QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXV, NUMBER 3
SEPTEMBER 2007, PAGES 471–498
S 0033-569X(07)01076-5
Article electronically published on August 1, 2007

THE ASYMPTOTIC PROBLEM FOR THE SPRINGLIKE MOTION OF A HEAVY PISTON IN A VISCOUS GAS

By

STUART S. ANTMAN (*Department of Mathematics, Institute for Physical Science and Technology, and Institute for Systems Research, University of Maryland, College Park, Maryland 20742-4015*)

AND

J. PATRICK WILBER (*Department of Theoretical and Applied Mathematics, University of Akron, Akron, Ohio 44325*)

Abstract. This paper treats the classical problem for the longitudinal motion of a piston separating two viscous gases in a closed cylinder of finite length. The motion of the gases is governed by singular initial-boundary-value problems for parabolic-hyperbolic partial differential equations depending on a small positive parameter ε, which characterizes the ratios of the masses of the gases to that of the piston. (The equation of state giving the pressure as a function of the specific volume need not be monotone and the viscosity may depend on the specific volume.) These equations are subject to a transmission condition, which is the equation of motion of the piston. The specific volumes of the gases are shown to have a positive lower bound at any finite time. This bound leads to the theorem asserting that (under mild smoothness restrictions) the initial-boundary-value problem has a unique classical solution defined for all time. The main emphasis of this paper is the treatment of the asymptotic behavior of solutions as $\varepsilon \searrow 0$. It is shown that this solution admits a rigorous asymptotic expansion in ε consisting of a regular expansion and an initial-layer expansion. The reduced problem, for the leading term of the regular expansion (which is obtained by setting $\varepsilon = 0$), is typically governed by an equation with memory, rather than by an ordinary differential equation of the sort governing the motion of a mass on a massless spring. The reduced problem nevertheless has a 2-dimensional attractor on which the dynamics is governed precisely by such an ordinary differential equation.

Received July 25, 2006.
2000 *Mathematics Subject Classification.* Primary 76N99; Secondary 35B41, 35C20, 35K55.
Key words and phrases. 1-dimensional gas dynamics, piston, viscous gas, quasilinear parabolic-hyperbolic system, asymptotics, transmission, attractors.
The work of the first author was supported in part by NSF Grant # DMS-0204505.
The work of the second author was supported in part by NSF Grant DMS-0407361.
E-mail address: ssa@math.umd.edu
E-mail address: pwilber@math.uakron.edu

1. Introduction. We study the classical problem for the forced or free longitudinal motion of a heavy piston (a rigid body) separating two viscous gases in a closed cylinder of finite length. The motion of the gases is governed by singular initial-boundary-value problems for parabolic-hyperbolic partial differential equations depending on a small positive parameter ε, which characterizes the ratios of the masses of the gases to that of the piston. These equations are subject to a transmission condition, which is the equation of motion of the piston. A straightforward energy estimate leads to standard bounds. These are used in a slightly tricky proof that the specific volumes of the gases have positive lower and upper bounds at any finite time. The former bound leads to the theorem asserting that (under mild smoothness restrictions) the initial-boundary-value problem has a unique classical solution defined for all time.

We then study the asymptotic behavior of solutions as $\varepsilon \searrow 0$. We show that (i) the solution of the initial-boundary-value problem admits a rigorous asymptotic expansion in ε consisting of a regular expansion and an initial-layer expansion, (ii) the reduced problem, governing the leading term of the regular expansion (which is obtained by setting the densities of the fluids equal to zero, i.e., by setting $\varepsilon = 0$), is typically governed by an equation with memory, rather than by an ordinary differential equation of the sort governing the motion of a mass on a massless spring (this equation has a globally defined solution), (iii) the reduced problem nevertheless has a 2-dimensional attractor on which the dynamics is governed precisely by such an ordinary differential equation.

This problem is mathematically similar to the formidable problem for the longitudinal motion of a heavy mass on a light nonlinearly viscoelastic spring [3, 4, 27, 28]. The main mathematical novelty of our piston problem vis-à-vis that for the spring is that the governing equations of motion for the gas are subject to a dynamical transmission condition, which is the equation of motion of the piston. Our study accordingly emphasizes the formulation of the governing equations and the analysis of those novel aspects of the problem due to the transmission condition. Its treatment differs considerably from that for the mass on a spring treated as a 1-dimensional continuum. We are fortunately able to invoke a number of technical results from [27, 28] to handle many of the remaining aspects of the problem.

For this problem we are able to treat a large family of constitutive equations (equations of state) for the pressure and the viscosity as functions of the specific volume (or, equivalently, of the density). For a discussion of the experimental evidence for such functions, see Bridgman [7]. We note that piston problems have played a fundamental role in gas dynamics (see [8, 18], e.g.).

Notation. We occasionally denote the function $u \mapsto f(u)$ by $f(\cdot)$ and denote the composite function $u \mapsto f(g(u))$ by $f \circ g$. The partial derivative of a function f with respect to a scalar argument t is denoted by either f_t or $\partial_t f$. The operator ∂_t is assumed to apply only to the term immediately following it. Obvious analogs of these notations will also be used.

We let c and C denote typical positive constants that are supplied as data or that can be estimated in terms of data. Their meanings usually change with each appearance (even in the same equation or inequality; indeed, C may be regarded as increasing and

c as decreasing with each appearance). Similarly, $t \mapsto \gamma(t)$ and $t \mapsto \Gamma(t)$ denote typical positive-valued continuous functions depending on the data. Tacit in the statement of an inequality of the form $\|u\| \le C$ is an assertion that there exists a positive number C such that this estimate holds.

Throughout this paper we use without comment the Cauchy-Bunyakovskiĭ-Schwarz inequality and the elementary inequality $2|ab| \le ca^2 + b^2/c$ for real a, b and for positive c. We may use the convention just discussed to write this last estimate as $2|ab| \le ca^2 + Cb^2$.

2. Formulation of the governing equations. A piston of scaled mass 1 moves in a cylinder of finite length under the action of the two viscous gases that it separates. (See Figure 1.) We begin this section with a brief formulation of the object of our study: the dimensionless form of the governing initial-boundary-value problem for the purely longitudinal motion of this system under simplifying symmetry assumptions. We then derive these equations from their dimensional form without the simplifying assumptions, an exercise that is not completely trivial, which shows that our analytic methods can handle this general case. The reader interested primarily in the analysis can skim over this material.

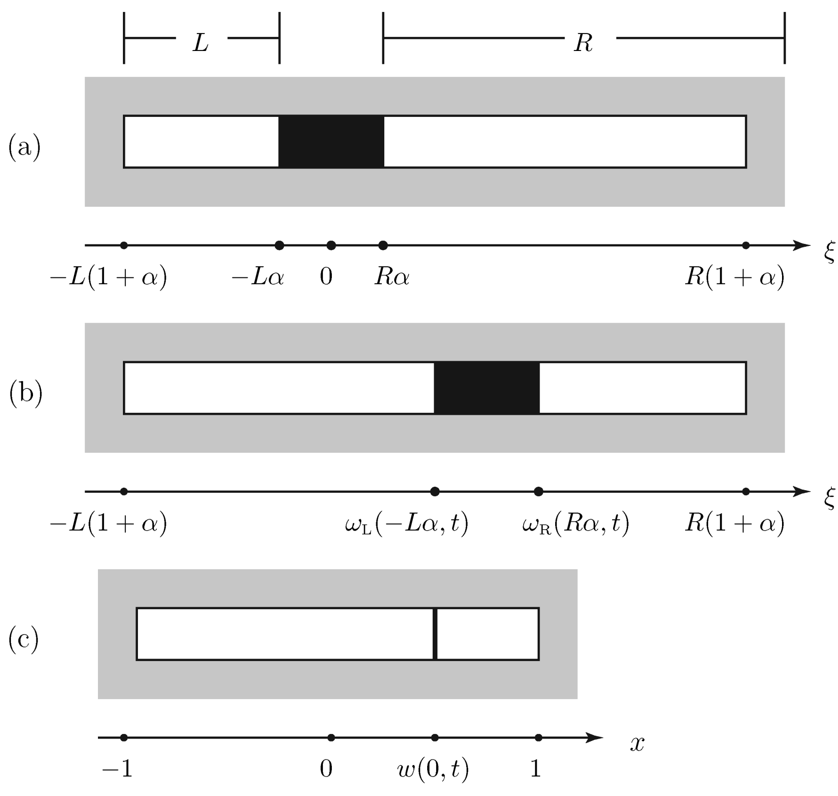

FIG. 1. (a) The reference configuration of the piston in the cylinder. (b) The configuration at time t. (c) The configuration at time t in the scaled variables.

We take the reference configuration of this system to be one in which the piston is centered between the regions occupied by the gases. We identify a typical material section (material point) of the gas to the left of the piston by the scaled coordinate $x \in (-1, 0)$ and a typical material section of the gas to the right of the piston by the scaled coordinate $x \in (0, 1)$. (Thus x is a material = Lagrangian coordinate.) We shall show that we can ignore the dimensions of the piston and accordingly take its reference position to be at 0. We assume that the reference density of the gas, to the left and right of the piston, is a small positive constant ε. Let $w(x, t)$ denote the position of the material section x of the gas at time t. We denote the position of the piston at time t by $w(0, t)$. The specific volume of the gas (which is the local ratio of deformed to reference length) at (x, t) is denoted $u(x, t) := w_x(x, t)$. The velocity of the gas at (x, t) is $v(x, t) := w_t(x, t)$. Thus u and v must satisfy the compatibility equation

$$u_t = v_x, \quad x \in (-1, 0) \cup (0, 1). \tag{2.1}$$

Let $p(u)$ be the scaled pressure in the gas due to the specific volume u. Let $\nu(u)$ be the scaled viscosity of the gas, allowed to depend on the specific volume. We set

$$\psi(u) := \int_1^u \frac{\nu(z)}{z}\, dz. \tag{2.2}$$

We assume that

$$p(u) > 0, \qquad p(u) \to \infty \quad \text{as} \quad u \to 0, \qquad p(u) \to 0 \quad \text{as} \quad u \to \infty, \tag{2.3}$$

$$\nu(u) > 0, \qquad \psi(u) \to -\infty \quad \text{as} \quad u \to 0, \qquad \frac{\nu(u)}{u} \to 0 \quad \text{as} \quad u \to \infty. \tag{2.4}$$

Our assumptions imply that ψ has a positive derivative everywhere and that ψ strictly increases from $-\infty$ either to a positive number or to ∞ as u increases from 0 to ∞. For much of our work, we require neither that p be decreasing (so that we allow van der Waal's gases) nor that ν be constant. The usual forms taken for p, ν, ψ are

$$p(u) = \text{const}\, u^{-\sigma} \text{ with } 1 < \sigma < 2, \quad \nu = \text{const}, \quad \psi(u) = \ln(u^\nu). \tag{2.5}$$

These clearly satisfy (2.3) and (2.4).

We assume that the gases to the left and right of the piston are the same compressible Newtonian fluids. Thus the internal contact force (stress) at time t exerted by the gas to the right of the material section x on that to the left is

$$-p(u(x, t)) + \psi'(u(x, t))v_x(x, t). \tag{2.6}$$

We assume for simplicity of exposition that there are no body forces acting on the gas. The momentum equation for the gas is the 1-dimensional Navier-Stokes equation in the material (= Lagrangian) formulation:

$$\varepsilon v_t = -p(u)_x + [\psi'(u)v_x]_x, \quad x \in (-1, 0) \cup (0, 1). \tag{2.7}$$

The piston is assumed to be subject not only to the contact forces exerted by the gases but also to an external force f, assumed to be a locally bounded function of t. The equation of motion of the piston is

$$v_t(0, t) = -[\![p(u)]\!](0, t) + [\![\psi'(u)v_x]\!](0, t) + f(t) \tag{2.8}$$

where $[\![h]\!](0,t) := \lim_{\eta \searrow 0}[h(\eta,t) - h(-\eta,t)]$. This equation is a transmission condition for the partial differential equations (2.1) and (2.7) at $x = 0$.

The requirement that the gas at the ends of the piston stay in contact with the ends leads to the boundary conditions

$$w(-1,t) = -1, \qquad w(1,t) = 1. \tag{2.9}$$

The requirement that the gas at the piston stay in contact with the piston yields another transmission condition

$$w(0-,t) = w(0,t) = w(0+,t). \tag{2.10}$$

These conditions imply that

$$v(-1,t) = 0, \quad v(1,t) = 0, \tag{2.11}$$

$$v(0-,t) = v(0+,t), \tag{2.12}$$

$$\int_{-1}^{1} u(x,t)\, dx = 2. \tag{2.13}$$

We supplement (2.1), (2.7), (2.8), (2.11)–(2.13) with the initial conditions

$$u(x,0) = \bar{u}(x) \quad \text{for} \quad x \in (-1,0) \cup (0,1), \quad v(x,0) = \bar{v}(x) \quad \text{for} \quad x \in (-1,1) \tag{2.14}$$

subject to the restriction that

$$\int_{-1}^{1} \bar{u}(x)\, dx = 2, \tag{2.15}$$

so that the initial conditions are compatible with (2.13). We require that the initial conditions be compatible with the boundary conditions (2.11)–(2.12) to whatever order is required in the analysis. We omit the details, referring to [28].

Our initial-boundary-value problem consisting of (2.1), (2.7), (2.8), (2.11)–(2.14) should be regarded as an abbreviation for a precise version of the Principle of Virtual Power, which is essentially equivalent to the weak formulation of these equations [4]. Note that the disposition of the small parameter ε in the evolution equations (2.1) and (2.7) is just like that which is standard for many asymptotic problems for ordinary differential equations.

Derivation of the general initial-boundary-value problem. As we shall show, our standard assumptions on the constitutive equations for the pressure ensure that the governing system admits at least one equilibrium configuration with the specific volume and density in each gas constant. We take one of these equilibrium configurations as a reference configuration. Let $\alpha \in (0,1]$, $L > 0$, $R > 0$. We identify each material section of the gas to the left or right of the piston by its position $\xi \in [-L(1+\alpha), -L\alpha]$ or $\xi \in [R\alpha, R(1+\alpha)]$ in this configuration. The length of the piston is $(L+R)\alpha$ and the length of the cylinder is $(L+R)(1+\alpha)$. Let ρ_{L} and ρ_{R} be the constant densities of the gases (masses per length) to the left and right of the piston in the reference configuration. Let $\omega(\xi,t)$ denote the position of material section ξ at time t. See Figure 1.

Thus

$$
\begin{aligned}
-L(1+\alpha) < \omega(\xi,t) < R(1+\alpha) - (R+L)\alpha = R - L\alpha \quad &\text{for} \quad \xi \in [-L(1+\alpha), -L\alpha], \\
-L + R\alpha = -L(1+\alpha) + (R+L)\alpha < \omega(\xi,t) < R(1+\alpha) \quad &\text{for} \quad \xi \in [R\alpha, R(1+\alpha)].
\end{aligned} \tag{2.16}
$$

The specific volume of the gas (local ratio of deformed to reference lengths) at (ξ, t) is $\omega_\xi(\xi,t)$. It should be positive everywhere.

Let $u \mapsto \pi_{\mathrm{L}}(u), \pi_{\mathrm{R}}(u)$ and $\lambda \mapsto \mu_{\mathrm{L}}(u), \mu_{\mathrm{R}}(u)$ be the constitutive functions giving the pressures and viscosities of the gases to the left and right of the piston as functions of the specific volumes for the gases. We assume that π_{L} and π_{R} have the same properties as p in (2.3) and that μ_{L} and μ_{R} have the same properties as ν in (2.4).

We assume that each gas is a compressible Newtonian fluid, so that the internal contact force at material section $\xi \in [-L(1+\alpha), -L\alpha]$ exerted by the gas to the right of the section on that to the left at time t is

$$
-\pi_{\mathrm{L}}(\omega_\xi(\xi,t)) + \frac{\mu_{\mathrm{L}}(\omega_\xi(\xi,t))\omega_{\xi t}(\xi,t)}{\omega_\xi(\xi,t)}, \tag{2.17}
$$

etc. We assume for simplicity of exposition that there are no body forces acting on the gas. In this case, the momentum equations are

$$
\rho_{\mathrm{L}}\omega_{tt} = -\partial_\xi \pi_{\mathrm{L}}(\omega_\xi) + \partial_\xi \left[\frac{\mu_{\mathrm{L}}(\omega_\xi)\omega_{\xi t}}{\omega_\xi} \right], \quad \xi \in (-L(1+\alpha), -L\alpha), \quad \text{etc.} \tag{2.18}
$$

The piston of mass m, which is assumed to be subject not only to the contact forces exerted by the gases but also to an external force $\frac{1}{2}m(L+R)f$, has the equation of motion

$$
\begin{aligned}
m\omega_{tt}(R\alpha,t) = {} & \tfrac{1}{2}m(L+R)f(t) + \pi_{\mathrm{L}}(\omega_\xi(-L\alpha,t)) - \pi_{\mathrm{R}}(\omega_\xi(R\alpha,t)) \\
& - \frac{\mu_{\mathrm{L}}(\omega_\xi(-L\alpha,t))\omega_{\xi t}(-L\alpha,t)}{\omega_\xi(-L\alpha,t)} + \frac{\mu_{\mathrm{R}}(\omega_\xi(R\alpha,t))\omega_{\xi t}(R\alpha,t)}{\omega_\xi(R\alpha,t)}.
\end{aligned} \tag{2.19}
$$

The requirement that the gas at the ends stay in contact with the ends leads to the boundary conditions

$$
\omega(-L(1+\alpha),t) = -L(1+\alpha), \quad \omega(R(1+\alpha),t) = R(1+\alpha). \tag{2.20}
$$

The requirement that the gas at the piston stay in contact with the piston yields the transmission condition

$$
\omega(R\alpha,t) = \omega(-L\alpha,t) + (R+L)\alpha. \tag{2.21}
$$

We shall be especially concerned with problems in which the piston is heavy, i.e., in which the masses of the gases are small with respect to the mass of the piston. We accordingly begin the nondimensionalization of our equations by introducing the dimensionless mass ratio ε and two unimportant dimensionless weight factors ϱ_{L} and ϱ_{R} by

$$
2\varepsilon = \frac{\rho_{\mathrm{L}}L}{m} + \frac{\rho_{\mathrm{R}}R}{m}, \qquad \frac{\rho_{\mathrm{L}}L}{m} =: \varepsilon\varrho_{\mathrm{L}}, \qquad \frac{\rho_{\mathrm{R}}R}{m} =: \varepsilon\varrho_{\mathrm{R}}. \tag{2.22}
$$

We assume that t is a dimensionless time. We introduce dimensionless independent coordinates and position variables by

$$
\begin{aligned}
&x := \frac{\xi + L\alpha}{L} \in [-1,0], \qquad w(x,t) := \frac{2\omega(L(x-\alpha),t) + L(1+2\alpha) - R}{L+R} \in (-1,1) \\
&\qquad \text{for} \quad \xi \in (-L(1+\alpha), -L\alpha), \\
&x := \frac{\xi - R\alpha}{R} \in [0,1], \qquad w(x,t) := \frac{2\omega(R(x+\alpha),t) + L - R(1+2\alpha)}{L+R} \in (-1,1) \\
&\qquad \text{for} \quad \xi \in (R\alpha, R(1+\alpha)).
\end{aligned} \tag{2.23}
$$

(These variables indicate that without loss of generality we can regard the piston as occupying a single section.) We define dimensionless pressures and viscosities by

$$
\begin{aligned}
p_{\mathrm{L}}(w_x) &:= \frac{2\pi_{\mathrm{L}}(\frac{L+R}{2L}w_x)}{m(L+R)\varrho_{\mathrm{L}}}, \quad p_{\mathrm{R}}(w_x) := \frac{2\pi_{\mathrm{R}}(\frac{L+R}{2R}w_x)}{m(L+R)\varrho_{\mathrm{R}}}, \\
\nu_{\mathrm{L}}(w_x) &:= \frac{2\mu_{\mathrm{L}}(\frac{L+R}{2L}w_x)}{m(L+R)\varrho_{\mathrm{L}}}, \quad \nu_{\mathrm{R}}(w_x) := \frac{2\mu_{\mathrm{R}}(\frac{L+R}{2R}w_x)}{m(L+R)\varrho_{\mathrm{R}}}.
\end{aligned} \tag{2.24}
$$

Then (2.18)–(2.21) reduce to

$$
\begin{aligned}
\varepsilon w_{tt} &= -\partial_x p_{\mathrm{L}}(w_x) + \partial_x \left[\frac{\nu_{\mathrm{L}}(w_x) w_{xt}}{w_x} \right], \quad x \in (-1,0), \\
\varepsilon w_{tt} &= -\partial_x p_{\mathrm{R}}(w_x) + \partial_x \left[\frac{\nu_{\mathrm{R}}(w_x) w_{xt}}{w_x} \right], \quad x \in (0,1),
\end{aligned} \tag{2.25}
$$

$$
\begin{aligned}
\partial_{tt} w(0,t) &= f(t) + \varrho_{\mathrm{L}} p_{\mathrm{L}}(w_x(0-,t)) - \varrho_{\mathrm{R}} p_{\mathrm{R}}(w_x(0+,t)) \\
&\quad - \frac{\varrho_{\mathrm{L}} \nu_{\mathrm{L}}(w_x(0-,t)) w_{xt}(0-,t)}{w_x(0-,t)} + \frac{\varrho_{\mathrm{R}} \nu_{\mathrm{R}}(w_x(0+,t)) w_{xt}(0+,t)}{w_x(0+,t)},
\end{aligned} \tag{2.26}
$$

$$
w(-1,t) = -1, \quad w(1,t) = 1, \tag{2.27}
$$

$$
w(0-,t) = w(0,t) = w(0+,t). \tag{2.28}
$$

Let us pause to show that our initial-boundary-value problem admits an equilibrium solution, which we chose to be the reference configuration and which we used to establish our original coordinate system. Let us denote the restriction of w to $[-1,0)$ by w_{L} and the restriction of w to $(0,1]$ by w_{R}. Clearly, the equations of (2.25) are satisfied by constant specific volumes $\partial_x w_{\mathrm{L}} = A$ and $\partial_x w_{\mathrm{R}} = B$, so that w_{L} and w_{R} have the forms $w_{\mathrm{L}} = Ax + M$ and $w_{\mathrm{R}} = Bx + N$. Requiring that these linear functions satisfy conditions (2.27) and (2.28) gives $A = M + 1$ and $B = 1 - M$. For equilibrium, (2.26) reduces to

$$
\varrho_{\mathrm{L}} p_{\mathrm{L}}(M+1) = \varrho_{\mathrm{R}} p_{\mathrm{R}}(1-M) \tag{2.29}
$$

with $M \in (-1,1)$ to ensure that these constant specific volumes are positive. Then (2.3) ensures that this equation has at least one solution $M \in (-1,1)$, and if these pressures are strictly decreasing functions of the specific volumes, then this equation has exactly one solution.

Solely for the purpose of simplifying notation, we assume that the reference states and the gases on each side of the piston are identical, so that $\varrho_{\mathrm{L}} = \varrho_{\mathrm{R}} = 1$, $p_{\mathrm{L}} = p_{\mathrm{R}} = p$, and $\nu_{\mathrm{L}} = \nu_{\mathrm{R}} = \nu$. In this case, (2.29) always has a solution $M = 0$, and we take the reference position of the piston to be $x = 0$. Then $u = w_x$ and $v = w_t$ satisfy the initial-boundary-value problem formulated at the beginning of this section.

3. Energy estimate. The global existence and regularity theory for parabolic-hyperbolic systems like (2.1), (2.7) can be achieved by several approaches, such as the Faedo-Galerkin method [17, 29] in Sobolev spaces [5, 6] or the Leray-Schauder theory [11, 15] in Hölder spaces [9] or semigroup theory [30]. These methods each rely on suitable a priori estimates. In this section we obtain the elementary energy estimate. In the next section, we use it to obtain much trickier estimates precluding the states of total compression and vacuum (the difficulty arising from the dynamic transmission condition (2.8)). This preclusion is central for the demonstration of existence and regularity.

Let

$$\varphi(u) := \int_u^1 p(z)\,dz \quad \text{so that} \quad -p(u) =: \varphi'(u). \tag{3.1}$$

We assume that the behavior of p for large u is such that φ is bounded below:

$$\varphi(u) > -C \quad \forall\, u > 0, \tag{3.2}$$

so that we could vary φ by a constant to ensure that $\varphi(u) > 0$ for $u > 0$. In particular, for the standard assumption of (2.5) that $p(u) = Ku^{-\sigma}$, we have

$$\varphi(u) = Ku^{-(\sigma-1)}/(\sigma - 1). \tag{3.3}$$

We multiply (2.7) by v, integrate the product with respect to x by parts over $(-1, 0)$ and $(0, 1)$ using (2.11) and (2.12), combine the result with the product of (2.8) with $v(0, t)$, and use (2.1) to obtain

$$\begin{aligned}
&\frac{\varepsilon}{2}\frac{d}{dt}\int_{-1}^1 v^2\,dx \\
&= -\int_{-1}^1 p(u)_x v\,dx + \int_{-1}^1 [\psi'(u)v_x]_x v\,dx \\
&= [-p(u(0-,\cdot)) + p(u(0+,\cdot)) + \psi'(u(0-,\cdot))v_x(0-,\cdot) - \psi'(u(0+,\cdot))v_x(0+,\cdot)]v(0,\cdot) \\
&\qquad - \int_{-1}^1 \varphi'(u)v_x\,dx - \int_{-1}^1 \psi'(u)v_x^2\,dx \\
&= -v(0,\cdot)v_t(0-,\cdot) + f(\cdot)v(0,\cdot) - \int_{-1}^1 \varphi'(u)u_t\,dx - \int_{-1}^1 \psi'(u)v_x^2\,dx,
\end{aligned} \tag{3.4}$$

whence

$$\frac{\varepsilon}{2}\frac{d}{dt}\int_{-1}^1 v^2\,dx + \frac{1}{2}\frac{d}{dt}v(0,\cdot)^2 + \frac{d}{dt}\int_{-1}^1 \varphi(u)\,dx + \int_{-1}^1 \psi'(u)v_x^2\,dx = f(\cdot)v(0,\cdot), \tag{3.5}$$

so that

$$\begin{aligned}&\frac{\varepsilon}{2}\int_{-1}^{1} v(x,t)^2\,dx+\frac{1}{2}v(0,t)^2+\int_{-1}^{1}\varphi(u(x,t))\,dx\\&+\int_0^t\int_{-1}^{1}\psi'(u(x,s))v_x(x,s)^2\,dx\,ds\\&\qquad=\frac{\varepsilon}{2}\int_{-1}^{1}\bar{v}^2\,dx+\frac{1}{2}\bar{v}(0)^2+\int_{-1}^{1}\varphi(\bar{u})\,dx+\int_0^t f(s)v(0,s)\,ds.\end{aligned} \tag{3.6}$$

Since f is locally bounded, it is locally square-integrable. Thus

$$\left|\int_0^t f(s)v(0,s)\,ds\right|\le\frac{1}{2}\int_0^t f(s)^2\,ds+\frac{1}{2}\int_0^t v(0,s)^2\,ds. \tag{3.7}$$

Then the Gronwall inequality applied to $v(0,t)^2$ implies that

$$\begin{aligned}&\frac{\varepsilon}{2}\int_{-1}^{1} v(x,t)^2\,dx+\frac{1}{2}v(0,t)^2+\int_{-1}^{1}\varphi(u(x,t))\,dx\\&+\int_0^t\int_{-1}^{1}\psi'(u(x,s))v_x(x,s)^2\,dx\,ds\le\Gamma(t).\end{aligned} \tag{3.8}$$

Note that if $f=0$, then the Γ on the right-hand side of (3.8) can be replaced by a constant C. For $\varepsilon>0$, (3.8) immediately yields

$$\left|\int_{x_1}^{x_2} v(x,t)\,dx\right|\le\sqrt{2}\sqrt{\left[\int_{-1}^{1} v(x,t)^2\,dx\right]}\le\Gamma(t). \tag{3.9}$$

4. Bounds on the specific volume. Let $-1<\xi<0<\eta<1$. Using (2.1), we integrate (2.7) over $[\xi,0)\cup(0,\eta]\times[0,t]$ to get

$$\varepsilon\left[\int_\xi^0+\int_0^\eta\right]\int_0^t v_t\,ds\,dz=\int_0^t\left[\int_\xi^0+\int_0^\eta\right][-p(u)+\psi'(u)u_t]_x\,dz\,ds, \tag{4.1}$$

so that (2.8) implies that

$$\psi(u(y,t))-\int_0^t p(u(y,s))\,ds=\psi(u(x,t))-\int_0^t p(u(x,s))\,ds+g(x,y,t) \tag{4.2}$$

for $x=\xi$, $y=\eta$ where

$$g(x,y,t):=\psi(\bar{u}(y))-\psi(\bar{u}(x))+\varepsilon\int_x^y[v(z,t)-\bar{v}(z)]\,dz+v(0,t)-\bar{v}(0)-\int_0^t f(s)\,ds. \tag{4.3}$$

Likewise, integrating (2.7) over $[\xi_1,\xi_2]\times[0,t]$ with $-1<\xi_1\le\xi_2<0$ yields (4.2) and

$$g(x,y,t):=\psi(\bar{u}(y))-\psi(\bar{u}(x))+\varepsilon\int_x^y[v(z,t)-\bar{v}(z)]\,dz \tag{4.4}$$

with $x=\xi_1$, $y=\xi_2$. Analogously we obtain (4.2) and (4.4) for $x=\eta_1$, $y=\eta_2$ with $0<\eta_1\le\eta_2<1$. Thus (4.2) holds for all x,y,t with $-1<x,y<1$ and with g appropriately defined. The inequality (3.8) implies that

$$|g(x,y,\tau)|\le\Gamma(t) \tag{4.5}$$

with $\Gamma\le C$ when $f=0$.

Our basic result is

THEOREM 4.1. Let (2.3) and (2.4) hold, so that ψ has the properties stated after (2.2), which ensure that ψ is invertible. Let p furthermore differ from a function that is nowhere increasing by a bounded function. Let (u, v) be a classical solution of the initial-boundary-value problem (2.1), (2.7), (2.8), (2.11)–(2.14) defined for $t \in [0, T]$. Then there is a function γ depending on the data such that

$$0 < \gamma(t) \leq u(x,t) \quad \text{for} \quad x \in (-1,0) \cup (0,1), \quad 0 \leq t \leq T. \tag{4.6}$$

If, furthermore,

$$\psi(u) \to \infty \quad \text{as} \quad u \to \infty, \tag{4.7}$$

then there is a function Γ depending on the data such that

$$u(x,t) \leq \Gamma(t) \quad \text{for} \quad x \in (-1,0) \cup (0,1), \quad 0 \leq t \leq T. \tag{4.8}$$

Proof. Set

$$\chi(x,t) := \psi(u(x,t)), \qquad \Pi(x,t) := \int_0^t p(\psi^{-1}(\chi(x,s)))\, ds \tag{4.9}$$

so that

$$\Pi_t(x,t) = p(\psi^{-1}(\chi(x,t))) \equiv p \circ \psi^{-1}(\chi(x,t)). \tag{4.10}$$

Thus (4.2) has the form

$$\chi(y,t) = \chi(x,t) + \Pi(y,t) - \Pi(x,t) + g(x,y,t). \tag{4.11}$$

We operate on this equation with $p \circ \psi^{-1}$ to get

$$\Pi_t(y,t) = p \circ \psi^{-1}\big(\chi(x,t) + \Pi(y,t) - \Pi(x,t) + g(x,y,t)\big). \tag{4.12}$$

Note that the domain of definition of $p \circ \psi^{-1}$ is $(-\infty, a)$ where $a = \infty$ if $\psi(u) \to \infty$ as $u \to \infty$, and a is a positive number if ψ is bounded above. Moreover, $p \circ \psi^{-1}(z) \to \infty$ as $z \to -\infty$ and $p \circ \psi^{-1}(z) \to 0$ as $z \to a$. The function $p \circ \psi^{-1}$ differs from a nowhere increasing function by a function whose absolute value is $\leq C$.

First suppose that $\Pi(y,t) - \Pi(x,t) + g(x,y,t) \geq 0$. If $p \circ \psi^{-1}$ were nowhere increasing, then $p \circ \psi^{-1}\big(\chi(x,t) + \Pi(y,t) - \Pi(x,t) + g(x,y,t)\big) \leq p \circ \psi^{-1}\big(\chi(x,t)\big) \equiv \Pi_t(x,t)$. Under our modification of the requirement that $p \circ \psi^{-1}$ be nowhere increasing, we obtain instead from (4.12) that

$$\Pi_t(y,t) \leq \Pi_t(x,t) + C \quad \text{when} \quad \Pi(y,t) - \Pi(x,t) + g(x,y,t) \geq 0. \tag{4.13}$$

Likewise,

$$\Pi_t(y,t) \geq \Pi_t(x,t) - C \quad \text{when} \quad \Pi(y,t) - \Pi(x,t) + g(x,y,t) \leq 0. \tag{4.14}$$

Since we are limiting our attention to classical solutions, the function $t \mapsto \Pi(y,t) - \Pi(x,t) + g(x,y,t)$ is continuous for any fixed x and y. Thus the sets on which this function is positive and on which it is negative are unions of countably many disjoint open intervals. Let (t_1, t_2) be one such interval on which $\Pi(y,t) - \Pi(x,t) + g(x,y,t)$ is positive, say. Then the integration of $(4.13)_1$ over (t_1, t) yields

$$\Pi(y,t) \leq \Pi(y,t_1) - \Pi(x,t_1) + \Pi(x,t) + C(t-t_1) = \Pi(x,t) + g(x,y,t_1) + C(t-t_1) \tag{4.15}$$

for $t \in (t_1, t_2)$. Thus

$$-g(x,y,t) \le \Pi(y,t) - \Pi(x,t) \le g(x,y,t_1) + C(t-t_1) \tag{4.16}$$

for $t \in (t_1, t_2)$. Accounting for the set where $\Pi(y,t) - \Pi(x,t) + g(x,y,t) = 0$, we thus find that

$$|\Pi(y,t) - \Pi(x,t)| \le \Gamma(t). \tag{4.17}$$

It then follows from $(4.9)_1$ and (4.11) that

$$|\psi(u(y,t)) - \psi(u(x,t))| \le 2\Gamma(t). \tag{4.18}$$

Let δ be a small positive number. Suppose that there were a material point x and a time T at which $\psi(u(x,T)) \le \psi(1-\delta) - 2\Gamma(T)$. Then (4.18) would imply that $\psi(u(y,T)) \le \psi(1-\delta)$ for all y. This means that $u(y,T) < 1$ for all y, which contradicts the requirement (2.13). Thus $u(x,t) \ge \psi^{-1}\big(\psi(1-\delta) - 2\Gamma(t)\big)$, which is (4.6). Likewise, suppose there were a material point x and a time T at which $\psi(u(x,T)) \ge \psi(1+\delta) + 2\Gamma(T)$. We could only be assured that such an inequality could hold when $\psi(u) \to \infty$ as $u \to \infty$. Then (4.18) would imply that $\psi(u(y,T)) \ge \psi(1+\delta)$ for all y, in contradiction to (2.13). □

For the usual case that $\psi(u) = \ln u^\nu$ and $p(u) = Ku^{-\sigma}$, we can give a more computational proof of this theorem: We define

$$\Pi(x,t) := K \int_0^t u(x,s)^{-\sigma} ds \tag{4.19}$$

so that $u(x,t) = [K^{-1}\Pi_t(x,t)]^{-1/\sigma}$. Then (4.2) becomes

$$\frac{\nu}{\sigma} \ln(K^{-1}\Pi_t(y,t)) + \Pi(y,t) = \frac{\nu}{\sigma} \ln(K^{-1}\Pi_t(x,t)) + \Pi(x,t) - g(x,y,t) \tag{4.20}$$

which is equivalent to

$$\ln\left(K^{-1} e^{\sigma\Pi(y,t)/\nu} \Pi_t(y,t)\right) = \ln\left(K^{-1} e^{\sigma\Pi(x,t)/\nu} \Pi_t(x,t) e^{\sigma g(x,y,t)/\nu}\right). \tag{4.21}$$

Let us set $H(x,t) := e^{\sigma\Pi(x,t)/\nu}$, so that (4.21) is equivalent to

$$H_t(y,t) = H_t(x,t) e^{\sigma g(x,y,t)/\nu}. \tag{4.22}$$

Note that (4.19) implies that $H(x,t)$ and $H_t(x,t)$ are positive for all x and t.

For any fixed x and y, the mapping $t \mapsto (H(x,t), H(y,t))$ defines a curve in the plane that starts at $(1,1)$ and, as a consequence of (4.22), moves rightward and upward. The integration of (4.22) with respect to time from 0 to t yields

$$H(y,t) - 1 = \int_0^t H_t(x,s) e^{\sigma g(x,y,s)/\nu} ds \le e^{\sigma\Gamma(t)/\nu} \int_0^t H_t(x,s)\, ds = e^{\sigma\Gamma(t)/\nu}[H(x,t) - 1], \tag{4.23}$$

so that

$$H(y,t) \le e^{\sigma \Gamma(t)/\nu} H(x,t). \tag{4.24}$$

Conditions (4.2), (4.19), and (4.24) imply that

$$\begin{aligned} \sigma|\psi(u(y,t)) - \psi(u(x,t))| &= \nu|\ln H(y,t) - \ln H(x,t) + \sigma g(x,y,t)/\nu| \\ &= \nu \left| \ln \frac{H(y,t) e^{\sigma g(x,y,t)/\nu}}{H(x,t)} \right| \le \nu \ln e^{2\sigma \Gamma(t)/\nu} = 2\sigma \Gamma(t), \end{aligned} \tag{4.25}$$

which is (4.18). Note that this $\psi(u) \to \infty$ as $u \to \infty$.

Note that Theorem 4.1 is valid for $\varepsilon = 0$. It is a much easier exercise to show that the vanishing of u at one point is impossible. But for the purpose of analyzing our initial-boundary-value problem, we need the a priori estimate (4.6). The estimate (4.9) is helpful but not crucial because it does not correspond to singular behavior in the problem.

The estimate (4.6) supports a global existence and regularity theorem for our initial-boundary-value problem for any $\varepsilon > 0$. The proof of this theorem follows by a straightforward adaptation of any of several works [1, 2, 5, 6, 10, 14, 19] devoted to systems of the form (2.1), (2.7) (or to generalizations thereof) with different boundary conditions, but subject to the requirements that $u > 0$ and that the system becomes singular at $u = 0$.

5. The reduced problem. We now study the dependence of solutions of the system (2.1), (2.7), (2.8), (2.11)–(2.14) on the small parameter ε. We first study the *reduced* problem obtained by setting $\varepsilon = 0$ in this system: From (2.1) and (2.7) we obtain

$$\begin{aligned} -p(u(\xi,t)) + \psi'(u(\xi,t))v_\xi(\xi,t) &= -p(u(0-,t)) + \psi'(u(0-,t))v_\xi(0-,t) =: h_{\mathrm{L}}(t), \\ -p(u(\eta,t)) + \psi'(u(\eta,t))v_\eta(\eta,t) &= -p(u(0+,t)) + \psi'(u(0+,t))v_\eta(0+,t) =: h_{\mathrm{R}}(t). \end{aligned} \tag{5.1}$$

In view of (2.11), these equations are equivalent to

$$\begin{aligned} v(\xi,t) &= \int_{-1}^{\xi} \frac{p(u(x,t))}{\psi'(u(x,t))}\,dx + h_{\mathrm{L}}(t) \int_{-1}^{\xi} \frac{1}{\psi'(u(x,t))}\,dx, \\ -v(\eta,t) &= \int_{\eta}^{1} \frac{p(u(x,t))}{\psi'(u(x,t))}\,dx + h_{\mathrm{R}}(t) \int_{\eta}^{1} \frac{1}{\psi'(u(x,t))}\,dx. \end{aligned} \tag{5.2}$$

We evaluate (5.2) at $\xi = 0$ and $\eta = 0$ to get alternative expressions for h_{L} and h_{R}:

$$\begin{aligned} h_{\mathrm{L}}(t) = \quad \tilde{h}_{\mathrm{L}}[u(\cdot,t), v(0,t)] :&= \frac{v(0,t) - \int_{-1}^{0} \frac{p(u(x,t))}{\psi'(u(x,t))}\,dx}{\int_{-1}^{0} \frac{1}{\psi'(u(x,t))}\,dx}, \\ -h_{\mathrm{R}}(t) = -\tilde{h}_{\mathrm{R}}[u(\cdot,t), v(0,t)] :&= \frac{v(0,t) + \int_{0}^{1} \frac{p(u(x,t))}{\psi'(u(x,t))}\,dx}{\int_{0}^{1} \frac{1}{\psi'(u(x,t))}\,dx}. \end{aligned} \tag{5.3}$$

Using (2.1), we replace (5.1) and (2.8) with

$$
\begin{aligned}
u_t(\xi,t) &= \frac{\tilde{h}_{\mathrm{L}}[u(\cdot,t),v(0,t)]+p(u(\xi,t))}{\psi'(u(\xi,t))},\\
u_t(\eta,t) &= \frac{\tilde{h}_{\mathrm{R}}[u(\cdot,t),v(0,t)]+p(u(\eta,t))}{\psi'(u(\eta,t))},\\
v_t(0,t) &= -h_{\mathrm{L}}(t)+h_{\mathrm{R}}(t)+f(t)
\end{aligned}
\tag{5.4}
$$

where we have employed (2.1). These equations form a system of "ordinary differential equations" for $u(\xi,\cdot), u(\eta,\cdot), v(0,\cdot)$ for $\xi \in [-1,0]$ and $\eta \in [0,1]$. (Note that (5.3) and (5.4) imply that $\int_{-1}^{1} u_t(x,t)\,dx = 0$, in consonance with (2.13).) These are subject to the relevant initial conditions of the form (2.14), namely,

$$
u(x,0) = \bar{u}(x), \qquad v(0,0) = \bar{v}(0). \tag{5.5}
$$

We integrate (5.4) from 0 to t and use (2.14) to get corresponding integral equations for u and $v(0,\cdot)$:

$$
u(\xi,t) = \bar{u}(\xi) + \int_0^t \frac{\tilde{h}_{\mathrm{L}}[u(\cdot,s),v(0,s)]+p(u(\xi,s))}{\psi'(u(\xi,s))}\,ds, \tag{5.6}
$$

etc. Note that (2.15) ensures that solutions of the integral equations (5.6) satisfy (2.13), so we need not make any provision for this condition.

Let (2.4) and (4.7) hold, so that the bounds (4.6) and (4.8) are valid. Let us fix any positive time T. We regard the right-hand sides of (5.6) as defining an operator taking $C([-1,0]\times[0,T])\times C([0,1]\times[0,T])\times C([0,T])$ into itself. Here $C([-1,0]\times[0,T])$ denotes the space of functions with values $u(\xi,t)$. Note that (4.8) prevents $\psi'(u(\xi,s))$ from approaching 0. A standard argument based on the Contraction Mapping Principle [27, 28] implies that (5.6) has a unique solution for $t \in [0,T]$ provided T is small enough. The a priori bound on $v(0,\cdot)$ given by (3.8) and the upper and lower bounds on u given in Section 4 imply that if $\psi(u) \to -\infty$ as $u \searrow 0$ and $\psi(u) \to \infty$ as $u \to \infty$, then this solution can be continued up to time T, and since T is arbitrary, the solution is globally defined. (See [13, Chap. A.II] or [20, Chap. 6] for discussions of continuation theorems.) Hence

THEOREM 5.1. Let $\psi(u) \to -\infty$ as $u \searrow 0$ and $\psi(u) \to \infty$ as $u \to \infty$. Then the reduced problem has a globally defined classical solution.

Note that we could have used the invertibility of ψ to replace the variable u with $\psi(u)$ to simplify (5.1), but doing so complicates (2.1) and its consequences in this section.

6. The impossibility for the motion of the piston to be governed by the standard ordinary differential equation. For the reduced problem, the gas has no inertia, so we might expect that it merely transmits elastic and viscous forces to the piston. We accordingly ask whether the motion of the piston for the reduced problem can be governed by a (nonlinear) ordinary differential equation of traditional type for the mass on a (nonlinearly elastic) spring (with internal dissipation) for any initial conditions governing the piston. Let us express (2.8) as

$$w_{tt}(0,t) = -[\![p(w_x)]\!](0,t) + [\![\psi'(w_x)w_{xt}]\!](0,t) + f(t). \tag{6.1}$$

This would be an ordinary differential equation for $w(0,\cdot)$ of the usual form if we could "cancel" the x-derivatives, i.e., if there were a function $w \mapsto j(w)$ such that

$$w_x(0\pm,t) = j(w(0,t)), \quad w_{xt}(0\pm,t) = j_w(w(0,t))w_t(0,t). \tag{6.2}$$

We substitute (6.2) into (5.2) evaluated at $\xi = 0 = \eta$, $t = 0$, and denote $v(0,0) \equiv w_t(0,0)$ by μ and $w(0,0)$ by λ, obtaining

$$\begin{aligned} \mu &= \int_{-1}^{0} \frac{p(\bar{u}(x))}{\psi'(\bar{u}(x))}\,dx + [-p(j(\lambda)) + \psi'(j(\lambda))j_w(\lambda)\mu] \int_{-1}^{0} \frac{1}{\psi'(\bar{u}(x))}\,dx, \\ -\mu &= \int_{0}^{1} \frac{p(\bar{u}(x))}{\psi'(\bar{u}(x))}\,dx + [-p(j(\lambda)) + \psi'(j(\lambda))j_w(\lambda)\mu] \int_{0}^{1} \frac{1}{\psi'(\bar{u}(x))}\,dx. \end{aligned} \tag{6.3}$$

Since we allow any initial conditions on the piston, these equations must hold for all μ and all λ. Since $\psi'(u) > 0$ for all u, the differentiation of (6.3) with respect to μ yields the contradiction that $j_w(\lambda)$ must be both positive and negative. Thus we conclude that *the motion of the piston in the reduced problem cannot be governed by a standard second-order ordinary differential equation valid for all initial conditions on the piston.* Indeed, if we treat $v(0,t)$ as a given function in $(5.4)_{1,2}$, then the solution u of any initial-value problem for this system would depend on the past history of $v(0,t)$, so that the substitution of this u into $(5.4)_3$ would convert it into a functional-ordinary differential equation with memory. (See [3] for related results.)

This conclusion is a consequence of the presence of the viscosity (cf. [3]). This viscosity, however, will enable us to produce a meaningful ordinary differential equation to approximate (5.4) and to give a precise mathematical position to the reduced problem.

7. Bounds for the solution of the reduced problem. Our immediate objective is to relate the reduced problem (5.4) to an appropriate ordinary differential equation. Specifically, we shall show in the next section that the dynamical system generated by (5.4) has a global attractor, which is contained in a finite-dimensional invariant manifold on which the dynamics is governed by a second-order ordinary differential equation like that governing the motion of a damped spring. For this purpose, we shall employ Theorem 7.9, which asserts the existence of uniform upper and lower bounds for solutions of (5.4). The proof of this theorem depends on a complicated sequence of elementary lemmas.

We strengthen several of our constitutive hypotheses solely for the analysis in this and the next section: We assume that

$$\psi'(u) \equiv \nu(u)/u \to \infty \text{ as } u \searrow 0, \qquad \psi(u) \to \infty \text{ as } u \to \infty, \qquad \psi''(u) < 0, \tag{7.1}$$

$$p'(u) \le 0 \text{ for all } u \in (0,\infty), \quad \varphi(u) \to \infty \text{ as } u \searrow 0. \tag{7.2}$$

Condition $(7.1)_1$ is a consequence of (2.4) (and (2.24)). Condition $(7.1)_2$ was invoked in (4.7). Conditions (7.2) are standard; cf. (2.5) and (3.2). We also strengthen our hypotheses on the external force f by assuming that f has period 1 and mean value

0. This periodicity is not essential but simplifies the dynamical-systems theory we shall apply. (We could get by with the assumption that $|\int_0^t f(\tau)\,d\tau| \le C$.) These assumptions are tacitly assumed to hold in this and the next section.

Our immediate goal is to prove several lemmas that lead to a uniform boundedness result for solutions of (5.4). The first lemma provides estimates on the evolution of the difference between the specific volumes at two material sections.

Lemma 7.1. Let (u, v) satisfy (5.4), (5.5) and let ξ_1, $\xi_2 \in [-1, 0]$. Then

$$u(\xi_1, t) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} u(\xi_2, t) \quad \forall\, t > 0 \quad \text{if} \quad \bar{u}(\xi_1) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \bar{u}(\xi_2), \tag{7.3}$$

$$\text{there are numbers } \xi^+,\ \xi^- \in [-1, 0] \text{ such that} \\ u(\xi^-, t) \le u(\xi, t) \le u(\xi^+, t) \quad \forall\, \xi \in [-1, 0], \quad \forall\, t \ge 0, \tag{7.4}$$

and

$$\begin{aligned} &\frac{d}{dt}\big\{\psi(u(\xi_2, t)) - \psi(u(\xi_1, t))\big\} \\ &\qquad = \left[\int_0^1 p'(\theta u(\xi_2, t) + (1-\theta) u(\xi_1, t))\, d\theta\right] [u(\xi_2, t) - u(\xi_1, t)]. \end{aligned} \tag{7.5}$$

Analogous results hold for $u(\eta, \cdot)$ with $\eta \in [0, 1]$.

Proof. For any fixed ξ_1, $\xi_2 \in [-1, 0]$, equation $(5.4)_1$ implies that $u(\xi_1, \cdot)$ and $u(\xi_2, \cdot)$ each satisfy

$$\dot{z}(t) = \frac{h_{\mathrm{L}}(t) + p(z(t))}{\psi'(z(t))} \tag{7.6}$$

on $[0, \infty)$. Since $(7.1)_2$ implies (4.8), we need not worry about $\psi'(z(t))$ approaching 0. We treat h_{L} as a given function of t, which is valid because of Theorem 5.1 on existence of the solution of the reduced problem. Thus the usual uniqueness theory for ordinary differential equations implies (7.3). The existence of ξ^+, $\xi^- \in [-1, 0]$ such that (7.4) holds follows immediately from (7.3). Equation $(5.4)_1$ and the Mean Value Theorem imply that

$$\begin{aligned} &\frac{d}{dt}\big\{\psi(u(\xi_2, t)) - \psi(u(\xi_1, t))\big\} \\ &\qquad \equiv \psi'(u(\xi_2, t)) u_t(\xi_2, t) - \psi'(u(\xi_1, t)) u_t(\xi_1, t) \\ &\qquad = p(u_t(\xi_2, t)) - p(u_t(\xi_1, t)) \\ &\qquad \equiv \left[\int_0^1 p'(\theta u(\xi_2, t) + (1-\theta) u(\xi_1, t))\, d\theta\right] [u(\xi_2, t) - u(\xi_1, t)]. \end{aligned} \tag{7.7}$$

(Note that the integral of (7.7) over time from 0 to t leads to an easy alternative proof of (7.3) and (7.4).) □

For a solution (u, v) to (5.4), the next lemma, a specialization of Theorem 4.1, bounds u uniformly from above for all $t \ge 0$.

Lemma 7.2. Let (u, v) satisfy (5.4), (5.5) on $[0, \infty)$. Let u_*, u^* be constants such that $0 < u_* \le \bar{u}(\xi) \le u^*$ for all $x \in [-1, 0]$. Then there is a constant U^* that depends only on u_*, u^* such that $u(\xi,t) \le U^*$ for all $\xi \in [-1,0]$ and for all $t \ge 0$. An analogous result holds for u on $[0,1]$.

Proof. Because u is nowhere negative and $\int_{-1}^1 u(x,t)\,dx = 2$ for all $t \ge 0$, inequality (7.4) implies that

$$2 \ge \int_{-1}^0 u(\xi,t)\,d\xi \ge u(\xi^-,t). \tag{7.8}$$

We define $H(\xi,t) := \psi(u(\xi,t)) - \psi(u(\xi^-,t))$ for $\xi \in [-1,0]$. By (7.5), $(7.2)_1$, and (7.4), $H_t \le 0$ and hence $H(\xi,t) \le H(\xi,0)$ for all $t \ge 0$. Note that we can bound $H(\xi,0)$ by a constant depending only on u_*, u^*. Since ψ is increasing, (7.8) implies that

$$H(\xi,0) \ge \psi(u(\xi,t)) - \psi(u(\xi^-,t)) \ge \psi(u(\xi,t)) - \psi(2). \tag{7.9}$$

Hypothesis $(7.1)_2$ completes the proof. □

We still need a uniform lower bound for u and a uniform bound on $|v|$. To establish these, we need several additional lemmas. After the first two, which are technical, follow several lemmas used in Theorem 7.9 to make phase-plane arguments that lead to uniform bounds.

LEMMA 7.3. Let (u,v) satisfy(5.4), (5.5). Let u_*, u^* be constants such that $0 < u_* \le \bar{u}(\xi) \le u^*$ for all $\xi \in [-1,0]$. Then there is a function $\mu \mapsto \hat{\delta}(\mu)$ depending only on u_*, u^* with $\hat{\delta}(\mu) \to 0$ as $\mu \searrow 0$ such that if $u(\xi^-,t) \le \mu$, then $u(\xi^+,t) - u(\xi^-,t) \le \hat{\delta}(\mu)$. An analogous result holds for u on $[0,1]$.

Proof. We define $\tilde{H}(\mu,\delta) := \psi(\mu+\delta) - \psi(\mu)$ for μ, $\delta > 0$. The properties (7.1) of ψ imply that

$$\tilde{H}(\mu,\delta) \to \begin{Bmatrix} \infty \\ 0 \end{Bmatrix} \text{ as } \begin{Bmatrix} \mu \\ \delta \end{Bmatrix} \searrow 0 \quad \text{for fixed} \quad \begin{Bmatrix} \delta \\ \mu \end{Bmatrix} > 0, \qquad \tilde{H}_\mu < 0, \qquad \tilde{H}_\delta > 0. \tag{7.10}$$

Since $\tilde{H}(\mu,\delta) \to \infty$ as $\delta \to \infty$ and $\tilde{H}(\mu,\delta) \to 0$ as $\delta \to 0$, the equation

$$\tilde{H}(\mu,\delta) = \tilde{H}(u_*, u^* - u_*) \tag{7.11}$$

for δ has a solution for each $\mu > 0$. The positivity of $\tilde{H}_\delta$ ensures that this solution is unique. We denote it by $\hat{\delta}(\mu)$. It satisfies $\tilde{H}(\mu,\hat{\delta}(\mu)) = \tilde{H}(u_*, u^* - u_*)$. The Local Implicit-Function Theorem then implies that $\hat{\delta}$ is continuously differentiable with $\hat{\delta}'(\mu) > 0$ for all $\mu > 0$, $\hat{\delta}(\mu) \to 0$ as $\mu \searrow 0$, and

$$\tilde{H}(\mu,\delta) \le \tilde{H}(\mu,\hat{\delta}(\mu)) \quad \text{if and only if} \quad 0 \le \delta \le \hat{\delta}(\mu). \tag{7.12}$$

Now we suppose that $u(\xi^-,t) \le \mu \le u_*$. We set

$$H(t) := \psi(u(\xi^+,t)) - \psi(u(\xi^-,t)) \equiv \tilde{H}(u(\xi^-,t), u(\xi^+,t) - u(\xi^-,t)). \tag{7.13}$$

By (7.5), (7.12), (7.11), and the hypotheses of this lemma, we obtain

$$\begin{aligned} H(t) &\equiv \tilde{H}(u(\xi^-,t), u(\xi^+,t) - u(\xi^-,t)) \\ &\le H(0) \equiv \tilde{H}(\bar{u}(\xi^-), \bar{u}(\xi^+) - \bar{u}(\xi^-)) \le \tilde{H}(u_*, u^* - u_*) \\ &= \tilde{H}(u(\xi^-,t), \hat{\delta}(u(\xi^-,t))). \end{aligned} \tag{7.14}$$

Finally, another application of (7.12) yields

$$0 \le u(\xi^+, t) - u(\xi^-, t) \le \hat{\delta}(u(\xi^-, t)) \le \hat{\delta}(\mu). \tag{7.15}$$

□

Lemma 7.4. Let (u, v) satisfy (5.4). Let $\hat{\delta}$ be the function from Lemma 7.3, and let μ_{L} and μ_{R} be positive constants that satisfy

$$\mu_{\mathrm{L}} + \hat{\delta}(\mu_{\mathrm{L}}) + \mu_{\mathrm{R}} + \hat{\delta}(\mu_{\mathrm{R}}) < 2. \tag{7.16}$$

If $u(\xi^-, t) \le \mu_{\mathrm{L}}$, then $u(\eta, t) \ge \mu_{\mathrm{R}}$ for all $\eta \in [0, 1]$. An analogous statement holds with the roles of ξ and η reversed.

Proof. If $u(\xi^-, t) \le \mu_{\mathrm{L}}$, then Lemma 7.3 implies that $u(\xi, t) \le \mu_{\mathrm{L}} + \hat{\delta}(\mu_{\mathrm{L}})$ for all $\xi \in [-1, 0]$. Hence (2.13) implies that

$$\int_0^1 u(\eta, t)\, d\eta = 2 - \int_{-1}^0 u(\xi, t)\, d\xi \ge 2 - (\mu_{\mathrm{L}} + \hat{\delta}(\mu_{\mathrm{L}})). \tag{7.17}$$

Now suppose that the conclusion were false, so that there would be an $\tilde{\eta} \in [0, 1]$ with $u(\tilde{\eta}, t) < \mu_{\mathrm{R}}$. Lemma 7.3 would then imply that

$$\int_0^1 u(\eta, t)\, d\eta \le \mu_{\mathrm{R}} + \hat{\delta}(\mu_{\mathrm{R}}). \tag{7.18}$$

Inequalities (7.16) and (7.17) would then be contradictory. □

Lemma 7.5. Let (u, v) satisfy (5.4).

$$\begin{aligned} &\text{If } v(0, t) \ge 0, \text{ then } u_t(\xi^-, t) \ge 0 \text{ and } u_t(\eta^+, t) \le 0. \\ &\text{If } v(0, t) \le 0, \text{ then } u_t(\xi^+, t) \le 0 \text{ and } u_t(\eta^-, t) \ge 0. \end{aligned} \tag{7.19}$$

Proof. Since $v(0, t) = \int_{-1}^0 u_t(\xi, t)\, d\xi$ by (2.1), the nonnegativity of $v(0, t)$ implies that there is a $\xi \in [-1, 0]$ such that $u_t(\xi, t) \ge 0$. Were $u_t(\xi^-, t) < 0$, then

$$\frac{d}{dt}\{\psi(u(\xi, t)) - \psi(u(\xi^-, t))\} = \psi'(u(\xi, t))u_t(\xi, t) - \psi'(u(\xi^-, t))u_t(\xi^-, t) > 0, \tag{7.20}$$

in contradiction to (7.4) and (7.5). The proofs of the other cases are similar. □

Lemma 7.6. Let (u, v) satisfy (5.4). If $v(0, t) \le 0$, then there is a sufficiently small $\mu_{\mathrm{L}} > 0$ such that $v_t(0, t) > 0$ when $u(\xi^-, t) \le \mu_{\mathrm{L}}$. If $v(0, t) \ge 0$, then there is a sufficiently small $\mu_{\mathrm{R}} > 0$ such that $v_t(0, t) < 0$ when $u(\eta^-, t) \le \mu_{\mathrm{R}}$.

Proof. Suppose that $v(0, t) \le 0$ and that $u(\xi^-, t) \le \mu_{\mathrm{L}}$ for some $\mu_{\mathrm{L}} > 0$. By Lemma 7.5, $u_t(\xi^+, t) \le 0$, and by Lemma 7.3, $u(\xi^+, t) \le \mu_{\mathrm{L}} + \hat{\delta}(\mu_{\mathrm{L}})$. Hence $(5.4)_1$ and $(7.2)_1$ imply that

$$0 \ge u_t(\xi^+, t)\psi'(u(\xi^+, t)) = h_{\mathrm{L}}(t) + p(u(\xi^+, t)) \ge h_{\mathrm{L}}(t) + p(\mu_{\mathrm{L}} + \hat{\delta}(\mu_{\mathrm{L}})). \tag{7.21}$$

Likewise, $u_t(\eta^-, t) \ge 0$ when $v(0, t) \le 0$, and therefore $h_{\mathrm{R}}(t) + p(u(\eta^-, t)) \ge 0$. Let us now fix a μ_{R} such that $0 < \mu_{\mathrm{R}} + \hat{\delta}(\mu_{\mathrm{R}}) < 2$. Then Lemma 7.4 implies that there is a μ_{L} so small that $u(\eta^-, t) \ge \mu_{\mathrm{R}}$ when $u(\xi^-, t) \le \mu_{\mathrm{L}}$. Thus

$$0 \le h_{\mathrm{R}}(t) + p(u(\eta^-, t)) \le h_{\mathrm{R}}(t) + p(\mu_{\mathrm{R}}). \tag{7.22}$$

Conditions $(5.4)_3$, (7.21), and (7.22) yield

$$v_t(0,t) = -h_{\mathrm{L}}(t) + h_{\mathrm{R}}(t) + f(t) \geq p(\mu_{\mathrm{L}} + \hat{\delta}(\mu_{\mathrm{L}})) - p(\mu_{\mathrm{R}}) - C. \tag{7.23}$$

Since $\mu + \hat{\delta}(\mu) \searrow 0$ as $\mu \to 0$, condition $(2.3)_2$ implies that the right-hand side of (7.23) is positive for μ_{L} sufficiently small. The proof of the other case is analogous. □

LEMMA 7.7. Let (u, v) satisfy (5.4) and let U^* be a constant such that $u(x,t) \leq U^*$ for all $x \in [-1,1]$ and for all $t \geq 0$. (Lemma 7.2 shows that there is such a U^* for every reasonable set of initial conditions.) Let $\mu_{\mathrm{L}} > 0$ and let t_0, t_1 be times with $t_1 > t_0$ such that $u(\xi^-, t_0) = \mu_{\mathrm{L}}$ and such that $u(\xi^-, t) \leq \mu_{\mathrm{L}}$ and $v(0,t) < 0$ for all $t \in [t_0, t_1)$. If μ_{L} is sufficiently small, then there is a $U_* > 0$ that depends only on μ_{L}, $v(0,t)$, and U^* such that $u(\xi^-, t) \geq U_*$ for all $t \in [t_0, t_1]$. An analogous statement holds with η^- replacing ξ^-.

Proof. System (5.4) implies that

$$\begin{aligned} v_t(0,t) = &- u_t(\xi^-, t)\psi'(u(\xi^-, t)) + p(u(\xi^-, t)) \\ &+ u_t(\eta^+, t)\psi'(u(\eta^+, t)) - p(u(\eta^+, t)) + f(t). \end{aligned} \tag{7.24}$$

We integrate (7.24) with respect to time from t_0 to $t \in [t_0, t_1]$ to get

$$\begin{aligned} v(0,t) - v(0,t_0) = &- \psi(u(\xi^-, t)) + \psi(u(\xi^-, t_0)) + \int_{t_0}^{t} p(u(\xi^-, s))\, ds \\ &+ \psi(u(\eta^+, t)) - \psi(u(\eta^+, t_0)) - \int_{t_0}^{t} p(u(\eta^+, s))\, ds + \int_{t_0}^{t} f(t)\, ds. \end{aligned} \tag{7.25}$$

We fix μ_{R} so small that $\mu_{\mathrm{R}} + \hat{\delta}(\mu_{\mathrm{R}}) < 2$. Lemma 7.4 implies that there is a μ_{L} so small that $\mu_{\mathrm{R}} \leq u(\eta^+, t) \leq U^*$ for all $t \in [t_0, t_1]$, the second inequality holding by hypothesis. Thus

$$\psi(u(\eta^+, t)) - \psi(u(\eta^+, t_0)) \geq \psi(\mu_{\mathrm{R}}) - \psi(U^*) \geq -C. \tag{7.26}$$

Note that C depends only on μ_{R}, which in turn depends on μ_{L} and on U^*. Next,

$$\int_{t_0}^{t} p(u(\xi^-, s))\, ds - \int_{t_0}^{t} p(u(\eta^+, s))\, ds \geq \int_{t_0}^{t} (p(\mu_{\mathrm{L}}) - p(\mu_{\mathrm{R}}))\, ds > 0, \tag{7.27}$$

where we ensure the final inequality by choosing $\mu_{\mathrm{L}} < \mu_{\mathrm{R}}$. The substitution of the estimates (7.26) and (7.27) into equation (7.25) yields $-v(0,t_0) \geq v(0,t) - v(0,t_0) \geq -\psi(u(\xi^-, t)) + \psi(\mu_{\mathrm{L}}) - C$, i.e.,

$$\psi(u(\xi^-, t)) \geq v(0, t_0) + \psi(\mu_{\mathrm{L}}) - C. \tag{7.28}$$

Since $\psi(u) \to -\infty$ as $u \searrow 0$, it follows that $u(\xi^-, t) \geq U_*$, where U_* depends only on $v(0, t_0)$, μ, and U^*. □

LEMMA 7.8. Let (u, v) satisfy (5.4). Let u_*, u^*, and v^* be positive constants. Let t_0 be a time such that $|v(0,t_0)| = v^*$, let ρ be a positive number such that $|v(0,t)| > v^*$ for $t \in (t_0, t_0 + \rho)$, and let $u_* \leq u(x, t_0) \leq u^*$ for all $x \in [-1, 1]$. If v^* is sufficiently large, then there is a first time $t_1 \in (t_0, \infty)$ such that $|v(0, t_1)| = v^*$, and there is a constant V^* depending only on u_*, u^*, and v^* such that $|v(0,t)| \leq V^*$ for all $t \in [t_0, t_1]$.

Proof. Lemma 7.2 implies that there is a constant U^* such that $u(x,t) \le U^*$ for all $x \in [-1,1]$ and for all $t \ge 0$. We define the total energy function E by

$$E(u,v) = \frac{v^2}{2} + \int_{-1}^{1} \int_{u}^{U^*} p(z)\,dz\,dx. \tag{7.29}$$

The function E is the sum of the kinetic energy of the piston and the potential energy of the gas (which has no kinetic energy). For any solution (u,v) of (5.4), $E(u(\cdot,t),v(0,t)) \ge 0$ and $E(u(\cdot,t_0),v(0,t_0)) \le E^*$, where E^* depends only on u_*, u^*, and v^*.

Because u is bounded above by U^*, condition $(7.1)_1$ implies that there is a constant $c > 0$ such that $\psi'(u(x,t)) \ge c > 0$. Thus, (3.5) with $\varepsilon = 0$ and (2.1) imply that

$$\begin{aligned} \frac{d}{dt}E(u(\cdot,t),v(0,t)) &\le -c\int_{-1}^{1} u_t(x,t)^2\,dx + v(0,t)f(t) \\ &\le -c\left[\int_{-1}^{0} u_t(\xi,t)\,d\xi\right]^2 - c\left[\int_{0}^{1} u_t(\eta,t)\,d\eta\right]^2 + v(0,t)f(t) \\ &= -2c\;v(0,t)^2 + v(0,t)f(t). \end{aligned} \tag{7.30}$$

Since f is bounded, the last line in (7.30) is less than -1, say, for $|v(0,t)| > v^*$ with v^* sufficiently large. Thus for t in a maximal interval of the form (t_0,t_1) on which $v(0,t) < -v^*$, the energy must satisfy $E(u(\cdot,t),v(0,t)) \le E(u(\cdot,t_0),v(0,t_0)) + t_0 - t$, so that

$$\tfrac{1}{2}v(0,t)^2 \le E^* + t_0 - t. \tag{7.31}$$

Hence, if $v(0,\cdot)$ leaves the interval $[-v^*,v^*]$, then it must return in finite time and, while not in the interval, it must satisfy $\frac{1}{2}v(0,t)^2 \le E^*$. □

Next we prove the fundamental

THEOREM 7.9 (Uniform Boundedness). Let (7.1) and (7.2) hold. If there are positive constants u_*, u^*, and v^* such that $u_* \le \bar{u}(x) \le u^*$ for all $x \in [-1,1]$ and $|\bar{v}(0)| \le v^*$ and if (u,v) satisfies (5.4) and (5.5), then there are positive constants U_*, U^*, and V^* depending only on u_*, u^*, v^* such that $U_* \le u(x,t) \le U^*$ and $|v(0,t)| \le V^*$ for all $x \in [-1,1]$ and all $t \ge 0$.

Proof. Lemma 7.2 says that there is a U^* such that $u(x,t) \le U^*$ for all $x \in [-1,1]$ and for all $t \ge 0$. We choose a positive constant $\mu < u_*, \mu_{\mathrm{L}}, \mu_{\mathrm{R}}$ where $\mu_{\mathrm{L}}, \mu_{\mathrm{R}}$ are the constants appearing in Lemmas 7.6 and 7.7. Thus the conclusions of these lemmas hold. We can choose v^* so large that the conclusion of Lemma 7.8 holds and then choose V^* so that if the solution v leaves the strip $|v| \le v^*$ with $\mu \le u(x,t) \le U^*$ for all $x \in [-1,1]$, then $|v| \le V^*$ until the solution crosses back into $|v| \le v^*$. Note that μ, U^*, and V^* depend only on u_*, u^*, and v^*.

We now find a $U_* > 0$ depending only on μ, U^*, and V^* such that $u(x,t) \ge U_*$ for all $x \in [-1,1]$ and all $t \ge 0$. Once this is done, we use Lemma 7.8 and choose V^* larger if necessary to conclude that $|v(0,t)| \le V^*$ for all $t \ge 0$.

Our strategy is to study the evolution of the curves

$$t \mapsto (\mathsf{u}_{\mathrm{L}}(t), \mathsf{v}(t)) := (u(\xi^-,t), v(0,t)), \qquad t \mapsto (\mathsf{u}_{\mathrm{R}}(t), \mathsf{v}(t)) := (u(\eta^-,t), v(0,t))$$

in the (u, v) phase plane. We need only find U_* so that these curves stay to the right of the line $\{\mathsf{u} = U_*\}$ in this plane.

Let t_0 be the first time at which either $(\mathsf{u}_{\rm L}, \mathsf{v})$ or $(\mathsf{u}_{\rm R}, \mathsf{v})$ hits the line $\{\mathsf{u} = \mu\}$. (If there is no such time t_0, then we are done.) Suppose that $v(0, t_0) < 0$, i.e., that both $(\mathsf{u}_{\rm L}(t), \mathsf{v}(t))$ and $(\mathsf{u}_{\rm R}(t), \mathsf{v}(t))$ lie below the u-axis. By Lemma 7.5, $(\mathsf{u}_{\rm R}, \mathsf{v})$ moves from left to right below the u-axis, and hence it must be $(\mathsf{u}_{\rm L}, \mathsf{v})$ that hits $\{\mathsf{u} = \mu\}$ at t_0. Because $\mu \leq \mathsf{u}_{\rm L} \leq U^*$ for all $t \in [0, t_0]$, Lemma 7.8 implies that $\mathsf{v}(t_0) \geq -V^*$. Now Lemma 7.7 implies that there is a constant $U_* > 0$ that depends only on μ, U^*, and V^* such that if $(\mathsf{u}_{\rm L}, \mathsf{v})$ stays below the u-axis and to the left of $\{\mathsf{u} = \mu\}$ on $[t_0, t_1)$, then $(\mathsf{u}_{\rm L}, \mathsf{v})$ stays to the right of the line $\{\mathsf{u} = U_*\}$ on $[t_0, t_1)$. Lemma 7.4 with $\mu_{\rm L} = \mu_{\rm R} = \mu$ implies that $(\mathsf{u}_{\rm R}, \mathsf{v})$ remains to the right of $\{\mathsf{u} = \mu\}$ on $[t_0, t_1]$. If, instead, $v(0, t_0) > 0$, then a similar argument shows that $(\mathsf{u}_{\rm L}, \mathsf{v})$ stays to the right of the line $\{\mathsf{u} = \mu\}$ and $(\mathsf{u}_{\rm R}, \mathsf{v})$ stays to the right of the line $\{\mathsf{u} = U_*\}$.

If $t_1 = \infty$, i.e., if $(\mathsf{u}_{\rm L}, \mathsf{v})$ stays in the strip below the u-axis and to the left of $\{\mathsf{u} = \mu\}$ for all $t \geq t_1$, then we are done. Otherwise, $(\mathsf{u}_{\rm L}, \mathsf{v})$ must leave this strip at a finite time t_1 and does so by crossing $\{\mathsf{u} = \mu\}$ below the u-axis or by crossing the u-axis.

Suppose first that $(\mathsf{u}_{\rm L}, \mathsf{v})$ moves back to the right of $\{\mathsf{u} = \mu\}$ at t_1 and that $(\mathsf{u}_{\rm L}(t_1), \mathsf{v}(t_1))$ and $(\mathsf{u}_{\rm R}(t_1), \mathsf{v}(t_1))$ are below the u-axis. We let t_2 denote the next time at which either $(\mathsf{u}_{\rm L}, \mathsf{v})$ or $(\mathsf{u}_{\rm R}, \mathsf{v})$ hits the line $\{\mathsf{u} = \mu\}$. An argument like that of the previous paragraph shows that both $(\mathsf{u}_{\rm L}, \mathsf{v})$ and $(\mathsf{u}_{\rm R}, \mathsf{v})$ stay to the right of $\{\mathsf{u} = U_*\}$ until both are to the right of $\{\mathsf{u} = \mu\}$ or until both cross the u-axis.

Suppose instead that $(\mathsf{u}_{\rm L}, \mathsf{v})$ and $(\mathsf{u}_{\rm R}, \mathsf{v})$ cross the u-axis at t_1. Because $(\mathsf{u}_{\rm R}, \mathsf{v})$ is crossing the u-axis from below to above, Lemma 7.6 implies that $(\mathsf{u}_{\rm R}, \mathsf{v})$ is to the right of the line $\{\mathsf{u} = \mu\}$. Lemma 7.5 implies that as long as $(\mathsf{u}_{\rm L}, \mathsf{v})$ stays above the u-axis, $(\mathsf{u}_{\rm L}, \mathsf{v})$ moves from left to right and hence stays to the right of the line $\{\mathsf{u} = U_*\}$. Now, arguments like those of the previous two paragraphs show that as long as $(\mathsf{u}_{\rm L}, \mathsf{v})$ and $(\mathsf{u}_{\rm R}, \mathsf{v})$ stay above the u-axis, both points stay to the right of $\{\mathsf{u} = U_*\}$. If $(\mathsf{u}_{\rm L}, \mathsf{v})$ and $(\mathsf{u}_{\rm R}, \mathsf{v})$ later cross the u-axis from above to below, Lemma 7.6 implies that $(\mathsf{u}_{\rm L}, \mathsf{v})$ is to the left of $\{\mathsf{u} = \mu\}$, and we can repeat the arguments given above to control $(\mathsf{u}_{\rm L}, \mathsf{v})$ and $(\mathsf{u}_{\rm R}, \mathsf{v})$ while they remain below the u-axis. Continuing in this fashion shows that $(\mathsf{u}_{\rm L}, \mathsf{v})$ and $(\mathsf{u}_{\rm R}, \mathsf{v})$ stay to the right of $\{\mathsf{u} = U_*\}$ for all $t \geq 0$. □

8. An attracting ordinary differential equation for the reduced problem. In this section we find a global attractor lying on an invariant manifold for the dynamical system generated by (5.4). Note that the reduced problem (5.4) is obtained by first formulating the governing equations when the gas in the cylinder has inertia and then letting this inertia go to zero. The attracting ordinary differential equation (8.4), on the other hand, can be directly based on the model that the gas is an inertialess spring that serves only to transmit elastic and viscous forces to the piston. We use abstract dynamical-systems theory to prove Theorem 8.2, which connects the dynamics of (5.4) to the dynamics on the invariant manifold. (The approach of this section is analogous to that in [27], which relates the reduced problem for the motion of a heavy mass on a light nonlinearly viscoelastic rod to an ordinary differential equation describing the motion of a particle on a massless spring. However, because of both the transmission condition and the changes in constitutive equations necessary for describing a viscous gas rather than a viscoelastic solid, the treatment here differs in important ways from that of [27].)

To discover this invariant manifold, suppose that (5.4) has a solution of the form

$$u(x,t) \equiv w_x(x,t) = \begin{cases} \zeta(t) & \text{if} \quad x \in [-1,0), \\ 2-\zeta(t) & \text{if} \quad x \in (0,1]. \end{cases} \tag{8.1}$$

This function satisfies (2.13) and

$$\zeta(t) = \int_{-1}^{0} w_x(\xi,t)\,d\xi = w(0,t)+1. \tag{8.2}$$

The requirement that $|w(0,t)| < 1$ implies that $0 < \zeta(t)$ and $2-\zeta(t) < 2$ for all $t \geq 0$. Equation (5.3) yields

$$h_{\mathrm{L}}(t) = \tilde{h}_{\mathrm{L}}[\zeta(t), \zeta_t(t)] = -p(\zeta(t)) + \psi'(\zeta(t))\zeta_t(t), \tag{8.3}$$

which, along with the corresponding relation between h_{R} and $2-\zeta(t)$, shows that the first two equations in (5.4) are satisfied automatically, while the third equation in (5.4) requires that ζ satisfy

$$\begin{aligned} \zeta_{tt}(t) &= -[p(2-\zeta(t)) - p(\zeta(t))] - [\psi'(2-\zeta(t)) + \psi'(\zeta(t))]\zeta_t(t) + f(t) \\ &=: -P(\zeta(t)) - Q(\zeta(t))\zeta_t(t) + f(t). \end{aligned} \tag{8.4}$$

Note that the system of two first-order ordinary differential equations equivalent to (8.4) is defined on the strip

$$\mathcal{S} := \{(\zeta, \dot{\zeta}) \in \mathbb{R}^2 : 0 < \zeta < 2\} \tag{8.5}$$

of the $(\zeta, \dot{\zeta})$ phase plane.

We have just shown that equation (8.4) must be satisfied by any solution to (5.4) with a spatially constant specific volume. Conversely, it is straightforward to show that given ζ satisfying (8.4), one can construct a solution (u, v) to (5.4) with the property that for all $t > 0$, $u(\xi,t) = \zeta(t)$ for $\xi \in [-1,0)$ and $u(\eta,t) = 2-\zeta(t)$ for $\eta \in (0,1]$. We call such solutions *constant-volume solutions*.

To define an appropriate dynamical system, we consider (5.4) as an ordinary differential equation on the set

$$\begin{aligned} \mathcal{X} := \{(y_{\mathrm{L}}, y_{\mathrm{R}}, v) : \quad & y_{\mathrm{L}} \in C([-1,0],\mathbb{R}), \quad y_{\mathrm{R}} \in C([0,1],\mathbb{R}), \quad z \in \mathbb{R}, \\ & y_{\mathrm{L}}(\xi) > 0 \quad \forall\, \xi \in [-1,0], \quad y_{\mathrm{R}}(\eta) > 0 \quad \forall\, \eta \in [0,1], \\ & \int_{-1}^{0} y_{\mathrm{L}}(\xi)\,d\xi + \int_{0}^{1} y_{\mathrm{R}}(\eta)\,d\eta = 2\}. \end{aligned} \tag{8.6}$$

We set $y := (y_{\mathrm{L}}, y_{\mathrm{R}})$. We may regard y as a real-valued function on $[-1,0) \cup (0,1]$ by setting $y(x) = y_{\mathrm{L}}(x)$ for $x \in [-1,0)$, $y(x) = y_{\mathrm{R}}(x)$ for $x \in (0,1]$ and taking $y(0-) = y_{\mathrm{L}}(0)$ and $y(0+) = y_{\mathrm{R}}(0)$. Since y_{L} and y_{R} are required to be positive, the usual distance functions for $C([-1,0],\mathbb{R})$ and $C([0,1],\mathbb{R})$ fail to make $\mathcal{X}$ a complete metric space. One that makes it complete is

$$d\left((y^1, z^1), (y^2, z^2)\right) := \max_{x\in[-1,0]} \left| \ln\left(y_{\mathrm{L}}^1(x)/y_{\mathrm{L}}^2(x)\right) \right| + \max_{x\in[0,1]} \left| \ln\left(y_{\mathrm{R}}^1(x)/y_{\mathrm{R}}^2(x)\right) \right| + |z^1 - z^2|. \tag{8.7}$$

We endow $\mathcal{X}$ with this distance function.

For $(y, z) \in \mathcal{X}$, we define the *time*-1 map $\mathfrak{T} : \mathcal{X} \to \mathcal{X}$ by

$$\mathfrak{T}(y, z) := (u(\cdot, 1), v(0, 1)) \tag{8.8}$$

where (u, v) satisfies (5.4) with $(u(\cdot, 0), v(0, 0)) = (y, z)$. Because f has period 1, standard arguments show

$$\mathfrak{T}^m\,(u(\cdot, 0), v(0, 0)) = (u(\cdot, m), v(0, m)) \tag{8.9}$$

for all nonnegative integers m. Hence the dynamical system $(\mathcal{X}, \mathfrak{T})$ describes the dynamics of (5.4).

Now define

$$\mathcal{X}_0 := \{(y_{\mathrm{L}}, y_{\mathrm{R}}, z) \in \mathcal{X} \,:\, y_{\mathrm{L}},\, y_{\mathrm{R}} \text{ are constant functions, } 0 < y_{\mathrm{L}} < 2,\ y_{\mathrm{L}} + y_{\mathrm{R}} = 2\}. \tag{8.10}$$

We note that a point $(y_{\mathrm{L}}, y_{\mathrm{R}}, z) \in \mathcal{X}_0$ can be identified with the point $(\zeta, \dot{\zeta}) = (y_{\mathrm{L}}(0), z)$ in the subset $\mathcal{S}$ of the $(\zeta, \dot{\zeta})$-plane. We consider (5.4) with initial data in $\mathcal{X}_0$. As noted above, we can use (8.4) to construct a constant-volume solution to (5.4) satisfying these initial data. By uniqueness, it follows that $\mathcal{X}_0$ is invariant for (5.4), and if we define $\mathfrak{T}_0$ as $\mathfrak{T}$ restricted to $\mathcal{X}_0$, then the dynamical system $(\mathcal{X}_0, \mathfrak{T}_0)$ is determined by (8.4).

The following theorem enables us to exploit the relation between solutions to (5.4) and solutions to (8.4).

THEOREM 8.1. There is a compact set $\mathcal{C}_0 \subset \mathcal{S}$ with the property that for any compact set $\mathcal{K}_0 \subset \mathcal{S}$, there is a time τ that depends only on $\mathcal{K}_0$ such that if $(\zeta(\cdot), \zeta_t(\cdot))$ is a solution to (8.4) with $(\zeta(0), \zeta_t(0)) \in \mathcal{K}_0$, then $(\zeta(t), \zeta_t(t)) \in \mathcal{C}_0$ for all $t \geq \tau$.

Sketch of Proof. Theorem XI.8.1 in Lefschetz [16] is a version of this result for an ordinary differential equation with the same structure as (8.4) but for functions P and Q that are defined on $\mathbb{R}$. To exploit Lefschetz's theorem, we note that $P(1) = 0$, that hypotheses $(2.3)_2$ and $(7.2)_2$ imply that

$$P(u) \to \begin{Bmatrix} -\infty \\ \infty \end{Bmatrix} \quad \text{as} \quad u \to \begin{Bmatrix} 0 \\ 2 \end{Bmatrix}, \qquad \int_1^\zeta P(u)\,du \to \begin{Bmatrix} \infty \\ \infty \end{Bmatrix} \quad \text{as} \quad \zeta \to \begin{Bmatrix} 0 \\ 2 \end{Bmatrix}, \tag{8.11}$$

and that hypotheses $(7.1)_1$ and $(2.4)_2$ imply that

$$Q(u) \to \begin{Bmatrix} \infty \\ \infty \end{Bmatrix} \quad \text{as} \quad u \to \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}, \qquad \int_1^\zeta Q(u)\,du \to \begin{Bmatrix} -\infty \\ \infty \end{Bmatrix} \quad \text{as} \quad \zeta \to \begin{Bmatrix} 0 \\ 2 \end{Bmatrix}. \tag{8.12}$$

These coercivity properties and our hypotheses on the external force f stated at the beginning of Section 7 support a straightforward exercise to check that the proof of Theorem XI.8.1 in [16] applies here *mutatis mutandis.* □

Some of the abstract theory for dynamical systems enables us to connect the dynamics of $(\mathcal{X}_0, \mathfrak{T}_0)$ to that of $(\mathcal{X}, \mathfrak{T})$. Let $\mathcal{Y}$ be a complete metric space with metric d. Let $\mathfrak{F}$ be a continuous map from $\mathcal{Y}$ to $\mathcal{Y}$. A set $\mathcal{A}$ is *invariant* under $\mathfrak{F}$ if $\mathfrak{F}(\mathcal{Y}) = \mathcal{Y}$. A set

$\mathcal{A}$ *attracts* a set $\mathcal{B}$ if for every $\varepsilon > 0$, there is an $N(\varepsilon)$ such that $\mathfrak{F}^n(\mathcal{B})$ belongs to the ε-neighborhood of $\mathcal{A}$ for all $n \geq N(\varepsilon)$. A compact invariant set $\mathcal{A}$ is a *maximal compact invariant set* if every compact invariant set of $\mathfrak{F}$ is contained in $\mathcal{A}$. Finally, a set $\mathcal{A}$ is a *global attractor* for the dynamical system $(\mathcal{Y}, \mathfrak{F})$ if $\mathcal{A}$ is a maximal compact invariant set that attracts each bounded set $\mathcal{B} \subset \mathcal{Y}$.

A subset $\mathcal{B}$ of $\mathcal{X}$ is said to have the *boundedness property* if there are positive constants u_*, u^*, v^* such that every (y, z) in $\mathcal{B}$ satisfies $u_* \leq y(x) \leq u^*$ for all $x \in [-1, 1]$ and also satisfies $|z| \leq v^*$.

We now state the main result of this section.

THEOREM 8.2. Let (7.1) and (7.2) hold. The dynamical system $(\mathcal{X}, \mathfrak{T})$ generated by (5.4) has a maximal, compact, invariant set $\mathcal{A}_0$ that attracts each set $\mathcal{B} \subset \mathcal{X}$ with the boundedness property. The set $\mathcal{A}_0$ is contained in $\mathcal{X}_0$ and equals the $\bigcap_{n\geq 0} \mathfrak{T}^n \mathcal{C}_0$, where $\mathcal{C}_0$ is the set introduced in Theorem 8.1.

Sketch of Proof. The proof is essentially the same as the proof of Theorem 4.6 in [27], so we omit the details. There are two basic steps: (i) the use of Theorem 7.9 together with (2.15) and (7.5) to show that if $\mathcal{B}$ is any set of initial conditions with the boundedness property, then the distance between $\mathfrak{T}^n \mathcal{B}$ and $\mathcal{X}_0$ goes to 0 as $n \to \infty$; (ii) the use of standard ideas from dynamical systems theory to show that $\bigcap_{n\geq 0} \mathfrak{T}^n \mathcal{C}_0$ is the global attractor for the dynamics restricted to $\mathcal{X}_0$. See [12, Lemma 2.4.2]. □

Theorem 8.2 says that the long-term dynamics for the reduced problem is determined by (8.4), which governs the dynamics on the invariant manifold $\mathcal{X}_0$. It thus gives a precise mathematical relation between the reduced problem and the ordinary differential equation that one would expect to govern the motions of a piston in a viscous gas with no inertia.

For a discretization of the problem treated in [27], which is closely related to the problem treated here, the question of how the invariant manifold $\mathcal{X}_0$ for the reduced problem perturbs to an invariant manifold for the governing equations when $0 < \varepsilon << 1$ is addressed in [26].

9. Asymptotic representation of solutions. In this section we assume that the data have as much regularity as is needed in the analysis. E.g., the second perturbation for the regular expansion is governed by (9.6), which involves p'', so in using this perturbation, we are tacitly assuming that p has two derivatives.

We introduce the stretched time variable τ by

$$t = \varepsilon\tau. \tag{9.1}$$

We seek asymptotic representations of the solutions (u, v) of the initial-boundary-value problem (2.1), (2.7), (2.8), (2.11)–(2.14) in the form

$$\begin{aligned} u(\cdot,\cdot;\varepsilon) &= u^k_{\mathrm{A}}(\cdot,\cdot;\varepsilon) + o(\varepsilon^k), \quad u^k_{\mathrm{A}}(x,t;\varepsilon) = u^k_{\mathrm{R}}(x,t;\varepsilon) + \varepsilon U^k_{\mathrm{L}}(x,t/\varepsilon;\varepsilon), \\ v(\cdot,\cdot;\varepsilon) &= v^k_{\mathrm{A}}(\cdot,\cdot;\varepsilon) + o(\varepsilon^k), \quad v^k_{\mathrm{A}}(x,t;\varepsilon) = v^k_{\mathrm{R}}(x,t;\varepsilon) + V^k_{\mathrm{L}}(x,t/\varepsilon;\varepsilon) \end{aligned} \tag{9.2}$$

where

$$
\begin{aligned}
u^k_{\mathrm{R}}(x,t;\varepsilon) &= \sum_{j=0}^{k} u_j(x,t)\frac{\varepsilon^j}{j!}, \qquad v^k_{\mathrm{R}}(x,t;\varepsilon) = \sum_{j=0}^{k} v_j(x,t)\frac{\varepsilon^j}{j!},\\
U^k_{\mathrm{L}}(x,\tau;\varepsilon) &= \sum_{j=0}^{k-1} U_j(x,\tau)\frac{\varepsilon^j}{j!}, \quad V^k_{\mathrm{L}}(x,\tau;\varepsilon) = \sum_{j=0}^{k} V_j(x,\tau)\frac{\varepsilon^j}{j!}
\end{aligned} \tag{9.3}
$$

with k a nonnegative integer. We shall precisely interpret (9.2) in terms of various norms. The presence of the extra factor ε in $(9.2)_2$ is merely for computational convenience. Here $(u^k_{\mathrm{R}}, v^k_{\mathrm{R}})$ is the *regular expansion*, $(U^k_{\mathrm{L}}, V^k_{\mathrm{L}})$ is the *initial-layer expansion*, and $(u^k_{\mathrm{A}}, v^k_{\mathrm{A}})$ is the *asymptotic expansion*. (The subscripts R and L have meanings different from those of Section 2.) In order that $(u^k_{\mathrm{L}}, v^k_{\mathrm{L}})$ truly represent an initial layer, its terms should come into play only for small t. We ensure this property by requiring that there be a positive number $\zeta(k)$ such that

$$
DU_j(x,\tau), DV_j(x,\tau) \le Ce^{-\zeta(k)\tau} \quad \text{for} \quad j = 0,1,2,\dots,k \tag{9.4}
$$

where $D = 1, \partial_x, \partial_{xx}, \partial_\tau, \dots,$ is a collection of partial derivatives depending on k that enter the analysis.

We require the regular expansion $(u^k_{\mathrm{R}}, v^k_{\mathrm{R}})$ to satisfy (2.1), (2.7), (2.8), (2.11)–(2.13) to within order $o(\varepsilon^k)$, but not necessarily all the initial conditions (2.14). Since $u_j(s,t) = \partial^j_\varepsilon u(s,t;0)$, etc., we get equations for (u_j, v_j), involving (u_0, v_0),..., (u_{j-1}, v_{j-1}), by substituting $(9.3)_{1,2}$ into (2.1), (2.7), (2.8), (2.11)–(2.13), differentiating the resulting equations j times with respect to ε, and then setting $\varepsilon = 0$: We find that (u_0, v_0) satisfies the reduced problem and that (u_1, v_1) and (u_2, v_2) satisfy the linear systems

$$
\begin{gathered}
u_{1t} = v_{1x},\\
v_{0t} = -[p'(u_0)u_1]_x + [\psi''(u_0)u_1 v_{0x} + \psi'(u_0)v_{1x}]_x,\\
v_1(-1,t) = 0, \quad v_1(1,t) = 0, \quad v_1(0,0) = 0, \quad \int_{-1}^{1} u_1(x,t)\,dx = 0,\\
v_{1t}(0,t) = -[\![p'(u_0)u_1]\!](0,t) + [\![\psi''(u_0)u_1 v_{0x} + \psi'(u_0)v_{1x}]\!](0,t),
\end{gathered} \tag{9.5}
$$

and

$$
\begin{aligned}
&u_{2t} = v_{2x},\\
&2v_{1t} = -[p'(u_0)u_2]_x + [\psi''(u_0)u_2 v_{0x} + \psi'(u_0)v_{2x}]_x\\
&\qquad\quad - [p''(u_0){u_1}^2]_x + [\psi'''(u_0){u_1}^2 v_{0x} + 2\psi''(u_0)u_1 v_{1x}]_x,\\
&v_2(-1,t) = 0, \quad v_2(1,t) = 0, \quad v_2(0,0) = 0, \quad \int_{-1}^{1} u_2(x,t)\,dx = 0,\\
&v_{2t}(0,t) = -[\![p'(u_0)u_2]\!](0,t) + [\![\psi''(u_0)u_2 v_{0x} + \psi'(u_0)v_{2x}]\!](0,t)\\
&\qquad\quad - [\![p''(u_0){u_1}^2]\!] + [\![\psi'''(u_0){u_1}^2 v_{0x} + 2\psi''(u_0)u_1 v_{1x}]\!].
\end{aligned} \tag{9.6}
$$

All the subsequent problems for the terms of the regular expansion are nonhomogeneous versions of the linear system (9.5) with the nonhomogeneous terms depending on the solutions of lower-order systems. We discuss the initial conditions for the regular expansion below.

Of course, we require the solution $(u(\cdot,\cdot,\varepsilon), v(\cdot,\cdot,\varepsilon))$ to satisfy the full initial-boundary-value problem, so that the initial-layer expansion $(U_{\mathrm{L}}^k, V_{\mathrm{L}}^k)$ must satisfy

$$\partial_\tau U_{\mathrm{L}}^k(x,\tau;\varepsilon) = \partial_x V_{\mathrm{L}}^k(x,\tau;\varepsilon) + o(\varepsilon^k), \tag{9.7}$$

$$\begin{aligned} \partial_\tau V_{\mathrm{L}}^k(x,\tau;\varepsilon) = -\partial_x \big[p\big(u_{\mathrm{A}}^k(x,\varepsilon\tau;\varepsilon) + o(\varepsilon^k)\big) - p\big(u_{\mathrm{R}}^k\big)\big] \\ +\partial_x \big[\psi'\big(u_{\mathrm{A}}^k(x,\varepsilon\tau;\varepsilon) + o(\varepsilon^k)\big)\big(\partial_x v_{\mathrm{A}}^k(x,\varepsilon\tau;\varepsilon) + o(\varepsilon^k)\big) \\ -\psi'\big(u_{\mathrm{R}}^k(x,\varepsilon\tau;\varepsilon)\big)\partial_x v_{\mathrm{R}}^k(x,\varepsilon\tau;\varepsilon)\big] + o(\varepsilon^k), \end{aligned} \tag{9.8}$$

$$V_{\mathrm{L}}^k(\pm 1,\tau;\varepsilon) = o(\varepsilon^k), \tag{9.9}$$

$$\int_{-1}^{1} U_{\mathrm{L}}^k(x,\varepsilon\tau;\varepsilon)\,dx = o(\varepsilon^{k-1}), \tag{9.10}$$

$$\begin{aligned} \partial_\tau V_{\mathrm{L}}^k(0,\tau;\varepsilon) = -\varepsilon \left[\!\left[p\big(u_{\mathrm{A}}^k + o(\varepsilon^k)\big) - p\big(u_{\mathrm{R}}^k\big)\right]\!\right] \\ + \varepsilon \left[\!\left[\psi'\big(u_{\mathrm{A}}^k + o(\varepsilon^k)\big)\big(\partial_x v_{\mathrm{A}}^k + o(\varepsilon^k)\big) - \psi'\big(u_{\mathrm{R}}^k\big)\partial_x v_{\mathrm{R}}^k\right]\!\right] + o(\varepsilon^k), \end{aligned} \tag{9.11}$$

$$u_{\mathrm{R}}^k(x,0;\varepsilon) + \varepsilon U_{\mathrm{L}}^k(x,0;\varepsilon) = \bar{u}(x) + o(\varepsilon^k), \tag{9.12}$$

$$v_{\mathrm{R}}^k(x,0;\varepsilon) + V_{\mathrm{L}}^k(x,0;\varepsilon) = \bar{v}(x) + o(\varepsilon^k). \tag{9.13}$$

In (9.11), the arguments of the $u_{\mathrm{R}}, v_{\mathrm{R}}$ are $(x,\varepsilon\tau;\varepsilon)$ and those of $u_{\mathrm{L}}, v_{\mathrm{L}}$ are $(x,\tau;\varepsilon)$.

Since $U_j(x,\tau) = \partial_\varepsilon^j U(x,\tau;0)$, etc., we find equations for (U_j, V_j) from (9.7)–(9.13) by the same process by which we found the equations for the regular expansion:

$$\begin{gathered} \partial_\tau U_0 = \partial_x V_0, \\ \partial_\tau V_0 = \partial_x \big[\psi'(\bar{u})\partial_x V_0\big], \\ V_0(\pm 1,\tau) = 0, \qquad \int_{-1}^{1} U_0(x,\tau)\,dx = 0, \qquad \partial_\tau V_0(0,\tau) = 0, \\ u_0(x,0) = \bar{u}(x), \qquad v_0(x,0) + V_0(x,0) = \bar{v}(x), \end{gathered} \tag{9.14}$$

and

$$\begin{gathered} \partial_\tau U_1 = \partial_x V_1, \\ \partial_\tau V_1 = \partial_x[\psi'(\bar{u})\partial_x V_1 + \psi''(\bar{u})u_1(x,0)\partial_x V_0], \\ V_1(\pm 1,\tau) = 0, \qquad \int_{-1}^{1} U_1(x,\tau)\,dx = 0, \qquad \partial_\tau V_1(0,\tau) = \left[\!\left[\psi'(\bar{u})\partial_x V_0\right]\!\right](0,\tau), \\ u_1(x,0) + U_0(x,0) = 0, \qquad v_1(x,0) + V_1(x,0) = 0. \end{gathered} \tag{9.15}$$

Note the linearity of (9.14). All subsequent initial-layer expansions are nonhomogeneous versions of this system. Note also that $(9.14)_2$ is just the heat equation (with a nonuniform conductivity).

For the reduced problem to be self-contained, i.e., for it to have a unique solution, we must prescribe for it the initial conditions $(9.14)_6$ and $v_0(0,0) = \bar{v}(0)$. As follows from Section 5, this condition gives v_0. Thus $(9.14)_7$ then gives an initial condition for V_0. From $(9.14)_5$ we then find $V_0(0,\tau) = V_0(0,0) = 0$. System (9.14) thus yields two well-posed initial-boundary-value problems for V_0, one for $x \in [-1,0]$ and one for $x \in [0,1]$.

We avoid using initial data for the U_j by exploiting (9.4) to deduce from $(9.14)_1$ that

$$U_0(x,\tau) = -\int_\tau^\infty \partial_x V_0(x,\sigma)\, d\sigma, \tag{9.16}$$

whence we obtain $U_0(s,0)$, which can now be inserted into $(9.15)_6$ to give an explicit initial condition for u_1. The same methods handle the higher-order problems.

The evolution equations (2.1), (2.7), (2.8), when viewed as ordinary differential equations in a Banach space, have a form that is standard for the asymptotic analysis of initial-value problems for the usual systems of ordinary differential equations (cf. [21, 23], e.g.). For our partial differential equations, there are, however, serious technical obstacles in obtaining the exponential bounds (9.4) and the essential error bounds of (9.2), upon which rests the justification of the asymptotics. Fortunately, these technical problems were resolved for the more complicated partial differential equations in [28]. There, e.g., our linear equation $(9.14)_2$ is replaced by a quasilinear equation. To demonstrate that its solution and many derivatives of its solution have exponential decay required a complicated modification of techniques based on the maximum principle for parabolic equations developed in [24]. (These were inspired by methods of S. N. Bernstein described in [15]. The equation in [28] is quasilinear because the tensile force depends nonlinearly on the strain rate (= velocity gradient) $u_t = v_x$.) The corresponding treatment for our problem is far simpler. Indeed, much of it can be based on the generalizations of the maximum principle given by [11]. We accordingly just informally state the fundamental theorem:

THEOREM 9.1. Let constitutive restrictions (2.3) and (2.4) hold. Let the initial functions $\bar{u}$ and $\bar{v}$ of (2.14) be sufficiently smooth and satisfy compatibility conditions of sufficiently high order. Let f be sufficiently smooth. Then for each $\varepsilon > 0$ the initial-boundary-value problem (2.1), (2.7), (2.8), (2.11)–(2.14) has a classical solution (u, v) defined for all time with a level of smoothness corresponding to that of the data, and u^k_{A} and v^k_{A} exist for all time for each positive integer k corresponding to the level of smoothness of the data. Let $\varepsilon_0 \in (0,1)$, $T > 0$, and let k be fixed. Then there is a constant $C(k,t)$ independent of ε that depends on the data of the problem such that

$$\begin{aligned} \sup_{-1\le x\le 1,\, 0\le t\le T} |\partial_x[u(x,t) - u^k_{\mathrm{A}}(x,t;\varepsilon)]| &\le C(k,T)\varepsilon^{k+1}, \\ \sup_{-1\le x\le 1,\, 0\le t\le T} |\partial_x[v(x,t) - v^k_{\mathrm{A}}(x,t;\varepsilon)]| &\le C(k,T)\varepsilon^{k+1}. \end{aligned} \tag{9.17}$$

The global existence of solutions depends crucially on the bounds of Sections 3 and 4 (cf. [1, 2, 5, 6, 14, 22]). The justification of the asymptotics in (9.17) depends on simplified versions of the proofs in [28] that the initial-layer expansion decays exponentially in time. It can be shown that other derivatives of the errors satisfy bounds just like (9.17); cf. [28].

10. Comments. Our work has exhibited the intricate relationship between the solution of the full initial-boundary-value problem for the parabolic-hyperbolic system (2.1), (2.7); the solution of reduced equations, which form a system with memory; and their attractor, which is governed by a standard second-order ordinary differential equation for the damped motion of a mass-spring system. It is important to note that the reduction effected by setting $\varepsilon = 0$ annihilates some but not all the inertia of the physical problem. Mechanical problems in which all the inertia is thus annihilated typically have asymptotic expansions in which there is an initial layer, but the behavior of the system differs considerably from ours.

Our effort in this paper was devoted to establishing the bounds for the general problem in Section 4 and for the reduced problem in Section 7. We then exploited available tools to establish the relationships between the full problem, the reduced problem, and its attractor.

Our methods could no doubt handle the motion of many pistons in a single cylinder. When combined with the methods of [28], they could handle the motion of a mass on two springs with their other ends attached to fixed points, with the springs treated as 1-dimensional nonlinear viscoelastic solids. They could presumably treat the motion of one or several mass points connected by several such springs.

Acknowledgment. The research reported here was supported in part by NSF grants DMS-0204505 and DMS-0407361.

References

[1] G. Andrews, On the existence of solutions to the equation $u_{tt} = u_{xxt} + \sigma(u_x)_x$, *J. Diff. Eqs.* **35** (1980) 200–231. MR561978 (81d:35073)

[2] G. Andrews and J. M. Ball, Asymptotic behaviour and changes of phase in one-dimensional nonlinear viscoelasticity, *J. Diff. Eqs.* **44** (1982) 306–341. MR657784 (83m:73046)

[3] S. S. Antman, The paradoxical asymptotic status of massless springs, *SIAM J. Appl. Math.* **48** (1988) 1319–1334. MR968832 (90b:35194)

[4] S. S. Antman, *Nonlinear Problems of Elasticity*, 2nd. edn., Springer, 2005. MR2132247 (2006e:74001)

[5] S. S. Antman and T. I. Seidman, Quasilinear hyperbolic-parabolic equations of nonlinear viscoelasticity, *J. Diff. Eqs.* **124** (1996) 132–185. MR1368064 (96k:35176)

[6] S. S. Antman and T. I. Seidman, Parabolic-hyperbolic systems governing the spatial motion of nonlinearly viscoelastic rods, *Arch. Rational Mech. Anal.* **175** (2005), 85–150. MR2106258 (2006b:74015)

[7] P. W. Bridgman, *The Physics of High Pressure*, G. Bell and Sons, 1931; Dover reprint, 1970.

[8] R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*, Interscience, 1948. MR0029615 (10:637c)

[9] C. M. Dafermos, The mixed initial-boundary value problem for the equations of nonlinear 1-dimensional viscoelasticity, *J. Diff. Eqs.* **6** (1969) 71–86. MR0241831 (39:3168)

[10] D. A. French, S. Jensen, and T. I. Seidman, A space-time finite element method for a class of nonlinear hyperbolic-parabolic equations, *Appl. Num. Math.* **31** (1999) 429–450. MR1719244 (2000k:65170)

[11] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, 1964. MR0181836 (31:6062)

[12] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Amer. Math. Soc. 1988. MR941371 (89g:58059)

[13] A. Haraux, *Nonlinear Evolution Equations—Global Behavior of Solutions*, Springer, 1981. MR610796 (83d:47066)

[14] Ya. I. Kanel', On a model system of equations of one-dimensional gas motion (in Russian), *Diff. Urav.* **4** (1969) 721–734. English translation: *Diff. Eqs.* **4** (1969), 374–380.

[15] O. A. Ladyženskaja, N. Ural'ceva, and V. A. Solonnikov, *Linear and Quasi-Linear Equations of Parabolic Type*, Amer. Math. Soc., 1968. MR0241821 (39:3159a)

[16] S. Lefschetz, *Differential Equations: Geometric Theory*, Interscience, 1963. MR0153903 (27:3864)

[17] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Dunod, Gauthier-Villars, 1969. MR0259693 (41:4326)

[18] T.-P. Liu, The free piston problem for gas dynamics, *J. Diff. Eqs.* **30** (1978), 175–191. MR513269 (80d:35091)

[19] R. C. MacCamy, Existence, uniqueness and stability of $u_{tt} = \frac{\partial}{\partial x}[\sigma(u_x) + \lambda(u_x)u_{xt}]$, *Indiana Univ. Math. J.* **20** (1970) 231–238. MR0265790 (42:699)

[20] R. H. Martin, Jr., *Nonlinear Operators and Differential Equations in Banach Spaces*, Wiley-Interscience, 1976. MR0492671 (58:11753)

[21] R. E. O'Malley, Jr., *Singular Perturbation Methods for Ordinary Differential Equations*, Springer, 1991. MR1123483 (92i:34071)

[22] R. L. Pego, Phase transitions in one-dimensional nonlinear viscoelasticity: Admissibility and stability, *Arch. Rational Mech. Anal.* **97** (1987) 353–394. MR865845 (87m:73037)

[23] D. R. Smith, *Singular Perturbation Theory*, Cambridge Univ. Pr., 1985. MR812466 (87d:34001)

[24] M. Wiegner, On the asymptotic behaviour of solutions of nonlinear parabolic equations, *Math. Z.* **188** (1984) 3–22. MR767358 (86b:35017)

[25] J. P. Wilber, Absorbing balls for equations modeling nonuniform deformable bodies with heavy rigid attachments. *J. Dynam. Diff. Eqs.* **14** (2002) 855–887. MR1940106 (2003j:37139)

[26] J.P. Wilber, Invariant manifolds describing the dynamics of a hyperbolic-parabolic equation from nonlinear viscoelasticity, *Dynam. Systems* **21** (2006), 465–489. MR2273689

[27] J. P. Wilber and S. S. Antman, Global attractors for a degenerate partial differential equation from nonlinear viscoelasticity, *Physica D* **150** (2001) 179–208. MR1820734 (2001m:74013)

[28] S.-C. Yip, S. S. Antman, and M. Wiegner, The motion of a particle on a light viscoelastic bar: Asymptotic analysis of the quasilinear parabolic-hyperbolic equation, *J. Math. Pures Appl.* **81** (2002), 283–309. MR1967351 (2004k:35378)

[29] E. Zeidler, *Nonlinear Functional Analysis and it Applications, Vol. II/B, Nonlinear Monotone Operators*, Springer, 1990.

[30] Songmu Zheng, *Nonlinear Parabolic Equations and Hyperbolic-Parabolic Coupled Systems*, Longman, 1995. MR1375458 (97g:35078)