$ and $\nabla \tau_k = \nabla \hat{\tau} = \nabla \tau$ in $Z_k := \{\hat{\tau} < k\}$. Taking $\theta = \tau_k$ in (2.13) we get

$$\begin{aligned} 0 \leq \int_{Z_k} \kappa |\nabla \tau_k|^2 &= \int_M \kappa \hat{c} \tau_k + \int_{\Gamma_\gamma \cup \Gamma_w} \kappa(\tau_w - \hat{\tau})\tau_k - \int_{Z_k} (\hat{\mathbf{u}} \cdot \nabla \tau_k)\tau_k \\ \text{(as } \hat{c}\tau_k,\ (\tau_w - \hat{\tau})\tau_k \leq 0 \text{ in } Z_k) \quad &\leq \int_{Z_k} |(\hat{\mathbf{u}} \cdot \nabla \tau_k)\tau_k| \\ &\leq \|\hat{\mathbf{u}}\|_{L^q(Z_k)} \|1\|_{L^{2p}(Z_k)} \|\tau_k\|_{L^{2p}(Z_k)} \|\nabla \tau_k\|_{L^2(Z_k)}. \end{aligned}$$

Next, we proceed exactly as we did in part a).

d) $\hat{\tau} \leq \tau_M$. The proof of this estimation is a little bit different, due to the boundary conditions. However, in the case $D = \kappa$ the proof is very easy by considering $v = c + \tau$, $\hat{v} = \hat{c} + \hat{\tau}$. Then

$$\int_{\Omega_\gamma} \kappa(\nabla v \cdot \nabla \theta) + (\mathbf{u} \cdot \nabla v)\theta = \int_{\Gamma \cup \Gamma_w} (2\tau_w - \hat{v})\theta, \quad \forall \theta \in \boldsymbol{\mathcal{T}}(A^\gamma).$$

We proceed as in a) and easily obtain an upper bound for $\hat{v}$ and also for $\hat{\tau}$.

In general, we use the result in [12] (Lemma 5.3, Chap. 2). Indeed, let $k_0 = \max\{\sup \tau_w + c_i, \tau_i\}$. For $k \geq k_0$ set $\tau_k = \max\{\hat{\tau} - k, 0\}$. From (2.13) we get

$$\begin{aligned} &\int_{Z_k} \kappa |\nabla \tau|^2 \\ = \ &\int_M \kappa \hat{c} \tau_k + \int_{\Gamma_\gamma \cup \Gamma_w} \kappa(\tau_w - \hat{\tau})\tau_k - \int_{Z_k} (\hat{\mathbf{u}} \cdot \nabla \tau_k)\tau_k \\ \leq \ &\kappa \left(c_i \int_M \tau_k + \int_{\Gamma_\gamma \cup \Gamma_w} (\tau_w - \hat{\tau})\tau_k \right) + \int_{Z_k} |(\hat{\mathbf{u}} \cdot \nabla \tau_k)\tau_k| \\ = \ &\kappa \left(c_i \int_{Z_k} \partial_2 \tau_k - c_i \int_{\Gamma_\gamma} \nu_2^\gamma \tau_k + \int_{\Gamma_\gamma \cup \Gamma_w} (\tau_w - \hat{\tau})\tau_k \right) + \int_{Z_k} |(\hat{\mathbf{u}} \cdot \nabla \tau_k)\tau_k| \\ = \ &\kappa \left(c_i \int_{Z_k} \partial_2 \tau_k + \int_{\Gamma_\gamma} (\tau_w - c_i \nu_2^\gamma - \hat{\tau})\tau_k + \int_{\Gamma_w} (\tau_w - \hat{\tau})\tau_k \right) + \int_{Z_k} |(\hat{\mathbf{u}} \cdot \nabla \tau_k)\tau_k| \\ \leq \ &\kappa c_i \|\partial_2 \tau_k\|_{L^1(Z_k)} + \|\hat{\mathbf{u}}\|_{L^q(Z_k)} \|1\|_{L^{2p}(Z_k)} \|\tau_k\|_{L^{2p}(Z_k)} \|\nabla \tau_k\|_{L^2(Z_k)} \\ \leq \ &\kappa c_i (\epsilon^{-1}|Z_k| + \epsilon \|\nabla \tau_k\|^2_{L^2(Z_k)}) + \epsilon^{-1}|Z_k|^{1/p} + \epsilon C(\boldsymbol{\alpha}, \boldsymbol{\beta}, G) \|\hat{\mathbf{u}}\|^2_{L^q(Z_k)} \|\nabla \tau_k\|^4_{L^2(Z_k)} \\ \leq \ &\epsilon^{-1}(\kappa c_i |Z_k| + |Z_k|^{1/p}) + \epsilon(\kappa c_i + C(\boldsymbol{\alpha}, \boldsymbol{\beta}, G) \|\hat{\mathbf{u}}\|^2_{L^q(\Omega_\gamma)} \|\nabla \tau\|^2_{L^2(\Omega_\gamma)}) \|\nabla \tau\|^2_{L^2(Z_k)}, \end{aligned}$$

because $\tau_w - \nu_2^\gamma c_i \leq \hat{\tau}$, $\tau_w \leq \hat{\tau}$ in Z_k. In the previous estimations we have used the Poincaré inequality in Ω_γ which makes the constant $C(k, \boldsymbol{\alpha}, \boldsymbol{\beta}, G)$ appear. For $\epsilon > 0$ small, it follows that $\|\nabla \tau\|^2_{L^2(Z_k)} \leq K|Z_k|^{1/p}$. From Lemma 5.3, Chap. 2, [12] follows the upper boundedness of τ, and consequently of $\hat{\tau}$. □

Proposition 3.4. For given $\hat{\mathbf{u}} \in L^q(A^\gamma \cup G)$, $q > 2$, the system (2.12), (2.13) has a unique solution $(c, \tau) \in \boldsymbol{\mathcal{C}}(A^\gamma) \times \boldsymbol{\mathcal{T}}(A^\gamma)$ satisfying

$$\begin{aligned} \|\nabla c\|_{L^2(\Omega_\gamma)} &\leq C(G) c_i (1 + \|\hat{\mathbf{u}}\|_{L^2(\Omega_\gamma)}), && (3.3) \\ \|\nabla \tau\|_{L^2(\Omega_\gamma)} &\leq C(k, \boldsymbol{\alpha}, \boldsymbol{\beta}, G)(c_i + |\tau_i - \tau_w| + \hat{\tau}_M \|\hat{\mathbf{u}}\|_{L^2(\Omega_\gamma)}). && (3.4) \end{aligned}$$

Proof. If we assume uniqueness, then the existence of the solution is obtained following the classical existence theory for second order elliptic linear PDE (systems), as in [6], [12]. For sake of completeness we will present a direct proof. Equation (2.12) is independent from (2.13), so we can solve c first. We set $L : \boldsymbol{\mathcal{C}} \mapsto \boldsymbol{\mathcal{C}}^*$, where $\boldsymbol{\mathcal{C}}^*$ is the dual space of $\boldsymbol{\mathcal{C}}$, defined by

$$Lc(\varphi) = \int_{\Omega_\gamma} D(\nabla c \cdot \nabla \varphi) + (\hat{\mathbf{u}} \cdot \nabla c)\varphi + \int_M D c \varphi, \qquad \forall\ c,\ \varphi \in \boldsymbol{\mathcal{C}}$$

and $\mathcal{L}(c, \varphi) = Lc(\varphi)$. We point out that (2.12) is equivalent to $L(c) = l$, where $l(\varphi) = -\int_M c_i \varphi$. To prove the existence of c we follow the technique used in [6] (Section 8.2). The bilinear form $\mathcal{L}$ is continuous in $\boldsymbol{\mathcal{C}}$ because

$$\begin{aligned} |\mathcal{L}(c, \varphi)| &\leq \textstyle\int_{\Omega_\gamma} D|\nabla c \cdot \nabla \varphi| + |\hat{\mathbf{u}} \cdot \nabla c||\varphi| \\ &\leq D\|\nabla c\|_{L^2(\Omega_\gamma)} \|\nabla \varphi\|_{L^2(\Omega_\gamma)} + \|\hat{\mathbf{u}}\|_{L^q(\Omega_\gamma)} \|\varphi\|_{L^p(\Omega_\gamma)} \|\nabla c\|_{L^2(\Omega_\gamma)} \quad (\tfrac{1}{p} + \tfrac{1}{q} = \tfrac{1}{2}) \\ &\leq D\|\nabla c\|_{L^2(\Omega_\gamma)} \|\nabla \varphi\|_{L^2(\Omega_\gamma)} \\ &\quad + \|\hat{\mathbf{u}}\|_{L^q(\Omega_\gamma)} C(\Omega_\gamma) \|\nabla \varphi\|^2_{L^2(\Omega_\gamma)} \|\nabla c\|^2_{L^2(\Omega_\gamma)} \quad \text{(Sobolev ineq.)} \\ &\leq C\|\nabla c\|_{L^2(\Omega_\gamma)} \|\nabla \varphi\|_{L^2(\Omega_\gamma)}. \end{aligned}$$

Also, we have $|\mathcal{L}(c, c)| \geq D\|\nabla\|^2_{L^2(\Omega_\gamma)} - \int_{\Omega_\gamma} |\hat{\mathbf{u}} \cdot \nabla c)c|$. From the estimation

$$\begin{aligned} \textstyle\int_{\Omega_\gamma} |\hat{\mathbf{u}} \cdot \nabla c)c| &\leq \|\hat{\mathbf{u}}\|_{L^q(\Omega_\gamma)} \|c\|_{L^p(\Omega_\gamma)} \|\nabla c\|_{L^2(\Omega_\gamma)} \ (\tfrac{1}{p} + \tfrac{1}{q} = \tfrac{1}{2}) \\ &\leq \|\hat{\mathbf{u}}\|_{L^q(\Omega_\gamma)} (C(\epsilon)\|c\|^2_{L^p(\Omega_\gamma)} + \epsilon\|\nabla c\|^2_{L^2(\Omega_\gamma)}) \\ &\leq \|\hat{\mathbf{u}}\|_{L^q(\Omega_\gamma)} \big(C(\epsilon)\big(K(\epsilon)\|c\|^2_{L^2(\Omega_\gamma)} + \tfrac{\epsilon}{C(\epsilon)}\|\nabla c\|^2_{L^2(\Omega_\gamma)}\big) \text{ (Ehrling's ineq.)} \\ &+ \epsilon\|\nabla c\|^2_{L^2(\Omega_\gamma)}\big) \\ &\leq \|\hat{\mathbf{u}}\|_{L^q(\Omega_\gamma)} \big(C(\epsilon)K(\epsilon)\|c\|^2_{L^2(\Omega_\gamma)} + 2\epsilon\|\nabla c\|^2_{L^2(\Omega_\gamma)}\big) \end{aligned}$$

where we have used Ehrling's inequality $\|c\|_{L^p(\Omega_\gamma)} \leq C(\epsilon)\|c\|_{L^2(\Omega_\gamma)} + \epsilon\|c\|_{H^1(\Omega_\gamma)}$, because the embedding $H^1(\Omega_\gamma) \subset L^p(\Omega_\gamma)$ is compact and $L^p(\Omega_\gamma) \subset L^2(\Omega_\gamma)$ is continuous, and $|ab| \leq K(\epsilon)a^2 + \epsilon b^2$. For ϵ small, it follows that $|\mathcal{L}(c, c)| \geq K\|\nabla c\|^2_{L^2(\Omega_\gamma)} - \lambda\|c\|^2_{L^2(\Omega_\gamma)}$, for any $K,\ \lambda > 0$. From the Lax-Milgram lemma it follows that the equation $L_\lambda c = l$, with $L_\lambda c = Lc + \lambda c$, has a unique solution in $\boldsymbol{\mathcal{C}}$. The equation $Lc = l$ is equivalent to $(L_\lambda - \lambda I)c = l$, or $(I - \lambda L_\lambda^{-1} I)c = L_\lambda^{-1} l$, where $I : \boldsymbol{\mathcal{C}} :\mapsto \boldsymbol{\mathcal{C}}^*$ is the identity operator, which is compact, and L_λ^{-1} is the inverse of L_λ, which is continuous. It follows that $L_\lambda^{-1} I$ is compact. Assuming that $Lc = l$ has at most one solution, it follows that the kernel of

$I - \lambda L_\lambda^{-1} I = L_\lambda^{-1} L$ is reduced to $\{0\}$. From Fredholm alternatives for compact operators follows the existence of c.

To prove the existence of τ we proceed in a similar way.

For uniqueness, let us assume that the system has at least two solutions and let $\delta_c \in \boldsymbol{\mathcal{C}}(A^\gamma)$, resp. $\delta_\tau \in \boldsymbol{\mathcal{T}}(A^\gamma)$, be the difference of two c, resp. τ solutions. Then δ_c satisfies

$$\int_{\Omega_\gamma} D(\nabla\delta_c \cdot \nabla\varphi) + (\hat{\mathbf{u}} \cdot \nabla\delta_c)\varphi + \int_M D\delta_c\varphi = 0, \quad \forall\varphi \in \boldsymbol{\mathcal{C}}(A^\gamma).$$

Using Proposition 3.3 for δ_c instead of $\hat{c}$, it follows that $\delta_c = 0$ a.e. in Ω_γ. Then, from (2.13) we get

$$\int_{\Omega_\gamma} \kappa(\nabla\delta_\tau \cdot \nabla\theta) + (\hat{\mathbf{u}} \cdot \nabla\delta_\tau)\theta + \int_{\Gamma_\gamma \cup \Gamma_w} \kappa\delta_\tau\theta \quad = \quad 0, \quad \forall\theta \in \boldsymbol{\mathcal{T}}(A^\gamma).$$

Following exactly the same technique as in the proof of Proposition 3.3 it is easy to deduce that $\delta_\tau = 0$, and thus uniqueness is proved.

Now let us prove (3.3), (3.4). Taking $\varphi = c$ in (2.12) we get

$$\begin{aligned}\int_{\Omega_\gamma} D|\nabla c|^2 + \int_M Dc^2 &= -\int_M Dc_i c - \int_{\Omega_\gamma} (\hat{\mathbf{u}} \cdot \nabla c)c \le c_i \left(\int_M |c| + \int_{\Omega_\gamma} |\hat{\mathbf{u}} \cdot \nabla c| \right) \\ &\le C(G)c_i(1 + \|\hat{\mathbf{u}}\|_{L^2(\Omega_\gamma)})\|\nabla c\|_{L^2(\Omega_\gamma)},\end{aligned}$$

where we have used trace inequality. Thus (3.3) is proved.
For the estimation of (3.4), taking $\theta = \tau$ in (2.13) implies

$$\begin{aligned}\int_{\Omega_\gamma} \kappa|\nabla\tau|^2 + \int_{\Gamma_\gamma \cup \Gamma_w} \kappa\tau^2 &= \int_{\Gamma_\gamma \cup \Gamma_w} \kappa\ (\tau_w - \tau_i)\tau + \int_M \kappa\ \hat{c}\tau - \int_{\Omega_\gamma} (\hat{\mathbf{u}} \cdot \nabla\tau)\tau \\ &\le (\kappa|\tau_w - \tau_i| + c_i) \int_{\partial\Omega_\gamma} |\tau| + \int_{\Omega_\gamma} |\hat{\mathbf{u}} \cdot \nabla\tau||\tau| \\ &\le C(k, \boldsymbol{\alpha}, \boldsymbol{\beta}, G) \left(|\tau_w - \tau_i| + c_i + \hat{\tau}_M \|\hat{\mathbf{u}}\|_{L^2(\Omega_\gamma)} \right) \|\nabla\tau\|_{L^2(\Omega_\gamma)},\end{aligned}$$

which proves the proposition. □

Now, let us turn our attention to equation (2.14). For given $c \in \boldsymbol{\mathcal{C}}(A^\gamma)$, the function $\mathbf{u}^c = (u_1^c, u_2^c)$ can be constructed similarly to [12], [18]. Namely, we have

PROPOSITION 3.5. Let $\hat{c} = c + c_i \in \hat{\boldsymbol{\mathcal{C}}}(A^\gamma)$ be given. There exists $\mathbf{u}^c = (\hat{u}_1, \hat{u}_2) \in \hat{\boldsymbol{\mathcal{U}}}(A^\gamma)$ satisfying $\hat{u}_2 = -g(\hat{c})$ on M in the $H^{1/2}(\Sigma)$-sense and

$$\|\mathbf{u}^c\|_{H^1(A^\gamma)} + \|\mathbf{u}^c\|_{L^p(G)} \le C(k, \boldsymbol{\alpha}, \boldsymbol{\beta}, G)(\phi + \|\hat{c}\|_{L^p(M)}), \quad 1 \le p < \infty. \tag{3.5}$$

Proof. For $\gamma \in \mathcal{O}$, if we set

$$\begin{aligned}A_0 &= \{(x_1, x_2), 0 < x_1 < -\tfrac{\alpha_1}{k},\ \gamma(x_1) < x_2 < 0\}, \\ A_l &= \{(x_1, x_2), l + \tfrac{\beta_1}{k} < x_1 < l,\ \gamma(x_1) < x_2 < 0\},\end{aligned}$$

then $A_0 \cup A_l \subset A^\gamma$. Now, let $\varphi \in \mathcal{D}(\mathbb{R}^2)$ be such that

$$\begin{aligned}&\mathrm{supp}(\varphi) \cap \{(0,0), (l,0), (0,\gamma(0)), (l,\gamma(l))\} = \text{empty}, \quad A^\gamma \cap \mathrm{supp}(\varphi) \subset A_0 \cup A_l, \\ &\textstyle\int_\Sigma \varphi = \int_{\Gamma_i} \varphi = \int_{\Gamma_o} \varphi = 1.\end{aligned}$$

We can choose the velocity $\mathbf{u}^c$ to satisfy the following boundary conditions:

$$\begin{array}{rclrcll} & & & u_2^c & = & -g(\hat{c}) & \text{on } M, \\ u_1^c & = & 0 & & & & \text{on } \Gamma_w, \\ u_1^c & = & 0, & u_2^c & = & -\varphi \int_M g(\hat{c}) & \text{on } \Sigma, \\ u_2^c & = & 0, & u_1^c & = & \phi\varphi & \text{on } \Gamma_i, \\ u_2^c & = & 0, & u_1^c & = & \left(\phi + \int_M g(\hat{c})\right)\varphi & \text{on } \Gamma_o, \\ u_1^c & = & 0, & u_2^c & = & 0 & \text{on } \Gamma. \end{array} \tag{3.6}$$

Let us point out that $\mathbf{u}^c$ satisfies the divergence-free compatibility conditions $\int_{\partial A^\gamma} \mathbf{u}^c \cdot \mathbf{n}^\gamma = \int_{\partial G} \mathbf{u}^c \cdot \mathbf{n} = 0$. We look for $\mathbf{u}^c$ in the form $\mathbf{u}^c = (\partial_2\psi, -\partial_1\psi)$. Then such a ψ must satisfy

$$\psi(x_1,x_2) = \begin{cases} \phi & \text{on } \Gamma_w \cap \{x_1 = 0\}, \\ \phi + \int_0^{x_1} g(\hat{c})(t,h)dt & \text{on } M, \\ \phi + \int_0^{l} g(\hat{c})(t,h)dt & \text{on } \Gamma_w \cap \{x_1 = l\}, \\ \phi + \int_M g(\hat{c}) \int_0^{x_1} \varphi(t,0)dt & \text{on } \Sigma, \\ \phi \int_{\gamma(0)}^{x_2} \varphi(0,t)dt & \text{on } \Gamma_i, \\ \left(\phi + \int_M g(\hat{c})\right) \int_{\gamma(l)}^{x_2} \varphi(l,t)dt & \text{on } \Gamma_o, \\ 0 & \text{on } \Gamma. \end{cases} \tag{3.7}$$

An extension of $\mathbf{u}^c$ in G can be constructed as follows. Let

$$\psi(x_1,x_2) = \phi + \xi(x_2)\int_0^{x_1} g(\hat{c})(t,h)dt + \xi(h-x_2)\int_0^{x_1} \varphi(t,0)dt \int_M g(\hat{c}),$$

where $\xi(t) \in C^\infty(\mathbb{R})$ is an appropriate function satisfying

$$\begin{array}{ll} \xi(t) = 1 - \xi(h-t) = \xi'(t) - \xi'(h-t), & t \in \mathbb{R}, \\ \xi(t) = \xi'(t) = 0, & t \le \frac{h}{6}. \end{array}$$

The function ξ may be constructed as follows. Let $\overline{\eta}(t)$ be given by

$$\begin{cases} \overline{\xi}(t) = \overline{\xi}(h-t) - 1 = 0, & t < \frac{h}{3}, \\ \overline{\xi}(t) = \frac{3}{h}t - 1, & \frac{1}{3}h \le t \le \frac{2}{3}h, \end{cases}$$

let $\eta(t)$ be the standard mollifier and let $\eta_n(t) = n^{-2}\eta(n^{-1}t)$. Then $\xi(t) = \eta_n * \overline{\xi}$ satisfies the requirements. It follows that

$$\begin{array}{rcl} u_1^c & = & \xi'(x_2)\int_0^{x_1} g(\hat{c})(t,h)dt - \xi'(h-x_2)\int_0^{x_1} \varphi(t,0)dt \int_M g(\hat{c}), \\ u_2^c & = & -\xi(x_2)g(\hat{c})(x_1,h) - \xi(h-x_2)\varphi(x_1,0)\int_M g(\hat{c}). \end{array}$$

As $\hat{c}$ is bounded and positive it follows that $|g(\hat{c})| \le \hat{c}$, $\|g(\hat{c})\|_{L^p(M)} \le \|\hat{c}\|_{L^p(M)}$ and

$$\mathbf{u}^c \in L^p(G), \quad \|\mathbf{u}^c\|_{L^p(G)} \le C(G)\|g(\hat{c})\|_{L^p(M)} \le C(G)\|\hat{c}\|_{L^p(M)}. \tag{3.8}$$

Let us point out that the previous estimation does not depend on γ. Moreover, u_2^c is differentiable w.r.t. x_2 and $u_c^2(x_1,h) = -g(\hat{c}) \in H^{1/2}(\Sigma)$.

The extension of ψ in A^γ may be constructed as follows. For $\mathbf{x} = (x_1, x_2) \in A^\gamma \backslash (A_0 \cup A_l)$ we set $\psi(\mathbf{x}) = 0$. In $A_0 \cup A_l$ we set

$$\psi(\mathbf{x}) = \begin{cases} \psi(0, x_2 - kx_1)\xi\left(\frac{x_2^2 h}{x_1^2+x_2^2}\right) + \psi(x_1 - \frac{x_2}{k}, 0)\xi\left(\frac{x_1^2 h}{x_1^2+x_2^2}\right), & \mathbf{x} \in A_0, \\ \psi(0, x_2 + k(l - x_1))\xi\left(\frac{x_2^2 h}{(x_1-l)^2+x_2^2}\right) + \psi(x_1 + \frac{x_2}{k}, 0)\xi\left(\frac{(x_1-l)^2 h}{(x_1-l)^2+x_2^2}\right), & \mathbf{x} \in A_l. \end{cases} \tag{3.9}$$

Let us point out that from the choice of $\operatorname{supp}(\varphi)$, in a neighborhood of $(0,0)$, resp. $(l,0)$, we have $\psi(x_1,x_2) = \psi(0,0) = \phi$, resp. $\psi(x_1,x_2) = \psi(l,0) = \phi + \int_M g(\hat{c})$. Also, $u_1^c = 0$ in a neighborhood of Σ because from the properties of ξ we have $\psi(x_1,x_2) = \psi(x_1,0)$. Similarly, we have $u_2^c = 0$ in a neighborhood of Γ_i and Γ_0. From the extension (3.9) it follows that $\|\mathbf{u}^c\|_{H^1(A^\gamma)}$ will be bounded only by $\|\psi(x_1,0)\|_{H^2}$, $\|\psi(0,x_2)\|_{H^2}$, $\|\psi(l,x_2)\|_{H^2}$. From (3.7) it follows that $\|\mathbf{u}^c\|_{H^1(A^\gamma)} \le C(k, \boldsymbol{\alpha}, \boldsymbol{\beta}, G)(\phi + \|\hat{c}\|_{L^2(M)})$ (as $\hat{c}$ is bounded and positive), which with (3.8) proves the estimation (3.5).

Finally, let us point out that $\mathbf{u}^c$ belongs to $\hat{\boldsymbol{\mathcal{U}}}(A^\gamma)$. Indeed, first we may extend $\mathbf{u}^c$ to an $H^1(\mathbb{R}^2)^2$ function with compact support, because $\partial\Omega$ is Lipschitz. Next, consider the sequence $\mathbf{u}^n = \eta_n * \mathbf{u}^c + \alpha_n \mathbf{v}$, with $\mathbf{v} \in \mathcal{D}(\mathbb{R}^2;\mathbb{R}^2)$, fixed, $\operatorname{supp}(\mathbf{v}) \cap \Sigma$ empty, $\nabla \cdot \mathbf{v} = 0$, $\int_{\Gamma_i} v_1 = 1$, and appropriate α_n such that $\int_{\Gamma_i} u_1^n = \phi$. Of course $\eta_n * \mathbf{u}^c \to \mathbf{u}^c$ in $\hat{\boldsymbol{\mathcal{U}}}(A^\gamma)$. It follows that $\int_{\Gamma_i} u_1^n \to \int_{\Gamma_i} u_1^c = \phi$, and thus $\alpha_n \to 0$. For n large we have $\mathbf{u}^n \in \hat{\mathcal{U}}(A^\gamma)$ because of the choice of the support of φ and the function ξ, which proves that $\mathbf{u}^c \in \hat{\boldsymbol{\mathcal{U}}}(A^\gamma)$. $\square$

REMARK 3.6. Assume $\hat{\mathbf{u}} \in L^q(A^\gamma \cup G)^2$, $q > 2$. From Proposition 3.1 we have $\hat{c} \in C^0(\overline{\Omega}_\gamma)$, and from Proposition 3.3 we get

$$\|\mathbf{u}^c\|_{H^1(A^\gamma)} + \|\mathbf{u}^c\|_{L^p(G)} \le C(k, \boldsymbol{\alpha}, \boldsymbol{\beta}, G)(\phi + c_i). \tag{3.10}$$

PROPOSITION 3.7. For given $\hat{c} \in \hat{\boldsymbol{\mathcal{C}}}(A^\gamma)$ let $\mathbf{u}^c = (u_1^c, u_2^c) \in \hat{\boldsymbol{\mathcal{U}}}(A^\gamma)$, $u_2^c = -g(\hat{c})$ as in Proposition 3.5. Then equation (2.14) has a unique solution $\mathbf{u} \in \boldsymbol{\mathcal{U}}(A^\gamma)$. Moreover, $\hat{\mathbf{u}} = \mathbf{u} + \mathbf{u}^c \in \hat{\boldsymbol{\mathcal{U}}}(A^\gamma) \cap H^1(A^\gamma \cup G)^2$ is the unique solution of (2.10) and

$$\|\hat{\mathbf{u}}\|_{H^1(A^\gamma)} + \|\hat{\mathbf{u}}\|_{H^1(G)} \le C(k, \boldsymbol{\alpha}, \boldsymbol{\beta}, G)(c_i + (1 + c_i)(\phi + \|\hat{c}\|_{L^2(M)})). \tag{3.11}$$

Proof. The existence of the solution $\mathbf{u} \in \boldsymbol{\mathcal{U}}(A^\gamma)$ follows immediately from the Lax-Milgram lemma. For the estimation (3.11), taking $\mathbf{v} = \mathbf{u}$ in (2.14) yields

$$\begin{aligned} \int_{A^\gamma} \mu|\nabla\mathbf{u}|^2 + \int_G \frac{\mu}{K}|\mathbf{u}|^2 &\le \mu\|\nabla\mathbf{u}^c\|_{L^2(A^\gamma)}\|\nabla\mathbf{u}\|_{L^2(A^\gamma)} + \frac{\mu}{K}\|\mathbf{u}^c\|_{L^2(G)}\|\mathbf{u}\|_{L^2(G)} \\ &\le \left(\mu\|\nabla\mathbf{u}\|_{L^2(A^\gamma)}^2 + \frac{\mu}{K}\|\mathbf{u}\|_{L^2(G)}^2\right)^{1/2} \\ &\quad \left(\mu\|\nabla\mathbf{u}^c\|_{L^2(A^\gamma)}^2 + \frac{\mu}{K}\|\mathbf{u}^c\|_{L^2(G)}^2\right)^{1/2}, \end{aligned} \tag{3.12}$$

which implies $\|\mathbf{u}\|_{\boldsymbol{\mathcal{U}}(A^\gamma)} \le C\|\mathbf{u}^c\|_{\boldsymbol{\mathcal{U}}(A^\gamma)}$. Combining this estimation with (3.5) gives

$$\|\mathbf{u}\|_{\boldsymbol{\mathcal{U}}(A^\gamma)} + \|\hat{\mathbf{u}}\|_{\boldsymbol{\mathcal{U}}(A^\gamma)} \le C(k, \boldsymbol{\alpha}, \boldsymbol{\beta}, G)(\phi + \|\hat{c}\|_{L^2(M)}). \tag{3.13}$$

Now, let us prove $\hat{\mathbf{u}} \in H^1(G)^2$ and let us find an estimation for $\|\nabla\hat{\mathbf{u}}\|_{L^2(G)}$. From [12], [15], [18], the decomposition $\mathbf{L}^2 = \mathbf{H} \oplus \mathbf{H}_1 \oplus \mathbf{H}_2$ is valid for a Lipschitz domain. Here

$$\begin{array}{ll} \mathbf{L}^2 = L^2(G) \times L^2(G), & \mathbf{H} \ = \{\hat{\mathbf{u}} \in \mathbf{L}^2,\ \nabla \cdot \hat{\mathbf{u}} = 0,\ \mathrm{tr}(\hat{\mathbf{u}}) = 0\}, \\ \mathbf{H}_1 = \{\hat{\mathbf{u}} = \nabla\hat{p},\ \hat{p} \in H^1(G),\ \Delta\hat{p} = 0\}, & \mathbf{H}_2 = \{\hat{\mathbf{u}} = \nabla\hat{q},\ \hat{q} \in H^1_0(G)\}, \end{array}$$

where "tr" is the trace operator on ∂G, well defined as G is Lipschitz domain, and $L^2(G)$, $H^1(G)$, $H^1_0(G)$ are the usual Sobolev spaces. For $\hat{\mathbf{u}} = \mathbf{u} + \mathbf{u}^c$, with $\mathbf{u}$ being the solution of (2.14), we have $\hat{\mathbf{u}} \in \mathbf{L}^2$ and $\int_G \hat{\mathbf{u}} \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in \mathbf{H}$. This implies $\hat{\mathbf{u}} \in \mathbf{H}_1 \oplus \mathbf{H}_2$, and thus $\hat{\mathbf{u}} = \nabla\hat{p}$, $\hat{p} \in H^1(G)$. As $\nabla \cdot \hat{\mathbf{u}} = 0$ it follows that $\Delta\hat{p} = 0$, so $\hat{\mathbf{u}} \in \mathbf{H}_1$. Following the construction of $\mathbf{H}_1$ in [15], [18] we find that

$$\hat{\mathbf{u}} = \nabla\hat{p}, \quad \hat{p} \in H^1(G), \quad \Delta\hat{p} = 0 \text{ in } G, \quad \partial_{\mathbf{n}}\hat{p} = \hat{\mathbf{u}} \cdot \mathbf{n} \text{ on } \partial G.$$

Let us recall that

$$\hat{\mathbf{u}} \cdot \mathbf{n} = -g(\hat{c}) \ \text{ on } M, \quad \hat{\mathbf{u}} \cdot \mathbf{n} = 0 \ \text{ on } \Gamma_w, \quad \hat{\mathbf{u}} \cdot \mathbf{n} = -\hat{u}_2(\cdot, 0^+) = -\hat{u}_2(\cdot, 0^-) \ \text{ on } \Sigma.$$

It follows that $\hat{p} \in H^2(G)$. Indeed, the function $\hat{p}$ can be extended by reflection to a harmonic function in a domain, say $R = \{(x_1, x_2), -l < x_1 < 2l,\ 0 < x_2 < h\}$. The extension is possible because $\partial_1\hat{p}(0, \cdot) = \partial_1\hat{p}(l, \cdot) = 0$. Let $\hat{\mathbb{p}}$ be the extension of $\hat{p}$ in R as follows:

$$\hat{\mathbb{p}}(x_1, \cdot) = p(-x_1, \cdot), \ \ x_1 \in (-l, 0), \qquad \hat{\mathbb{p}}(x_1, \cdot) = p(2l - x_1, \cdot), \ \ x_1 \in (0, 2l).$$

As we have

$$\partial_2\hat{\mathbb{p}}(x_1, 0) = \begin{cases} \hat{u}_2(-x_1, 0), & x_1 \in (-l, 0), \\ \hat{u}_2(x_1, 0), & x_1 \in (0, l), \\ \hat{u}_2(2l - x_1, 0), & x_1 \in (l, 2l), \end{cases} \quad -\partial_2\hat{\mathbb{p}}(x_1, h) = \begin{cases} g(\hat{c})(-x_1, h), & x_1 \in (-l, 0), \\ g(\hat{c})(x_1, h), & x_1 \in (0, l), \\ g(\hat{c})(2l - x_1, h), & x_1 \in (l, 2l), \end{cases}$$

from the construction of the spaces $\boldsymbol{\mathcal{C}}(A^\gamma)$ and $\boldsymbol{\mathcal{U}}(A^\gamma)$ it follows that $\partial_2\hat{\mathbb{p}}(x_1, 0)$, $\partial_2\hat{\mathbb{p}}(x_1, h)$ belong to $H^{1/2}(-l, 2l)$ because $g(\hat{c}) \in H^1(\Omega_\gamma)$, $\hat{\mathbf{u}} \in H^1(A^\gamma)$. From regularity results for the Neumann problem, [7], [15], it follows that $\hat{\mathbb{p}} \in H^2(R)$. Thus $\hat{p} \in H^2(G)$ and $\|\hat{p} - ⨍2_G\hat{p}\|_{H^2(G)} \leq C(G)\|\hat{\mathbf{u}} \cdot \mathbf{n}\|_{H^{1/2}(\partial G)}$. From the boundedness of $\hat{c}$ it follows that $\|g(\hat{c})\|_{H^1(G)} \leq \|\hat{c}\|_{H^1(G)}$ and we get

$$\begin{array}{rcl} \|\hat{\mathbf{u}}\|_{H^1(G)} & \leq & \|\nabla\hat{p}\|_{H^1(G)} \leq C(G)\|\hat{\mathbf{u}} \cdot \mathbf{n}\|_{H^{1/2}(\partial G)} \\ & \leq & C(G)(\|\hat{c}\|_{H^1(G)} + \|\hat{\mathbf{u}}\|_{H^1(A^\gamma)}) \\ \text{(using (3.3),} & \leq & C(G)(c_i(1 + \|\hat{\mathbf{u}}\|_{L^2(\Omega_\gamma)}) + \|\hat{\mathbf{u}}\|_{H^1(A^\gamma)}) \\ & \leq & C(G)(c_i + (1 + c_i)\|\hat{\mathbf{u}}\|_{\boldsymbol{\mathcal{U}}(A^\gamma)}) \\ \text{using (3.13))} & \leq & C(k, \boldsymbol{\alpha}, \boldsymbol{\beta}, G)(c_i + (1 + c_i)(\phi + \|\hat{c}\|_{L^2(M)})), \end{array} \tag{3.14}$$

which completes (3.11). □

Remark 3.8. For K large, estimation (3.11) is independent of K. Indeed, estimation (3.12) gives an estimation for $\|\nabla\mathbf{u}\|_{L^2(A^\gamma)}$ independent of K because of the bounds (3.5) or (3.10). We can proceed with the estimation of $\|\mathbf{u}\|_{H^1(G)}$ given by (3.14), which is given only in terms of $\hat{\mathbf{u}}$ on Σ and of $\hat{c}$ on M, independent of K.

Proposition 3.9. *The system (2.8)–(2.10) has a solution $(\hat{c}, \hat{\tau}, \hat{\mathbf{u}}) \in \hat{\boldsymbol{\mathcal{C}}}(A^\gamma) \times \hat{\boldsymbol{\mathcal{T}}}(A^\gamma) \times \hat{\boldsymbol{\mathcal{U}}}(A^\gamma)$, $(\hat{c}, \hat{\tau}, \hat{\mathbf{u}}) = (c + c_i, \tau + \tau_i, \mathbf{u} + \mathbf{u}^c)$. If c_i, ϕ are small enough, then the solution is unique.*

Proof. The existence of a solution follows by using a classical compactness argument. Indeed, let $(\hat{c}^0, \hat{\tau}^0) = (c_i, \tau_i)$ be given. For $n \in \mathbb{N}$, we assume that $(\hat{c}^n, \hat{\tau}^n)$ is given and we set $\hat{\mathbf{u}}^n = \mathbf{u}^n + \mathbf{u}^{c^n}$, where $\mathbf{u}^{c^n}$ is given by Proposition 3.5 for $\hat{c} = \hat{c}^n$ and $\mathbf{u}^n$ is the solution of (2.14). Moreover, we set $(\hat{c}^{n+1}, \hat{\tau}^{n+1}) = (c^{n+1} + c_i, \tau^{n+1} + \tau_i)$ where (c^{n+1}, τ^{n+1}) is the solution of (2.12), (2.13) for $\hat{\mathbf{u}} = \hat{\mathbf{u}}^n$. The estimations (3.1), (3.2), (3.3), (3.4), (3.5), (3.10), (3.11) give uniform bounds for the sequence $(\hat{c}^n, \hat{\tau}^n, \hat{\mathbf{u}}^n)$ in $H^1(\Omega_\gamma)^4$. It follows that the sequence of $(\hat{c}, \hat{\tau}, \hat{\mathbf{u}}^n)$ will converge weakly in $H^1(\Omega_\gamma)^4$, and strongly in $H^s(\Omega_\gamma)^4$, $s < 1$. It also follows that the sequence of the traces on $\partial\Omega_\gamma$ of $\hat{c}^n$ and $\hat{\tau}^n$ will converge strongly in $H^{s-1/2}(\partial\Omega_\gamma)$, which also implies the convergence in $L^1(M)$ of the sequence of $g(\hat{c}^n)$. Then, we can pass to the limit in equations (2.8)–(2.10).

For uniqueness, let us assume that the system has at least two solutions $(\hat{c}_m, \hat{\tau}_m, \hat{\mathbf{u}}_m) = (c_m + c_i, \tau_m + \tau_i, \mathbf{u}_m + \mathbf{u}^{c_m})$, $m = 1, 2$, and let $\delta_c = \hat{c}_1 - \hat{c}_2$, $\delta_\tau = \hat{\tau}_1 - \hat{\tau}_2$, $\delta_{\hat{\mathbf{u}}} = \hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2$, $\delta_{\mathbf{u}^c} = \mathbf{u}^{c_1} - \mathbf{u}^{c_2}$. From (2.12), (2.14) it follows that δ_c and $\delta_{\mathbf{u}}$ satisfy

$$\int_{\Omega_\gamma} D(\nabla\delta_c \cdot \nabla\varphi) + \int_M D\delta_c\varphi = -\int_{\Omega_\gamma} (\hat{\mathbf{u}}_1 \cdot \nabla\delta_c)\varphi + (\delta_{\hat{\mathbf{u}}} \cdot \nabla c_2)\varphi, \tag{3.15}$$

$$\int_{A^\gamma} \mu(\nabla\delta_{\mathbf{u}} \cdot \nabla\mathbf{v}) + \int_G \frac{\mu}{K}(\delta_{\mathbf{u}} \cdot \mathbf{v}) = -\int_{A^\gamma} \mu(\nabla\delta_{\mathbf{u}^c} \cdot \nabla\mathbf{v}) - \int_G \frac{\mu}{K}(\delta_{\mathbf{u}^c} \cdot \mathbf{v}), \tag{3.16}$$

for all $\varphi \in \boldsymbol{\mathcal{C}}(A^\gamma)$ and $\mathbf{v} \in \boldsymbol{\mathcal{U}}(A^\gamma)$. Now, with $\varphi = \delta_c$ in (3.15) we obtain

$$\begin{aligned} D\|\nabla\delta_c\|^2_{L^2(\Omega_\gamma)} &\leq \int_{\Omega_\gamma} |(\hat{u}_1 \cdot \nabla\delta_c)\delta_c| + |(\delta_{\hat{\mathbf{u}}} \cdot \nabla c_2)\delta_c| \\ &\leq \|\hat{\mathbf{u}}\|_{L^4(\Omega_\gamma)}\|\nabla\delta_c\|_{L^2(\Omega_\gamma)}\|\delta_c\|_{L^4(\Omega_\gamma)} + \|\delta_{\hat{\mathbf{u}}}\|_{L^4(\Omega_\gamma)}\|\nabla c_2\|_{L^2(\Omega_\gamma)}\|\delta_c\|_{L^4(\Omega_\gamma)} \\ &\leq C\left(\|\hat{\mathbf{u}}\|_{L^4(\Omega_\gamma)}\|\nabla\delta_c\|_{L^2(\Omega_\gamma)} + \|\delta_{\hat{\mathbf{u}}}\|_{L^4(\Omega_\gamma)}\|\nabla c_2\|_{L^2(\Omega_\gamma)}\right)\|\nabla\delta_c\|_{L^2(\Omega_\gamma)}. \end{aligned} \tag{3.17}$$

Let us estimate $\|\delta_{\hat{\mathbf{u}}}\|_{L^4(\Omega_\gamma)}$ in terms of $\|\nabla\delta_c\|_{L^2(\Omega_\gamma)}$. Equation (3.16) with $\mathbf{v} = \delta_{\mathbf{u}}$ gives $\|\delta_{\mathbf{u}}\|_{\boldsymbol{\mathcal{U}}(A^\gamma)} \leq C\|\delta_{\mathbf{u}^c}\|_{\boldsymbol{\mathcal{U}}(A^\gamma)}$. Proposition 3.5 with δ_c instead of $\hat{c}$ gives

$$\|\delta_{\mathbf{u}^c}\|_{H^1(A^\gamma)} + \|\delta_{\mathbf{u}^c}\|_{L^p(G)} \leq C(k, \boldsymbol{\alpha}, \boldsymbol{\beta}, G)\|\delta_c\|_{L^p(M)}, \quad 1 \leq p < \infty, \tag{3.18}$$

because $\delta_c = 0$ on Γ_i. It follows that

$$\|\delta_{\hat{\mathbf{u}}}\|_{H^1(A^\gamma)} + \|\delta_{\hat{\mathbf{u}}}\|_{L^2(G)} \leq C(k, \boldsymbol{\alpha}, \boldsymbol{\beta}, G)\|\delta_c\|_{L^2(M)} \leq C(k, \boldsymbol{\alpha}, \boldsymbol{\beta}, G)\|\nabla\delta_c\|_{L^2(\Omega_\gamma)}. \tag{3.19}$$

It remains to find an estimation for $\|\delta_{\hat{\mathbf{u}}}\|_{L^4(G)}$ in terms of $\|\nabla\delta_c\|_{L^2(\Omega_\gamma)}$. Let us remember that

$$\begin{aligned} &\nabla \cdot \delta_{\hat{\mathbf{u}}} = 0 \ \text{ in } \Omega_\gamma, \\ &\delta_{\hat{\mathbf{u}}} \cdot \mathbf{n} = -\delta_g := -\tfrac{\delta_c}{(1+\hat{c}_1)(1+\hat{c}_2)} \ \text{ on } M, \\ &\delta_{\hat{\mathbf{u}}} \cdot \mathbf{n} = 0 \ \text{ on } \Gamma_w, \quad \delta_{\hat{\mathbf{u}}} \cdot \mathbf{n} = -\delta_{\hat{u}_2}(\cdot, 0^+) = -\delta_{\hat{u}_2}(\cdot, 0^-) \ \text{ on } \Sigma. \end{aligned}$$

From Proposition 3.3, 3.4, 3.7, it's easy to prove that for $1 < q < 2$ we have

$$\begin{aligned} \|\delta_g\|_{W^{1,q}(G)} &\leq C(q)(\|\delta_c\|_{L^2(G)} + \|\nabla c_2\|_{L^2(G)}\|\nabla\delta_c\|_{L^2(G)}) \\ &\leq C(q, k, \boldsymbol{\alpha}, \boldsymbol{\beta}, G)Q(c_i, \phi)\|\nabla\delta_c\|_{L^2(\Omega_\gamma)}, \end{aligned} \tag{3.20}$$

where $Q(c_i, \phi)$ is a polynomial function of (c_i, ϕ). Then, from L^q regularity results for the Neumann problem, as in Proposition 3.7 (but with $1 < q < 2$ instead of $q = 2$, [1]), it follows that $\delta_{\hat{\mathbf{u}}} \in W^{1,q}(G)$ and

$$\begin{array}{rcl} \|\delta_{\hat{\mathbf{u}}}\|_{W^{1,q}(G)} & \leq & C(G)(\|\delta_g\|_{W^{1,q}(G)} + \|\delta_{\hat{\mathbf{u}}}\|_{W^{1,q}(A^\gamma)}) \\ \text{(using (3.19), (3.20))} & \leq & C(q,k,\boldsymbol{\alpha},\boldsymbol{\beta},G)Q(c_i,\phi)\|\nabla\delta_c\|_{L^2(\Omega_\gamma)}, \end{array}$$

which from the Sobolev inequality for $q \approx 2$ gives

$$\|\delta_{\hat{\mathbf{u}}}\|_{L^4(G)} \leq C(q,k,\boldsymbol{\alpha},\boldsymbol{\beta},G)Q(c_i,\phi)\|\nabla\delta_c\|_{L^2(\Omega_\gamma)}. \tag{3.21}$$

From (3.17) we obtain

$$\begin{array}{rcl} D\|\nabla\delta_c\|^2_{L^2(\Omega_\gamma)} & \leq & C(\cdot)\left(\|\hat{\mathbf{u}}_1\|_{L^4(\Omega_\gamma)}\|\nabla\delta_c\|_{L^2(\Omega_\gamma)} + \|\delta_{\hat{\mathbf{u}}}\|_{L^4(\Omega_\gamma)}\|\nabla c_2\|_{L^2(\Omega_\gamma)}\right)\|\nabla\delta_c\|_{L^2(\Omega_\gamma)} \\ \text{(using (3.21),} & \leq & C(\cdot)Q(\cdot)\left(\|\hat{\mathbf{u}}\|_{L^4(\Omega_\gamma)} + \|\nabla c_2\|_{L^2(\Omega_\gamma)}\right)\|\nabla\delta_c\|^2_{L^2(\Omega_\gamma)} \\ \text{using (3.3),} & \leq & C(\cdot)Q(\cdot)(\|\hat{\mathbf{u}}\|_{L^4(\Omega_\gamma)} + (1+c_i)\|\hat{\mathbf{u}}\|_{L^2(\Omega_\gamma)})\|\nabla\delta_c\|^2_{L^2(\Omega_\gamma)} \\ & \leq & C(\cdot)Q(\cdot)\|\hat{\mathbf{u}}\|_{H^1(A_\gamma\cup G)}\|\nabla\delta_c\|^2_{L^2(\Omega_\gamma)} \\ \text{using (3.11), (3.1))} & \leq & C(\cdot)Q(\cdot)(c_i + (1+c_i)(\phi+c_i))\|\nabla\delta_c\|^2_{L^2(\Omega_\gamma)}, \end{array}$$

with $C(\cdot) = C(q,k,\boldsymbol{\alpha},\boldsymbol{\beta},G)$ and $Q(\cdot) = Q(c_i,\phi)$ a polynomial function. If $\|\nabla\delta_c\|^2_{L^2(\Omega_\gamma)} \neq 0$, this implies

$$D \leq C(\cdot)Q(\cdot)(c_i + (1+c_i)(\phi+c_i)).$$

For c_i and ϕ small enough, this inequality is impossible. It follows that $\|\nabla\delta_c\|^2_{L^2(\Omega_\gamma)} = 0$, and thus $\delta_c = 0$. The estimation (3.19) implies that $\delta_{\hat{\mathbf{u}}} = 0$. Writing the equation for δ_τ and using a similar technique as for δ_c, it's easy to conclude that $\delta_\tau = 0$, which proves the theorem. □

Now let us describe in more detail some properties of the weak solution $(\hat{c},\hat{\tau},\hat{\mathbf{u}}) \in \hat{\boldsymbol{\mathcal{C}}}(A^\gamma) \times \hat{\boldsymbol{\mathcal{T}}}(A^\gamma) \times \hat{\boldsymbol{\mathcal{U}}}(A^\gamma)$ of (2.8)–(2.10).

PROPOSITION 3.10. There exists $\hat{p} \in L^2(\Omega_\gamma)$ such that $(\hat{c},\hat{\tau},\hat{\mathbf{u}},\hat{p})$ satisfies (1.2)–(1.5) in the distribution sense.

Proof. The first two equations of the proposition follow immediately from (2.12)–(2.13). The third equation follows from (2.3) and Lemma 2.1, [18]. □

The verification of boundary conditions in a stronger sense than that given by (2.8)–(2.10) is a matter of regularity results, which is not the purpose of this paper. However, in order to address the shape optimization problem (1.8), we will describe the boundary conditions related to $\hat{p}$ and prove the formula giving $\hat{p}$ on Γ_i.

PROPOSITION 3.11. There exists $\hat{p} \in L^2(A^\gamma) \cap H^2(G)$ satisfying (1.2)–(1.5) in the distribution sense and

i) $\hat{p} = p_i$ on Γ_i, $p_i \in \mathbb{R}$;

ii) $\hat{p} = p_o (= 0)$ on Γ_o;

iii) the trace of $\hat{p} \in L^2(A^\gamma)$ on Σ is well defined in the $H^{-1/2}(\Sigma) \times H_0^{1/2}(\Sigma)$-sense and $[\hat{p}] = 0$ on Σ in $H^{-1/2}(\Sigma)$;

iv) the constant p_i is given by

$$p_i = \frac{1}{\phi}\left(\mu\int_{A^\gamma}|\nabla\hat{\mathbf{u}}|^2 + \frac{\mu}{K}\int_G \hat{\mathbf{u}}^2 + \int_M \hat{p}\hat{u}_2\right). \tag{3.22}$$

Proof. Equality (2.14) and Remark 1.9, [18], imply the existence of $\hat{p} \in L^2(\Omega_\gamma)$ such that $\mu\Delta\hat{\mathbf{u}} = \nabla\hat{p}$ in $\mathcal{D}'(A^\gamma)$ and $\mu\hat{\mathbf{u}} + K\nabla\hat{p} = 0$ in $\mathcal{D}'(G)$. Moreover, from the interior regularity results for the Stokes equation it follows that $\hat{\mathbf{u}} \in C^\infty_{\text{loc}}(A^\gamma \cup G)$, $\hat{p} \in C^\infty_{\text{loc}}(A^\gamma \cup G)$, and from Proposition 3.7 we have $\hat{p} \in H^2(G)$.

Now, let us prove that the trace of $\hat{p} \in L^2(A^\gamma)$ on $\Gamma_i \cup \Gamma_o \cup \Sigma$ exists. The function $\hat{\mathbf{u}} \in H^1(A^\gamma)^2$ can be extended by reflections as follows. Let

$$\mathbb{g}(x_1) = \begin{cases} \gamma(-x_1), & x_1 \in (-l, 0), \\ \gamma(x_1), & x_1 \in (0, l), \\ \gamma(2l - x_1), & x_1 \in (l, 2l), \end{cases}$$

and set $\mathbb{A}^\gamma = \{(x_1, x_2),\ x_1 \in (-l, 2l),\ \mathbb{g}(x_1) < x_2 < 0\}$. We can define $\hat{\mathbb{u}} = (\hat{\mathbb{u}}_1, \hat{\mathbb{u}}_2)$, an extension of $\hat{\mathbf{u}}$ in $\mathbb{A}^\gamma$, by

$$\hat{\mathbb{u}}_1(x_1, x_2) = \begin{cases} \hat{u}_1(-x_1, x_2), & x_1 \in (-l, 0), \\ \hat{u}_1(x_1, x_2), & x_1 \in (0, l), \\ \hat{u}_1(2l - x_1, x_2), & x_1 \in (l, 2l), \end{cases} \quad \hat{\mathbb{u}}_2(x_1, x_2) = \begin{cases} -\hat{u}_2(-x_1, x_2), & x_1 \in (-l, 0), \\ \hat{u}_2(x_1, x_2), & x_1 \in (0, l), \\ -\hat{u}_2(2l - x_1, x_2), & x_1 \in (l, 2l). \end{cases}$$

It is trivial to prove that $\hat{\mathbb{u}} \in \mathbb{H}^1(\mathbb{A}^\gamma)^2$, $\nabla \cdot \hat{\mathbb{u}} = 0$. Moreover, for $\mathbb{v} = (\mathbb{v}_1, \mathbb{v}_2) \in \mathcal{D}(\mathbb{A}^\gamma)^2$, $\nabla\mathbb{v} = 0$, we have $\int_{\mathbb{A}^\gamma} \nabla\hat{\mathbb{u}} \cdot \nabla\mathbb{v} = 0$. Indeed, let us focus on the case $\text{supp}(\mathbb{v}) \subset \{x_1 < l\}$ (the general case being similar). We have (in all of the following calculus, all of the partial derivatives ∂_i are w.r.t. x, unless otherwise noted)

$$\begin{aligned}
\int_{\mathbb{A}^\gamma} \nabla\hat{\mathbb{u}} \cdot \nabla\mathbb{v} &= \int_{A^\gamma} \nabla\hat{\mathbb{u}}(x_1, x_2) \cdot \nabla\mathbb{v}(x_1, x_2) \\
&\quad + \int_{\mathbb{A}^\gamma \setminus A^\gamma} \nabla\hat{\mathbb{u}}_1(x_1, x_2) \cdot \nabla\mathbb{v}_1(x_1, x_2) + \nabla\hat{\mathbb{u}}_2(x_1, x_2) \cdot \nabla\mathbb{v}_2(x_1, x_2) \\
&= \int_{A^\gamma} \nabla\hat{\mathbf{u}}(x_1, x_2) \cdot \nabla\mathbb{v}(x_1, x_2) \\
&\quad + \int_{\mathbb{A}^\gamma \setminus A^\gamma} \partial_1\hat{u}_1(-x_1, x_2)\partial_1\mathbb{v}_1(x_1, x_2) + \partial_2\hat{u}_1(-x_1, x_2)\partial_2\mathbb{v}_1(x_1, x_2) \\
&\qquad -\partial_1\hat{u}_2(-x_1, x_2)\partial_1\mathbb{v}_2(x_1, x_2) - \partial_2\hat{u}_2(-x_1, x_2)\partial_2\mathbb{v}_2(x_1, x_2) \\
&= \int_{A^\gamma} \nabla\hat{\mathbf{u}}(x_1, x_2) \cdot \nabla\mathbb{v}(x_1, x_2) \\
\text{(subs. } y_1 = -x_1) &\quad + \int_{\mathbb{A}^\gamma} \partial_1\hat{u}_1(y_1, x_2)\partial_1\mathbb{v}_1(-y_1, x_2) + \partial_2\hat{u}_1(y_1, x_2)\partial_2\mathbb{v}_1(-y_1, x_2) \\
&\qquad -\partial_1\hat{u}_2(y_1, x_2)\partial_1\mathbb{v}_2(-y_1, x_2) - \partial_2\hat{u}_2(y_1, x_2)\partial_2\mathbb{v}_2(-y_1, x_2) \\
&= \int_{A^\gamma} \nabla\hat{\mathbf{u}}(x_1, x_2) \cdot \nabla\mathbb{v}(x_1, x_2) \\
&\quad + \int_{\mathbb{A}^\gamma} \partial_{y_1}\hat{u}_1(y_1, x_2)\partial_{y_1}\mathbb{v}_1(-y_1, x_2) + \partial_2\hat{u}_1(y_1, x_2)\partial_2\mathbb{v}_1(-y_1, x_2) \\
&\qquad +\partial_{y_1}\hat{u}_2(y_1, x_2)\partial_{y_1}(-\mathbb{v}_2(-y_1, x_2)) \\
&\qquad +\partial_2\hat{u}_2(y_1, x_2)\partial_2(-\mathbb{v}_2(-y_1, x_2)) \\
\text{(from (2.10))} &= \int_{A^\gamma} \nabla\hat{\mathbf{u}}(x_1, x_2) \cdot \nabla\mathbf{v}(x_1, x_2) = 0,
\end{aligned}$$

because $\mathbf{v} = (v_1, v_2) = (\mathbb{v}_1(x_1, x_2) + \mathbb{v}_1(-x_1, x_2), \mathbb{v}_2(x_1, x_2) - \mathbb{v}_2(-x_1, x_2))$ belongs to $\boldsymbol{\mathcal{U}}(A^\gamma)$ as $v_2 = 0$ and

$$0 = \int_{A^\gamma} \nabla \cdot \mathbf{v} = \int_{\partial A^\gamma} \mathbf{v} \cdot \nu^\gamma = \int_{\Gamma_\gamma \cup \Gamma_o \cup \Sigma} \mathbf{v} \cdot \nu^\gamma - \int_{\Gamma_i} v_1 = - \int_{\Gamma_i} v_1.$$

In previous equalities ∂_i, resp. ∂_{*_i}, denotes the derivative w.r.t. the ith variable, resp. $*_i$ variable. Then, there exists $\hat{\mathbb{p}} \in L^2(\mathbb{A}^\gamma)$ such that $-\mu\Delta\hat{\mathbb{u}} + \hat{\mathbb{p}} = 0$ in $\mathcal{D}'(\mathbb{A}^\gamma)$. From the interior regularity results for the Stokes equation it follows that $\hat{\mathbb{u}} \in C^\infty_{\text{loc}}(\mathbb{A}^\gamma)^2$ and $\hat{\mathbb{p}} \in C^\infty_{\text{loc}}(\mathbb{A}^\gamma)$. Thus, $\hat{p} \in C^\infty(\overline{A^\gamma} \backslash (\overline{\Sigma} \cup \overline{\Gamma_\gamma}))$, which implies $\hat{p} \in C^\infty_{\text{loc}}(\Gamma_i \cup \Gamma_o)$.

For the trace of $\hat{p} \in L^2(A^\gamma)$ on Σ, we proceed as follows. Let $\epsilon < 0$ and set $A^{\gamma,\epsilon} = A^\gamma \cap \{x_2 < \epsilon\}$, $\Sigma^\epsilon = (0, l) \times \{\epsilon\}$. For $v \in \mathcal{D}(\Sigma)$ let $\mathbf{v} = (v_1, v_2) \in \mathcal{U}(A^\gamma)$, $v_2 = v$ on Σ, $\mathbf{v} = 0$ on $\Gamma_i \cup \Gamma_\gamma \cup \Gamma_o$. From $-\mu\Delta\hat{\mathbf{u}} + \nabla\hat{p} = 0$ in $C^\infty_{\text{loc}}(\mathbb{A}^\gamma)^2$, it follows that

$$\begin{aligned} \int_{\Sigma^\epsilon} \hat{p} v \quad &= \quad \int_{\partial A^{\gamma,\epsilon}} \mu(\mathbf{v} \cdot \partial_2 \hat{\mathbf{u}}) - \int_{\Gamma_i \cup \Gamma_\gamma \cup \Gamma_o} \hat{p}(\mathbf{v} \cdot \mathbf{n}^\gamma) - \int_{A^{\gamma,\epsilon}} \mu(\nabla\hat{\mathbf{u}} \cdot \nabla\mathbf{v}) \\ &\to \quad \int_{\Sigma} \mu(\mathbf{v} \cdot \partial_2 \hat{\mathbf{u}}) - \int_{A^\gamma} \mu(\nabla\hat{\mathbf{u}} \cdot \nabla\mathbf{v}) = - \int_{A^\gamma} \mu(\nabla\hat{\mathbf{u}} \cdot \nabla\mathbf{v}), \quad \text{as} \quad \epsilon \to 0, \end{aligned} \tag{3.23}$$

because $\partial_2 \hat{u}_2 = 0$ in $H^{-1/2}(\Sigma)$ and $v_1 = 0$ on Σ. By continuity, we define

$$\int_\Sigma \hat{p} v = - \int_{A^\gamma} \mu \nabla\hat{\mathbf{u}} \cdot \nabla\mathbf{v}, \quad \forall v \in H^{1/2}_0(\Sigma), \quad \mathbf{v} = (v_1, v_2) \in \boldsymbol{\mathcal{U}}(A^\gamma), \quad v_2|_\Sigma = v. \tag{3.24}$$

The definition is consistent because if $\mathbf{w} \in \boldsymbol{\mathcal{U}}(A^\gamma)$, $w_2 = v$ on Σ, from (2.14) we have $\int_{A^\gamma} \nabla\hat{\mathbf{u}} \cdot \nabla(\mathbf{v} - \mathbf{w}) = 0$. Equation (3.24) defines $\int_\Sigma \hat{p} v$ in the $H^{-1/2}(\Sigma)$-sense.

Now, let $\mathbf{v} \in \mathcal{U}(A^\gamma)$. For $\epsilon < 0$ small, from (2.14) and local regularity of $\hat{\mathbf{u}}$, $\hat{p}$ in A^γ, we have

$$\begin{aligned} -\int_{\Gamma_o} p_o v_1 \quad &= \quad \int_{A^\gamma} \mu(\nabla\hat{\mathbf{u}} \cdot \nabla\mathbf{v}) + \int_G \frac{\mu}{K}(\hat{\mathbf{u}} \cdot \mathbf{v}) \\ &= \quad \lim_{\epsilon \to 0} \int_{A^{\gamma,\epsilon}} \mu(\nabla\hat{\mathbf{u}} \cdot \nabla\mathbf{v}) + \int_G \frac{\mu}{K}(\hat{\mathbf{u}} \cdot \mathbf{v}) \\ &= \quad \int_{\Gamma_i} -\mu(\mathbf{v} \cdot \partial_1 \hat{\mathbf{u}}) + \hat{p} v_1 + \int_{\Gamma_o} \mu(\mathbf{v} \cdot \partial_1 \hat{\mathbf{u}}) - \hat{p} v_1 + \lim_{\epsilon \to 0} \int_{\Sigma^\epsilon} \mu(\mathbf{v} \cdot \partial_2 \hat{\mathbf{u}}) - \hat{p} v_2 \\ &- \quad \int_G \nabla\hat{p} \cdot \mathbf{v} \\ &= \quad \int_{\Gamma_i} \hat{p} v_1 - \int_{\Gamma_o} \hat{p} v_1 - \int_\Sigma \hat{p}(\cdot, 0^-) v_2 - \int_{\partial G} \hat{p}(\mathbf{v} \cdot \mathbf{n}) \\ &= \quad \int_{\Gamma_i} \hat{p} v_1 - \int_{\Gamma_o} \hat{p} v_1 - \int_\Sigma (\hat{p}(\cdot, 0^-) - \hat{p}(\cdot, 0^+)) v_2. \end{aligned}$$

By continuity it follows that

$$\int_{\Gamma_i} \hat{p} v_1 = \int_{\Gamma_o} (\hat{p} - p_o) v_1 = \int_\Sigma [\hat{p}] v_2 = 0, \quad \forall \mathbf{v} \in \boldsymbol{\mathcal{U}}(A^\gamma). \tag{3.25}$$

As $\hat{p} \in C^\infty_{\text{loc}}(\Gamma_i \cup \Gamma_o)$, (i) and (ii) follow. Moreover, we get $[\hat{p}] = 0$ on Σ in the $H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)$-sense.

The proof of (3.22) starts with the estimation of $\int_{A^\gamma} \mu|\nabla\hat{\mathbf{u}}|^2 + \int_G \frac{\mu}{K}\hat{\mathbf{u}}^2$. Namely, let $\hat{\mathbf{v}}^n \in \hat{\mathcal{U}}(A^\gamma)$, $\hat{\mathbf{v}}^n \to \hat{\mathbf{u}}$. Then

$$\begin{aligned}\int_{A^\gamma} \mu(\nabla\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{v}}^n) \quad &+ \quad \int_G \frac{\mu}{K}(\hat{\mathbf{u}}\cdot\hat{\mathbf{v}}^n) \\ &= \lim_{\epsilon\to 0}\int_{A^{\gamma,\epsilon}} \mu(\nabla\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{v}}^n) + \int_G \frac{\mu}{K}(\hat{\mathbf{u}}\cdot\hat{\mathbf{v}}^n) \\ &= \int_{\Gamma_i} -\mu(\hat{\mathbf{v}}^n\cdot\partial_1\hat{\mathbf{u}}) + \hat{p}\hat{v}_1^n + \int_{\Gamma_o} \mu(\hat{\mathbf{v}}^n\cdot\partial_1\hat{\mathbf{u}}) - \hat{p}\hat{v}_1^n \\ &+ \lim_{\epsilon\to 0}\int_{\Sigma^\epsilon} \mu(\hat{\mathbf{v}}^n\cdot\partial_2\hat{\mathbf{u}}) - \hat{p}\hat{v}_2^n - \int_G \nabla\hat{p}\cdot\hat{\mathbf{v}}^n \\ &= \int_{\Gamma_i} \hat{p}\hat{v}_1^n - \int_{\Gamma_o} \hat{p}\hat{v}_1^n - \int_\Sigma [\hat{p}]\hat{v}_2^n - \int_M \hat{p}\hat{v}_2^n \\ &= \int_{\Gamma_i} \hat{p}\hat{v}_1^n - \int_{\Gamma_o} \hat{p}\hat{v}_1^n - \int_M \hat{p}\hat{v}_2^n.\end{aligned}$$

This gives

$$p_1\int_{\Gamma_i} \hat{v}_1^n = \int_{A^\gamma} \mu(\nabla\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{v}}^n) + \int_G \frac{\mu}{K}(\hat{\mathbf{u}}\cdot\hat{\mathbf{v}}^n) + p_o\int_{\Gamma_o} \hat{v}_1^n + \int_M \hat{p}\hat{v}_2^n,$$

which by letting $n \to \infty$ gives (3.22). □

Now, let us return to the shape optimization problem (1.8). Let us consider $(c(\gamma) + c_i, \tau + \tau_i(\gamma), \mathbf{u}(\gamma) + \mathbf{u}^{c(\gamma)})$ a solution of (2.8)–(2.10) in Ω_γ, $\hat{p}(\gamma) = \hat{p}$ with $\hat{p}$ given by Proposition 3.11 and $\hat{c}^v(\gamma) = 1 - \hat{c}(\gamma)$, and consider the shape functional $E(\gamma)$ given by (1.10). We have the following results.

THEOREM 3.12. There exists a solution $\gamma_* \in \mathcal{O}$ of (1.8).

Proof. Let $\gamma_n \in \mathcal{O}$ be a minimizing sequence of $E(\gamma)$, and as $\gamma(0)$, $\gamma(l)$ are bounded and γ_n are uniformly Lipschitz functions, there exists a rectangle $R = (0, l) \times (0, h)$, for any $r < 0$, such that $\Omega^n := A^{\gamma_n} \cup \Sigma \cup G \subset R \cup \Sigma \cup G =: \Omega^R$. As γ_n is Lipschitz, we can extend $\hat{z}^n := (\hat{c}^n, \hat{\tau}^n, \hat{\mathbf{u}}^n)$ to $V := H^1(\Omega^R)^2 \times H^1(R\cup G) \times H^1(\Omega^R)$ functions. In fact, $\hat{\mathbf{u}}^n$ is extended simply by zero in $\Omega^R\backslash\Omega^n$. We denote these extensions with the same letters.

From estimations (3.3), (3.4), (3.5), (3.10), and (3.11), the sequence $\hat{z}^n$ is uniformly bounded in V, and from the fact that γ_n are uniformly Lipschitz functions, there exists a subsequence of $\hat{z}^n$ converging to $\hat{z}^*$ strongly in $V^s := H^s(\Omega^R)^2 \times H^s(R\cup G) \times H^s(\Omega^R)$, $s < 1$, [14], weakly in V, and a subsequence of γ_n converging to $\gamma_* \in \mathcal{O}$ strongly in $C^0([0, l])$, so that the sequence of domains Ω^n converges to $\Omega^* = A^{\gamma_*} \cup \Sigma \cup G$ in the sense of Hausdorff, of compacts and of characteristic functions, [8], [9]. We use the same notations for these subsequences as for the original sequences.

It is easy to prove that $(\hat{c}^*, \hat{\tau}^*, \hat{\mathbf{u}}^*) \in \hat{\mathcal{C}}(A^{\gamma_*}) \times \hat{\mathcal{T}}(A^{\gamma_*}) \times \hat{\mathcal{U}}(A^{\gamma_*})$. Indeed, from the construction of the spaces $\hat{\mathcal{C}}(A^{\gamma_n})$, $\hat{\mathcal{T}}(A^{\gamma_n})$, $\hat{\mathcal{U}}(A^{\gamma_n})$, we can find $(\tilde{c}^n, \tilde{\tau}^n, \tilde{\mathbf{u}}^n) \in \hat{\mathcal{C}}(\tilde{A}^{\gamma_n}) \times \hat{\mathcal{T}}(\tilde{A}^{\gamma_n}) \times \hat{\mathcal{U}}(\tilde{A}^{\gamma_n})$, where $\tilde{A}^{\gamma_n} = \{(x_1, x_2),\ x_1 \in (0, l),\ \gamma_n(x_1) + \delta_n < x_2 < 0\}$, where $\delta_n = \min\{\|\gamma_* - \gamma_n\|_{C^0([0,l])}, 0\}$, such that

$$\|\hat{c}^n - \tilde{c}^n\|_{\mathcal{C}(A^{\gamma_n})} + \|\hat{\tau}^n - \tilde{\tau}^n\|_{\mathcal{T}(A^{\gamma_n})} + \|\hat{\mathbf{u}}^n - \tilde{\mathbf{u}}^n\|_{\mathcal{U}(A^{\gamma_n})} \leq \sigma_n,$$

with δ_n, σ_n tending to zero as $n \to \infty$. It follows that for n large $(\tilde{c}^n, \tilde{\tau}^n, \tilde{\mathbf{u}}^n) \in \hat{\boldsymbol{\mathcal{C}}}(A^{\gamma_*}) \times \hat{\boldsymbol{\mathcal{T}}}(A^{\gamma_*}) \times \hat{\boldsymbol{\mathcal{U}}}(A^{\gamma_*})$ and converges to $(\hat{c}^*, \hat{\tau}^*, \hat{\mathbf{u}}^*)$ in $\hat{\boldsymbol{\mathcal{C}}}(A^{\gamma_*}) \times \hat{\boldsymbol{\mathcal{T}}}(A^{\gamma_*}) \times \hat{\boldsymbol{\mathcal{U}}}(A^{\gamma_*})$, which proves the claim.

As $\hat{z}^n$ converges strongly in V^s, $s < 1$, and weakly in V, it follows trivially that $\hat{c}^*$, $\hat{\tau}^*$ solve (2.8), (2.9) and $\hat{\mathbf{u}}^*$ satisfies (2.10) with A^{γ_*} rather than A^γ. Of course, $u_2^* = -g(c^*)$ on M due to the strong convergence of the sequence $\hat{z}^n$ in V^s, $s < 1$. Thus, we have proved that $(\hat{c}^*, \hat{\tau}^*, \hat{\mathbf{u}}^*)$ satisfy (2.8)–(2.10) with A^{γ_*} instead of A^γ. This means that the map $\gamma \to (\hat{c}(\gamma), \hat{\tau}(\gamma), \hat{\mathbf{u}}(\gamma))$ is compact from $\mathcal{O}$ to V weakly, and to V^s, $s < 1$, strongly. From the continuity of the trace operator $\varphi \in V^s \to \varphi \in L^2(M) \times L^1(\Gamma_o)$, $s < 1$, $s \approx 1$, and lower semi-continuity of $\int_{A^\gamma} |\nabla \hat{\mathbf{u}}(\gamma)|^2$, it follows that the functional $\gamma \to E(\gamma)$ is lower semi-continuous in $\mathcal{O}$, which proves that γ_* is a minimizer of $E(\gamma)$ in $\mathcal{O}$. ☐

Let us finish this paper with a result giving more information on the optimal domain A^{γ_*}.

Proposition 3.13. Let γ_K be the solution of (1.8) for a given K. Then, for K large enough we have $\gamma_K < 0$; thus the boundary Γ_{γ_K} does not intersect Σ.

Proof. For a given K, let γ_K be the solution of (1.8). Assume for a moment that there exists a sequence $\{K\}$, $K \to \infty$, such that $\Gamma_{\gamma_K} \cap \Sigma$ is not empty. As in Theorem 3.12, let $\hat{z}^K := (\hat{c}^K, \hat{\tau}^K, \hat{\mathbf{u}}^K)$ be the solution of (2.8)–(2.10) in A^{γ_K}.

Like in Theorem 3.12, from the uniformity of the bounds (3.3), (3.4), (3.5), (3.10) and (3.11) (also Remark 3.8), there exists a subsequence of $\hat{z}^K$ converging weakly in V and strongly in V^s, $s < 1$, and a subsequence of γ_K converging in $C^0([0, l])$, so that the sequence of domains $\Omega^K := A^{\gamma_K} \cup \Sigma \cup G$ converges to $\Omega^* = A^{\gamma_*} \cup \Sigma \cup G$ in the sense of Hausdorff, of compacts and of characteristic functions, [8], [9]. We use the same notations for these subsequences as for the original sequences.

From (2.10) for $\gamma = \gamma_*^K$ we get

$$\mu \int_{A^{\gamma_K}} (\nabla \hat{\mathbf{u}}^K \cdot \nabla \mathbf{v}) + \frac{\mu}{K} \int_G (\hat{\mathbf{u}}^K \cdot \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \boldsymbol{\mathcal{U}}(A^{\gamma_K}).$$

Letting K tend to ∞, from the weak convergence of $\mathbf{u}^K$ and uniform (w.r.t. K, Remark 3.8) bound (3.11) we get

$$\int_{A^{\gamma_*}} \nabla \hat{\mathbf{u}}^* \cdot \nabla \mathbf{v} = 0, \quad \forall \mathbf{v} \in \boldsymbol{\mathcal{U}}(A^{\gamma_*}).$$

Again, like in Theorem 3.12, we have $\hat{\mathbf{u}}^* \in \hat{\boldsymbol{\mathcal{U}}}(A^{\gamma_*})$, and similarly to Proposition 3.7 we have $\mathbf{u}^* \in H^1(A^{\gamma_*})^2$. Moreover, it follows trivially that $\Gamma_{\gamma_*} \cap \Sigma$ is not empty, $\int_{\Gamma_i} \hat{u}_1^* = \phi$. We can extend $\hat{\mathbf{u}}^*$ to $H^1(\mathbb{A}^{\gamma_*})^2$, where $\mathbb{A}^{\gamma_*} = \{(x_1, x_2),\ -l_0 < x_1 < l_0, |x_2| < |\gamma_*(|x_1|)|\}$, $l_0 = \min\{y_1, \gamma_*(y_1) = 0\}$. Namely, we set

$$\begin{array}{rcl} \hat{\mathbb{u}}_1(x_1, x_2) & = & -\operatorname{sign}(x_2) \hat{u}_1(|x_1|, |x_2|), \\ \hat{\mathbb{u}}_2(x_1, x_2) & = & \operatorname{sign}(x_1) \hat{u}_2(|x_1|, |x_2|). \end{array}$$

As $\hat{\mathbf{u}}^* = 0$ on Γ_{γ_*}, it follows that $\hat{\mathbb{u}}^* \in H_0^1(\mathbb{A}^{\gamma_*})^2$ because $(0, 0)$ has zero capacity, [9]. As in Proposition 3.11, it is easy to prove that $\int_{\mathbb{R}^2} \nabla \hat{\mathbb{u}}^* \cdot \nabla \mathbf{v} = 0$, for $\mathbf{v} \in H_0^1(\mathbb{A}^{\gamma_*})^2$. It follows that $\hat{\mathbb{u}}^* = 0$, which implies $\int_{\Gamma_i} \hat{\mathbb{u}}_1^* = 0$. This is a contradiction and completes the proof. ☐

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