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DECAY OF THE ELECTROMAGNETIC FIELD IN A MAXWELL BLOCH SYSTEM

By

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Abstract. This paper is concerned with the initial-boundary value problem for the Maxwell-Bloch system which describes the propagation of electromagnetic waves in a polarized quantum-mechanical medium with two energy levels. The main goal is the investigation of the large-time asymptotic behavior of the solutions if there are no relaxation terms in the equations governing the polarization field and the density.

1. Introduction. This paper is concerned with the initial-boundary value problem for the Maxwell-Bloch system which describes the propagation of electromagnetic waves in a polarized quantum-mechanical medium with two energy levels. The system under consideration consists of a generally nonlinear second order system of differential equations for the dielectrical polarization and the density coupled with Maxwell's equations for the electromagnetic field:

$$\varepsilon\partial_t\mathbf{E} = \text{ curl }\mathbf{H} - \partial_t\tilde{\mathbf{P}} - \sigma\mathbf{E}, \quad \mu\partial_t\mathbf{H} = -\text{ curl }\mathbf{E}, \tag{1.1}$$

on $(0,\infty)\times\Omega$ coupled with the equations

$$\partial_t^2\mathbf{P} + \alpha\mathbf{P} = N\mathbf{E} \tag{1.2}$$

on $(0,\infty)\times G$ and

$$\partial_t N = -\gamma\mathbf{E}\partial_t\mathbf{P} \tag{1.3}$$

on $(0,\infty)\times G$. The initial-boundary conditions

$$\vec{n}\wedge\mathbf{E} = 0 \text{ on } (0,\infty)\times\Gamma_1, \quad \vec{n}\wedge\mathbf{H} = 0 \text{ on } (0,\infty)\times\Gamma_2, \tag{1.4}$$

$$\mathbf{E}(0,x) = \mathbf{E}_0(x), \mathbf{H}(0,x) = \mathbf{H}_0(x), \tag{1.5}$$

$$N(0,x) = N_0(x), \quad \mathbf{P}(0,x) = \mathbf{P}_0(x), \text{ and } \partial_t\mathbf{P}(0,x) = \mathbf{P}_1(x) \text{ on } G \tag{1.6}$$

are imposed. The electromagnetic field is governed by the classical Maxwell equations, whereas the polarizable medium occupying the set G is modeled as a gas of quantum mechanical systems with two energy levels as described in [2], [3] and [11]. Here $\Omega\subset\mathbb{R}^3$

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is an arbitrary spatial domain, $\Gamma_1 \subset \partial\Omega$, $\Gamma_2 \overset{\text{def}}{=} \partial\Omega \setminus \Gamma_1$ and $G \subset \Omega$ is a nonempty open subset. The whole-space case $\Omega = \mathbb{R}^3$ without boundary condition (1.4) is also considered.

The unknown functions are the electric and magnetic fields $\mathbf{E}, \mathbf{H}$, which depend on the time $t \geq 0$ and the space-variable $x \in \Omega$, as well as the dielectric polarization $\mathbf{P}$ and N, the difference of the densities of the electrons in the excited and in the ground state, which are defined defined on $(0, \infty) \times G$. In (1.1) the function $\tilde{\mathbf{P}}$ is the extension of $\mathbf{P}$ on $(0, \infty) \times \Omega$ defined by zero on the set $(0, \infty) \times (\Omega \setminus G)$.

The coefficients α, γ take into account the possibly variable dipole moment and density of the medium. In this paper there are no damping terms in the equations (1.2) and (1.3) governing the polarization field and N. This model serves as an approximation of the physically reasonable case of large relaxation time for the polarization and N. The conductivity term $\sigma\mathbf{E}$ in (1.1) describes losses of the medium causing an attenuation of the electromagnetic wave as suggested in [11], ch.2.4.5. The system is closely related to the anharmonic oscillator model from nonlinear optics, [7], [10], in which a generally nonlinear restoring force for the dielectric polarization $\mathbf{P}$ occurs.

In [3] it is shown that (1.1)-(1.6) admit a unique strong solution if $G = \Omega = \mathbb{R}^3$ and the coefficients do not depend on the space variables. This result has been generalized in [4] to the case where the medium has a finite, but arbitrary number of energy levels. The main goal of [4] is to prove the existence of finite-energy solutions and to obtain the uniqueness of the solution under additional smoothness assumptions on the initial data. A theory for weak finite-energy solutions for arbitrary spatial domains with additional boundary conditions is developed in [8].

The main topic of this paper is the investigation of the large-time asymptotic behavior of the solutions to the system (1.1)-(1.6). To compare with earlier work, consider the equations

$$\partial_t^2\mathbf{P} + \beta\partial_t\mathbf{P} + \alpha\mathbf{P} = N\mathbf{E}, \quad \partial_t N = -\nu(N - N_e) - \gamma\mathbf{E}\partial_t\mathbf{P} \tag{1.7}$$

instead of (1.2) and (1.3). Here N_e and β are some positive functions on G. In [8] the decay of $\mathbf{E}, \mathbf{H}$ and $\mathbf{P}$ has been shown if there is no conductivity, but $\beta > 0$ and $\nu > 0$ in (1.7). The presence of these damping terms is vital for the proof of that decay in [8]. In [9], only the decay of polarization field $\mathbf{P}$ is shown if there is no conductivity, $\beta > 0$ and $\nu = 0$. There is no information about the asymptotic behavior of $\mathbf{E}$ and $\mathbf{H}$ in [9].

The main goal of this paper is to show that the conductivity alone causes enough dissipation to identify the asymptotic behavior of $\mathbf{E}$ and $\mathbf{H}$ as well as that of $\mathbf{P}$, although there is no direct damping for $\mathbf{P}$. In particular, it is shown that, if $\beta = \nu = 0$, the electric and magnetic fields $\mathbf{E}, \mathbf{H}$ as well as the polarization field $\mathbf{P}$ decay with respect to the weak topology in the case of vanishing space charges in the nonconducting region where $\sigma = 0$. However, there is no information about the asymptotic behavior of N.

For the system (1.1)-(1.6) the energy dissipation law

$$\frac{1}{2}\frac{d}{dt}\left[\int_\Omega (|\varepsilon^{1/2}\mathbf{E}|^2 + |\mu^{1/2}\mathbf{H}|^2)dx + \int_G \gamma^{-1}N dx\right] = -\int_\Omega \sigma|\mathbf{E}|^2 dx \leq 0 \tag{1.8}$$

holds which includes a dissipative term arising from the electrical conductivity σ. This dissipation law implies $\sigma^{1/2}\mathbf{E} \in L^2((0,\infty), L^2(\Omega))$ and shows that $\|(\mathbf{E}(t), \mathbf{H}(t))\|_{L^2(\Omega)}$

and $\|\partial_t \mathbf{P}(t)\|_{L^2(G)}$ remain bounded as $t \to \infty$. However, this alone does not provide any information about the asymptotic behavior of $\mathbf{P}$, the magnetic field $\mathbf{H}$, and the electric field $\mathbf{E}$ on the set of vanishing electrical conductivity, since there is no direct damping for these quantities.

This paper is organized as follows. In section 3 it is shown that the electromagnetic field converges weakly as $t \to \infty$ to an asymptotic state that is determined by the prescribed initial data $\mathbf{E}_0, \mathbf{H}_0, \mathbf{P}_0$ (Theorem 3.1). Furthermore, it is shown that the polarization field $\mathbf{P}$ decays weakly to zero provided that all of the set G, in which the polarization is located, is conducting.

The strong L^q_{loc}- convergence of the electromagnetic field as $t \to \infty$ is shown in section 4 for all $q \in [1,2)$ (Theorem 4.1). For this purpose it is shown that $\|\text{ curl } \mathbf{E}(t)\|_{L^2(\mathbb{R}^3)}$ and $\|\text{ curl } \mathbf{H}(t)\|_{L^2(\mathbb{R}^3)}$ remain bounded for $t \to \infty$ if the initial data $\mathbf{E}_0, \mathbf{H}_0$ are sufficiently regular (Lemma 2.3). Furthermore, the local strong convergence of the electromagnetic field with respect to the energy norm as well as the strong convergence of the curl-free part of the electric field in $L^2(\Omega)$ in the exterior-domain case is proved (Theorem 4.2).

2. Definitions, notation and auxiliary results. All assumptions stated in this section hold throughout this paper.

The dielectric and magnetic susceptibilities $\varepsilon, \mu \in L^\infty(\mathbb{R}^3, \mathbb{R}^{3\times 3})$ are assumed to be uniformly positive matrix-valued functions, which means that $\varepsilon(x), \mu(x) \in \mathbb{R}^{3\times 3}$ are symmetric and

$$y \cdot \varepsilon(x) y, y \cdot \mu(x) y \geq c_1 |y|^2 \text{ for all } x \in \mathbb{R}^3, y \in \mathbb{R}^3 \text{ with some } c_1 > 0. \tag{2.1}$$

Next, let $\alpha \in L^\infty(G)$ and $\gamma \in L^\infty(G)$ be functions on the open subset G which have strictly positive lower bounds. Finally, $\sigma \in L^\infty(\Omega)$ is assumed to be nonnegative.

It is assumed that the conducting region

$$G_\sigma \stackrel{\text{def}}{=} \{x \in \Omega : \sigma(x) > 0\} \tag{2.2}$$

satisfies

$$G \subset G_\sigma \tag{2.3}$$

which means that at least the set G, on which the polarization is located, is conducting. Furthermore,

$$\varepsilon(x) = \mu(x) = 1 \text{ on } \Omega \setminus G. \tag{2.4}$$

The physical meaning of this condition is that the set $\Omega \setminus G$ represents a vacuum region.

For an arbitrary open set $K \subset \mathbb{R}^3$ the space of all infinitely differentiable functions with compact support contained in K is denoted by $C_0^\infty(K)$. Let $H_{curl}(K)$ be defined as the space of all $\mathbf{E} \in L^2(K, \mathbb{R}^3)$ with curl $\mathbf{E} \in L^2(K)$ in the sense of distributions.

As in [5] and [7], let W_E and W_H be the spaces of all vector fields in $H_{curl}(K)$ that satisfy the boundary conditions (1.4).

In what follows the first three and the last three components of a vector $\mathbf{u} \in \mathbb{C}^6$ are denoted by $\underline{\mathbf{u}}_1 \in \mathbb{C}^3$ and $\underline{\mathbf{u}}_2 \in \mathbb{C}^3$ respectively. Now, the following operators are defined. Let $D(B) \stackrel{\text{def}}{=} W_E \times W_H$ and

$$B(\mathbf{E}, \mathbf{H}) \stackrel{\text{def}}{=} (\varepsilon^{-1} \text{ curl } \mathbf{H}, -\mu^{-1} \text{ curl } \mathbf{E}) \text{ for } (\mathbf{E}, \mathbf{H}) \in D(B).$$

Then B is a densely defined skew-adjoint operator in the Hilbert-space $X \stackrel{\text{def}}{=} L^2(\Omega, \mathbb{R}^6)$ endowed with the weighted scalar-product

$$\langle (\mathbf{E},\mathbf{H}),(\mathbf{F},\mathbf{G})\rangle_X \stackrel{\text{def}}{=} \int_\Omega \left(\varepsilon \mathbf{E}\cdot\overline{\mathbf{F}} + \mu \mathbf{H}\cdot\overline{\mathbf{G}}\right) dx.$$

In what follows $W_{E,0}$ and $W_{H,0}$ denote the spaces of all $\mathbf{f} \in W_E$ and $\mathbf{g} \in W_H$ with $\operatorname{curl}\mathbf{f} = \operatorname{curl}\mathbf{g} = 0$ respectively. Let $W_{E,0}^\perp$ and $W_{H,0}^\perp$, denote the orthogonal complements of $W_{E,0}$ and $W_{H,0}$, respectively, with respect to the weighted scalar products

$$\langle \mathbf{E},\mathbf{F}\rangle_\varepsilon \stackrel{\text{def}}{=} \int_\Omega \varepsilon \mathbf{E}\cdot\overline{\mathbf{F}}dx \text{ and } \langle \mathbf{G},\mathbf{H}\rangle_\mu \stackrel{\text{def}}{=} \int_\Omega \mu \mathbf{G}\cdot\overline{\mathbf{H}}dx.$$

Let Q denote the orthogonal projector on $(\ker B)^\perp = \overline{\text{ ran } B}$. In particular

$$Q(\mathbf{f},\mathbf{g}) = (Q_E\mathbf{f}, Q_H\mathbf{g}) \text{ for all } (\mathbf{f},\mathbf{g}) \in X, \tag{2.5}$$

where Q_E and Q_E are the orthogonal projectors on $W_{E,0}^\perp$ and $W_{H,0}^\perp$ with respect to $\langle\cdot,\cdot\rangle_\varepsilon$ and $\langle\cdot,\cdot\rangle_\mu$ respectively.

Furthermore, let $\mathcal{N}_0$ be the set of all $\mathbf{a} \in \ \ker B = W_{E,0}\times W_{H,0}$ with $\underline{\mathbf{a}}_1(x) = 0$ for all $x \in G_\sigma$, where G_σ is given in (2.2).

Let $\mathcal{P}$ be the orthogonal-projector on $\mathcal{N}_0$ in X. Note that $\mathcal{P}$ is of the form

$$\mathcal{P}(\mathbf{f},\mathbf{g}) = (\mathcal{P}_1\mathbf{f}, (1-Q_H)\mathbf{g}) \text{ for } (\mathbf{f},\mathbf{g}) \in X, \tag{2.6}$$

where $\mathcal{P}_1$ is the orthogonal projector on the space

$$\mathcal{Z}_1 \stackrel{\text{def}}{=} \{\mathbf{e} \in W_{E,0} : \mathbf{e} = 0 \text{ almost everywhere on } G_\sigma\}$$

with respect to the scalar-product $\langle\cdot,\cdot\rangle_\varepsilon$.

The assumptions on the initial data are

$$(\mathbf{E}_0,\mathbf{H}_0) \in X \stackrel{\text{def}}{=} L^2(\Omega,\mathbb{R}^6), \quad \mathbf{P}_0,\mathbf{P}_1 \in L^\infty(G)\cap L^1(G) \tag{2.7}$$
$$\text{and } N_0 \in L^\infty(G)\cap L^1(G).$$

Let $W^{k,p}_{loc}([0,\infty),Y)$ denote for a Banach-space Y the space of all functions defined on $[0,\infty)$ with values in Y, whose derivatives up to order k belong to $L^p((0,T),Y)$ for all $T>0$.

First it follows from the contraction mapping principle as in [8] that problem (1.1)–(1.6) has a unique global weak solution on $(0,\infty)$.

Proposition 2.1. Problem (1.1)-(1.6) has a unique weak solution $(\mathbf{E},\mathbf{H},\mathbf{P},N)$ with the properties $(\mathbf{E},\mathbf{H}) \in C([0,\infty),X)$, $\mathbf{P} \in W^{2,2}_{loc}([0,\infty),L^2(G,\mathbb{R}^3)) \cap W^{1,2}_{loc}([0,\infty),L^\infty(G))$ and $N \in W^{1,2}_{loc}([0,\infty),L^2(G)) \cap L^\infty_{loc}([0,\infty),L^\infty(G))$.

In particular (1.1) is satisfied in the sense that

$$\begin{aligned}&(\mathbf{E}(t),\mathbf{H}(t)) \qquad (2.8)\\ &= \exp(tB)(\mathbf{E}_0,\mathbf{H}_0) - \int_0^t \exp((t-s)B)[\mathcal{R}\partial_t\mathbf{P}(s) + (\varepsilon^{-1}\sigma\mathbf{E}(s),0)]ds.\end{aligned}$$

Here $(\exp(tB))_{t\in\mathbb{R}}$ is the unitary group generated by B, and $\mathcal{R} : L^2(G) \to X$ is defined by

$$(\mathcal{R}\mathbf{p})(x) \stackrel{\text{def}}{=} (\varepsilon(x)^{-1}\mathbf{p}(x),0) \text{ if } x \in G \text{ and } (\mathcal{R}\mathbf{p})(x) \stackrel{\text{def}}{=} 0 \text{ if } x \in \Omega\setminus G.$$

First some a-priori bounds on the solution of the system (1.1) - (1.6) are given.

LEMMA 2.2. The solution has the properties

$$\gamma(x)|\partial_t \mathbf{P}(t,x)|^2 + \gamma(x)\alpha(x)|\mathbf{P}(t,x)|^2 + N(t,x)^2 \tag{2.9}$$
$$= \gamma(x)|\mathbf{P}_1(x)|^2 + \gamma(x)\alpha(x)|\mathbf{P}_0(x)|^2 + N_0(x)^2$$

for all $x \in G$ and $t > 0$,

$$\mathbf{P} \in L^\infty([0,\infty), L^1(G)) \cap L^\infty([0,\infty), L^\infty(G)) \subset L^\infty([0,\infty), L^2(G)), \tag{2.10}$$

$$\partial_t \mathbf{P} \in L^\infty([0,\infty), L^1(G)) \cap L^\infty([0,\infty), L^\infty(G)) \subset L^\infty([0,\infty), L^2(G)), \tag{2.11}$$

$$\partial_t^2 \mathbf{P} \in L^\infty([0,\infty), L^2(G)), \tag{2.12}$$

and

$$N \in L^\infty([0,\infty), L^1(G)) \cap L^\infty([0,\infty), L^\infty(G)). \tag{2.13}$$

Furthermore :

$$\sigma^{1/2}\mathbf{E} \in L^2([0,\infty), L^2(G)) \tag{2.14}$$

and

$$(\mathbf{E}, \mathbf{H}) \in L^\infty((0,\infty), X). \tag{2.15}$$

Proof. First

$$\frac{1}{2}\partial_t \left[\gamma(x)|\partial_t \mathbf{P}(t,x)|^2 + \gamma(x)\alpha(x)|\mathbf{P}(t,x)|^2 + N(t,x)^2\right]$$
$$= \gamma(x)\partial_t \mathbf{P}(t,x)\left[\partial_t^2 \mathbf{P}(t,x) + \alpha(x)\mathbf{P}(t,x)\right] - N(t,x)\gamma(x)\partial_t \mathbf{P}(t,x)\mathbf{E}(t,x) = 0,$$

whence (2.9). From (2.7) and (2.9) one obtains immediately (2.10) - (2.13) by integration of both sides of (2.9) over G. In view of (2.11) and (2.13) equation (1.2) gives

$$\frac{d}{dt}\int_G \gamma^{-1} N(t)dx = -\int_G \mathbf{E}\partial_t \mathbf{P} dx, \tag{2.16}$$

whereas the standard energy estimate for (2.8) yields

$$\frac{1}{2}\frac{d}{dt}\|(\mathbf{E}(t), \mathbf{H}(t))\|_X^2 \tag{2.17}$$
$$= -\langle \mathcal{R}\partial_t \mathbf{P}(t), (\mathbf{E}(t), \mathbf{H}(t))\rangle_X - \langle (\varepsilon^{-1}\sigma \mathbf{E}(t), 0), (\mathbf{E}(t), \mathbf{H}(t))\rangle_X$$
$$= -\int_G \mathbf{E}\partial_t \mathbf{P} dx - \int_G \sigma|\mathbf{E}|^2 dx.$$

From (2.16) and (2.17) one obtains

$$\frac{d}{dt}\left(\frac{1}{2}\|(\mathbf{E}(t), \mathbf{H}(t))\|_X^2 - \int_G \gamma^{-1}N(t)dx\right) = -\int_G \sigma|\mathbf{E}|^2 dx \leq 0. \tag{2.18}$$

(Here and in what follows the space variable x is often omitted in the notation for brevity's sake.) Finally, assertions (2.14) and (2.15) follow from (2.13) and (2.18). $\square$

By a classical result from semigroup theory, [12], it follows from (2.12) that $(\mathbf{E}, \mathbf{H}) \in C^1([0,\infty), X)$ and

$$(\partial_t \mathbf{E}(t), \partial_t \mathbf{H}(t)) = \exp(tB)[B(\mathbf{E}_0, \mathbf{H}_0) - \mathcal{R}\mathbf{P}_1 - (\varepsilon^{-1}\sigma \mathbf{E}_0, 0)] \tag{2.19}$$
$$-\int_0^t \exp((t-s)B)[\mathcal{R}\partial_t^2 \mathbf{P}(s) + (\varepsilon^{-1}\sigma\partial_t \mathbf{E}(s), 0)]ds,$$

provided that $(\mathbf{E}_0, \mathbf{H}_0) \in D(B)$. Next, a global bound on the time derivatives of $(\mathbf{E}, \mathbf{H})$ will be given.

LEMMA 2.3. Suppose that

$$\gamma(x)|\mathbf{P}_1(x)|^2 + \gamma(x)\alpha(x)|\mathbf{P}_0(x)|^2 + N_0(x)^2 \leq C^2\sigma(x)^2 \tag{2.20}$$

with some constant C independent of $x \in G$, and

$$(\mathbf{E}_0, \mathbf{H}_0) \in D(B). \tag{2.21}$$

Then

$$(\partial_t\mathbf{E}, \partial_t\mathbf{H}) \in L^\infty((0,\infty), X), \tag{2.22}$$

$$(\mathbf{E}, \mathbf{H}) \in L^\infty((0,\infty), D(B)) \tag{2.23}$$

and

$$\sigma^{1/2}\partial_t\mathbf{E} \in L^2([0,\infty), L^2(\Omega)). \tag{2.24}$$

REMARK 2.4. Assumption (2.20) is satisfied if one assumes that σ has a positive lower bound on G.

Proof. The standard energy estimate for (2.19) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(\partial_t\mathbf{E}(t), \partial_t\mathbf{H}(t))\|_X^2 &= -\langle\mathcal{R}\partial_t^2\mathbf{P}(t), (\partial_t\mathbf{E}(t), \partial_t\mathbf{H}(t))\rangle_X \\ &\quad -\langle(\varepsilon^{-1}\sigma\partial_t\mathbf{E}(t), 0), (\partial_t\mathbf{E}(t), \partial_t\mathbf{H}(t))\rangle_X \\ &= -\int_G \partial_t\mathbf{E}\cdot\partial_t^2\mathbf{P}dx - \int_\Omega \sigma|\partial_t\mathbf{E}|^2dx \\ &= -\int_G \partial_t\mathbf{E}\cdot[N\mathbf{E} - \alpha\mathbf{P}]dx - \int_\Omega \sigma|\partial_t\mathbf{E}|^2dx \\ &= -\int_G N\partial_t\mathbf{E}\cdot\mathbf{E}dx + \frac{d}{dt}\int_G \alpha\mathbf{E}\mathbf{P}dx - \int_G \alpha\mathbf{E}\partial_t\mathbf{P}dx - \int_\Omega \sigma|\partial_t\mathbf{E}|^2dx \\ &= \frac{d}{dt}\int_G \alpha[\mathbf{E}\mathbf{P} + \gamma^{-1}N]dx - \int_G N\partial_t\mathbf{E}\cdot\mathbf{E}dx - \int_\Omega \sigma|\partial_t\mathbf{E}|^2dx.\end{aligned} \tag{2.25}$$

By (2.9) and (2.20) one has

$$|N(t,x)| \leq C\sigma(x) \text{ for all } t \geq 0, x \in \Omega. \tag{2.26}$$

Now, one obtains from (2.25) and (2.26)

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\|(\partial_t\mathbf{E}(t), \partial_t\mathbf{H}(t))\|_X^2 - \int_G \alpha[\mathbf{E}\mathbf{P} + \gamma^{-1}N]dx\right) \\ \leq C^2\int_G \sigma|\mathbf{E}|^2dx - 1/2\int_G \sigma|\partial_t\mathbf{E}|^2dx.\end{aligned} \tag{2.27}$$

Lemma 2.2 yields

$$\sup_{t>0}\left|\int_\Omega \alpha[\mathbf{E}\mathbf{P} + \gamma^{-1}N]dx\right| < \infty. \tag{2.28}$$

Finally, the assertion follows from (2.14), (2.27) and (2.28). $\square$

LEMMA 2.5. There hold

$$(1-Q)\,(\mathbf{E}(t),\mathbf{H}(t)) = (1-Q)\left((\mathbf{E}_0,\mathbf{H}_0) + \mathcal{R}\mathbf{P}_0 - \mathcal{R}\mathbf{P}(t) - \int_0^t (\varepsilon^{-1}\sigma\mathbf{E}(s),0)ds\right)$$

and

$$\mathcal{P}(\mathbf{E}(t),\mathbf{H}(t)) = \mathcal{P}(\mathbf{E}_0,\mathbf{H}_0).$$

Proof. For all $\mathbf{a} \in \ker B$ one has by (2.8)

$$\begin{aligned} &\langle(\mathbf{E}(t),\mathbf{H}(t)),\mathbf{a}\rangle_X \\ &= \left\langle \exp(tB)(\mathbf{E}_0,\mathbf{H}_0) - \int_0^t \exp((t-s)B)[\mathcal{R}\partial_s\mathbf{P}(s) + (\varepsilon^{-1}\sigma\mathbf{E}(s),0)]ds, \mathbf{a}\right\rangle_X \\ &= \left\langle (\mathbf{E}_0,\mathbf{H}_0) + \mathcal{R}\mathbf{P}_0 - \mathcal{R}\mathbf{P}(t) - \int_0^\infty (\varepsilon^{-1}\sigma\mathbf{E}(s),0)ds), \mathbf{a}\right\rangle_X , \end{aligned}$$

which proves the first assertion.

Let $\mathbf{a} \in \mathcal{N}_0$, that means $\mathbf{a} \in \ker B$ and $\underline{\mathbf{a}}_1 = 0$ on G_σ.

Then one has, by (2.3), $\langle \mathcal{R}\mathbf{P}(t),\mathbf{a}\rangle_X = \langle(\varepsilon^{-1}\sigma\mathbf{E}(s),0),\mathbf{a}\rangle_X = 0$ and, hence,

$$\langle(\mathbf{E}(t),\mathbf{H}(t)),\mathbf{a}\rangle_X = \langle(\mathbf{E}_0,\mathbf{H}_0),\mathbf{a}\rangle_X ,$$

from which the second assertion follows. □

3. Weak convergence for $t \to \infty$. This section is concerned with the system (1.1)-(1.6) for arbitrary spatial domains Ω.

THEOREM 3.1. Suppose that (2.1)-(2.4) and (2.7) are fulfilled. Then

$$(\mathbf{E}(t),\mathbf{H}(t)) \overset{t\to\infty}{\longrightarrow} \mathcal{P}(\mathbf{E}_0,\mathbf{H}_0) \text{ in } X \text{ weakly},$$
$$\mathbf{P}(t) \overset{t\to\infty}{\longrightarrow} 0 \text{ and } \partial_t\mathbf{P}(t) \overset{t\to\infty}{\longrightarrow} 0 \text{ in } L^2(G) \text{ weakly}.$$

COROLLARY 3.2. Under the assumptions of Theorem 3.1 it follows that

$$Q(\mathbf{E}(t),\mathbf{H}(t)) \overset{t\to\infty}{\longrightarrow} 0 \text{ in } X \text{ weakly}.$$

Since ran $\mathcal{P} \subset \ker B$ one has $Q\mathcal{P} = 0$. Hence, Corollary 3.2 follows immediately from Theorem 3.1.

COROLLARY 3.3. Suppose that, in addition to the assumptions of Theorem 3.1,

$$\int_\Omega (\varepsilon\mathbf{E}_0\mathbf{f} + \mu\mathbf{H}_0\mathbf{g})dx = 0 \text{ for all } (\mathbf{f},\mathbf{g}) \in \mathcal{N}_0. \tag{3.1}$$

Then

$$(\mathbf{E}(t),\mathbf{H}(t)) \overset{t\to\infty}{\longrightarrow} 0 \text{ in } X \text{ weakly}.$$

This follows directly from Theorem 3.1, since condition (3.1) is equivalent to

$$\mathcal{P}(\mathbf{E}_0,\mathbf{H}_0) = 0.$$

Condition (3.1) includes

$$\text{div } (\varepsilon\mathbf{E}_0) = 0 \text{ on } \Omega\setminus\overline{G_\sigma}, \quad \text{div } (\mu\mathbf{H}_0) = 0 \text{ on } \Omega$$

for which the physical meaning is that the space charge $\rho \overset{\text{def}}{=}$ div $(\varepsilon\mathbf{E})$ vanishes on the nonconducting region $\Omega \setminus G_\sigma$. This property is invariant under the nonlinear flow, i.e. $\operatorname{div}(\varepsilon\mathbf{E}) = 0$ on $\Omega \setminus G_\sigma$ is fulfilled for all times as soon as it is satisfied at $t = 0$.

Remark 3.4. If the medium is not conducting the electromagnetic field $(\mathbf{E}, \mathbf{H})$ does not decay in general. Consider for example the case where $\sigma = 0$, $\mathbf{P}_0 = 0$, $\mathbf{P}_1 = 0$ and $N_0 = 0$. Then $(\mathbf{E}, \mathbf{H}, \mathbf{P}, N)$ solves (1.1)-(1.6) if and only if $\mathbf{P}(t) = 0$, $N(t) = 0$ and $(\mathbf{E}, \mathbf{H})$ solves the linear undamped Maxwell equations

$$\varepsilon\partial_t\mathbf{E} = \text{ curl } \mathbf{H}, \quad \mu\partial_t\mathbf{H} = -\text{ curl } \mathbf{E}.$$

In particular there is no decay of $(\mathbf{E}, \mathbf{H})$ in the bounded domain case.

As in [7] the weak ω-limit set of the solution of (1.1)–(1.6) is identified.

Proposition 3.5. Assume (2.1) and (2.4). Then every $\mathbf{g} \in X$ with $\underline{(\exp(tB)\mathbf{g})}_1 = 0$ on G_σ for all $t \in \mathbb{R}$ satisfies $\mathbf{g} \in \ker B$.

This result has been proved in [6] for arbitrary, not necessarily bounded, spatial domains. In [13] this has been proved for the scalar wave equation and Maxwell's equations with constant coefficients.

Remark 3.6. If the whole domain is conducting, i.e. $G_\sigma = \Omega$, Proposition 3.5 follows immediately. To see this, assume that $\mathbf{g} \in X$ with $\underline{\mathbf{f}}_1 = 0$ on $\mathbb{R} \times \Omega$, where $\mathbf{f}(t) \overset{\text{def}}{=} \exp(tB)\mathbf{g}$. Then

$$\partial_t(\mu\underline{\mathbf{f}}_2) = -\text{ curl } \underline{\mathbf{f}}_1 = 0 \text{ on } \mathbb{R} \times \Omega$$

in the sense of distributions, which implies that $\underline{\mathbf{f}}_2$ is independent of t. Hence, $\mathbf{f}(t) = \exp(tB)\mathbf{g}$ is constant with respect to t on $\mathbb{R} \times \Omega$, which is only possible if $\mathbf{g} \in \ker B$, since B is the generator of $\{\exp(tB)\}_{t\in\mathbb{R}}$.

Proof of Theorem 3.1. By Lemma 2.5 it suffices to show that

$$(1 - \mathcal{P})(\mathbf{E}(t), \mathbf{H}(t)) \overset{t\to\infty}{\longrightarrow} 0 \text{ in } X \text{ weakly}, \tag{3.2}$$
$$\mathbf{P}(t) \overset{t\to\infty}{\longrightarrow} 0 \text{ and } \partial_t\mathbf{P}(t) \overset{t\to\infty}{\longrightarrow} 0 \text{ in } L^2(G) \text{ weakly}.$$

Note that, by Lemma 2.2, $\|(\mathbf{E}(t), \mathbf{H}(t))\|_X$, $\|\mathbf{P}(t))\|_{L^2(G)}$ and $\|\partial_t\mathbf{P}(t))\|_{L^2(G)}$ remain bounded as $t \to \infty$. Suppose $\mathbf{g} \in X$, $\mathbf{Q}_1, \mathbf{Q}_2 \in L^2(G)$ and $t_n \overset{n\to\infty}{\longrightarrow} \infty$ with

$$(\mathbf{E}(t_n), \mathbf{H}(t_n)) \overset{n\to\infty}{\longrightarrow} \mathbf{g} \text{ in } X \text{ weakly}, \tag{3.3}$$

$$\mathbf{P}(t_n) \overset{n\to\infty}{\longrightarrow} \mathbf{Q}_1 \text{ and } \partial_t\mathbf{P}(t_n) \overset{n\to\infty}{\longrightarrow} \mathbf{Q}_2 \text{ in } L^2(G) \text{ weakly}. \tag{3.4}$$

Let $\mathbf{u}^{(n)}(t) \overset{\text{def}}{=} (\mathbf{E}(t_n+t), \mathbf{H}(t_n+t))$ for $t \in (-t_n, \infty)$ and $\mathbf{u}^{(n)}(t) \overset{\text{def}}{=} 0$ for $t \le -t_n$. Next, let $\mathbf{p}^{(n)}(t) \overset{\text{def}}{=} \mathbf{P}(t_n + t)$ for $t \in (-t_n, \infty)$ and $\mathbf{p}^{(n)}(t) \overset{\text{def}}{=} \mathbf{P}_0$ for $t \le -t_n$. After passing to a further subsequence one has by (2.10) - (2.12):

$$\mathbf{p}^{(n)} \overset{n\to\infty}{\longrightarrow} \mathbf{p}^{(\infty)} \text{ in } W^{1,\infty}(\mathbb{R}, L^2(G)) \text{ weak } -*, \tag{3.5}$$

i.e.

$$\mathbf{p}^{(n)} \overset{n\to\infty}{\longrightarrow} \mathbf{p}^{(\infty)}, \quad \partial_t\mathbf{p}^{(n)} \overset{n\to\infty}{\longrightarrow} \partial_t\mathbf{p}^{(\infty)} \text{ in } L^\infty(\mathbb{R}, L^2(G)) \text{ weak } -*.$$

Let $t \in \mathbb{R}$. By (2.8) one has

$$\mathbf{u}^{(n)}(t) = \exp(tB)(\mathbf{E}(t_n), \mathbf{H}(t_n)) - \int_0^t \exp((t-s)B) \Big[\mathcal{R}\partial_t \mathbf{p}^{(n)}(s) + (\varepsilon^{-1}\sigma \mathbf{E}(s+t_n), 0)\Big]\, ds$$

for all $n \in \mathbb{N}$ with $t_n + t \geq 0$. It follows from (2.14), (3.3) and (3.5) that

$$\mathbf{u}^{(n)}(t) \overset{\mathrm{n}\to\infty}{\longrightarrow} \mathbf{u}^{(\infty)}(t) \tag{3.6}$$

$$\overset{\mathrm{def}}{=} \exp(tB)\mathbf{g} - \int_0^t \exp((t-s)B)\mathcal{R}\partial_t \mathbf{p}^{(\infty)}(s) ds \text{ in } X \text{ weakly for all } t \in \mathbb{R}.$$

In particular, one has $\mathbf{u}^{(\infty)} \in C(\mathbb{R}, X)$ with $\mathbf{u}^{(\infty)}(0) = \mathbf{g}$ and

$$\partial_t(\varepsilon \underline{\mathbf{u}^{(\infty)}}_1) = \operatorname{curl} \underline{\mathbf{u}^{(\infty)}}_2 - \partial_t \widetilde{\mathbf{p}^{(\infty)}} \text{ and } \partial_t(\mu \underline{\mathbf{u}^{(\infty)}}_2) = - \operatorname{curl} \underline{\mathbf{u}^{(\infty)}}_1 \tag{3.7}$$

on $\mathbb{R} \times \Omega$ in the sense of distributions. For all $a, b \in \mathbb{R}$ with $a < b$ it follows from (2.14) that

$$\left\| \int_a^b \sigma^{1/2} \underline{\mathbf{u}^{(n)}}_1(t) dt \right\|_{L^2(\Omega)} \leq C_{K,1}(b-a)^{1/2} \left(\int_{a+t_n}^{b+t_n} \|\sigma^{1/2}\mathbf{E}(t)\|^2_{L^2(\Omega)} dt \right)^{1/2} \overset{\mathrm{n}\to\infty}{\longrightarrow} 0,$$

whence, by (3.6),

$$\underline{\mathbf{u}^{(\infty)}}_1(t) = 0 \text{ a.e. on } G_\sigma \text{ for all } t \in \mathbb{R}. \tag{3.8}$$

Furthermore, for all $a < b$, one obtains from (1.2), (2.13) and (2.14) that

$$\int_a^b \|\sigma^{1/2}[\partial_t^2 \mathbf{p}^{(n)}(t) + \alpha \mathbf{p}^{(n)}(t)]\|_{L^2(G)} dt$$
$$= \int_a^b \|\sigma^{1/2} N(t+t_n)\mathbf{E}(t+t_n)\|_{L^2(G)} dt \leq C \int_a^b \|\sigma^{1/2}\mathbf{E}(t+t_n)\|_{L^2(G)} dt \overset{\mathrm{n}\to\infty}{\longrightarrow} 0,$$

which implies by (2.2), (2.3) and (3.5) that

$$\partial_t^2 \mathbf{p}^{(\infty)} + \alpha \mathbf{p}^{(\infty)} = 0 \text{ on } \mathbb{R} \times G. \tag{3.9}$$

Next, it follows from (2.3), (3.7) and (3.8) that $\partial_t(\mu \underline{\mathbf{u}^{(\infty)}}_2) = 0$ on $\mathbb{R} \times G$, which means that $\underline{\mathbf{u}^{(\infty)}}_2$ is independent of t. Since $\mathbf{u}^{(\infty)}(0) = \mathbf{g}$, it follows that

$$\underline{\mathbf{u}^{(\infty)}}_2(t) = \underline{\mathbf{g}}_2 \text{ a.e. on } G \text{ for all } t \in \mathbb{R}. \tag{3.10}$$

Next, (3.7), (3.8) and (3.10) yield, by condition (2.3),

$$\partial_t \mathbf{p}^{(\infty)} = \operatorname{curl} \underline{\mathbf{u}^{(\infty)}}_2 - \partial_t(\varepsilon \underline{\mathbf{u}^{(\infty)}}_1) = \operatorname{curl} \underline{\mathbf{g}}_2$$

on $\mathbb{R} \times G$ in the sense of distributions, in particular $\partial_t \mathbf{p}^{(\infty)}$ is constant on $\mathbb{R} \times G$ with respect to time. But, by (3.9), this is only possible if $\mathbf{p}^{(\infty)} = 0$. Hence, by (3.4) and (3.5),

$$\mathbf{Q}_1 = \mathbf{Q}_2 = 0. \tag{3.11}$$

Furthermore, one obtains from 3.6

$$\mathbf{u}^{(\infty)}(t) = \exp(tB)\mathbf{g} \text{ for all } t \in \mathbb{R}. \tag{3.12}$$

By Proposition 3.5, (3.8) and (3.12) one has $\mathbf{g} \in \ker B$ and, hence, $\mathbf{g} \in \mathcal{N}_0$, i.e.

$$(1 - \mathcal{P})\mathbf{g} = 0, \tag{3.13}$$

since $\mathcal{P}$ is the orthogonal-projector on $\mathcal{N}_0$ in X. Finally, by (3.11) and (3.13), zero is the only possible accumulation point of $(1-\mathcal{P})(\mathbf{E}(t), \mathbf{H}(t))$, $\mathbf{P}(t)$ and $\partial_t \mathbf{P}(t)$ for $t \to \infty$ with respect to the weak topology. This completes the proof of (3.2). $\square$

4. Strong decay of the electromagnetic field.

THEOREM 4.1. Assume (2.20), (2.21) and the assumptions of Theorem 3.1. Then

$$\|\mathbf{E}(t) - \mathcal{P}_1\mathbf{E}_0\|_{L^2(K)} \stackrel{t\to\infty}{\longrightarrow} 0 \text{ for all compact } K \subset \Omega \setminus \overline{G_\sigma},$$

$$\|\mathbf{H}(t) - (1 - P_H)\mathbf{H}_0\|_{L^2(K)} \stackrel{t\to\infty}{\longrightarrow} 0 \text{ for all compact } K \subset \Omega.$$

and, if ∂G_σ is a set of measure zero,

$$\|(\mathbf{E}(t), \mathbf{H}(t)) - \mathcal{P}(\mathbf{E}_0, \mathbf{H}_0)\|_{L^q(K)} \stackrel{t\to\infty}{\longrightarrow} 0 \text{ for all } q \in [1, 2) \text{ and all compact } K \subset \Omega.$$

Proof. Since $\nabla\varphi \in \mathcal{Z}_1 \subset W_{E,0}$ for all $\varphi \in C_0^\infty(\Omega \setminus \overline{G_\sigma}) \subset C_0^\infty(\mathbb{R}^3 \setminus \overline{\Gamma_1})$, it follows that $(\nabla\varphi, 0) \in \mathcal{N}_0 \subset \ker B$ and, hence, by Lemma 2.5:

$$\begin{aligned}\int_\Omega \varepsilon\mathbf{E}(t) \cdot \nabla\varphi dx &= \langle(\mathbf{E}(t), \mathbf{H}(t)), (\nabla\varphi, 0)\rangle_X = \langle\mathcal{P}(\mathbf{E}(t), \mathbf{H}(t)), (\nabla\varphi, 0)\rangle_X \\ &= \langle\mathcal{P}(\mathbf{E}_0, \mathbf{H}_0), (\nabla\varphi, 0)\rangle_X = \int_{\Omega\setminus G_\sigma} \varepsilon(\mathcal{P}_1\mathbf{E}_0) \cdot \nabla\varphi dx,\end{aligned}$$

and, analogously,

$$\forall\psi \in C_0^\infty(\Omega) : \int_\Omega \mu\mathbf{H}(t) \cdot \nabla\psi dx = \int_\Omega \mu[(1 - Q_H)\mathbf{H}_0] \cdot \nabla\psi dx.$$

Therefore, $\mathbf{F}(t) \stackrel{\text{def}}{=} (\mathbf{E}(t), \mathbf{H}(t)) - \mathcal{P}(\mathbf{E}_0, \mathbf{H}_0)$ satisfies

$$\text{div } (\varepsilon\underline{\mathbf{F}}_1(t)) = 0 \text{ on } \Omega \setminus \overline{G_\sigma} \text{ and } \text{div } (\mu\underline{\mathbf{F}}_2(t)) = 0 \text{ on } \Omega. \tag{4.1}$$

Furthermore, since $\mathcal{P}(\mathbf{E}_0, \mathbf{H}_0) \in \ker B$, $\mathbf{F} \in L^\infty((0, \infty), D(B))$ by Lemma 2.3 , in particular

$$\underline{\mathbf{F}}_1 \in L^\infty((0, \infty), H_{curl}(\Omega)) \text{ and } \underline{\mathbf{F}}_2 \in L^\infty((0, \infty), H_{curl}(\Omega)). \tag{4.2}$$

Let $K \subset \Omega$ and $M \subset \Omega \setminus \overline{G_\sigma}$ be compact. From (4.1), (4.2) and the well-known compactness result in [5] or [14] it follows that

$$\{\underline{\mathbf{F}}_1(t) : t \geq 0\} \text{ is precompact in } L^2(M)$$
$$\text{and } \{\underline{\mathbf{F}}_2(t) : t \geq 0\} \text{ is precompact in } L^2(K).$$

Therefore, Theorem 3.1 yields

$$\|\underline{\mathbf{F}}_1(t)\|_{L^2(M)} + \|\underline{\mathbf{F}}_2(t)\|_{L^2(K)} \stackrel{t\to\infty}{\longrightarrow} 0 \tag{4.3}$$

whence the first two assertions.

Next, it follows from (2.14) and (2.24) that

$$\|\sigma^{1/2}\mathbf{E}(t)\|_{L^2(G)} \leq \|\sigma^{1/2}\mathbf{E}(t)\|_{L^2(G_\sigma)} \overset{\mathrm{t}\to\infty}{\longrightarrow} 0. \tag{4.4}$$

Let $K_1 \overset{\text{def}}{=} K \cap G_\sigma$ and $K_2 \overset{\text{def}}{=} K \setminus G_\sigma$. Furthermore, let $A_k \overset{\text{def}}{=} \{x \in K_1 : \sigma(x) > 1/k\}$. From Hölder's inequality and (4.4) one obtains

$$\begin{aligned} \int_{A_k} |\mathbf{E}(t)|^q dx &\leq |K|^{1-q/2} \left(\int_{A_k} |\mathbf{E}(t)|^2 dx \right)^{q/2} \\ &\leq |K|^{1-q/2} \left(k \int_{A_k} \sigma |\mathbf{E}(t)|^2 dx \right)^{q/2} \overset{\mathrm{t}\to\infty}{\longrightarrow} 0. \end{aligned} \tag{4.5}$$

(Here $|K|$ denotes the Lebesgue measure of K.) Furthermore,

$$\begin{aligned} \int_{K_1\setminus A_k} |\mathbf{E}(t)|^q dx &\leq |K_1 \setminus A_k|^{1-q/2} \left(\int_{K} |\mathbf{E}(t)|^2 dx \right)^{q/2} \\ &\leq |K_1 \setminus A_k|^{1-q/2} \|\mathbf{E}\|^q_{L^\infty((0,\infty),L^2(\Omega))}. \end{aligned}$$

Since $K_1 = \bigcup_{k=1}^\infty A_k$ this implies

$$\int_{K_1\setminus A_k} |\mathbf{E}(t)|^q dx \overset{\mathrm{k}\to\infty}{\longrightarrow} 0 \text{ uniformly w.r.t. } t > 0. \tag{4.6}$$

Now, (4.5) and (4.6) yield $\int_{K_1} |\mathbf{E}(t)|^q dx \overset{\mathrm{t}\to\infty}{\longrightarrow} 0$ and, thus,

$$\int_{K_1} |\underline{\mathbf{F}}_1(t)|^q dx \overset{\mathrm{t}\to\infty}{\longrightarrow} 0. \tag{4.7}$$

By (4.3) one has for every compact set $L \subset \Omega \setminus \overline{G_\sigma}$:

$$\begin{aligned} &\limsup_{t\to\infty} \int_{K_2} |\underline{\mathbf{F}}_1(t)|^q dx \\ &\leq |K|^{1-q/2} \limsup_{t\to\infty} \|\underline{\mathbf{F}}_1(t)\|^q_{L^2(K_2\cap L)} + |K_2 \setminus L|^{1-q/2} \|\underline{\mathbf{F}}_1\|^q_{L^\infty((0,\infty),L^2(\Omega))} \\ &\leq |K_2 \setminus L|^{1-q/2} \|\underline{\mathbf{F}}_1\|^q_{L^\infty((0,\infty),L^2(\Omega))}. \end{aligned}$$

Since $K_2 \subset K \setminus \overline{G_\sigma} \cup \partial G_\sigma$ and ∂G_σ is a set of measure zero, one finds to every given $\delta > 0$ some compact set $L \subset \Omega \setminus \overline{G_\sigma}$ with $|K_2 \setminus L| \leq \delta$. Hence, it follows from the previous estimate that

$$\int_{K_2} |\underline{\mathbf{F}}_1(t)|^q dx \overset{\mathrm{t}\to\infty}{\longrightarrow} 0. \tag{4.8}$$

Finally, (4.7) and (4.8) complete the proof. □

In what follows Ω is an exterior domain, i.e. an open set with bounded complement, and G_σ is assumed to be bounded. Furthermore, only the boundary condition $\vec{n} \wedge \mathbf{E} = 0$ on $(0,\infty) \times \partial\Omega$ is considered.

THEOREM 4.2. Let $\Omega \subset \mathbb{R}^3$ be an exterior domain such that $\mathbb{R}^3 \setminus \overline{\Omega}$ is a bounded Lipschitz-domain and $\Gamma_1 = \partial\Omega$. It is also assumed that $\mathcal{C} \overset{\text{def}}{=} G_\sigma \cup (\mathbb{R}^3 \setminus \overline{\Omega}) = \mathcal{C}_1 \cup ... \cup \mathcal{C}_N$ is a bounded Lipschitz-domain, where $\mathcal{C}_k$ are the connected components of $\mathcal{C}$. Furthermore, let σ have a positive lower bound on G_σ. Then one obtains under the assumptions of Theorem 4.1

$$\|(1-Q_E)\mathbf{E}(t)-\mathcal{P}_1\mathbf{E}_0\|_{L^2(\Omega)} \overset{t\to\infty}{\longrightarrow} 0$$
$$\text{and } \|\mathbf{E}(t)-\mathcal{P}_1\mathbf{E}_0\|_{L^2(\Omega\cap B_R(0))} \overset{t\to\infty}{\longrightarrow} 0 \text{ for all } R\in(0,\infty).$$

Lemma 4.3. Under the assumptions of Theorem 4.2 there holds

$$\|Q_E\mathbf{E}(t)\|_{L^2(\Omega\cap B_R(0))} \overset{t\to\infty}{\longrightarrow} 0 \text{ for all } R\in(0,\infty).$$

Proof. Let $R\in(0,\infty)$ such that $\mathbb{R}^3\backslash\overline{\Omega}\subset B_R(0)$. Since $\nabla\varphi\in W_{E,0}$ for all $\varphi\in C_0^\infty(\Omega)$, it follows that

$$\text{div }(\varepsilon\mathbf{e})=0 \text{ for all } \mathbf{e}\in W_{E,0}^\perp = \text{ ran } Q_E. \tag{4.9}$$

Since $\mathbb{R}^3\setminus\overline{\Omega}$ is assumed to be a Lipschitz-domain, by the compactness result in [14], the space of all $\mathbf{e}\in W_E\subset \overset{0}{H}_{curl}(\Omega)$ (i.e. $\mathbf{e}\in H_{curl}(\Omega)$ and $\vec{n}\wedge\mathbf{e}=0$ on $\partial\Omega$) which obey (4.9) is compactly embedded in $L^2(\Omega\cap B_R(0))$. That means that the embedding $W_E\cap W_{E,0}^\perp\hookrightarrow L^2(\Omega\cap B_R(0))$ is compact. Thus, it follows from Lemma 2.3 that

$$\{Q_E\mathbf{E}(t): t\geq 0\} \text{ is precompact in } L^2(\Omega\cap B_R(0)).$$

Finally, the assertion follows from Corollary 3.2. □

Note that, under the assumptions of Theorem 4.2, $\mathcal{Z}_1$ is the space of all $\mathbf{f}\in H_{curl}(\Omega)$ with curl $\mathbf{f}=0$ on Ω , $\vec{n}\wedge\mathbf{f}=0$ on $\partial\Omega$ and $\mathbf{f}=0$ on G_σ. Next, let

$$\mathcal{Y}\overset{\text{def}}{=} W_{E,0}\cap\mathcal{Z}_1^\perp$$

with respect to the scalar-product $\langle\cdot,\cdot\rangle_\varepsilon$, i.e. $\mathcal{Y}$ is defined as the space of all $\mathbf{e}\in H_{curl}(\Omega)$ with curl $\mathbf{e}=0$ on Ω , $\vec{n}\wedge\mathbf{e}=0$ on $\partial\Omega$ and

$$\forall\mathbf{f}\in\mathcal{Z}_1: \int_\Omega \varepsilon\mathbf{e}\mathbf{f}dx=0.$$

Lemma 4.4. Under the assumptions of Theorem 4.2 there exists a constant K_1 such that

$$\forall\mathbf{e}\in\mathcal{Y}: \|\mathbf{e}\|_{L^2(\Omega)}\leq K_1\|\mathbf{e}\|_{L^2(G_\sigma)}.$$

Proof. Let $\mathbf{e}\in\mathcal{Y}$. Since curl $\mathbf{e}=0$ on Ω and $\vec{n}\wedge\mathbf{e}=0$ on $\partial\Omega$, one has curl $\tilde{\mathbf{e}}=0$ on $\mathbb{R}^3$ in the sense of distributions where $\tilde{\mathbf{e}}\in L^2(\mathbb{R}^3)$ denotes the extension by zero on $\mathbb{R}^3\setminus\Omega$ of $\mathbf{e}$. By a classical result there exists a $\varphi\in L^6(\mathbb{R}^3)\cap H^1_{loc}(\mathbb{R}^3)$ with

$$\tilde{\mathbf{e}}=\nabla\varphi. \tag{4.10}$$

Now, let $\psi\in H^1(\mathcal{C})$ be defined as

$$\psi(x)\overset{\text{def}}{=}\varphi(x)-\beta_k \text{ if } x\in\mathcal{C}_k.$$

Here

$$\beta_k\overset{\text{def}}{=}|\mathcal{C}_k|^{-1}\int_{\mathcal{C}_k}\varphi dx$$

where $|\mathcal{C}_k|$ denotes the Lebesgue measure of $\mathcal{C}_k$. Recall that $\mathcal{C}_k$ are the connected components of $\mathcal{C} = G_\sigma \cup (\mathbb{R}^3 \setminus \overline{\Omega})$. Since the average of ψ taken over each $\mathcal{C}_k$ is zero, Poincare's inequality yields

$$\|\psi\|_{H^1(\mathcal{C})} \le C_1 \|\nabla \psi\|_{L^2(\mathcal{C})} = C_1 \|\nabla \varphi\|_{L^2(\mathcal{C})} = C_1 \|\tilde{\mathbf{e}}\|_{L^2(\mathcal{C})} = C_1 \|\mathbf{e}\|_{L^2(G_\sigma)} \tag{4.11}$$

with some constant C_1 independent of $\mathbf{e}$. Let $T : H^1(\mathcal{C}) \to H^1(\mathbb{R}^3)$ be an H^1-extension operator, see [1], and let $\varphi_1 \stackrel{\text{def}}{=} T(\psi) \in H^1(\mathbb{R}^3)$. By the definition of ψ the function $\psi - \varphi$ and, hence, $\varphi_1 - \varphi$ are constant on each $\mathcal{C}_k$, whence $\nabla \varphi_1 - \nabla \varphi = 0$ on $\mathcal{C} = G_\sigma \cup (\mathbb{R}^3 \setminus \overline{\Omega})$. Therefore,

$$\nabla \varphi_1 - \nabla \varphi \in \mathcal{Z}_1. \tag{4.12}$$

(To be precise, the restriction of $\nabla \varphi_1 - \nabla \varphi$ to Ω belongs to $\mathcal{Z}_1$.) Furthermore, (4.11) yields

$$\|\nabla \varphi_1\|_{L^2(\Omega)} \le \|T(\psi)\|_{H^1(\Omega)} \le C_2 \|\psi\|_{H^1(\mathcal{C})} \le C_3 \|\mathbf{e}\|_{L^2(G_\sigma)}. \tag{4.13}$$

Now, (4.12) and (4.13) yield

$$\begin{aligned} \|\varepsilon^{1/2}\mathbf{e}\|^2_{L^2(\Omega)} &\le \int_\Omega \varepsilon \mathbf{e} \cdot [\nabla \varphi_1 - \nabla \varphi] dx + \|\varepsilon^{1/2}\mathbf{e}\|_{L^2(\Omega)} \|\varepsilon^{1/2} \nabla \varphi_1\|_{L^2(\Omega)} \\ &= \|\varepsilon^{1/2}\mathbf{e}\|_{L^2(\Omega)} \|\varepsilon^{1/2} \nabla \varphi_1\|_{L^2(\Omega)} \le C_4 \|\varepsilon^{1/2}\mathbf{e}\|_{L^2(\Omega)} \|\mathbf{e}\|_{L^2(G_\sigma)}, \end{aligned}$$

which completes the proof. $\square$

Proof of Theorem 4.2. Let $\mathbf{u}(t) \stackrel{\text{def}}{=} (1 - Q_E)\mathbf{E}(t) - \mathcal{P}_1 \mathbf{E}_0$. By Lemma 2.5 one has for all $\mathbf{f} \in \mathcal{Z}_1$:

$$\langle (1 - Q_E)\mathbf{E}(t), \mathbf{f} \rangle_\varepsilon = \langle \mathcal{P}_1 \mathbf{E}(t), \mathbf{f} \rangle_\varepsilon = \langle \mathcal{P}_1 \mathbf{E}_0, \mathbf{f} \rangle_\varepsilon$$

and, thus, $\mathbf{u}(t) \in W_{E,0} \cap \mathcal{Z}_1^\perp = \mathcal{Y}$. Hence, Lemma 4.4 yields

$$\begin{aligned} \|(1 - Q_E)\mathbf{E}(t) - \mathcal{P}_1 \mathbf{E}_0\|_{L^2(\Omega)} &= \|\mathbf{u}(t)\|_{L^2(\Omega)} \le K_1 \|\mathbf{u}(t)\|_{L^2(G_\sigma)} \\ &\le K_1 (\|\mathbf{E}(t)\|_{L^2(G_\sigma)} + \|Q_E \mathbf{E}(t)\|_{L^2(G_\sigma)}). \end{aligned} \tag{4.14}$$

Since σ has a positive lower bound on G_σ, one obtains from (4.4) that

$$\|\mathbf{E}(t)\|_{L^2(G_\sigma)} \stackrel{t \to \infty}{\longrightarrow} 0. \tag{4.15}$$

Furthermore, since G_σ is bounded, Lemma 4.3 yields

$$\|Q_E \mathbf{E}(t)\|_{L^2(G_\sigma)} \stackrel{t \to \infty}{\longrightarrow} 0. \tag{4.16}$$

Finally, the assertion follows from (4.14)-(4.16) and Lemma 4.3 again. $\square$

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