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PARABOLIC-HYPERBOLIC TIME-DEPENDENT GINZBURG-LANDAU-MAXWELL EQUATIONS

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Abstract. This article is devoted to the long-term dynamics of a parabolic-hyperbolic system arising in superconductivity. In the literature, the existence and uniqueness of the solution have been investigated but, to our knowledge, no asymptotic result is available. For the bidimensional model we prove that the system generates a dissipative semigroup in a proper phase-space where it possesses a (regular) global attractor. Then, we show the existence of an exponential attractor whose basin of attraction coincides with the whole phase-space. Thus, in particular, this exponential attractor contains the global attractor which, as a consequence, is of finite fractal dimension.

1. Introduction. This paper is concerned with the asymptotic behavior of a time-dependent Ginzburg-Landau-Maxwell model of superconductivity. When the temperature of a superconducting material stays close to a critical temperature, the evolution of the system can be described by the Ginzburg-Landau equations in the state variables $(\psi, \tilde{\mathbf{A}}, \Phi)$. Here ψ is the complex order parameter, whose squared modulus represents the concentration of the superconducting electrons, while $\tilde{\mathbf{A}}$ and Φ are the magnetic and the electric potentials, respectively. The differential system, proposed by Gor′kov and Èliashberg (cf. [7]) and widely studied in literature (cf., e.g., [3, 10, 13]), consists of two coupled parabolic equations for ψ and $\tilde{\mathbf{A}}$. These are proved to be gauge-invariant. Hence, the choice of a particular gauge, which results in an additional equation, determines univocally $\tilde{\mathbf{A}}$, Φ, and consequently ψ, making the problem well-posed. The

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gauge-invariance of the model is examined in [1], where a physical interpretation of the equations is achieved by formulating the problem in terms of observable variables. The introduction of gauge-independent variables shows that the equation for $\tilde{\mathbf{A}}$ coincides with one of the Maxwell equations, in the so-called quasi-steady approximation, namely when the displacement current is negligible.

Here we remove this restriction and consider the general evolution model characterized by a hyperbolic equation for $\tilde{\mathbf{A}}$ (the equation for ψ is unchanged). For this problem with the London gauge, the existence and uniqueness of the solution are established in [12]. However, no asymptotic result seems to have appeared in the literature, even if the long-time behavior of the solutions is investigated for the quasi-steady model (cf. [9, 10]). In both papers, the superconductor occupies a bounded domain $\Omega \subset \mathbb{R}^2$ and the phase-space is $L^2(\Omega) \times L^2(\Omega)$. Although this choice makes some *a priori* estimates difficult to obtain (cf. [10]), it allows one to apply Hilbert space techniques. In particular, the classical method developed by Constantin, Foias and Temam (see, e.g., [11]) was used by Tang and Wang to prove the finite fractal dimension of the global attractor, while Rodriguez-Bernal, Wang and Willie can show the existence of an exponential attractor by means of the so-called squeezing property. Unfortunately, the Banach structure of our phase-space prevents us from applying similar arguments.

The main steps of our analysis are the following: in the bidimensional case, we see that the system generates a dissipative strongly continuous semigroup that admits a global attractor $\mathcal{A}$. Next, relying on the regularity of this set, we provide the existence of an exponential attractor $\mathcal{E}$ whose basin of attraction extends to the whole phase-space. Here we apply a recent method, working in Banach spaces, due to [4] (see also [6]) as well as the property known as transitivity of the exponential attraction devised in [5]. The finite fractal dimension of $\mathcal{A}$ is a consequence of this last result, since the global attractor is the minimal compact attracting set. Thus the dynamics on the global attractor are described by a finite number of parameters.

2. The dynamical system. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$, whose unit outward normal is denoted by $\boldsymbol{n}$. As anticipated in the Introduction, the state variables are the order parameter $\psi : \Omega \times \mathbb{R}^+ \to \mathbb{C}$, the magnetic potential $\tilde{\mathbf{A}} : \Omega \times \mathbb{R}^+ \to \mathbb{R}^2$ and the electric potential $\Phi : \Omega \times \mathbb{R}^+ \to \mathbb{R}$. Following the literature, we introduce the operator

$$D_{\tilde{\mathbf{A}}}\psi = \nabla\psi - i\tilde{\mathbf{A}}\psi,$$

then we denote $\mathbf{A} = \tilde{\mathbf{A}} + \mathbf{A}_{\rm ext}$, where $\mathbf{A}_{\rm ext}$ is the (time independent) vector field whose curl corresponds to the external magnetic field such that

$$\mathrm{div}\mathbf{A}_{\rm ext} = 0 \quad \text{in}\,\Omega \qquad \text{and} \qquad \mathbf{A}_{\rm ext} \cdot \boldsymbol{n}|_{\partial\Omega} = 0.$$

We can now state our problem (cf. [12]):

$$\psi_t - i\Phi\psi - D^2_{\tilde{\mathbf{A}}}\psi - \lambda^2(1 - |\psi|^2)\psi = 0, \tag{2.1}$$

$$\varepsilon(\mathbf{A}_t - \nabla\Phi)_t + \sigma(\mathbf{A}_t - \nabla\Phi) + \mathrm{curl}^2\mathbf{A} + \frac{i}{2}(\bar{\psi}D_{\tilde{\mathbf{A}}}\psi - \psi\overline{D_{\tilde{\mathbf{A}}}\psi}) = 0, \tag{2.2}$$

where λ, ε and σ are positive constants representing the Ginzburg-Landau parameter, the dielectric constant and the electric conductivity, respectively. The problem is supplemented with the boundary conditions

$$\partial_{\boldsymbol{n}}\psi|_{\partial\Omega} = 0, \quad \mathbf{A}\cdot\boldsymbol{n}|_{\partial\Omega} = 0, \quad \mathrm{curl}\mathbf{A}\times\boldsymbol{n}|_{\partial\Omega} = \mathbf{0}, \quad \partial_{\boldsymbol{n}}\Phi|_{\partial\Omega} = 0 \tag{2.3}$$

and with the initial conditions

$$\psi(0) = \psi_0, \quad \mathbf{A}(0) = \mathbf{A}_0, \quad \mathbf{A}_t(0) = \dot{\mathbf{A}}_0, \quad \Phi(0) = \Phi_0, \qquad \text{in}\,\Omega. \tag{2.4}$$

It is apparent that the solution to this system is not unique (indeed there are two equations but three unknowns); moreover, the problem is gauge-invariant, so that choosing the London gauge

$$\mathrm{div}\mathbf{A} = 0 \quad \text{in}\,\Omega \qquad \text{and} \qquad \int_\Omega \Phi = 0, \tag{2.5}$$

from the divergence of (2.2), keeping (2.1) into account, we obtain a third equation for Φ:

$$-\varepsilon\Delta\Phi_t - \sigma\Delta\Phi + \frac{i}{2}(\bar{\psi}\psi_t - \psi\bar{\psi}_t) + |\psi|^2\Phi = 0, \tag{2.6}$$

which makes the problem well-posed (cf. [12]). Besides, with this gauge $\tilde{\mathbf{A}}$ is a solenoidal vector field, and hence

$$-D^2_{\tilde{\mathbf{A}}}\psi = -\Delta\psi + 2i\tilde{\mathbf{A}}\cdot\nabla\psi + |\tilde{\mathbf{A}}|^2\psi.$$

Our first task is to make precise the phase-space, but since the model involves vector fields as well as complex phase fields, some notations are in order.

2.1. *Notations and function spaces.* As usual, $L^p(\Omega)$ and $W^{k,p}(\Omega)$ stand for the Lebesgue and the Sobolev spaces of real valued functions, with the convention that $H^k(\Omega) = W^{k,2}(\Omega)$. We denote by bold letters the spaces of vector-valued functions, whereas a subscript $\mathbb{C}$ characterizes those of complex-valued functions. Without further specifications, $\|\cdot\|$ stands for the $L^2_{\mathbb{C}}(\Omega)$, $\mathbf{L}^2(\Omega)$ or $L^2(\Omega)$-norm, according to the context. The brackets correspond to

$$\langle u, v\rangle = \int_\Omega u(x)v(x)dx,$$

indifferently for complex and vector-valued functions. In particular, the inner product in $L^2_{\mathbb{C}}(\Omega)$ is $(u,v) = \langle u, \bar{v}\rangle$. We next consider the spaces

$$\mathbf{X}_0 = \{\mathbf{u}\in\mathbf{H}^1(\Omega): \quad \mathrm{div}\mathbf{u} = 0 \quad \text{in}\,\Omega, \quad \mathbf{u}\cdot\boldsymbol{n}|_{\partial\Omega} = 0, \quad \mathrm{curl}\mathbf{u}\times\boldsymbol{n}|_{\partial\Omega} = \mathbf{0}\}$$

$$\mathbf{H}_0(\mathrm{div}0,\Omega) = \{\mathbf{u}\in\mathbf{L}^2(\Omega): \quad \mathrm{div}\mathbf{u} = 0 \quad \text{in}\,\Omega, \quad \mathbf{u}\cdot\boldsymbol{n}|_{\partial\Omega} = 0\}$$

$$H^1_{0m}(\Omega) = \Big\{u\in H^1(\Omega): \quad \int_\Omega u = 0\Big\}$$

$$H^2_{0m}(\Omega) = \Big\{u\in H^2(\Omega): \quad \partial_{\boldsymbol{n}}u|_{\partial\Omega} = 0, \quad \int_\Omega u = 0\Big\},$$

all endowed with their usual topologies.

With the notations defined above, the phase-space is

$$\mathbb{X}^0_\infty = \{\psi\in H^2_{\mathbb{C}}(\Omega): \quad \partial_{\boldsymbol{n}}\psi|_{\partial\Omega} = 0, \quad \|\psi\|_{L^\infty_{\mathbb{C}}(\Omega)} \le 1\}\times[\mathbf{X}_0\cap\mathbf{H}^2(\Omega)]\times\mathbf{X}_0\times H^2_{0m}(\Omega),$$

which is a closed subset of the Banach space

$$\mathbb{X}^0 = \{\psi \in H^2_{\mathbb{C}}(\Omega) : \quad \partial_{\boldsymbol{n}}\psi|_{\partial\Omega} = 0\} \times [\mathbf{X}_0 \cap \mathbf{H}^2(\Omega)] \times \mathbf{X}_0 \times H^2_{0m}(\Omega),$$

equipped with the standard norm $\|\cdot\|_0$. As will be clear in a while, in order to find proper estimates in $\mathbb{X}^0_\infty$, in some cases we first need to obtain a control in the weaker norm $\|\cdot\|_{-1}$ of the space

$$\mathbb{X}^{-1} = H^1_{\mathbb{C}}(\Omega) \times \mathbf{X}_0 \times \mathbf{L}^2(\Omega) \times H^1_{0m}(\Omega) \supset \mathbb{X}^0_\infty.$$

In particular, since we are in dimension two, we have at our disposal the well-known Ladyzhenskaya and Agmon inequalities (see, e.g., [11])

$$\|u\|^2_{L^4} \leq c_L \|u\| \|u\|_{H^1}, \qquad \forall u \in H^1(\Omega), \tag{2.7}$$

$$\|u\|^2_{L^\infty} \leq c_A \|u\| \|u\|_{H^2}, \qquad \forall u \in H^2(\Omega), \tag{2.8}$$

as well as

$$\|\mathbf{A}\|^2 \leq \kappa \|\mathrm{curl}\,\mathbf{A}\|^2, \qquad \forall \mathbf{A} \in \mathbf{X}_0, \tag{2.9}$$

and the Poincaré inequality

$$\|\Phi\|^2 \leq c_P \|\nabla\Phi\|^2, \qquad \forall \Phi \in H^1_{0m}(\Omega). \tag{2.10}$$

The four positive constants c_L, c_A, κ and c_P depend only on Ω. We shall often use these inequalities, along with the Young and the Hölder inequalities, without mentioning it.

In the forthcoming estimates, we denote by c any positive constant depending only on the structural data of the problem. Further dependencies will be specified on occurrence, and c may vary even within the same formula.

Our first task is the generation of a dynamical system in $\mathbb{X}^0_\infty$. It is shown in [12, Theorem 2] that problem (2.1)–(2.6) with initial data in $\mathbb{X}^0_\infty$ admits a unique solution in this phase-space, provided that $\mathbf{A}_{\mathrm{ext}} \in \mathbf{H}_0(\mathrm{div}0, \Omega) \cap \mathbf{W}^{1,6}(\Omega)$. Moreover, taking advantage of the subsequent Lemma 3.3 and Remark 3.4, we can obtain at any fixed time the following continuous dependence on the initial data.

Proposition 2.1. For any $R \geq 0$, there exists $K = K(R) \geq 0$ such that, for any two initial data $z_1, z_2 \in \mathbb{X}^0_\infty$ with $\|z_i\|_0 \leq R$, $i = 1, 2$, there holds

$$\begin{aligned}&\|S(t)z_1 - S(t)z_2\|^2_0 + \int_0^t [\|\partial_t\psi_1(y) - \partial_t\psi_2(y)\|^2_{H^1_{\mathbb{C}}} + \|\nabla\partial_t\Phi_1(y) - \nabla\partial_t\Phi_2(y)\|^2] dy\\ &\leq K^2 e^{2Kt}\|z_1 - z_2\|^2_0, \qquad t \geq 0.\end{aligned}$$

This result, along with [12, Theorem 2], lead to

Theorem 2.1. Problem (2.1)–(2.6) generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on the phase-space $\mathbb{X}^0_\infty$.

Proof of Proposition 2.1. Here and in the sequel, we perform formal computations that are justified within the same Galerkin scheme adopted in [12].

Let $(\psi_i(t), \mathbf{A}_i(t), \partial_t\mathbf{A}_i(t), \Phi_i(t)) = S(t)z_i$ and $\tilde{\mathbf{A}}_i(t) = \mathbf{A}_i(t) - \mathbf{A}_{\mathrm{ext}}$, for any $z_i = (\psi_{0i}, \mathbf{A}_{0i}, \dot{\mathbf{A}}_{0i}, \Phi_{0i}) \in \mathbb{X}^0_\infty$, with $\|z_i\|_0 \leq R$, $i = 1, 2$. For further convenience, notice that

Lemma 3.3 yields

$$\sup_{t\geq 0}\big(\|S(t)z_i\|_0+\|\partial_t\psi_i(t)\|\big)\leq c,\qquad i=1,2,$$

where, throughout this proof, the constant c is allowed to depend on R. It is straightforward to check that the difference $(\psi(t),\mathbf{A}(t),\mathbf{A}_t(t),\Phi(t))=S(t)z_1-S(t)z_2$ solves the following system:

$$\begin{aligned}&\psi_t-\Delta\psi-i\Phi\psi_1-i\Phi_2\psi+2i\mathbf{A}\cdot\nabla\psi_1+2i\tilde{\mathbf{A}}_2\cdot\nabla\psi+\mathbf{A}\cdot(\tilde{\mathbf{A}}_1+\tilde{\mathbf{A}}_2)\psi_1+|\tilde{\mathbf{A}}_2|^2\psi\\&\quad-\lambda^2\psi+\lambda^2\psi(|\psi_1|^2+|\psi_2|^2)+\lambda^2\psi_1\psi_2\bar\psi=0,\end{aligned}\tag{2.11}$$

$$\begin{aligned}&\varepsilon(\mathbf{A}_t-\nabla\Phi)_t+\sigma(\mathbf{A}_t-\nabla\Phi)+\mathrm{curl}^2\mathbf{A}+\frac{i}{2}[\bar\psi_1\nabla\psi-\psi_1\nabla\bar\psi+\bar\psi\nabla\psi_2-\psi\nabla\bar\psi_2]\\&\quad+\mathbf{A}|\psi_1|^2+\tilde{\mathbf{A}}_2\psi_1\bar\psi+\tilde{\mathbf{A}}_2\psi\bar\psi_2=0,\end{aligned}\tag{2.12}$$

$$-\varepsilon\Delta\Phi_t-\sigma\Delta\Phi+\frac{i}{2}(\bar\psi_1\psi_t-\psi_1\bar\psi_t+\bar\psi\partial_t\psi_2-\psi\partial_t\bar\psi_2)+|\psi_1|^2\Phi+(\psi_1\bar\psi+\psi\bar\psi_2)\Phi_2=0,\tag{2.13}$$

supplemented with homogeneous boundary conditions and initial data z_1-z_2. Adding together the product of (2.11) by $\bar\psi_t+\bar\psi$ and the conjugate of (2.11) by $\psi_t+\psi$, integrating the result in Ω, we obtain

$$\begin{aligned}&\frac12\frac{d}{dt}\|\psi\|^2_{H^1_{\mathbb{C}}}+\|\nabla\psi\|^2+\|\psi_t\|^2+\lambda^2\langle|\psi|^2,|\psi_1|^2+|\psi_2|^2\rangle+\langle|\tilde{\mathbf{A}}_2|^2,|\psi|^2\rangle\\
&=\frac{i}{2}\langle\Phi,\psi_1\bar\psi-\bar\psi_1\psi\rangle-i\langle\mathbf{A},\bar\psi\nabla\psi_1-\psi\nabla\bar\psi_1\rangle-i\langle\tilde{\mathbf{A}}_2,\bar\psi\nabla\psi-\psi\nabla\bar\psi\rangle\\
&\quad-\frac12\langle\mathbf{A}\cdot(\tilde{\mathbf{A}}_1+\tilde{\mathbf{A}}_2),\psi_1\bar\psi+\bar\psi_1\psi\rangle+\lambda^2\|\psi\|^2-\frac{\lambda^2}{2}[\langle\psi_1\psi_2,\bar\psi^2\rangle+\langle\bar\psi_1\bar\psi_2,\psi^2\rangle]\\
&\quad+\frac{i}{2}\langle\Phi,\psi_1\bar\psi_t-\bar\psi_1\psi_t\rangle+\frac{i}{2}\langle\Phi_2,\psi\bar\psi_t-\bar\psi\psi_t\rangle-i\langle\mathbf{A},\bar\psi_t\nabla\psi_1-\psi_t\nabla\bar\psi_1\rangle\\
&\quad-i\langle\tilde{\mathbf{A}}_2,\bar\psi_t\nabla\psi-\psi_t\nabla\bar\psi\rangle-\frac12\langle\mathbf{A}\cdot(\tilde{\mathbf{A}}_1+\tilde{\mathbf{A}}_2),\psi_1\bar\psi_t+\bar\psi_1\psi_t\rangle\\
&\quad-\frac12\langle|\tilde{\mathbf{A}}_2|^2,\psi\bar\psi_t+\bar\psi\psi_t\rangle+\frac{\lambda^2}{2}[\langle\psi,\bar\psi_t\rangle+\langle\bar\psi,\psi_t\rangle]\\
&\quad-\frac{\lambda^2}{2}\langle|\psi_1|^2+|\psi_2|^2,\psi\bar\psi_t+\bar\psi\psi_t\rangle-\frac{\lambda^2}{2}[\langle\psi_1\psi_2,\bar\psi\bar\psi_t\rangle+\langle\bar\psi_1\bar\psi_2,\psi\psi_t\rangle].\end{aligned}\tag{2.14}$$

Then, standard inequalities lead us to

$$\begin{aligned}&\frac12\frac{d}{dt}\|\psi\|^2_{H^1_{\mathbb{C}}}+\|\nabla\psi\|^2+\|\psi_t\|^2+\lambda^2\langle|\psi_1|^2+|\psi_2|^2,|\psi|^2\rangle+\langle|\tilde{\mathbf{A}}_2|^2,|\psi|^2\rangle\\&\leq\frac12\|\psi_t\|^2+c(\|\psi\|^2_{H^1_{\mathbb{C}}}+\|\mathrm{curl}\mathbf{A}\|^2+\|\nabla\Phi\|^2).\end{aligned}$$

The product of (2.12) with $\mathbf{A}_t$ in $\mathbf{L}^2(\Omega)$ yields

$$\begin{aligned}&\frac12\frac{d}{dt}(\|\mathrm{curl}\mathbf{A}\|^2+\varepsilon\|\mathbf{A}_t\|^2)+\sigma\|\mathbf{A}_t\|^2\\&=-\frac{i}{2}\langle\bar\psi_1\nabla\psi-\psi_1\nabla\bar\psi,\mathbf{A}_t\rangle-\frac{i}{2}\langle\bar\psi\nabla\psi_2-\psi\nabla\bar\psi_2,\mathbf{A}_t\rangle-\langle|\psi_1|^2,\mathbf{A}\cdot\mathbf{A}_t\rangle\\&\quad-\langle\psi_1\bar\psi+\bar\psi_2\psi,\tilde{\mathbf{A}}_2\cdot\mathbf{A}_t\rangle,\end{aligned}\tag{2.15}$$

which yields the inequality

$$\frac{1}{2}\frac{d}{dt}(\|\mathrm{curl}\mathbf{A}\|^2+\varepsilon\|\mathbf{A}_t\|^2)\leq c\big(\|\psi\|^2_{H^1_{\mathbb{C}}}+\|\mathrm{curl}\mathbf{A}\|^2+\|\mathbf{A}_t\|^2\big).$$

Finally, we multiply (2.13) by Φ_t in $L^2(\Omega)$, getting

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\sigma\|\nabla\Phi\|^2\big)+\varepsilon\|\nabla\Phi_t\|^2 \qquad (2.16)\\ &=-\frac{i}{2}\langle\bar\psi_1\psi_t-\psi_1\bar\psi_t+\bar\psi\partial_t\psi_2-\psi\partial_t\bar\psi_2,\Phi_t\rangle-\langle|\psi_1|^2,\Phi\Phi_t\rangle-\langle\psi_1\bar\psi+\psi\psi_2,\Phi_2\Phi_t\rangle.\end{aligned}$$

Notice that

$$\frac{1}{2}\frac{d}{dt}\big(\sigma\|\nabla\Phi\|^2\big)+\varepsilon\|\nabla\Phi_t\|^2\leq\frac{\varepsilon}{2}\|\nabla\Phi_t\|^2+c\|\psi_t\|^2+c\big(\|\psi\|^2_{H^1_{\mathbb{C}}}+\|\nabla\Phi\|^2\big).$$

Adding (2.14) and (2.15) to (2.16), multiplied by a suitably small constant $\eta>0$, in light of the above inequalities, leads to

$$\begin{aligned}&\frac{d}{dt}\big(\|\psi\|^2_{H^1_{\mathbb{C}}}+\|\mathrm{curl}\mathbf{A}\|^2+\varepsilon\|\mathbf{A}_t\|^2+\eta\sigma\|\nabla\Phi\|^2\big)+\|\psi_t\|^2+\eta\varepsilon\|\nabla\Phi_t\|^2\\ &\leq c\big(\|\psi\|^2_{H^1_{\mathbb{C}}}+\|\mathrm{curl}\mathbf{A}\|^2+\varepsilon\|\mathbf{A}_t\|^2+\eta\sigma\|\nabla\Phi\|^2\big),\end{aligned}$$

and thus we easily obtain

$$\|S(t)z_1-S(t)z_2\|^2_{-1}+\int_0^t\big(\|\psi_t(y)\|^2+\|\nabla\Phi_t(y)\|^2\big)dy\leq ce^{ct}\|z_1-z_2\|^2_{-1}, \qquad (2.17)$$

for some constant c depending on R, but independent of time.

To obtain the corresponding inequality in $\mathbb{X}^0_\infty$, we differentiate (2.11) with respect to time:

$$\begin{aligned}&\psi_{tt}-\Delta\psi_t+|\tilde{\mathbf{A}}_2|^2\psi_t+\lambda^2(|\psi_1|^2+|\psi_2|^2)\psi_t=(i\Phi_2+\lambda^2)\psi_t\\ &\quad+[i\partial_t\Phi_2-2\tilde{\mathbf{A}}_2\cdot\partial_t\mathbf{A}_2-\lambda^2(\psi_1\partial_t\bar\psi_1+\bar\psi_1\partial_t\psi_1+\psi_2\partial_t\bar\psi_2+\bar\psi_2\partial_t\psi_2)]\psi\\ &\quad-\lambda^2(\psi_2\partial_t\psi_1+\psi_1\partial_t\psi_2)\bar\psi-2i\partial_t\mathbf{A}_2\cdot\nabla\psi-2i\tilde{\mathbf{A}}_2\cdot\nabla\psi_t-\lambda^2\psi_1\psi_2\bar\psi_t\\ &\quad-[2i\nabla\partial_t\psi_1+(\partial_t\mathbf{A}_1+\partial_t\mathbf{A}_2)\psi_1+(\tilde{\mathbf{A}}_1+\tilde{\mathbf{A}}_2)\partial_t\psi_1]\cdot\mathbf{A}\\ &\quad-[2i\nabla\psi_1+(\tilde{\mathbf{A}}_1+\tilde{\mathbf{A}}_2)\psi_1]\cdot\mathbf{A}_t+i\partial_t\psi_1\Phi+i\psi_1\Phi_t.\end{aligned}$$

Adding together this equation multiplied by $\bar{\psi}_t$ and its conjugate multiplied by ψ_t, integrating the result in Ω, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\psi_t\|^2 + \|\nabla\psi_t\|^2 + \langle|\tilde{\mathbf{A}}_2|^2, |\psi_t|^2\rangle + \lambda^2\langle|\psi_1|^2 + |\psi_2|^2, |\psi_t|^2\rangle \\
&= \lambda^2\|\psi_t\|^2 + \frac{i}{2}\langle\partial_t\Phi_2, \psi\bar{\psi}_t - \bar{\psi}\psi_t\rangle \\
&\quad - \langle\tilde{\mathbf{A}}_2\cdot\partial_t\mathbf{A}_2 + \frac{\lambda^2}{2}(\psi_1\partial_t\bar{\psi}_1 + \bar{\psi}_1\partial_t\psi_1 + \psi_2\partial_t\bar{\psi}_2 + \bar{\psi}_2\partial_t\psi_2), \psi\bar{\psi}_t + \bar{\psi}\psi_t\rangle \\
&\quad - \frac{\lambda^2}{2}[\langle(\psi_2\partial_t\psi_1 + \psi_1\partial_t\psi_2)\bar{\psi}, \bar{\psi}_t\rangle + \langle(\bar{\psi}_2\partial_t\bar{\psi}_1 + \bar{\psi}_1\partial_t\bar{\psi}_2)\psi, \psi_t\rangle] \\
&\quad - i\langle\partial_t\mathbf{A}_2, \bar{\psi}_t\nabla\psi - \psi_t\nabla\bar{\psi}\rangle - i\langle\tilde{\mathbf{A}}_2, \bar{\psi}_t\nabla\psi_t - \psi_t\nabla\bar{\psi}_t\rangle \\
&\quad - \frac{\lambda^2}{2}[\langle\psi_1\psi_2, \bar{\psi}_t^2\rangle + \langle\bar{\psi}_1\bar{\psi}_2, \psi_t^2\rangle] - i\langle\bar{\psi}_t\nabla\partial_t\psi_1 - \psi_t\nabla\partial_t\bar{\psi}_1, \mathbf{A}\rangle \\
&\quad - \frac{1}{2}\langle(\partial_t\mathbf{A}_1 + \partial_t\mathbf{A}_2)\cdot\mathbf{A}, \psi_1\bar{\psi}_t + \bar{\psi}_1\psi_t\rangle - \frac{1}{2}\langle(\tilde{\mathbf{A}}_1 + \tilde{\mathbf{A}}_2)\cdot\mathbf{A}, \partial_t\psi_1\bar{\psi}_t + \partial_t\bar{\psi}_1\psi_t\rangle \\
&\quad - i\langle\mathbf{A}_t, \bar{\psi}_t\nabla\psi_1 - \psi_t\nabla\bar{\psi}_1\rangle - \frac{1}{2}\langle(\tilde{\mathbf{A}}_1 + \tilde{\mathbf{A}}_2)\cdot\mathbf{A}_t, \psi_1\bar{\psi}_t + \bar{\psi}_1\psi_t\rangle \\
&\quad + \frac{i}{2}\langle\Phi, \partial_t\psi_1\bar{\psi}_t - \partial_t\bar{\psi}_1\psi_t\rangle + \frac{i}{2}\langle\Phi_t, \psi_1\bar{\psi}_t - \bar{\psi}_1\psi_t\rangle.
\end{aligned}$$

In particular, by standard inequalities we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\psi_t\|^2 + \|\nabla\psi_t\|^2 + \langle|\tilde{\mathbf{A}}_2|^2, |\psi_t|^2\rangle + \lambda^2\langle|\psi_1|^2 + |\psi_2|^2, |\psi_t|^2\rangle \\
&\le \frac{1}{2}\|\nabla\psi_t\|^2 + c\big(\|\psi\|^2_{H^1_{\mathbb{C}}} + \|\mathbf{A}_t\|^2 + \|\mathrm{curl}\mathbf{A}\|^2 + \|\nabla\Phi\|^2 + \|\nabla\Phi_t\|^2\big) \\
&\quad + c\|\nabla\partial_t\psi_1\|^2\|\mathrm{curl}\mathbf{A}\|^2 + c(1 + \|\nabla\partial_t\Phi_2\|^2)\|\psi_t\|^2.
\end{aligned}$$

Now, thanks to (2.17) and the forthcoming Remark 3.4, the Gronwall lemma yields

$$\|\psi_t(t)\|^2 + \int_0^t \|\nabla\psi_t(y)\|^2 dy \le ce^{ct}\|z_1 - z_2\|_0^2, \tag{2.18}$$

which, in view of the estimate

$$\|\Delta\psi\| \le c\big(\|\psi_t\| + \|\psi\|_{H^1_{\mathbb{C}}} + \|\mathrm{curl}\mathbf{A}\| + \|\nabla\Phi\|\big)$$

and of (2.17), gives as a consequence

$$\|\psi(t)\|^2_{H^2_{\mathbb{C}}} \le ce^{ct}\|z_1 - z_2\|_0^2. \tag{2.19}$$

Next, a differentiation of (2.12) with respect to time leads to

$$\begin{aligned}
&\varepsilon(\mathbf{A}_{tt} - \nabla\Phi_t)_t + \sigma(\mathbf{A}_{tt} - \nabla\Phi_t) + \mathrm{curl}^2\mathbf{A}_t + |\psi_1|^2\mathbf{A}_t \\
&= \Big[\frac{i}{2}\nabla\partial_t\bar{\psi}_2 - \partial_t\mathbf{A}_2\bar{\psi}_2 - \tilde{\mathbf{A}}_2\partial_t\bar{\psi}_2\Big]\psi - \Big[\frac{i}{2}\nabla\partial_t\psi_2 + \partial_t\mathbf{A}_2\psi_1 + \tilde{\mathbf{A}}_2\partial_t\psi_1\Big]\bar{\psi} \\
&\quad + \Big[\frac{i}{2}\nabla\bar{\psi}_2 - \tilde{\mathbf{A}}_2\bar{\psi}_2\Big]\psi_t - \Big[\frac{i}{2}\nabla\psi_2 + \tilde{\mathbf{A}}_2\psi_1\Big]\bar{\psi}_t - \frac{i}{2}\partial_t\bar{\psi}_1\nabla\psi + \frac{i}{2}\partial_t\psi_1\nabla\bar{\psi} \\
&\quad - \frac{i}{2}(\bar{\psi}_1\nabla\psi_t - \psi_1\nabla\bar{\psi}_t) - (\psi_1\partial_t\bar{\psi}_1 + \bar{\psi}_1\partial_t\psi_1)\mathbf{A}.
\end{aligned}$$

Multiplying this equation by $\mathbf{A}_{tt}$ in $\mathbf{L}^2(\Omega)$, we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|\mathrm{curl}\mathbf{A}_t\|^2+\varepsilon\|\mathbf{A}_{tt}\|^2\big)+\sigma\|\mathbf{A}_{tt}\|^2=-\langle|\psi_1|^2,\mathbf{A}_t\cdot\mathbf{A}_{tt}\rangle \qquad (2.20)\\
&\quad+\langle\frac{i}{2}\nabla\partial_t\bar{\psi}_2-\partial_t\mathbf{A}_2\bar{\psi}_2-\tilde{\mathbf{A}}_2\partial_t\bar{\psi}_2,\psi\mathbf{A}_{tt}\rangle-\langle\frac{i}{2}\nabla\partial_t\psi_2+\partial_t\mathbf{A}_2\psi_1+\tilde{\mathbf{A}}_2\partial_t\psi_1,\bar{\psi}\mathbf{A}_{tt}\rangle\\
&\quad+\langle\frac{i}{2}\nabla\bar{\psi}_2-\tilde{\mathbf{A}}_2\bar{\psi}_2,\psi_t\mathbf{A}_{tt}\rangle-\langle\frac{i}{2}\nabla\psi_2+\tilde{\mathbf{A}}_2\psi_1,\bar{\psi}_t\mathbf{A}_{tt}\rangle-\frac{i}{2}\langle\partial_t\bar{\psi}_1,\nabla\psi\cdot\mathbf{A}_{tt}\rangle\\
&\quad+\frac{i}{2}\langle\partial_t\psi_1,\nabla\bar{\psi}\cdot\mathbf{A}_{tt}\rangle-\frac{i}{2}\langle\bar{\psi}_1\nabla\psi_t-\psi_1\nabla\bar{\psi}_t,\mathbf{A}_{tt}\rangle-\langle\psi_1\partial_t\bar{\psi}_1+\bar{\psi}_1\partial_t\psi_1,\mathbf{A}\cdot\mathbf{A}_{tt}\rangle.
\end{aligned}$$

Taking advantage of the continuous embedding $H^2(\Omega)\subset L^\infty(\Omega)$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|\mathrm{curl}\mathbf{A}_t\|^2+\varepsilon\|\mathbf{A}_{tt}\|^2\big)\\
&\le c(1+\|\nabla\partial_t\psi_1\|^2+\|\nabla\partial_t\psi_2\|^2)\|\mathbf{A}_{tt}\|^2+c\|\mathrm{curl}\mathbf{A}_t\|^2+c(\|\psi\|^2_{H^2_{\mathbb{C}}}+\|\psi_t\|^2_{H^1_{\mathbb{C}}}+\|\mathrm{curl}\mathbf{A}\|^2).
\end{aligned}$$

The product of (2.13) with $-\Delta\Phi$ in $L^2(\Omega)$ furnishes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\varepsilon\|\Delta\Phi\|^2\big)+\sigma\|\Delta\Phi\|^2+\frac{i}{2}\langle\bar{\psi}_1\psi_t-\psi_1\bar{\psi}_t+\bar{\psi}\partial_t\psi_2-\psi\partial_t\bar{\psi}_2,-\Delta\Phi\rangle \qquad (2.21)\\
&\quad+\langle|\psi_1|^2\Phi,-\Delta\Phi\rangle+\langle\psi_1\bar{\psi}\Phi_2,-\Delta\Phi\rangle+\langle\psi\bar{\psi}_2\Phi_2,-\Delta\Phi\rangle=0,
\end{aligned}$$

which easily entails

$$\frac{d}{dt}\big(\varepsilon\|\Delta\Phi\|^2\big)+\sigma\|\Delta\Phi\|^2\le c(\|\psi\|^2_{H^1_{\mathbb{C}}}+\|\psi_t\|^2+\|\nabla\Phi\|^2).$$

Adding together (2.20) and (2.21), on account of (2.17), (2.18), (2.19) and Remark 3.4, the Gronwall Lemma gives

$$\|\mathrm{curl}\mathbf{A}_t\|^2+\varepsilon\|\mathbf{A}_{tt}\|^2+\varepsilon\|\Delta\Phi\|^2\le ce^{ct}\|z_1-z_2\|_0^2;$$

then, by the estimate

$$\|\mathrm{curl}^2\mathbf{A}\|^2\le c\big(\|\mathbf{A}_{tt}\|^2+\|\mathbf{A}_t\|^2+\|\mathrm{curl}\mathbf{A}\|^2+\|\psi\|^2_{H^1_{\mathbb{C}}}\big),$$

which is easily derived from (2.12), we accomplish our purpose. □

3. Dissipativity. This section is devoted to the proof of the following

PROPOSITION 3.1. There exists $R_0\ge 0$ such that the ball $\mathcal{B}_0$ of $\mathbb{X}^0_\infty$ centered at zero of radius R_0 is absorbing for $S(t)$ in $\mathbb{X}^0_\infty$.

As we shall see, in order to prove the dissipation of $S(t)$ in $\mathbb{X}^0_\infty$, we need to pass through a weaker dissipativity result in this space endowed with the $\mathbb{X}^{-1}$-norm.

LEMMA 3.1. There exist two positive constants C and ϑ_{-1} such that, given $z\in\mathbb{X}^0_\infty$, there holds

$$\|S(t)z\|_{-1}\le Ce^{-\vartheta_{-1}t}\|z\|_{-1}+C,\qquad \forall t\ge 0.$$

As a consequence, it is possible to find $R_{-1}>0$ such that, given $R\ge 0$,

$$\sup_{z\in\mathbb{X}^0_\infty,\|z\|_{-1}\le R}\|S(t)z\|_{-1}\le R_{-1},$$

for any $t\ge t_0$ where $t_0=t_0(R)\ge 0$.

Proof. We introduce the notation

$$|||z|||^2 = \|D_{\tilde{\mathbf{A}}}\psi\|^2 + \frac{\lambda^2}{2}\|1-|\psi|^2\|^2 + \|\mathrm{curl}\mathbf{A}\|^2 + \varepsilon\|\mathbf{A}_t\|^2 + \varepsilon\|\nabla\Phi\|^2, \qquad z \in \mathbb{X}^{-1}.$$

Although this is not a norm, nevertheless it satisfies

$$c\|z\|_{-1}^2 - c \le |||z|||^2 \le c\|z\|_{-1}^2 + c, \qquad \forall z \in \mathbb{X}_\infty^0, \tag{3.1}$$

thanks to the bound $\|\psi\|^2 \le |\Omega|$. Indeed, from the Young inequality and from (2.9), we deduce

$$\begin{aligned} |i\langle\tilde{\mathbf{A}}, \bar{\psi}\nabla\psi - \psi\nabla\bar{\psi}\rangle| &\le 2\|\tilde{\mathbf{A}}\psi\|\|\nabla\psi\| \le \frac{1}{1+\eta}\|\nabla\psi\|^2 + (1+\eta)\|\tilde{\mathbf{A}}\psi\|^2 \\ &\le \frac{1}{1+\eta}\|\nabla\psi\|^2 + \langle|\tilde{\mathbf{A}}|^2, |\psi|^2\rangle + 2\eta\|\mathbf{A}\|^2 + 2\eta\|\mathbf{A}_{\mathrm{ext}}\|^2 \\ &\le \frac{1}{1+\eta}\|\nabla\psi\|^2 + \langle|\tilde{\mathbf{A}}|^2, |\psi|^2\rangle + 2\eta\kappa\|\mathrm{curl}\,\mathbf{A}\|^2 + c, \end{aligned}$$

for any $\eta > 0$. Thus, choosing $\eta = -\frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{2}{\kappa}}$, we have

$$\begin{aligned} \|D_{\tilde{\mathbf{A}}}\psi\|^2 + \|\mathrm{curl}\mathbf{A}\|^2 &= \|\nabla\psi\|^2 + i\langle\tilde{\mathbf{A}}, \bar{\psi}\nabla\psi - \psi\nabla\bar{\psi}\rangle + \langle|\tilde{\mathbf{A}}|^2, |\psi|^2\rangle + \|\mathrm{curl}\mathbf{A}\|^2 \\ &\ge \frac{\eta}{1+\eta}(\|\nabla\psi\|^2 + \|\mathrm{curl}\mathbf{A}\|^2) - c. \end{aligned}$$

For any $\alpha \in (0,1]$, we introduce the functional

$$E_{-1}(z) = |||z|||^2 + \|\psi\|^2 + 2\alpha\varepsilon\langle\mathbf{A}_t, \mathbf{A}\rangle + \alpha\sigma\|\mathbf{A}\|^2$$

which, provided that α is small enough, on account of (3.1), satisfies

$$c\|z\|_{-1}^2 + c \ge c|||z|||^2 + c \ge E_{-1}(z) \ge \frac{1}{2}|||z|||^2 \ge c\|z\|_{-1}^2 - c, \qquad \forall z \in \mathbb{X}_\infty^0. \tag{3.2}$$

Our next task is a differential inequality for $E_{-1}(t) = E_{-1}(S(t)z)$, for any $z \in \mathbb{X}_\infty^0$ with $\|z\|_{-1} \le R$. Adding together (2.1) multiplied by $\bar{\psi}_t + \bar{\psi}$ and the conjugate of (2.1) multiplied by $\psi_t + \psi$, integrating the resulting equation in Ω, since $\tilde{\mathbf{A}}_t = \mathbf{A}_t$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|D_{\tilde{\mathbf{A}}}\psi\|^2 + \|\psi\|^2 + \frac{\lambda^2}{2}\|1-|\psi|^2\|^2\Big) + \|D_{\tilde{\mathbf{A}}}\psi\|^2 + \frac{\lambda^2}{2}\|1-|\psi|^2\|^2 + \frac{\lambda^2}{2}\|\psi\|_{L^4_{\mathbb{C}}}^4 + \|\psi_t\|^2 \\ &\quad + \frac{i}{2}\langle\Phi, \bar{\psi}\psi_t - \psi\bar{\psi}_t\rangle - \langle\tilde{\mathbf{A}}\cdot\mathbf{A}_t, |\psi|^2\rangle - \frac{i}{2}\langle\mathbf{A}_t, \bar{\psi}\nabla\psi - \psi\nabla\bar{\psi}\rangle = \frac{\lambda^2}{2}|\Omega|. \end{aligned}$$

Next, we multiply (2.2) by $\mathbf{A}_t + \alpha\mathbf{A}$ in $\mathbf{L}^2(\Omega)$, getting

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big(\|\mathrm{curl}\mathbf{A}\|^2 + \varepsilon\|\mathbf{A}_t\|^2 + 2\alpha\varepsilon\langle\mathbf{A}_t, \mathbf{A}\rangle + \alpha\sigma\|\mathbf{A}\|^2\big) + \alpha\|\mathrm{curl}\mathbf{A}\|^2 + (\sigma - \alpha\varepsilon)\|\mathbf{A}_t\|^2 \\ &\quad + \frac{i}{2}\langle\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi}, \mathbf{A}_t\rangle + \langle\tilde{\mathbf{A}}\cdot\mathbf{A}_t, |\psi|^2\rangle + \frac{i}{2}\alpha\langle\bar{\psi}D_{\tilde{\mathbf{A}}}\psi - \psi\overline{D_{\tilde{\mathbf{A}}}\psi}, \mathbf{A}\rangle = 0. \end{aligned}$$

As a third step, the product of (2.6) with Φ in $L^2(\Omega)$ yields

$$\frac{1}{2}\frac{d}{dt}\big(\varepsilon\|\nabla\Phi\|^2\big) + \sigma\|\nabla\Phi\|^2 + \langle\Phi^2, |\psi|^2\rangle + \frac{i}{2}\langle\bar{\psi}\psi_t - \psi\bar{\psi}_t, \Phi\rangle = 0.$$

Adding these three equations, it is readily seen that $E_{-1}(t)$ satisfies

$$\frac{1}{2}\frac{dE_{-1}}{dt} + \|D_{\tilde{\mathbf{A}}}\psi\|^2 + \frac{\lambda^2}{2}\|1-|\psi|^2\|^2 + \alpha\|\mathrm{curl}\mathbf{A}\|^2 \tag{3.3}$$
$$+ (\sigma - \alpha\varepsilon)\|\mathbf{A}_t\|^2 + \sigma\|\nabla\Phi\|^2 + \|\psi_t\|^2 + \frac{\lambda^2}{2}\|\psi\|^4_{L^4_{\mathbb{C}}} + \langle\Phi^2, |\psi|^2\rangle$$
$$= -i\langle\bar{\psi}\psi_t - \psi\bar{\psi}_t, \Phi\rangle - \frac{i}{2}\alpha\langle\bar{\psi}D_{\tilde{\mathbf{A}}}\psi - \psi\overline{D_{\tilde{\mathbf{A}}}\psi}, \mathbf{A}\rangle + \frac{\lambda^2}{2}|\Omega|.$$

In order to control the right-hand side, notice that

$$|-i\langle\bar{\psi}\psi_t - \psi\bar{\psi}_t, \Phi\rangle| \le 2\|\psi\Phi\|\|\psi_t\| \le \|\psi_t\|^2 + \langle\Phi^2, |\psi|^2\rangle.$$

Moreover,

$$\Big| -\frac{i}{2}\alpha\langle\bar{\psi}D_{\tilde{\mathbf{A}}}\psi - \psi\overline{D_{\tilde{\mathbf{A}}}\psi}, \mathbf{A}\rangle\Big| \le \alpha\|D_{\tilde{\mathbf{A}}}\psi\|\|\mathbf{A}\| \le \frac{\alpha\kappa}{2}\|D_{\tilde{\mathbf{A}}}\psi\|^2 + \frac{\alpha}{2}\|\mathrm{curl}\mathbf{A}\|^2.$$

Thus, possibly reducing α, we obtain

$$\frac{dE_{-1}}{dt} + \alpha\|\!|S(t)z|\!\|^2 \le \lambda^2|\Omega|,$$

which, on account of (3.2), gives

$$\frac{dE_{-1}}{dt} + 2\vartheta_{-1}E_{-1} \le c,$$

for some $\vartheta_{-1} > 0$. Finally, the Gronwall Lemma yields

$$E_{-1}(t) \le E_{-1}(z)e^{-2\vartheta_{-1}t} + c, \qquad \forall t \ge 0,$$

and (3.2) allows us to complete the proof. □

Remark 3.2. It is worth mentioning that

$$\sup_{z\in\mathbb{X}^0_\infty,\, \|z\|_{-1}\le R}\ \sup_{t\ge 0}\int_t^{t+1}\|\psi_t(y)\|^2 dy \le C(R). \tag{3.4}$$

This can be easily seen integrating (3.3) with $\alpha = 0$, taking Lemma 3.1 into account.

Proposition 3.1 is a straightforward consequence of

Lemma 3.3. There exists $\vartheta_0 > 0$ such that, given $\rho_{-1}, \rho_0 \ge 0$, it is possible to find $C_{-1} = C_{-1}(\rho_{-1}) > 0$ and $C_0 = C_0(\rho_0) > 0$ satisfying

$$\|S(t)z\|_0 \le C_0 e^{-\vartheta_0 t} + C_{-1}, \quad \forall t \ge 0,$$

for any $z \in \mathbb{X}^0_\infty$ with $\|z\|_{-1} \le \rho_{-1}$ and $\|z\|_0 \le \rho_0$.

Proof. In this proof c stands for a positive constant possibly depending on ρ_{-1}. Notice that (2.1) and (2.2) yield

$$\|\Delta\psi\| \le \|\psi_t\| + \|\Phi\| + 2\|\tilde{\mathbf{A}}\cdot\nabla\psi\| + \||\tilde{\mathbf{A}}|^2\| + 2\lambda^2, \tag{3.5}$$
$$\|\mathrm{curl}^2\,\mathbf{A}\| \le \varepsilon\|\mathbf{A}_{tt}\| + \sigma\|\mathbf{A}_t\| + \|\nabla\psi\| + \|\tilde{\mathbf{A}}\|. \tag{3.6}$$

Besides, on account of Lemma 3.1, our goal is a dissipativity estimate for

$$\|\Delta\psi(t)\|^2 + \|\mathrm{curl}^2\,\mathbf{A}(t)\|^2 + \|\mathrm{curl}\,\mathbf{A}_t(t)\|^2 + \|\Delta\Phi(t)\|^2.$$

Hence we shall accomplish this purpose controlling the first two terms by the corresponding estimates for $\|\psi_t\|$ and $\|\mathbf{A}_{tt}\|$, while the other two contributions are directly estimated. In particular, we shall obtain a differential inequality for the functional $E_0(t) = E_0(S(t)z)$, defined as

$$E_0 = \alpha\|\psi_t\|^2 + \|\mathrm{curl}\mathbf{A}_t\|^2 + \varepsilon\|\mathbf{A}_{tt}\|^2 + 2\beta\varepsilon\langle\mathbf{A}_t, \mathbf{A}_{tt}\rangle + \beta\sigma\|\mathbf{A}_t\|^2 + \varepsilon\|\Delta\Phi\|^2 + \sigma\|\nabla\Phi\|^2,$$

where $\alpha \geq 1$ and $\beta \in (0, 1]$ will be properly fixed later. This functional is related to the norms we want to control by

$$\frac{1}{2}\Big(\alpha\|\psi_t\|^2 + \|\mathrm{curl}\mathbf{A}_t\|^2 + \varepsilon\|\mathbf{A}_{tt}\|^2 + \varepsilon\|\Delta\Phi\|^2 + \sigma\|\nabla\Phi\|^2\Big) \leq E_0, \tag{3.7}$$

$$E_0 \leq \frac{3}{2}\Big(\alpha\|\psi_t\|^2 + \|\mathrm{curl}\mathbf{A}_t\|^2 + \varepsilon\|\mathbf{A}_{tt}\|^2 + \varepsilon\|\Delta\Phi\|^2 + \sigma\|\nabla\Phi\|^2\Big), \tag{3.8}$$

provided that $\beta \leq \min\Big\{\dfrac{\sigma}{2\varepsilon}, \dfrac{1}{4\sigma\kappa}\Big\}$.

Formally differentiating (2.1) and (2.2) with respect to time, by definition of $D_{\tilde{\mathbf{A}}}$, we obtain

$$\begin{aligned}&\psi_{tt} - i\Phi\psi_t - i\Phi_t\psi - \Delta\psi_t + 2i\mathbf{A}_t\cdot\nabla\psi + 2i\tilde{\mathbf{A}}\cdot\nabla\psi_t + |\tilde{\mathbf{A}}|^2\psi_t + 2\tilde{\mathbf{A}}\cdot\mathbf{A}_t\psi \\ &\quad - \lambda^2(1-2|\psi|^2)\psi_t + \lambda^2\psi^2\bar{\psi}_t = 0,\end{aligned} \tag{3.9}$$

$$\begin{aligned}&\varepsilon(\mathbf{A}_t - \nabla\Phi)_{tt} + \sigma(\mathbf{A}_t - \nabla\Phi)_t + \mathrm{curl}^2\mathbf{A}_t + \frac{i}{2}(\bar{\psi}\nabla\psi_t - \psi\nabla\bar{\psi}_t) \\ &\quad + \frac{i}{2}(\bar{\psi}_t\nabla\psi - \psi_t\nabla\bar{\psi}) + \tilde{\mathbf{A}}(\psi\bar{\psi}_t + \bar{\psi}\psi_t) + \mathbf{A}_t|\psi|^2 = 0.\end{aligned} \tag{3.10}$$

The sum of (3.9) multiplied by $\bar{\psi}_t$ with the conjugate of (3.9) multiplied by ψ_t, by an integration on Ω, gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\psi_t\|^2) + \|\nabla\psi_t\|^2 - \frac{i}{2}\langle\Phi_t, \psi\bar{\psi}_t - \bar{\psi}\psi_t\rangle - \lambda^2\langle(1-2|\psi|^2), |\psi_t|^2\rangle \\ &\quad + \frac{\lambda^2}{2}[\langle\psi^2, \bar{\psi}_t^2\rangle + \langle\bar{\psi}^2, \psi_t^2\rangle] + \langle|\tilde{\mathbf{A}}|^2, |\psi_t|^2\rangle - i\langle\mathbf{A}_t, \psi\nabla\bar{\psi}_t - \bar{\psi}\nabla\psi_t\rangle \\ &\quad + \langle\tilde{\mathbf{A}}\cdot\mathbf{A}_t, \psi\bar{\psi}_t + \bar{\psi}\psi_t\rangle + i\langle\tilde{\mathbf{A}}, \bar{\psi}_t\nabla\psi_t - \psi_t\nabla\bar{\psi}_t\rangle = 0.\end{aligned}$$

The product of (3.10) by $\mathbf{A}_{tt} + \beta\mathbf{A}_t$ in $\mathbf{L}^2(\Omega)$, for some $\beta \in (0, 1]$ to be chosen later, leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\mathrm{curl}\mathbf{A}_t\|^2 + \varepsilon\|\mathbf{A}_{tt}\|^2 + 2\beta\varepsilon\langle\mathbf{A}_t, \mathbf{A}_{tt}\rangle + \beta\sigma\|\mathbf{A}_t\|^2\big) + \beta\|\mathrm{curl}\mathbf{A}_t\|^2 \\ &\quad + (\sigma - \beta\varepsilon)\|\mathbf{A}_{tt}\|^2 + \beta\langle|\mathbf{A}_t|^2, |\psi|^2\rangle \\ &= -i\langle\bar{\psi}\nabla\psi_t - \psi\nabla\bar{\psi}_t, \mathbf{A}_{tt}\rangle - \langle\mathbf{A}_t\cdot\mathbf{A}_{tt}, |\psi|^2\rangle - \langle\tilde{\mathbf{A}}\cdot\mathbf{A}_{tt}, \bar{\psi}_t\psi + \bar{\psi}\psi_t\rangle \\ &\quad - \beta i\langle\bar{\psi}\nabla\psi_t - \psi\nabla\bar{\psi}_t, \mathbf{A}_t\rangle \\ &\quad - \beta\langle|\mathbf{A}_t|^2, |\psi|^2\rangle - \beta\langle\tilde{\mathbf{A}}\cdot\mathbf{A}_t, \bar{\psi}\psi_t + \bar{\psi}_t\psi\rangle.\end{aligned}$$

It is left to consider (2.6), which is multiplied by $-\Delta\Phi + \Phi_t$, yielding

$$\frac{1}{2}\frac{d}{dt}\big(\varepsilon\|\Delta\Phi\|^2 + \sigma\|\nabla\Phi\|^2\big) + \sigma\|\Delta\Phi\|^2 + \varepsilon\|\nabla\Phi_t\|^2$$
$$= -\frac{i}{2}\langle\bar\psi\psi_t - \psi\bar\psi_t, -\Delta\Phi\rangle - \langle|\psi|^2\Phi, -\Delta\Phi\rangle - \frac{i}{2}\langle\bar\psi\psi_t - \psi\bar\psi_t, \Phi_t\rangle - \langle|\psi|^2\Phi, \Phi_t\rangle.$$

The sum of the last three equations yields

$$\frac{1}{2}\frac{dE_0}{dt} + \alpha\|\psi_t\|^2_{H^1_{\mathbb{C}}} + \beta\|\mathrm{curl}\mathbf{A}_t\|^2 + (\sigma - \beta\varepsilon)\|\mathbf{A}_{tt}\|^2 + \sigma\|\Delta\Phi\|^2 + \varepsilon\|\nabla\Phi_t\|^2$$
$$+ \alpha\langle|\tilde{\mathbf{A}}|^2, |\psi_t|^2\rangle + \beta\langle|\mathbf{A}_t|^2, |\psi|^2\rangle$$
$$= \sum_{j=1}^{5} I_j(t),$$

where, for the sake of simplicity, we have set

$$I_1 = i\frac{\alpha}{2}\langle\Phi_t, \psi\bar\psi_t - \bar\psi\psi_t\rangle - \frac{i}{2}\langle\bar\psi\psi_t - \psi\bar\psi_t, \Phi_t\rangle - \langle|\psi|^2\Phi, \Phi_t\rangle$$
$$I_2 = -i\langle\bar\psi\nabla\psi_t - \psi\nabla\bar\psi_t, \mathbf{A}_{tt}\rangle - \langle\mathbf{A}_t\cdot\mathbf{A}_{tt}, |\psi|^2\rangle - \langle\tilde{\mathbf{A}}\cdot\mathbf{A}_{tt}, \bar\psi_t\psi + \bar\psi\psi_t\rangle$$
$$I_3 = -\frac{i}{2}\langle\bar\psi\psi_t - \psi\bar\psi_t, -\Delta\Phi\rangle - \langle|\psi|^2\Phi, -\Delta\Phi\rangle$$
$$I_4 = i\alpha\langle\mathbf{A}_t, \nabla\bar\psi_t\psi - \nabla\psi_t\bar\psi\rangle - i\beta\langle\bar\psi\nabla\psi_t - \psi\nabla\bar\psi_t, \mathbf{A}_t\rangle - \alpha\langle\tilde{\mathbf{A}}\cdot\mathbf{A}_t, \psi\bar\psi_t + \bar\psi\psi_t\rangle$$
$$- \beta\langle\tilde{\mathbf{A}}\cdot\mathbf{A}_t, \bar\psi\psi_t + \bar\psi_t\psi\rangle$$
$$I_5 = -i\alpha\langle\tilde{\mathbf{A}}, \nabla\psi_t\bar\psi_t - \nabla\bar\psi_t\psi_t\rangle + \alpha\lambda^2\langle 1 - 2|\psi|^2, |\psi_t|^2\rangle$$
$$- \frac{\alpha}{2}\lambda^2[\langle\psi^2, \bar\psi_t^2\rangle + \langle\bar\psi^2, \psi_t^2\rangle] + \alpha\|\psi_t\|^2.$$

By standard inequalities and, in particular, by (2.7)–(2.10), we obtain

$$|I_1| \le (\alpha+1)\|\Phi_t\|\|\psi_t\| + \|\Phi\|\|\Phi_t\| \le \frac{\varepsilon}{2}\|\nabla\Phi_t\|^2 + c\|\psi_t\|^2 + c$$
$$|I_2| \le \|\mathbf{A}_{tt}\|(2\|\nabla\psi_t\| + \|\mathbf{A}_t\| + 2\|\tilde{\mathbf{A}}\|_{\mathbf{L}^4(\Omega)}\|\psi_t\|_{L^4_{\mathbb{C}}})$$
$$\le \frac{\sigma}{2}\|\mathbf{A}_{tt}\|^2 + \Big(\frac{6}{\sigma} + \frac{\alpha}{6}\Big)\|\psi_t\|^2_{H^1_{\mathbb{C}}} + c\|\psi_t\|^2 + c$$
$$|I_3| \le \|\Delta\Phi\|(\|\psi_t\| + \|\Phi\|) \le \frac{\sigma}{2}\|\Delta\Phi\|^2 + c\|\psi_t\|^2 + c$$
$$|I_4| \le 2(\alpha+\beta)\|\mathbf{A}_t\|\|\nabla\psi_t\| + 2(\alpha+\beta)\|\tilde{\mathbf{A}}\|_4\|\mathbf{A}_t\|\|\psi_t\|_4 \le \frac{\alpha+\beta}{6}\|\psi_t\|^2_{H^1_{\mathbb{C}}} + c$$
$$|I_5| \le 2\alpha\|\tilde{\mathbf{A}}\|_4\|\nabla\psi_t\|\|\psi_t\|_4 + 2(\lambda^2+1)\alpha\|\psi_t\|^2 \le \frac{\alpha}{6}\|\psi_t\|^2_{H^1_{\mathbb{C}}} + c\|\psi_t\|^2 + c.$$

Replacing these estimates in the differential equation, we end up with

$$\frac{1}{2}\frac{dE_0}{dt} + \Big(\frac{\alpha}{2} - \frac{\beta}{6} - \frac{6}{\sigma}\Big)\|\psi_t\|^2_{H^1_{\mathbb{C}}} + \beta\|\mathrm{curl}\mathbf{A}_t\|^2 + \Big(\frac{\sigma}{2} - \beta\varepsilon\Big)\|\mathbf{A}_{tt}\|^2 + \frac{\sigma}{2}\|\Delta\Phi\|^2$$
$$+ \frac{\varepsilon}{2}\|\nabla\Phi_t\|^2 \le c + c\|\psi_t\|^2.$$

Thus, provided that α is large enough and possibly reducing β, from (3.8) we have

$$\frac{dE_0}{dt} + 2\vartheta_0 E_0 + \frac{\alpha}{2}\|\nabla\psi_t\|^2 + \varepsilon\|\nabla\Phi_t\|^2 \le c + c\|\psi_t\|^2,$$

for some $\vartheta_0 > 0$ depending only on ε and σ. The Gronwall lemma applies thanks to (3.4), yielding

$$E_0(S(t)z) \leq E_0(z)e^{-2\vartheta_0 t} + c.$$

Since, on account of (2.1) and (2.2), a further application of (3.7)–(3.8) gives

$$\alpha\|\psi_t(t)\|^2 + \|\mathrm{curl}\mathbf{A}_t(t)\|^2 + \varepsilon\|\mathbf{A}_{tt}(t)\|^2 + \varepsilon\|\Delta\Phi(t)\|^2 + \sigma\|\nabla\Phi(t)\|^2 \leq 3e^{-2\vartheta_0 t}C(\rho_0^2) + c,$$

(3.5)–(3.6) lead to the claim. $\square$

Remark 3.4. Integrating the last differential inequality for E_0, it is apparent that, for any $R \geq 0$, there exists a positive constant $C(R)$ such that

$$\sup_{z\in\mathbb{X}^0_\infty,\, \|z\|_0\leq R}\ \sup_{t\geq 0} \int_t^{t+1} \big(\|\nabla\psi_t(y)\|^2 + \varepsilon\|\nabla\Phi_t(y)\|^2\big)dy \leq C(R).$$

Besides,

$$\sup_{z\in\mathbb{X}^0_\infty,\, \|z\|_0\leq R}\ \sup_{t\geq 0} \int_t^{t+1} \big(\|\nabla\Delta\psi(y)\|^2 + \|\Delta\Phi_t(y)\|^2\big)dy \leq C(R).$$

Indeed, taking the gradient of (2.1), we easily obtain

$$\|\nabla\Delta\psi(t)\| \leq C(R)(1 + \|\nabla\psi_t\|).$$

Moreover, multiplying (2.6) by $-\Delta\Phi_t$ in $L^2(\Omega)$, we can see by standard computations that

$$\varepsilon\|\Delta\Phi_t\|^2 + \sigma\frac{d}{dt}\|\Delta\Phi\|^2 \leq C(R).$$

We accomplish our purpose integrating in time these two inequalities.

4. The global attractor. This section is devoted to the proof of the existence of a compact attracting set. For this purpose we introduce the space $\mathbb{X}^1_\infty$ compactly embedded in $\mathbb{X}^0_\infty$. Namely, $\mathbb{X}^1_\infty = \mathbb{X}^0_\infty \cap \mathbb{X}^1$, where

$$\mathbb{X}^1 = H^3_{\mathbb{C}}(\Omega) \times [\mathbf{X}_0 \cap \mathbf{H}^3(\Omega)] \times [\mathbf{X}_0 \cap \mathbf{H}^2(\Omega)] \times [H^2_{0m}(\Omega) \cap H^3(\Omega)]$$

is equipped with the norm $\|\cdot\|_1$.

Theorem 4.1. The dynamical system $(S(t), \mathbb{X}^0_\infty)$ admits a connected global attractor $\mathcal{A} \subset \mathbb{X}^1_\infty$.

The claim follows from the existence of a bounded set $\hat{\mathcal{B}}^1$ in $\mathbb{X}^1$ which (exponentially) attracts any bounded set in $\mathbb{X}^0_\infty$; that is, denoting by dist the usual Hausdorff semidistance,

$$\lim_{t\to\infty} \mathrm{dist}_{\mathbb{X}^0}(S(t)\mathcal{B}, \hat{\mathcal{B}}^1) = 0,$$

for any bounded $\mathcal{B} \subset \mathbb{X}^0_\infty$. Since $S(t)\mathcal{B} \subset \mathbb{X}^0_\infty$, for any $t \geq 0$ and any $\mathcal{B}$, this implies

$$\lim_{t\to\infty} \mathrm{dist}_{\mathbb{X}^0}(S(t)\mathcal{B}, \hat{\mathcal{B}}^1 \cap \mathbb{X}^0_\infty) = 0. \tag{4.1}$$

In other words, there exists a bounded set $\tilde{\mathcal{B}}^1 = \hat{\mathcal{B}}^1 \cap \mathbb{X}^0_\infty \subset \mathbb{X}^1_\infty$ which (exponentially) attracts any bounded set in $\mathbb{X}^0_\infty$. Since $\mathbb{X}^1_\infty \Subset \mathbb{X}^0_\infty$, this yields the existence of a compact attracting set. The exponential attraction rate will be used in the next section. The proof of Theorem 4.1 is divided into two lemmas. Given any bounded set $\mathcal{B} \subset \mathbb{X}^0_\infty$, for any $z \in \mathcal{B}$ such that $\|z\|_0 \le R$, we decompose $S(t)z$ as

$$S(t)z = z_d(t) + z_c(t),$$

where $z_d(t) = (\psi^d(t), \mathbf{A}^d(t), \mathbf{A}^d_t(t), \Phi^d(t))$ solves

$$\begin{cases} \psi^d_t - \Delta\psi^d + \psi^d = 0, \\ \varepsilon(\mathbf{A}^d_t - \nabla\Phi^d)_t + \sigma(\mathbf{A}^d_t - \nabla\Phi^d) + \mathrm{curl}^2\mathbf{A}^d = 0, \\ -\varepsilon\Delta\Phi^d_t - \sigma\Delta\Phi^d = 0, \\ \mathrm{div}\mathbf{A}^d = 0, \quad \int_\Omega \Phi^d = 0, \\ \partial_{\boldsymbol{n}}\psi^d|_{\partial\Omega} = 0, \quad \mathbf{A}^d \cdot \boldsymbol{n}|_{\partial\Omega} = 0, \quad \mathrm{curl}\mathbf{A}^d \times \boldsymbol{n}|_{\partial\Omega} = 0, \quad \partial_{\boldsymbol{n}}\Phi^d|_{\partial\Omega} = 0, \\ \psi^d(0) = \psi_0, \quad \mathbf{A}^d(0) = \mathbf{A}_0, \quad \mathbf{A}^d_t(0) = \dot{\mathbf{A}}_0, \quad \Phi^d(0) = \Phi_0, \end{cases} \tag{4.2}$$

whereas $z_c(t) = (\psi^c(t), \mathbf{A}^c(t), \mathbf{A}^c_t(t), \Phi^c(t))$ is the solution to

$$\begin{cases} \psi^c_t - \Delta\psi^c + \psi^c = F(\psi, \tilde{\mathbf{A}}, \Phi), \\ \varepsilon(\mathbf{A}^c_t - \nabla\Phi^c)_t + \sigma(\mathbf{A}^c_t - \nabla\Phi^c) + \mathrm{curl}^2\mathbf{A}^c = \mathbf{G}(\psi, \tilde{\mathbf{A}}), \\ -\varepsilon\Delta\Phi^c_t - \sigma\Delta\Phi^c = H(\psi, \Phi), \\ \mathrm{div}\mathbf{A}^c = 0, \quad \int_\Omega \Phi^c = 0, \\ \partial_{\boldsymbol{n}}\psi^c|_{\partial\Omega} = 0, \quad \mathbf{A}^c \cdot \boldsymbol{n}|_{\partial\Omega} = 0, \quad \mathrm{curl}\mathbf{A}^c \times \boldsymbol{n}|_{\partial\Omega} = 0, \quad \partial_{\boldsymbol{n}}\Phi^c|_{\partial\Omega} = 0, \\ \psi^c(0) = 0, \quad \mathbf{A}^c(0) = 0, \quad \mathbf{A}^c_t(0) = 0, \quad \Phi^c(0) = 0. \end{cases} \tag{4.3}$$

Here we have set

$$\begin{aligned} F(\psi, \tilde{\mathbf{A}}, \Phi) &= i\Phi\psi - |\tilde{\mathbf{A}}|^2\psi - 2i\tilde{\mathbf{A}} \cdot \nabla\psi + \lambda^2(1 - |\psi|^2)\psi + \psi, \\ \mathbf{G}(\psi, \tilde{\mathbf{A}}) &= \frac{i}{2}(\psi\nabla\bar{\psi} - \bar{\psi}\nabla\psi) - \tilde{\mathbf{A}}|\psi|^2, \\ H(\psi, \Phi) &= -\frac{i}{2}(\bar{\psi}\psi_t - \psi\bar{\psi}_t) - |\psi|^2\Phi. \end{aligned}$$

We shall see that the $\mathbb{X}^0$-norm of $z_d(t)$ exponentially decays to zero as $t \to \infty$, while $z_c(t)$ is confined in a compact set of $\mathbb{X}^0$, since it remains in a bounded set of $\mathbb{X}^1$. This program is carried out in the next two lemmas.

Lemma 4.2. There exist two positive constants γ and ν, both independent of R, such that the solution z_d of problem (4.2) satisfies

$$\|z_d(t)\|_0^2 \le \gamma^2 e^{-2\nu t} R^2, \qquad t \ge 0. \tag{4.4}$$

Proof. It is immediate to check that the energy functional

$$\begin{aligned} E(z_d) = {} & \|\Delta\psi^d\|^2 + 2\|\nabla\psi^d\|^2 + \|\psi^d\|^2 + \|\mathrm{curl}^2\mathbf{A}^d\|^2 + \varepsilon\|\mathrm{curl}\mathbf{A}^d_t\|^2 \\ & + \sigma\langle \mathrm{curl}\mathbf{A}^d_t, \mathrm{curl}\mathbf{A}^d\rangle + \frac{\sigma^2}{2\varepsilon}\|\mathrm{curl}\mathbf{A}^d\|^2 + \varepsilon(\|\Delta\Phi^d\|^2 + \|\nabla\Phi^d\|^2) \end{aligned}$$

satisfies

$$\frac{1}{\gamma}\|z_d\|_0^2 \le E(z_d) \le \gamma\|z_d\|_0^2,$$

for some $\gamma > 0$ determined by ε, σ and Ω. Besides, $E(z_d(t))$ satisfies the differential inequality

$$\frac{1}{2}\frac{d}{dt}E(z_d) + \alpha\|z_d\|_0^2 \leq 0.$$

Thus, exploiting the relation between $E(z_d)$ and $\|z_d\|_0$, we get (4.4) with $\nu = \alpha/\gamma$. $\square$

LEMMA 4.3. There exists a set $\hat{\mathcal{B}}^1 \subset \mathbb{X}^1$ such that

$$\lim_{t\to\infty} \operatorname{dist}_{\mathbb{X}^0}(S(t)\mathcal{B}, \hat{\mathcal{B}}^1) = 0,$$

for any bounded set $\mathcal{B} \subset \mathbb{X}^0_\infty$.

Proof. Allowing c to depend on R, the claim amounts to finding a constant $c > 0$ such that the solution z_c of system (4.3) satisfies the inequality

$$\|z_c(t)\|_1 \leq c, \qquad t \geq 0.$$

Since Lemmas 3.3 and 4.2 ensure

$$\|z_c(t)\|_0 \leq c, \qquad \forall t \geq 0, \tag{4.5}$$

we accomplish our purpose by proving

$$\|\nabla\Delta\psi^c(t)\| + \|\operatorname{curl}^3 \mathbf{A}^c(t)\| + \|\operatorname{curl}^2 \mathbf{A}^c_t(t)\| + \|\nabla\Delta\Phi^c(t)\| \leq c, \qquad t \geq 0.$$

The control on the first two norms is not obtained directly, but by considering the corresponding time derivatives. Namely, exploiting

$$\nabla\Delta\psi^c = \nabla\psi^c_t + \nabla\psi^c - \nabla F(\psi, \tilde{\mathbf{A}}, \Phi),$$
$$\operatorname{curl}^3\mathbf{A}^c = -\varepsilon\operatorname{curl}\mathbf{A}^c_{tt} - \sigma\operatorname{curl}\mathbf{A}^c_t + \operatorname{curl}\mathbf{G}(\psi, \tilde{\mathbf{A}}),$$

we reach our goal by controlling $\|\nabla\psi^c_t\|$ and $\|\operatorname{curl}\mathbf{A}^c_{tt}\|$. Indeed, take

$$\psi^c_{tt} - \Delta\psi^c_t + \psi^c_t = \partial_t F(\psi, \tilde{\mathbf{A}}, \Phi), \tag{4.6}$$

supplemented with the initial condition

$$\psi^c_t(0) = F(\psi_0, \tilde{\mathbf{A}}_0, \Phi_0) \in H^1_{\mathbb{C}}.$$

Multiplying (4.6) by $-\Delta\bar{\psi}^c_t$ and summing the resulting equality to the product of the conjugate of (4.6) by $-\Delta\psi^c_t$, by an integration on Ω, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\psi^c_t\|^2 + \|\Delta\psi^c_t\|^2 + \|\nabla\psi^c_t\|^2 &= -\frac{1}{2}\big(\langle\partial_t F(\psi, \tilde{\mathbf{A}}, \Phi), \Delta\bar{\psi}^c_t\rangle + \langle\partial_t \bar{F}(\psi, \tilde{\mathbf{A}}, \Phi), \Delta\psi^c_t\rangle\big)\\ &\leq \frac{1}{2}\big(\|\partial_t F(\psi, \tilde{\mathbf{A}}, \Phi)\|^2 + \|\Delta\psi^c_t\|^2\big).\end{aligned}$$

On account of (4.5), it is readily seen that

$$\|\partial_t F(\psi(t), \tilde{\mathbf{A}}(t), \Phi(t))\| \leq c(1 + \|\nabla\psi_t(t)\| + \|\nabla\Phi_t(t)\|);$$

hence, thanks to Remark 3.4, from the Gronwall lemma we deduce

$$\|\nabla\psi^c_t(t)\|^2 \leq \|\nabla\psi^c_t(0)\|^2 e^{-2t} + c \leq c. \tag{4.7}$$

Next, the time derivative of the second equation in (4.3) is

$$\varepsilon(\mathbf{A}^c_{tt} - \nabla\Phi^c_t)_t + \sigma(\mathbf{A}^c_{tt} - \nabla\Phi^c_t) + \operatorname{curl}^2\mathbf{A}^c_t = \partial_t\mathbf{G}(\psi, \tilde{\mathbf{A}}),$$

so the product by $\mathrm{curl}^2\mathbf{A}^c_{tt}+\frac{\sigma}{4\varepsilon}\mathrm{curl}^2\mathbf{A}^c_t$ in $\mathbf{L}^2(\Omega)$ gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\mathrm{curl}^2\mathbf{A}^c_t\|^2+\varepsilon\|\mathrm{curl}\mathbf{A}^c_{tt}\|^2+\frac{\sigma}{2}\langle\mathrm{curl}\mathbf{A}^c_{tt},\mathrm{curl}\mathbf{A}^c_t\rangle+\frac{\sigma^2}{4\varepsilon}\|\mathrm{curl}\mathbf{A}^c_t\|^2\big)\\&\quad+\frac{\sigma}{4\varepsilon}\|\mathrm{curl}^2\mathbf{A}^c_t\|^2+\frac{3\sigma}{4}\|\mathrm{curl}\mathbf{A}^c_{tt}\|^2=\langle\partial_t\mathbf{G}(\psi,\tilde{\mathbf{A}}),\mathrm{curl}^2\mathbf{A}^c_{tt}+\frac{\sigma}{4\varepsilon}\mathrm{curl}^2\mathbf{A}^c_t\rangle\\&\le\frac{\sigma}{8\varepsilon}\|\mathrm{curl}^2\mathbf{A}^c_t\|^2+\frac{\sigma}{8\varepsilon}\|\partial_t\mathbf{G}(\psi,\tilde{\mathbf{A}})\|^2+\|\mathrm{curl}\partial_t\mathbf{G}(\psi,\tilde{\mathbf{A}})\|\|\mathrm{curl}\mathbf{A}^c_{tt}\|.\end{aligned}$$

Here, for the sake of simplicity, we have multiplied $\mathrm{curl}^2\mathbf{A}^c_t$ by $\frac{\sigma}{4\varepsilon}$, but any positive constant, provided that it is small enough, will do. In order to estimate the right-hand side, notice that

$$\partial_t\mathbf{G}(\psi,\tilde{\mathbf{A}})=\frac{i}{2}\big(\psi_t\nabla\bar\psi-\bar\psi_t\nabla\psi+\psi\nabla\bar\psi_t-\bar\psi\nabla\psi_t\big)-\mathbf{A}_t|\psi|^2-\tilde{\mathbf{A}}(\psi\bar\psi_t+\bar\psi\psi_t)$$

satisfies the estimate

$$\|\partial_t\mathbf{G}(\psi(t),\tilde{\mathbf{A}}(t))\|\le c(1+\|\nabla\psi_t(t)\|).$$

Analogous computations, (2.8) and Remark 3.4 lead us to control

$$\begin{aligned}\mathrm{curl}\partial_t\mathbf{G}(\psi,\tilde{\mathbf{A}})=&i(\nabla\psi_t\times\nabla\bar\psi+\nabla\psi\times\nabla\bar\psi_t)-(\psi_t\bar\psi+\psi\bar\psi_t)\mathrm{curl}\tilde{\mathbf{A}}-|\psi|^2\mathrm{curl}\mathbf{A}_t\\&+\mathbf{A}_t\times(\bar\psi\nabla\psi+\psi\nabla\bar\psi)+\tilde{\mathbf{A}}\times(\bar\psi_t\nabla\psi+\bar\psi\nabla\psi_t+\psi_t\nabla\bar\psi+\psi\nabla\bar\psi_t)\end{aligned}$$

as

$$\|\mathrm{curl}\partial_t\mathbf{G}(\psi(t),\tilde{\mathbf{A}}(t))\|\le c(1+\|\nabla\psi_t(t)\|^{3/2}).$$

Replacing these estimates in the above differential inequality, by a generalized Gronwall Lemma (see [8, Appendix]) together with Remark 3.4, we obtain

$$\|\mathrm{curl}^2\mathbf{A}^c_t(t)\|^2+\varepsilon\|\mathrm{curl}\mathbf{A}^c_{tt}(t)\|^2\le2\varepsilon\|\mathrm{curl}\mathbf{A}^c_{tt}(0)\|^2e^{-\frac{\sigma}{6\varepsilon}t}+c,$$

where the initial datum

$$\varepsilon\mathrm{curl}\mathbf{A}^c_{tt}(0)=\mathrm{curl}\mathbf{G}(\psi_0,\tilde{\mathbf{A}}_0)=i\nabla\psi_0\times\nabla\bar\psi_0-|\psi_0|^2\mathrm{curl}\tilde{\mathbf{A}}_0+\tilde{\mathbf{A}}_0\times(\bar\psi_0\nabla\psi_0+\psi_0\nabla\bar\psi_0),$$

recovered from the second equation in (4.3), satisfies $\|\mathrm{curl}\mathbf{A}^c_{tt}(0)\|\le c$. As a consequence,

$$\|\mathrm{curl}^2\mathbf{A}^c_t(t)\|^2+\varepsilon\|\mathrm{curl}\mathbf{A}^c_{tt}(t)\|^2\le c.\tag{4.8}$$

Finally we multiply the third equation of (4.3) by $\Delta^2\Phi^c$ in $L^2(\Omega)$, getting

$$\frac{\varepsilon}{2}\frac{d}{dt}\|\nabla\Delta\Phi^c\|^2+\sigma\|\nabla\Delta\Phi^c\|^2=-\langle\nabla H(\psi,\Phi),\nabla\Delta\Phi^c\rangle\le\|\nabla H(\psi,\Phi)\|\,\|\nabla\Delta\Phi^c\|.$$

As above, it is readily seen that

$$\nabla H(\psi,\Phi)=-\frac{i}{2}(\psi_t\nabla\bar\psi+\bar\psi\nabla\psi_t-\bar\psi_t\nabla\psi-\psi\nabla\bar\psi_t)-(\bar\psi\nabla\psi+\psi\nabla\bar\psi)\Phi-|\psi|^2\nabla\Phi$$

is controlled by

$$\|\nabla H(\psi(t),\Phi(t))\|\le c(1+\|\nabla\psi_t(t)\|);$$

hence, a third application of the Gronwall inequality in view of Remark 3.4 gives

$$\|\nabla\Delta\Phi^c(t)\|^2\le c.\tag{4.9}$$

Collecting (4.7)–(4.9), we get the desired conclusion. □

5. Exponential attractors.

THEOREM 5.1. The semigroup $S(t)$ possesses a regular exponential attractor, namely a bounded set $\mathcal{E} \subset \mathbb{X}^1_\infty$, closed and of finite fractal dimension in $\mathbb{X}^0_\infty$, positively invariant for $S(t)$ and satisfying the following exponential attraction property:

(EA) There exist $\omega > 0$ and a positive increasing function J such that, for every bounded set $\mathcal{B} \subset \mathbb{X}^1_\infty$ with $R = \sup_{z\in\mathcal{B}} \|z\|_0$, there holds

$$\mathrm{dist}_{\mathbb{X}^0}(S(t)\mathcal{B}, \mathcal{E}) \leq J(R)e^{-\omega t}.$$

Since the global attractor is the minimal compact attracting set, we have $\mathcal{A} \subset \mathcal{E}$. Thus, Theorem 5.1 gives as a byproduct the finite fractal dimension of the global attractor.

Our first task is to confine the dynamics to a regular but significant set, whose existence is guaranteed by the next

PROPOSITION 5.1. There exists a closed ball $\mathcal{B}_1 \subset \mathbb{X}^1_\infty$ such that

(i) there is a positive increasing function M such that, for every bounded set $\mathcal{B} \subset \mathbb{X}^0_\infty$ with $R = \sup_{z\in\mathcal{B}} \|z\|_0$, there holds

$$\mathrm{dist}_{\mathbb{X}^0}(S(t)\mathcal{B}, \mathcal{B}_1) \leq M(R)e^{-\nu t},$$

with $\nu > 0$ as in Lemma 4.2;

(ii) there is a time $t_1 \geq 0$ such that

$$S(t)\mathcal{B}_1 \subset \mathcal{B}_1, \qquad \forall t \geq t_1.$$

In view of (4.1), this result holds true once we prove the dissipativity of the dynamical system restricted to $\mathbb{X}^1_\infty$. Indeed the existence of a bounded exponentially attracting set $\mathcal{K} \subset \mathbb{X}^1_\infty$ is given by Lemmas 4.2 and 4.3. Next, possibly enlarging $\mathcal{K}$, we obtain $\mathcal{B}_1$ satisfying Proposition 5.1.

PROPOSITION 5.2. There exists $R_1 \geq 0$ such that the ball of $\mathbb{X}^1_\infty$ of radius R_1 is an absorbing set for $S(t)$ in $\mathbb{X}^1_\infty$.

Proof. Let $\mathcal{B}$ be a bounded set in $\mathbb{X}^1_\infty$: in particular, there exist ρ_0 and ρ_1 such that

$$\sup_{z\in\mathcal{B}} \|z\|_0 \leq \rho_0 \qquad \text{and} \qquad \sup_{z\in\mathcal{B}} \|z\|_1 \leq \rho_1.$$

As in Lemma 3.3, we shall prove the existence of $\vartheta_1 > 0$ and of two positive constants $\Lambda_0(\rho_0)$ and $\Lambda_1(\rho_1)$ such that

$$\sup_{z\in\mathcal{B}} \|S(t)z\|_1 \leq \Lambda_1 e^{-\vartheta_1 t} + \Lambda_0.$$

Throughout this proof, the constant c is supposed to depend also on ρ_0. Notice that, by Lemma 3.3, we learn

$$\sup_{t\geq 0}\sup_{z\in\mathcal{B}} \|S(t)z\|_0 \leq c,$$

so we are left to control $(\nabla\Delta\psi(t), \mathrm{curl}^3\,\mathbf{A}(t), \mathrm{curl}^2\,\mathbf{A}_t(t), \nabla\Delta\Phi(t))$. As in the previous sections, the desired dissipativity estimates for the first two norms are not directly obtained, but are recovered from the controls on $\|\nabla\psi_t\|$ and $\|\mathrm{curl}\,\mathbf{A}_{tt}\|$. For this purpose, we consider the equations obtained, formally differentiating with respect to time (2.1) and (2.2), that is,

$$\psi_{tt} - i\Phi_t\psi - i\Phi\psi_t - \Delta\psi_t + 2i\mathbf{A}_t\cdot\nabla\psi + 2i\tilde{\mathbf{A}}\cdot\nabla\psi_t + 2\tilde{\mathbf{A}}\cdot\mathbf{A}_t\psi + |\tilde{\mathbf{A}}|^2\psi_t \tag{5.1}$$
$$- \lambda^2(1-2|\psi|^2)\psi_t + \lambda^2\psi^2\bar{\psi}_t = 0,$$

$$\varepsilon(\mathbf{A}_{tt} - \nabla\Phi_t)_t + \sigma(\mathbf{A}_{tt} - \nabla\Phi_t) + \operatorname{curl}^2\mathbf{A}_t + \frac{i}{2}(\bar{\psi}_t\nabla\psi - \psi_t\nabla\bar{\psi} + \bar{\psi}\nabla\psi_t - \psi\nabla\bar{\psi}_t) \tag{5.2}$$
$$+ \mathbf{A}_t|\psi|^2 + \tilde{\mathbf{A}}(\bar{\psi}\psi_t + \psi\bar{\psi}_t) = 0,$$

supplemented with the initial conditions

$$\psi_t(0) = i\Phi_0\psi_0 + \Delta\psi_0 - 2i\tilde{\mathbf{A}}_0\cdot\nabla\psi_0 - |\tilde{\mathbf{A}}_0|^2\psi_0 + \lambda^2(1-|\psi_0|^2)\psi_0,$$
$$\varepsilon\operatorname{curl}\mathbf{A}_{tt}(0) = -\sigma\operatorname{curl}\dot{\mathbf{A}}_0 - \operatorname{curl}^3\mathbf{A}_0 - i\nabla\bar{\psi}_0\times\nabla\psi_0 - (\psi_0\nabla\bar{\psi}_0 + \bar{\psi}_0\nabla\psi_0)\times\tilde{\mathbf{A}}_0$$
$$- |\psi_0|^2\operatorname{curl}\tilde{\mathbf{A}}_0.$$

Adding together the product of (5.1) by $-\Delta\bar{\psi}_t$ and the conjugate of (5.1) by $-\Delta\psi_t$, by an integration on Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\psi_t\|^2 + \|\nabla\psi_t\|^2 + \|\Delta\psi_t\|^2 \tag{5.3}$$
$$= -\frac{i}{2}\langle\Phi_t, \psi\Delta\bar{\psi}_t - \bar{\psi}\Delta\psi_t\rangle - \frac{i}{2}\langle\nabla\Phi, \bar{\psi}_t\nabla\psi_t - \psi_t\nabla\bar{\psi}_t\rangle - i\langle\mathbf{A}_t, \nabla\bar{\psi}\Delta\psi_t - \nabla\psi\Delta\bar{\psi}_t\rangle$$
$$- i\langle\tilde{\mathbf{A}}, \nabla\bar{\psi}_t\Delta\psi_t - \nabla\psi_t\Delta\bar{\psi}_t\rangle + \langle\tilde{\mathbf{A}}\cdot\mathbf{A}_t, \psi\Delta\bar{\psi}_t + \bar{\psi}\Delta\psi_t\rangle + \frac{1}{2}\langle|\tilde{\mathbf{A}}|^2, \psi_t\Delta\bar{\psi}_t + \bar{\psi}_t\Delta\psi_t\rangle$$
$$- \frac{\lambda^2}{2}\langle 1-2|\psi|^2, \psi_t\Delta\bar{\psi}_t + \bar{\psi}_t\Delta\psi_t\rangle + \frac{\lambda^2}{2}\big(\langle\psi^2, \bar{\psi}_t\Delta\bar{\psi}_t\rangle + \langle\bar{\psi}^2, \psi_t\Delta\psi_t\rangle\big) + \|\nabla\psi_t\|^2.$$

Notice that we have also added to both sides $\|\nabla\psi_t\|^2$ and that the right-hand side can be estimated by

$$c(1 + \|\nabla\psi_t\|^2 + \|\nabla\Phi_t\|^2) + \frac{1}{2}\|\Delta\psi_t\|^2.$$

Next, in view of

$$\operatorname{curl}(\bar{\psi}_t\nabla\psi - \psi_t\nabla\bar{\psi} + \bar{\psi}\nabla\psi_t - \psi\nabla\bar{\psi}_t) = 2(\nabla\bar{\psi}_t\times\nabla\psi - \nabla\psi_t\times\nabla\bar{\psi}),$$

the product of (5.2) by $\operatorname{curl}^2\mathbf{A}_{tt} + \frac{\sigma}{4\varepsilon}\operatorname{curl}^2\mathbf{A}_t$ in $\mathbf{L}^2(\Omega)$ leads us to

$$\frac{1}{2}\frac{d}{dt}\big(\|\operatorname{curl}^2\mathbf{A}_t\|^2 + \varepsilon\|\operatorname{curl}\mathbf{A}_{tt}\|^2 + \frac{\sigma}{2}\langle\operatorname{curl}\mathbf{A}_{tt}, \operatorname{curl}\mathbf{A}_t\rangle + \frac{\sigma^2}{4\varepsilon}\|\operatorname{curl}\mathbf{A}_t\|^2\big) \tag{5.4}$$
$$+ \frac{\sigma}{4\varepsilon}\|\operatorname{curl}^2\mathbf{A}_t\|^2 + \frac{3\sigma}{4}\|\operatorname{curl}\mathbf{A}_{tt}\|^2 = -\langle|\psi|^2\operatorname{curl}\mathbf{A}_t, \operatorname{curl}\mathbf{A}_{tt}\rangle$$
$$- \langle(\bar{\psi}\nabla\psi + \psi\nabla\bar{\psi})\times\mathbf{A}_t, \operatorname{curl}\mathbf{A}_{tt}\rangle - \langle(\bar{\psi}\psi_t + \psi\bar{\psi}_t)\operatorname{curl}\tilde{\mathbf{A}}, \operatorname{curl}\mathbf{A}_{tt}\rangle$$
$$- \langle(\bar{\psi}\nabla\psi_t + \psi\nabla\bar{\psi}_t + \psi_t\nabla\bar{\psi} + \bar{\psi}_t\nabla\psi)\times\tilde{\mathbf{A}}, \operatorname{curl}\mathbf{A}_{tt}\rangle - \frac{\sigma}{4\varepsilon}\langle|\psi|^2\mathbf{A}_t, \operatorname{curl}^2\mathbf{A}_t\rangle$$
$$- \frac{\sigma}{4\varepsilon}\langle(\bar{\psi}\psi_t + \psi\bar{\psi}_t)\tilde{\mathbf{A}}, \operatorname{curl}^2\mathbf{A}_t\rangle - i\langle\nabla\bar{\psi}_t\times\nabla\psi - \nabla\psi_t\times\nabla\bar{\psi}, \operatorname{curl}\mathbf{A}_{tt}\rangle$$
$$- \frac{i\sigma}{8\varepsilon}\langle\bar{\psi}_t\nabla\psi - \psi_t\nabla\bar{\psi} + \bar{\psi}\nabla\psi_t - \psi\nabla\bar{\psi}_t, \operatorname{curl}^2\mathbf{A}_t\rangle.$$

By standard inequalities, the right-hand side is controlled by

$$c(1+\|\nabla\psi_t\|^2)+c\|\Delta\psi_t\|^2+\frac{\sigma}{2}\|\mathrm{curl}\,\mathbf{A}_{tt}\|^2+\frac{\sigma}{8\varepsilon}\|\mathrm{curl}^2\,\mathbf{A}_t\|^2.$$

Finally, the product of (2.6) by $\Delta^2\Phi$ in $L^2(\Omega)$ gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\varepsilon\|\nabla\Delta\Phi\|^2\big)+\sigma\|\nabla\Delta\Phi\|^2\\&=\frac{i}{2}\langle\bar\psi\nabla\psi_t-\psi\nabla\bar\psi_t+\psi_t\nabla\bar\psi-\bar\psi_t\nabla\psi,\nabla\Delta\Phi\rangle+\langle|\psi|^2\nabla\Phi,\nabla\Delta\Phi\rangle\\&\quad+\langle(\psi\nabla\bar\psi+\bar\psi\nabla\psi)\Phi,\nabla\Delta\Phi\rangle.\end{aligned}$$

The Young inequality yields

$$\frac{1}{2}\frac{d}{dt}\big(\varepsilon\|\nabla\Delta\Phi\|^2\big)+\sigma\|\nabla\Delta\Phi\|^2\le c(1+\|\nabla\psi_t\|^2)+\frac{\sigma}{2}\|\nabla\Delta\Phi\|^2. \tag{5.5}$$

Let us introduce the functional

$$\begin{aligned}E_1(t)=\|\nabla\psi_t\|^2&+\eta\|\mathrm{curl}^2\,\mathbf{A}_t\|^2+\eta\varepsilon\|\mathrm{curl}\,\mathbf{A}_{tt}\|^2+\varepsilon\|\nabla\Delta\Phi\|^2+\frac{\eta\sigma^2}{4\varepsilon}\|\mathrm{curl}\,\mathbf{A}_t\|^2\\&+\frac{\eta\sigma}{2}\langle\mathrm{curl}\,\mathbf{A}_{tt},\mathrm{curl}\,\mathbf{A}_t\rangle,\end{aligned}$$

where $\eta\in(0,1)$ will be suitably chosen later. E_1 is readily seen to satisfy

$$\frac{1}{2}\big(\|\nabla\psi_t\|^2+\eta\|\mathrm{curl}^2\,\mathbf{A}_t\|^2+\eta\varepsilon\|\mathrm{curl}\,\mathbf{A}_{tt}\|^2+\varepsilon\|\nabla\Delta\Phi\|^2\big)\le E_1, \tag{5.6}$$

$$E_1\le\frac{3}{2}\big(\|\nabla\psi_t\|^2+\eta\|\mathrm{curl}^2\,\mathbf{A}_t\|^2+\eta\varepsilon\|\mathrm{curl}\,\mathbf{A}_{tt}\|^2+\varepsilon\|\nabla\Delta\Phi\|^2\big)+c. \tag{5.7}$$

Besides, adding together (5.4) multiplied by η to (5.3) and (5.5), on account of the previous estimates, it is apparent that

$$\begin{aligned}&\frac{1}{2}\frac{dE_1}{dt}+\|\nabla\psi_t\|^2+\frac{\eta\sigma}{8\varepsilon}\|\mathrm{curl}^2\,\mathbf{A}_t\|^2+\frac{\eta\sigma}{4}\|\mathrm{curl}\,\mathbf{A}_{tt}\|^2+\frac{\sigma}{2}\|\nabla\Delta\Phi\|^2+\frac{1}{2}\|\Delta\psi_t\|^2\\&\le\eta c\|\Delta\psi_t\|^2+c(1+\|\nabla\psi_t\|^2+\|\nabla\Phi_t\|^2).\end{aligned}$$

Hence, provided that η is small enough (notice that η depends on ρ_0), thanks to (5.7), we obtain

$$\frac{dE_1}{dt}+2\vartheta_1E_1\le c(1+\|\nabla\psi_t\|^2+\|\nabla\Phi_t\|^2),$$

for some positive ϑ_1, which, by the Gronwall lemma, yields

$$E_1(t)\le E_1(0)e^{-2\vartheta_1t}+c.$$

Finally, by (5.6)–(5.7), we accomplish our purpose. □

Remark 5.2. As a byproduct, this lemma gives the following integral estimate

$$\sup_{\|z\|_1\le R}\sup_{t\ge0}\int_t^{t+1}\big(\|\psi_t(y)\|^2_{H^2_{\mathbb{C}}}+\|\mathrm{curl}^2\,\mathbf{A}_t(y)\|^2+\|\mathrm{curl}\,\mathbf{A}_{tt}(y)\|^2\big)dy\le C(R).$$

We now come to the proof of Theorem 5.1, which leans on the next abstract result from [4, 6], adapted to the present case. We use the notation of Proposition 5.1.

Lemma 5.3. Let there exist $t^\star\ge t_1$ such that the following conditions hold.

(C1) The map

$$(t,z) \mapsto S(t)z : [t^\star, 2t^\star] \times \mathcal{B}_1 \to \mathcal{B}_1$$

is $\frac{1}{2}$-Hölder continuous in time and Lipschitz continuous in the initial data, when $\mathcal{B}_1$ is endowed with the $\mathbb{X}^0$-topology.

(C2) Setting $S = S(t^\star)$, there are $\lambda \in (0, \frac{1}{2})$ and $\Lambda \geq 0$ such that, for every $z_1, z_2 \in \mathcal{B}_1$,

$$Sz_1 - Sz_2 = D(z_1, z_2) + K(z_1, z_2),$$

where

$$\|D(z_1, z_2)\|_0 \leq \lambda \|z_1 - z_2\|_0$$

and

$$\|K(z_1, z_2)\|_1 \leq \Lambda \|z_1 - z_2\|_0.$$

Then there exists a set $\mathcal{E} \subset \mathcal{B}_1$, closed and of finite fractal dimension in $\mathbb{X}^0$, positively invariant for $S(t)$, such that

$$\mathrm{dist}_{\mathbb{X}^0}(S(t)\mathcal{B}_1, \mathcal{E}) \leq J_0 e^{-\omega_0 t},$$

for some $\omega_0 > 0$ and $J_0 \geq 0$.

Proof of Theorem 5.1. Our first task is to verify that the assumptions of Lemma 5.3 hold true. Condition (C1) is satisfied by virtue of Proposition 2.1 and of the integral estimate

$$\sup_{z \in \mathcal{B}_1} \int_0^t \|\partial_t S(y) z\|_0^2 dy \leq C(t),$$

which easily follows from Remarks 3.4 and 5.2. Here and throughout this proof, $C(t)$ stands for a positive constant depending increasingly on t. Indeed, for any $2t^\star \geq t \geq \tau \geq t^\star$, given $z_1, z_2 \in \mathcal{B}_1$, we have

$$\begin{aligned}
\|S(t)z_1 - S(\tau)z_2\|_0 &\leq \|S(t)z_1 - S(t)z_2\|_0 + \|S(t)z_2 - S(\tau)z_2\|_0 \\
&\leq K e^{2Kt^\star} \|z_1 - z_2\|_0 + \int_\tau^t \|\partial_t S(y) z_2\|_0 dy \\
&\leq C(t^\star)[\|z_1 - z_2\|_0 + \sqrt{t - \tau}\,].
\end{aligned}$$

Concerning (C2), fix a pair of initial data $z_1, z_2 \in \mathcal{B}_1$ and set $z = z_1 - z_2 = (\psi_0, \mathbf{A}_0, \dot{\mathbf{A}}_0, \Phi_0)$. The difference of the solutions $S(t)z_i = (\psi_i(t), \mathbf{A}_i(t), \partial_t \mathbf{A}_i(t), \Phi_i(t))$ can be decomposed as

$$S(t)z_1 - S(t)z_2 = \tilde{z}_d(t) + \tilde{z}_c(t),$$

where $\tilde{z}_d(t) = (\psi^d(t), \mathbf{A}^d(t), \mathbf{A}^d_t(t), \Phi^d(t))$ solves

$$\begin{cases}
\psi^d_t - \Delta \psi^d + \psi^d = 0, \\
\varepsilon(\mathbf{A}^d_t - \nabla \Phi^d)_t + \sigma(\mathbf{A}^d_t - \nabla \Phi^d) + \mathrm{curl}^2 \mathbf{A}^d = 0, \\
-\varepsilon \Delta \Phi^d_t - \sigma \Delta \Phi^d = 0, \\
\mathrm{div} \mathbf{A}^d = 0, \quad \int_\Omega \Phi^d = 0, \\
\partial_{\boldsymbol{n}} \psi^d|_{\partial\Omega} = 0, \quad \mathbf{A}^d \cdot \boldsymbol{n}|_{\partial\Omega} = 0, \quad \mathrm{curl} \mathbf{A}^d \times \boldsymbol{n}|_{\partial\Omega} = 0, \quad \partial_{\boldsymbol{n}} \Phi^d|_{\partial\Omega} = 0, \\
\psi^d(0) = \psi_0, \quad \mathbf{A}^d(0) = \mathbf{A}_0, \quad \mathbf{A}^d_t(0) = \dot{\mathbf{A}}_0, \quad \Phi^d(0) = \Phi_0,
\end{cases} \tag{5.8}$$

while $\tilde{z}_c(t) = (\psi^c(t), \mathbf{A}^c(t), \mathbf{A}^c_t(t), \Phi^c(t))$ is the solution to

$$\begin{cases} \psi^c_t - \Delta\psi^c + \psi^c = F(\psi_1, \tilde{\mathbf{A}}_1, \Phi_1) - F(\psi_2, \tilde{\mathbf{A}}_2, \Phi_2), \\ \varepsilon(\mathbf{A}^c_t - \nabla\Phi^c)_t + \sigma(\mathbf{A}^c_t - \nabla\Phi^c) + \mathrm{curl}^2\mathbf{A}^c = \mathbf{G}(\psi_1, \tilde{\mathbf{A}}_1) - \mathbf{G}(\psi_2, \tilde{\mathbf{A}}_2), \\ -\varepsilon\Delta\Phi^c_t - \sigma\Delta\Phi^c = H(\psi_1, \Phi_1) - H(\psi_2, \Phi_2), \\ \mathrm{div}\mathbf{A}^c = 0, \quad \int_\Omega \Phi^c = 0, \\ \partial_{\boldsymbol{n}}\psi^c|_{\partial\Omega} = 0, \quad \mathbf{A}^c\cdot\boldsymbol{n}|_{\partial\Omega} = 0, \quad \mathrm{curl}\mathbf{A}^c\times\boldsymbol{n}|_{\partial\Omega} = 0, \quad \partial_{\boldsymbol{n}}\Phi^c|_{\partial\Omega} = 0, \\ \psi^c(0) = 0, \quad \mathbf{A}^c(0) = 0, \quad \mathbf{A}^c_t(0) = 0, \quad \Phi^c(0) = 0. \end{cases} \tag{5.9}$$

Here $F, \mathbf{G}, H$ are as in Lemma 4.3. Since the exponential decay

$$\|\tilde{z}_d(t)\|_0^2 \le \gamma^2 e^{-2\nu t}\|z_1 - z_2\|_0^2 \tag{5.10}$$

follows easily from Lemma 4.2, we focus on system (5.9). First we observe that Proposition 2.1 and (5.10) ensure

$$\sup_{z\in\mathcal{B}_1} \|\tilde{z}_c(t)\|_0^2 \le C(t)\|z_1 - z_2\|_0^2.$$

In order to complete the estimate of the $\mathbb{X}^1$ norm, we consider the formal time derivative of the first two equations of (5.9), that is,

$$\psi^c_{tt} - \Delta\psi^c_t + \psi^c_t = \partial_t F(\psi_1, \tilde{\mathbf{A}}_1, \Phi_1) - \partial_t F(\psi_2, \tilde{\mathbf{A}}_2, \Phi_2), \tag{5.11}$$

$$\varepsilon(\mathbf{A}^c_{tt} - \nabla\Phi^c_t)_t + \sigma(\mathbf{A}^c_{tt} - \nabla\Phi^c_t) + \mathrm{curl}^2\mathbf{A}^c_t = \partial_t\mathbf{G}(\psi_1, \tilde{\mathbf{A}}_1) - \partial_t\mathbf{G}(\psi_2, \tilde{\mathbf{A}}_2). \tag{5.12}$$

Arguing as in Lemma 4.3, we find

$$\frac{1}{2}\frac{d}{dt}\|\nabla\psi^c_t\|^2 + \|\nabla\psi^c_t\|^2 \le \frac{1}{2}\|\partial_t F(\psi_1, \tilde{\mathbf{A}}_1, \Phi_1) - \partial_t F(\psi_2, \tilde{\mathbf{A}}_2, \Phi_2)\|^2.$$

Some long but easy computations lead us to

$$\begin{aligned} &\|\partial_t F(\psi_1(t), \tilde{\mathbf{A}}_1(t), \Phi_1(t)) - \partial_t F(\psi_2(t), \tilde{\mathbf{A}}_2(t), \Phi_2(t))\|^2 \\ &\le c\big(\|S(t)z_1 - S(t)z_2\|_0^2 + \|\nabla\partial_t\psi_1(t) - \nabla\partial_t\psi_2(t)\|^2 + \|\nabla\partial_t\Phi_1(t) - \nabla\partial_t\Phi_2(t)\|^2\big). \end{aligned}$$

Hence, integrating the above differential inequality, by virtue of Proposition 2.1, we end up with

$$\|\nabla\psi^c_t(t)\|^2 \le C(t)\|z_1 - z_2\|_0^2.$$

Next we consider (5.12), which we multiply in $\mathbf{L}^2(\Omega)$ by $\mathrm{curl}^2\,\mathbf{A}_{tt}$, getting

$$\frac{1}{2}\frac{d}{dt}(\|\mathrm{curl}^2\,\mathbf{A}^c_t\|^2 + \varepsilon\|\mathrm{curl}\,\mathbf{A}^c_{tt}\|^2) + \sigma\|\mathrm{curl}\,\mathbf{A}^c_{tt}\|^2 = \langle\partial_t\mathbf{G}(\psi_1, \tilde{\mathbf{A}}_1) - \partial_t\mathbf{G}(\psi_2, \tilde{\mathbf{A}}_2), \mathrm{curl}^2\,\mathbf{A}^c_{tt}\rangle.$$

Since

$$\begin{aligned} &\partial_t\mathbf{G}(\psi_1, \tilde{\mathbf{A}}_1) - \partial_t\mathbf{G}(\psi_2, \tilde{\mathbf{A}}_2) \\ &= \frac{i}{2}[\psi_t\nabla\bar{\psi}_1 + \partial_t\psi_2\nabla\bar{\psi} - \bar{\psi}_t\nabla\psi_1 - \partial_t\bar{\psi}_2\nabla\psi + \psi\nabla\partial_t\bar{\psi}_1 + \psi_2\nabla\bar{\psi}_t - \bar{\psi}\nabla\partial_t\psi_1 - \bar{\psi}_2\nabla\psi_t] \\ &\quad - \mathbf{A}_t|\psi_1|^2 - \partial_t\mathbf{A}_2(\psi\bar{\psi}_1 + \psi_2\bar{\psi}) - \tilde{\mathbf{A}}(\psi_1\partial_t\bar{\psi}_1 + \bar{\psi}_1\partial_t\psi_1) \\ &\quad - \tilde{\mathbf{A}}_2[\psi\partial_t\bar{\psi}_1 + \psi_2\bar{\psi}_t + \bar{\psi}\partial_t\psi_1 + \bar{\psi}_2\psi_t], \end{aligned}$$

it is straightforward to obtain

$$\|\partial_t \mathbf{G}(\psi_1(t), \tilde{\mathbf{A}}_1(t)) - \partial_t \mathbf{G}(\psi_2(t), \tilde{\mathbf{A}}_2(t))\|^2 \le c\big(\|S(t)z_1 - S(t)z_2\|_0^2 + \|\partial_t\psi_1(t) - \partial_t\psi_2(t)\|^2_{H^1_{\mathbb{C}}}\big).$$

Thus, by the Young inequality,

$$\|\mathrm{curl}^2\, \mathbf{A}_t^c(t)\|^2 + \varepsilon \|\mathrm{curl}\, \mathbf{A}_{tt}^c(t)\|^2 \le C(t)\|z_1 - z_2\|_0^2.$$

Finally, arguing as in Lemma 4.3, the product in $L^2(\Omega)$ of the third equation of (5.9) by $\Delta^2\Phi^c$ gives

$$\frac{\varepsilon}{2}\frac{d}{dt}\|\nabla\Delta\Phi^c\|^2 \le \frac{1}{4\sigma}\|\nabla H(\psi_1, \Phi_1) - \nabla H(\psi_2, \Phi_2)\|^2.$$

Again

$$\|\nabla H(\psi_1(t), \Phi_1(t)) - \nabla H(\psi_2(t), \Phi_2(t))\|^2 \le c\big(\|S(t)z_1 - S(t)z_2\|_0^2 + \|\partial_t\psi_1(t) - \partial_t\psi_2(t)\|^2_{H^1_{\mathbb{C}}}\big);$$

thus, integrating the above differential inequality we have

$$\|\nabla\Delta\Phi^c(t)\|^2 \le C(t)\|z_1 - z_2\|_0^2.$$

By (5.10), we choose $t^\star \ge t_1$ large enough such that

$$\|(\psi^d(t^\star), \mathbf{A}^d(t^\star), \mathbf{A}_t^d(t^\star), \Phi^d(t^\star))\|_0 \le \frac{\lambda^2}{2}\|z_1 - z_2\|_0,$$

for a fixed $\lambda < \frac{1}{2}$. Then, taking

$$D(z_1, z_2) = (\psi^d(t^\star), \mathbf{A}^d(t^\star), \mathbf{A}_t^d(t^\star), \Phi^d(t^\star))$$

and

$$K(z_1, z_2) = (\psi^c(t^\star), \mathbf{A}^c(t^\star), \mathbf{A}_t^c(t^\star), \Phi^c(t^\star)),$$

condition (C2) is satisfied.

In conclusion, we have the claim of Theorem 5.1, except that the basin of exponential attraction is $\mathcal{B}_1$, and not the whole space $\mathbb{X}^0_\infty$, as required. To fill this gap, we shall exploit the transitivity of the exponential attraction, devised in [5, Theorem 5.1]. Namely, if Proposition 2.1 holds, and

$$\mathrm{dist}_{\mathbb{X}^0}(S(t)\mathcal{B}, \mathcal{B}_1) \le M(R)e^{-\nu t} \qquad \text{and} \qquad \mathrm{dist}_{\mathbb{X}^0}(S(t)\mathcal{B}_1, \mathcal{E}) \le J_0 e^{-\omega_0 t},$$

then the desired property (EA) follows. □

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