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ERRATUM TO "THIN ELASTIC FILMS: THE IMPACT OF HIGHER ORDER PERTURBATIONS"

By

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1. Introduction. In the paper [2] the main objective was to identify

$$E_-^\gamma(u,b;A) := \inf \left\{ \liminf_{\varepsilon\to 0^+} E_\varepsilon^\gamma(u_\varepsilon;A) :\ u_\varepsilon \in W^{2,2}\left(\Omega;\mathbb{R}^3\right), \right.$$
$$\left. u_\varepsilon \rightharpoonup u \text{ in } W^{1,q}\left(\Omega;\mathbb{R}^3\right), \quad \frac{1}{\varepsilon}D_3 u_\varepsilon \rightharpoonup b \text{ in } L^q\left(\Omega;\mathbb{R}^3\right) \right\},$$

where

$$E_\varepsilon^\gamma(u;A) := \int_{A\times I} W\left(D_p u \left| \frac{1}{\varepsilon} D_3 u\right.\right) dx$$
$$+ \varepsilon^\gamma \int_{A\times I} \left(|D_p^2 u|^2 + \frac{1}{\varepsilon^2}|D_{p3}u|^2 + \frac{1}{\varepsilon^4}|D_{33}u|^2 \right) dx$$

if $u \in W^{2,2}(\Omega;\mathbb{R}^3)$, and $E_\varepsilon^\gamma(u;A) := \infty$ otherwise. Here $q > 1$, $\Omega = \omega \times I$, $\mathcal{A}(\omega)$ is the family of open subsets of ω, and W satisfies the condition

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$(H_1)'$ $W : \mathbb{R}^{3\times 3} \to [0,\infty)$ is continuous and there exists $C > 0$ such that

$$\frac{1}{C}\left|F\right|^q - C \leq W\left(F\right) \leq C\left(1 + \left|F\right|^q\right)$$

for all $F \in \mathbb{R}^{3\times 3}$.

We recall that

$$\begin{aligned} \mathcal{V}^\gamma := \big\{ &(u,b) \in W^{1,q}\left(\Omega;\mathbb{R}^3\right) \times L^q\left(\Omega;\mathbb{R}^3\right) : \, D_3 u = 0 \ \mathcal{L}^3 \text{ a.e. in } \Omega, \\ &D_3 b = 0 \ \mathcal{L}^3 \text{ a.e. in } \Omega \text{ if } \gamma < 2, \ D_3 b \in L^2\left(\Omega;\mathbb{R}^3\right) \text{ if } \gamma = 2 \big\} \end{aligned} \tag{1.1}$$

and for $(u,b) \in \mathcal{V}^\gamma$ and $A \in \mathcal{A}(\omega)$,

$$\begin{aligned} H^\gamma\left(u,b,A\right) := \inf \Big\{ &\liminf_{\varepsilon\to 0^+} \int_{A\times I} W\left(D_p u_\varepsilon\left(x\right) | b_\varepsilon\left(x\right)\right) dx : \, \{u_\varepsilon\} \subset W^{1,q}(\Omega;\mathbb{R}^3), \\ &\{b_\varepsilon\} \subset L^q\left(\Omega;\mathbb{R}^3\right), \, u_\varepsilon \rightharpoonup u \text{ in } W^{1,q}(\Omega;\mathbb{R}^3), \ b_\varepsilon \rightharpoonup b \text{ in } L^q(\Omega;\mathbb{R}^3) \Big\} \end{aligned}$$

if $\gamma \neq 2$, and

$$\begin{aligned} H^2\left(u,b,A\right) := \inf \Big\{ &\liminf_{\varepsilon\to 0^+} \int_{A\times I} \left[W\left(D_p u_\varepsilon\left(x\right) | b_\varepsilon\left(x\right)\right) + \left|D_3 b_\varepsilon\left(x\right)\right|^2 \right] dx : \\ &\{u_\varepsilon\} \subset W^{1,q}(\Omega;\mathbb{R}^3), \{b_\varepsilon\} \subset L^q\left(\Omega;\mathbb{R}^3\right) \text{ with } D_3 b_\varepsilon \in L^2\left(\Omega;\mathbb{R}^3\right), \\ &u_\varepsilon \rightharpoonup u \text{ in } W^{1,q}(\Omega;\mathbb{R}^3), \ b_\varepsilon \rightharpoonup b \text{ in } L^q(\Omega;\mathbb{R}^3) \Big\}. \end{aligned}$$

In this erratum we slightly change notation and denote H^γ and H^2 by $H^\gamma_{3,3}$ and $H^2_{3,3}$, respectively. We also introduce $H^\gamma_{2,3}$ and $H^2_{2,3}$, which we define analogously except for the fact that the approximating sequences $\{u_\varepsilon\}$ belong to $W^{1,q}(\omega;\mathbb{R}^3)$.

The proof of Theorem 3.1 in [2] has an error in the upper bound. To date this upper bound can only be established if $\gamma < 2$ or if $D_3 b = 0$ $\mathcal{L}^3$ a.e. in Ω. Therefore, Theorem 3.1 in [2] should be replaced by

Theorem A (New Theorem 3.1). *Assume that condition $(H_1)'$ is satisfied. Then,*

$$H^\gamma_{3,3}\left(u,b,A\right) \leq E^\gamma_-(u,b;A) \leq H^\gamma_{2,3}\left(u,b,A\right) \tag{1.2}$$

for all $(u,b) \in \mathcal{V}^\gamma$ and $A \in \mathcal{A}(\omega)$. If $\gamma < 2$ or $\gamma \geq 2$ and $D_3 b = 0$ $\mathcal{L}^3$ a.e. in Ω, then

$$\begin{aligned} E^\gamma_-(u,b;A) &= H^\gamma_{3,3}\left(u,b,A\right) = H^\gamma_{2,3}\left(u,b,A\right) = H^\gamma_{2,2}\left(u,b,A\right) \\ &= \int_A \left(Q_2 \times C_2\right)[W]\left(D_p u\left(x_\alpha\right) | b\left(x_\alpha\right)\right) dx_\alpha. \end{aligned}$$

The lower bound in the original Theorem 3.1 in [2] is correct. The mistake in the proof of Theorem 3.1 was in trying to show that the upper bound is still $H^\gamma_{3,3}$. The correct upper bound is $H^\gamma_{2,3}$, as justified next, and in general there may be a gap between $H^\gamma_{3,3}$ and $H^\gamma_{2,3}$. However, when $D_3 b = 0$ $\mathcal{L}^3$ a.e. in Ω (a condition that is automatically satisfied when $\gamma < 2$ in the finite energy regime), then indeed

$$H^\gamma_{2,3}\left(u,b,A\right) = H^\gamma_{3,3}\left(u,b,A\right).$$

To see this, we refer the reader to the proof of Theorem 4.1 in [2], which still works unchanged for any $\gamma > 0$ when $D_3 b = 0$ $\mathcal{L}^3$ a.e. in Ω.

To prove the second inequality in (1.2) we remark that the estimates involving $\{u * \varphi_j, b * \varphi_j\}$ led to (3.2) and (3.3) for (u, b, A), and this analysis relied heavily on the fact that $D_3 u = 0\ \mathcal{L}^3$ a.e. in Ω. Therefore, given $(u, b) \in \mathcal{V}^\gamma$, $A \in \mathcal{A}(\omega)$, if $\{u_j\} \subset W^{1,q}(\omega; \mathbb{R}^3)$ converges weakly to u in $W^{1,q}(\omega; \mathbb{R}^3)$ and $\{b_j\} \subset L^q(\Omega; \mathbb{R}^3)$ converges weakly to b in $L^q(\Omega; \mathbb{R}^3)$, then we may apply (3.2) and (3.3) to (u_j, b, A), and we deduce that

$$E_-^\gamma(u, b; A) \leq \liminf_{j\to\infty} E_-^\gamma(u_j, b_j; A) \leq \liminf_{j\to\infty} \int_{A\times I} W\left(D_p u_j\left(x_\alpha\right)|b_j\left(x\right)\right) dx$$

if $\gamma \neq 2$, and if $\gamma = 2$ that

$$\begin{aligned} E_-^2(u, b; A) &\leq \liminf_{j\to\infty} E_-^2(u_j, b_j; A) \\ &\leq \liminf_{j\to\infty} \int_{A\times I} \left[W\left(D_p u_j\left(x_\alpha\right)|b_j\left(x\right)\right) + \left|D_3 b_j\left(x\right)\right|^2\right] dx. \end{aligned}$$

Taking the infimum over all such sequences $\{u_j\}$ and $\{b_j\}$ yields

$$E_-^\gamma\left(u, b; A\right) \leq H_{2,3}^\gamma\left(u, b, A\right).$$

Theorem A covers completely the case $\gamma < 2$ but leaves (partially) open the case $\gamma \geq 2$ when $D_3 b \not\equiv 0\ \mathcal{L}^3$ a.e. in Ω. We close the gap in this erratum. To be precise, if $\gamma = 2$ and if W satisfies the additional q-Lipschitz condition

$$\left|W\left(F\right) - W\left(G\right)\right| \leq C\left(1 + |F|^{q-1} + |G|^{q-1}\right)|F - G| \tag{1.3}$$

for all F, $G \in \mathbb{R}^{3\times 3}$, then we characterize the Γ-limit in Theorem B, while for $\gamma > 2$ we refer to Theorem G.

When $\gamma = 2$, we introduce the functional

$$\overline{\mathcal{W}}_2 : \mathbb{R}^{2\times 3} \times W^{1,2}(I; \mathbb{R}^3) \to [0, \infty)$$

defined for $\overline{F} \in \mathbb{R}^{2\times 3}$ and $b \in W^{1,2}(I; \mathbb{R}^3)$ by

$$\begin{aligned} \overline{\mathcal{W}}_2(\overline{F}|b) := \inf_{\varphi, L} \Bigg\{ \int_Q \Bigg(& W\left(\overline{F} + D_p\varphi(x)|b(x_3) + LD_3\varphi\left(x\right)\right) + \left|\frac{1}{L} D_p^2 \varphi(x)\right|^2 \\ & + \left|D_{p3}\varphi(x)\right|^2 + \left|b'(x_3) + LD_{33}\varphi(x)\right|^2 \Bigg) \, dx : \\ & L > 0,\ \varphi \in W^{2,2}(Q; \mathbb{R}^3),\ \varphi(\cdot, x_3)\ Q'\text{-periodic for } \mathcal{L}^1 \text{ a.e. } x_3, \\ & \int_{Q'} D_3\varphi(x_\alpha, x_3)\, dx_\alpha = 0 \text{ for } \mathcal{L}^1 \text{ a.e. } x_3 \Bigg\}. \end{aligned} \tag{1.4}$$

We have the following representation.

Theorem B. Assume that $\gamma = 2$ and that conditions $(H_1)'$ and (1.3) are satisfied. Then for all $(u, b) \in \mathcal{V}^2$ and $A \in \mathcal{A}(\omega)$,

$$E_-^2(u, b; A) = \int_A \overline{\mathcal{W}}_2(D_p u(x_\alpha)|b(x_\alpha, \cdot))\, dx_\alpha, \tag{1.5}$$

where $\overline{\mathcal{W}}_2$ is defined in (1.4).

To prove the theorem above, we start by showing that under the q-Lipschitz condition (1.3), minimizing sequences for $\overline{\mathcal{W}}_2$ prefer scales L diverging to infinity.

Proposition C. Assume that $\gamma = 2$ and that conditions $(H_1)'$ and (1.3) are satisfied. Then for all $\overline{F} \in \mathbb{R}^{2\times 3}$ and $b \in W^{1,2}(I;\mathbb{R}^3)$ the $\inf_{\varphi,L}$ in the definition of $\overline{\mathcal{W}}_2(\overline{F}|b)$ may be replaced by $\lim_{L\to\infty}\inf_{\varphi}$.

Proof. For $\overline{F} \in \mathbb{R}^{2\times 3},\ b \in W^{1,2}(I;\mathbb{R}^3)$, and $L > 0$ let

$$\begin{aligned}\mathcal{H}_L(\overline{F}|b) := \inf_{\varphi}\Bigg\{ \int_Q \Bigg(& W\left(\overline{F} + D_p\varphi(x)|b(x_3) + LD_3\varphi\left(x\right)\right) + \left|\frac{1}{L}D_p^2\varphi(x)\right|^2 \\ & + \left|D_{p3}\varphi(x)\right|^2 + \left|b'(x_3) + LD_{33}\varphi(x)\right|^2\Bigg)\, dx : \\ & \varphi \in W^{2,2}(Q;\mathbb{R}^3), \varphi(\cdot, x_3)\ Q'\text{-periodic for } \mathcal{L}^1 \text{ a.e. } x_3, \\ & \int_{Q'} D_3\varphi(x_\alpha, x_3)\, dx_\alpha = 0 \text{ for } \mathcal{L}^1 \text{ a.e. } x_3 \Bigg\}.\end{aligned}$$

Clearly,

$$\overline{\mathcal{W}}_2(\overline{F}|b) \le \liminf_{L\to\infty} \mathcal{H}_L(\overline{F}|b).$$

We now prove that

$$\limsup_{L\to\infty} \mathcal{H}_L(\overline{F}|b) \le \overline{\mathcal{W}}_2(\overline{F}|b).$$

Consider a sequence $\{L_n\}$ converging to infinity such that

$$\limsup_{L\to\infty} \mathcal{H}_L(\overline{F}|b) = \lim_{n\to\infty} \mathcal{H}_{L_n}(\overline{F}|b).$$

Let $\varphi \in W^{2,\infty}(Q;\mathbb{R}^3)$ and $L > 0$ be admissible for $\overline{\mathcal{W}}_2(\overline{F}|b)$, and define

$$\varphi_n\left(x\right) := \frac{1}{m_n}\varphi\left(m_n x_\alpha, x_3\right),$$

where

$$m_n := \left\lceil \frac{L_n}{L} \right\rceil$$

with $\lceil L_n/L \rceil$ the integer part of L_n/L. Note that φ_n is admissible for $\mathcal{H}_{L_n}$, and so

$$\begin{aligned}
&\lim_{n\to\infty} \mathcal{H}_{L_n}(\overline{F}|b) \\
&\quad \le \liminf_{n\to\infty} \left\{ \int_Q \left(W\left(\overline{F} + D_p\varphi_n(x)\,|b(x_3) + L_n D_3\varphi_n(x)\right) + \left| \frac{1}{L_n} D_p^2 \varphi_n(x) \right|^2 \right.\right. \\
&\qquad \left.\left. + \left|D_{p3}\varphi_n(x)\right|^2 + \left|b'(x_3) + L_n D_{33}\varphi_n(x)\right|^2 \right) dx \right\} \\
&\quad = \liminf_{n\to\infty} \left\{ \int_Q \left(W\left(\overline{F} + D_p\varphi(m_n x_\alpha, x_3) \left| b(x_3) + \frac{L_n}{m_n} D_3\varphi(m_n x_\alpha, x_3) \right) \right.\right.\right. \\
&\qquad + \left| \frac{m_n}{L_n} D_p^2\varphi(m_n x_\alpha, x_3) \right|^2 + \left|D_{p3}\varphi(m_n x_\alpha, x_3)\right|^2 \\
&\qquad \left.\left. + \left| b'(x_3) + \frac{L_n}{m_n} D_{33}\varphi(m_n x_\alpha, x_3) \right|^2 \right) dx \right\}.
\end{aligned}$$

By (1.3), together with the facts that $L_n \to \infty$ and that $\varphi \in W^{2,\infty}(Q;\mathbb{R}^3)$, we may rewrite the right-hand side of the previous inequality as

$$\begin{aligned}
\liminf_{n\to\infty} &\left\{ \int_Q \left(W\left(\overline{F} + D_p\varphi(m_n x_\alpha, x_3)\,|b(x_3) + L D_3\varphi(m_n x_\alpha, x_3)\right) \right.\right. \\
&+ \left| \frac{1}{L} D_p^2\varphi(m_n x_\alpha, x_3) \right|^2 + \left|D_{p3}\varphi(m_n x_\alpha, x_3)\right|^2 \\
&\left.\left. + \left|b'(x_3) + L D_{33}\varphi(m_n x_\alpha, x_3)\right|^2 \right) dx \right\} + o(1).
\end{aligned}$$

Since $\varphi(\cdot, x_3)$ is Q'-periodic for $\mathcal{L}^1$ a.e. $x_3 \in I$, it follows from Lebesgue's Dominated Convergence Theorem, Fubini's Theorem, and the Riemann-Lebesgue Lemma that

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{H}_{L_n}(\overline{F}|b) \le \int_Q \Bigg(&\left(W\left(\overline{F} + D_p\varphi(x_\alpha, x_3)\,|b(x_3) + L D_3\varphi(x_\alpha, x_3)\right)\right) \\
&+ \left| \frac{1}{L} D_p^2\varphi(x_\alpha, x_3) \right|^2 \\
&+ \left|D_{p3}\varphi(x_\alpha, x_3)\right|^2 + \left|b'(x_3) + L D_{33}\varphi(x_\alpha, x_3)\right|^2 \Bigg) dx.
\end{aligned}$$

Using the arbitrariness of φ and L, the density of smooth functions in the set of test functions for (1.4), and the growth hypothesis $(H_1)'$, we conclude that

$$\lim_{n\to\infty} \mathcal{H}_{L_n}(\overline{F}|b) \le \overline{\mathcal{W}}_2(\overline{F}|b). \qquad \square$$

To prove Theorem B, it is enough to show that for any given sequence $\{\varepsilon_n\}$, with $\varepsilon_n \to 0^+$, there exists a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ such that the Γ-lower limit, defined by

$$E^-_{\{\varepsilon_{n_k}\}}(u,b;A) := \inf\left\{\liminf_{k\to\infty} E^2_{\varepsilon_{n_k}}(u_k;A) :\ u_k \in W^{2,2}\left(\Omega;\mathbb{R}^3\right),\right. \tag{1.6}$$

$$\left. u_k \rightharpoonup u \text{ in } W^{1,q}\left(\Omega;\mathbb{R}^3\right), \quad \frac{1}{\varepsilon_{n_k}} D_3 u_{n_k} \rightharpoonup b \text{ in } L^q\left(\Omega;\mathbb{R}^3\right)\right\},$$

coincides with

$$\int_A \overline{\mathcal{W}}_2(D_p u(x_\alpha)|b(x_\alpha,\cdot))\,dx_\alpha$$

for all $(u,b)\in\mathcal{V}^2$ and $A\in\mathcal{A}(\omega)$.

To choose the subsequence $\{\varepsilon_{n_k}\}$, let $\mathcal{R}(\omega)$ be the countable subfamily of $\mathcal{A}(\omega)$ obtained by taking all finite unions of open squares in ω with faces parallel to the axes, centered at $x_\alpha\in\omega\cap\mathbb{Q}^2$ and with rational edge length. Since $L^1(\Omega;\mathbb{R}^3)$ is a separable metric space, using Kuratowski's Compactness Theorem and a diagonal argument, we may find a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ such that

$$\Gamma\text{-}\lim_{k\to\infty} E^2_{\varepsilon_{n_k}}(u,b;A) \text{ exists for all } (u,b)\in\mathcal{V}^2 \text{ and for all } A'\in\mathcal{R}(\omega). \tag{1.7}$$

Theorem D. Assume that condition $(H_1)'$ is satisfied and that $\gamma=2$. Then for every $(u,b)\in\mathcal{V}^2$ the set function $E^-_{\{\varepsilon_{n_k}\}}(u,b;\cdot)$ is the trace of a Radon measure absolutely continuous with respect to $\mathcal{L}^2\lfloor\omega$.

Proof. The proof is very similar to that of Theorem 4.2 in [2], and thus we only indicate the main changes.

Step 1: Fix $(u,b)\in\mathcal{V}^2$. We claim that

$$E^-_{\{\varepsilon_{n_k}\}}(u,b;A_1) \le E^-_{\{\varepsilon_{n_k}\}}(u,b;A_2) + E^-_{\{\varepsilon_{n_k}\}}(u,b;A_1\setminus\overline{A_3}) \tag{1.8}$$

for all A_1, A_2, $A_3\in\mathcal{A}(\omega)$, with $A_3\Subset A_2\subset A_1$.

Without loss of generality we may assume that the right-hand side of the previous inequality is finite.

Fix $\eta>0$ and find $\{u_k\}\subset W^{2,2}\left(\Omega;\mathbb{R}^3\right)$ converging weakly to u in $W^{1,q}(\Omega;\mathbb{R}^3)$ and such that $\frac{1}{\varepsilon_{n_k}}D_3u_k \rightharpoonup b$ in $L^q\left(\Omega;\mathbb{R}^3\right)$, and

$$\liminf_{k\to\infty} E^2_{\varepsilon_{n_k}}(u_k;\left(A_1\setminus\overline{A_3}\right)) \le E^-_{\{\varepsilon_{n_k}\}}(u,b;A_1\setminus\overline{A_3})+\eta.$$

Extract a subsequence $\{n_{k_j}\}$ for which

$$\lim_{j\to\infty} E^2_{\varepsilon_{n_{k_j}}}(u_{k_j};\left(A_1\setminus\overline{A_3}\right)) \le E^-_{\{\varepsilon_{n_k}\}}(u,b;A_1\setminus\overline{A_3})+\eta. \tag{1.9}$$

Let $A'\in\mathcal{R}(\omega)$ be such that $A_3\Subset A'\Subset A_2$. By (1.7) there exists a sequence $\{v_k\}\subset W^{2,2}\left(\Omega;\mathbb{R}^3\right)$ converging weakly to u in $W^{1,q}(\Omega;\mathbb{R}^3)$ and such that $\frac{1}{\varepsilon_{n_k}}D_3v_k \rightharpoonup b$ in $L^q\left(\Omega;\mathbb{R}^3\right)$, and

$$E^-_{\{\varepsilon_{n_k}\}}(u,b;A') = \lim_{k\to\infty} E^2_{\varepsilon_{n_k}}(v_k;A').$$

In particular,

$$E^-_{\{\varepsilon_{n_k}\}}(u,b;A') = \lim_{j\to\infty} E^2_{\varepsilon_{n_{k_j}}}\left(v_{k_j};A'\right). \tag{1.10}$$

For every $v \in W^{2,2}(\Omega;\mathbb{R}^3)$, for every Borel set $E \subset \omega$, and for every $j \in \mathbb{N}$ define

$$\begin{aligned}\mathcal{G}_j(v;E) := &\int_{E\times I}\left(1+|D_p v|^q+\frac{1}{\varepsilon^q_{n_{k_j}}}|D_3 v|^q\right)dx\\&+\int_{E\times I}\left(\varepsilon^2_{n_{k_j}}|D^2_p v|^2+|D_{p3}v|^2+\frac{1}{\varepsilon^2_{n_{k_j}}}|D_{33}v|^2\right)dx.\end{aligned}$$

Due to the coercivity hypothesis $(H_1)'$ we may extract a bounded subsequence from the sequence of measures $\nu_j := \mathcal{G}_j(u_{k_j};\cdot)+\mathcal{G}_j(v_{k_j};\cdot)$ restricted to $A'\setminus\overline{A_3}$ converging $\star$-weakly to some Radon measure ν defined on $A'\setminus\overline{A_3}$.

Find $t>0$ such that $\nu(S_t)=0$, where

$$S_t := \{x_\alpha \in A' :\ \mathrm{dist}(x_\alpha,\partial A_3)=t\}.$$

For $\delta>0$ define

$$L_\delta := \{x_\alpha \in A' :\ \mathrm{dist}(x_\alpha,S_t)<\delta\}.$$

Choose δ so small that $L_\delta \subset A'\setminus\overline{A_3}$. Consider a smooth cut-off function $\varphi_\delta \in C^\infty_0(A_2;[0,1])$ such that $\varphi_\delta=1$ in

$$\{x_\alpha\in A' :\ \mathrm{dist}(x_\alpha,\partial A_3)<t-\delta\}$$

and $\varphi_\delta=0$ in

$$\{x_\alpha\in A' :\ \mathrm{dist}(x_\alpha,\partial A_3)>t+\delta\},$$

with

$$\|D_p\varphi_\delta\|_{L^\infty(\omega)}\le C/\delta,\quad \|D^2_p\varphi_\delta\|_{L^\infty(\omega)}\le C/\delta^2.$$

Define

$$\tilde{u}_k(x) := \begin{cases}(1-\varphi_\delta(x_\alpha))u_{k_j}(x)+\varphi_\delta(x_\alpha)v_{k_j}(x) & \text{if } k=k_j \text{ for some } j\in\mathbb{N},\\ u(x_\alpha)+\varepsilon_{n_k}\int_0^{x_3}(b*\psi_k)(x_\alpha,s)\,ds & \text{otherwise,}\end{cases}$$

where $\psi_k=\psi_k(x_\alpha)$ is a standard mollifier. Then $\tilde{u}_k \rightharpoonup u$ in $W^{1,q}(\Omega;\mathbb{R}^3)$ and since φ_δ does not depend on x_3, we also have that $\frac{1}{\varepsilon_{n_k}}D_3\tilde{u}_k \rightharpoonup b$ in $L^q(\Omega;\mathbb{R}^3)$ as $k\to\infty$. Hence

$$E^-_{\{\varepsilon_{n_k}\}}(u,b;A_1)\le\liminf_{k\to\infty}E^2_{\varepsilon_{n_k}}(u_k;A_1)\le\liminf_{j\to\infty}E^2_{\varepsilon_{n_{k_j}}}(\tilde{u}_{k_j};A_1). \tag{1.11}$$

Thus it remains to estimate the right-hand side of the previous inequality. By the growth condition $(H_1)'$, we have the estimate

$$\begin{aligned}E^2_{\varepsilon_{n_{k_j}}}(\tilde{u}_{k_j};A_1)\le&E^2_{\varepsilon_{n_{k_j}}}(u_{k_j};A_1\setminus\overline{A_3})+E^2_{\varepsilon_{n_{k_j}}}(v_{k_j};A')\\&+C\left(\mathcal{G}_j(u_{k_j};L_\delta)+\mathcal{G}_j(v_{k_j};L_\delta)\right)\\&+\frac{C}{\delta^q}\int_{L_\delta\times I}|u_{k_j}-v_{k_j}|^q\,dx+\frac{C\varepsilon^2_{n_{k_j}}}{\delta^4}\int_{L_\delta\times I}|u_{k_j}-v_{k_j}|^2\,dx\\&+\frac{C\varepsilon^2_{n_{k_j}}}{\delta^2}\int_{L_\delta\times I}|D_pu_{k_j}-D_pv_{k_j}|^2\,dx+\frac{C}{\delta^2}\int_{L_\delta\times I}|D_3u_{k_j}-D_3v_{k_j}|^2\,dx.\end{aligned} \tag{1.12}$$

Since

$$\sup_{j\in\mathbb{N}}\int_{L_\delta\times I}|\varepsilon_{n_{k_j}}D^2u_{k_j}|^2\,dx<\infty,$$

by Poincaré's inequality we have that

$$\int_{L_\delta \times I} |\varepsilon_{n_{k_j}} Du_{k_j} - c_j|^2 \, dx \leq C_\delta \int_{L_\delta \times I} |\varepsilon_{n_{k_j}} D^2 u_{k_j}|^2 \, dx,$$

where

$$c_j := \frac{1}{\mathcal{L}^2 (L_\delta)} \int_{L_\delta \times I} \varepsilon_{n_{k_j}} Du_{k_j} \, dx \to 0.$$

Hence

$$\sup_{j \in \mathbb{N}} \int_{L_\delta \times I} |\varepsilon_{n_{k_j}} Du_{k_j}|^2 \, dx < \infty.$$

It follows by the Rellich-Kondrachov Theorem and the fact that $u_{k_j} \rightharpoonup u$ in $W^{1,q}(\Omega; \mathbb{R}^3)$ that $\varepsilon_{n_{k_j}} Du_{k_j} \to 0$ in $L^2 \left(L_\delta \times I; \mathbb{R}^{3\times 3} \right)$. Again by Poincaré's inequality we have that

$$\int_{L_\delta \times I} |\varepsilon_{n_{k_j}} u_{k_j} - d_j|^2 \, dx \leq C_\delta \int_{L_\delta \times I} |\varepsilon_{n_{k_j}} Du_{k_j}|^2 \, dx,$$

where

$$d_j := \frac{1}{\mathcal{L}^2 (L_\delta)} \int_{L_\delta \times I} \varepsilon_{n_{k_j}} u_{k_j} \, dx \to 0,$$

and so $\varepsilon_{n_{k_j}} u_{k_j} \to 0$ in $L^2 \left(L_\delta \times I; \mathbb{R}^{3\times 3} \right)$. Similar conclusions hold for v_{k_j}. Hence, letting $j \to \infty$ in (1.12) and using (1.9) and (1.10), we have

$$\begin{aligned} E^-_{\{\varepsilon_{n_k}\}}(u, b; A_1) &\leq E^-_{\{\varepsilon_{n_k}\}}(u, b; A_1 \setminus \overline{A_3}) + E^-_{\{\varepsilon_{n_k}\}}(u, b; A') + \eta + C\nu(\overline{L_\delta}) \\ &\leq E^-_{\{\varepsilon_{n_k}\}}(u, b; A_1 \setminus \overline{A_3}) + E^-_{\{\varepsilon_{n_k}\}}(u, b; A_2) + \eta + C\nu(\overline{L_\delta}) \end{aligned}$$

and letting δ go to zero we obtain

$$\begin{aligned} E^-_{\{\varepsilon_{n_k}\}}(u, b; A_1) &\leq E^-_{\{\varepsilon_{n_k}\}}(u, b; A_2) + E^-_{\{\varepsilon_{n_k}\}}(u, b; A_1 \setminus \overline{A_3}) + 2\eta + C\nu(S_t) \\ &= E^-_{\{\varepsilon_{n_k}\}}(u, b; A_2) + E^-_{\{\varepsilon_{n_k}\}}(u, b; A_1 \setminus \overline{A_3}) + 2\eta. \end{aligned}$$

It suffices to let $\eta \to 0^+$. □

As an immediate consequence of the previous theorem we have

$$E^-_{\{\varepsilon_{n_k}\}}(u, b; A) = \int_A \frac{dE^-_{\{\varepsilon_{n_k}\}}(u, b; \cdot)}{d\mathcal{L}^2} (x_\alpha) \, dx_\alpha,$$

where $\frac{dE^-_{\{\varepsilon_{n_k}\}}(u,b;\cdot)}{d\mathcal{L}^2}$ is the Radon-Nikodým derivative of $E^-_{\{\varepsilon_{n_k}\}}(u, b; \cdot)$ with respect to the Lebesgue measure in $\mathbb{R}^2$.

Remark E. A proof almost identical to that of Remark 4.3 in [2] shows that $\overline{\mathcal{W}}_2(\cdot, \cdot)$ is upper semi-continuous on $\mathbb{R}^{2\times 3} \times W^{1,2}(I; \mathbb{R}^3)$ equipped with its strong topology.

We now turn to the proof of Theorem B. The argument is very similar to that of Theorem 4.4 in [2] with the exception that in the proof of the lower bound the additional hypothesis (1.3) allows us to avoid the use of equi-integrable sequences.

Proof. Fix $(u, b) \in \mathcal{V}^2$ and $A \in \mathcal{A}(\omega)$. As usual, we identify u with a function in $W^{1,q}\left(\omega; \mathbb{R}^3\right)$. Also, for simplicity of notation, from now on we write ε_k in place of ε_{n_k}.

Lower bound. We claim that

$$E^-_{\{\varepsilon_k\}} (u, b; A) \geq \int_A \overline{\mathcal{W}}_2(D_p u(x_\alpha) | b(x_\alpha, \cdot)) \, dx_\alpha. \tag{1.13}$$

Consider any sequence $\{u_k\} \subset W^{2,2}\left(\Omega;\mathbb{R}^3\right)$ such that $u_k \rightharpoonup u$ in $W^{1,q}(\Omega;\mathbb{R}^3)$, $\frac{1}{\varepsilon_k}D_3u_k \rightharpoonup b$ in $L^q(\Omega;\mathbb{R}^3)$. Extracting a subsequence, if necessary, we may assume, without loss of generality, that

$$\liminf_{k\to\infty} E^2_{\varepsilon_k}(u_k;A) = \lim_{k\to\infty} E^2_{\varepsilon_k}(u_k;A) \tag{1.14}$$

and that the bounded sequence

$$\mu_k := \left(W\left(D_pu_k\left|\frac{1}{\varepsilon_k}D_3u_k\right.\right) + \varepsilon_k^2\left|D_p^2u_k\right|^2 + \left|D_{p3}u_k\right|^2 + \frac{1}{\varepsilon_k^2}\left|D_{33}u_k\right|^2\right)\mathcal{L}^3\lfloor(A\times I)$$

satisfies

$$\mu_k \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}\left(A\times I\right)$$

for some nonnegative finite Radon measure μ on $A\times I$. Denote by $\hat\mu$ the finite Radon measure on A defined by

$$\hat\mu(B) := \mu\left(B\times I\right),$$

for all Borel sets $B\subset A$. We will show below that the Radon-Nikodým derivative of $\hat\mu$ with respect to the Lebesgue measure on $\mathbb{R}^2$ satisfies

$$\frac{d\hat\mu}{d\mathcal{L}^2}(x_\alpha) \geq \overline{\mathcal{W}}_2(D_pu(x_\alpha)|b(x_\alpha,\cdot)) \tag{1.15}$$

for $\mathcal{L}^2$ a.e. every point $x_\alpha\in A$.

Note that if (1.15) holds, then from (1.14),

$$\begin{aligned}\lim_{k\to\infty} E^2_{\varepsilon_k}(u_k;A) &= \lim_{k\to\infty}\mu_k\left(A\times I\right) \geq \hat\mu(A)\\ &\geq \int_A \frac{d\hat\mu}{d\mathcal{L}^2}(x_\alpha)\,dx_\alpha \geq \int_A \overline{\mathcal{W}}_2(D_pu(x_\alpha)|b(x_\alpha,\cdot))\,dx_\alpha.\end{aligned}$$

Taking the infimum over all admissible sequences $\{u_k\}$ we obtain (1.13).

Step 2: As in Lemma 5.1 in [2], it can be shown that, up to the extraction of a subsequence,

$$\frac{1}{\varepsilon_k}D_3u_k(\cdot,x_3) \rightharpoonup b(\cdot,x_3) \text{ in } L^q(A;\mathbb{R}^3) \text{ for all } x_3\in I \tag{1.16}$$

and that for any Borel subset $B\subset A$ and for all $x_3\in I$,

$$\sup_{k\in\mathbb{N}}\left|\int_B \frac{1}{\varepsilon_k}D_3u_k\left(x_\alpha,x_3\right)\,dx_\alpha\right| < \infty. \tag{1.17}$$

We now address the proof of (1.15).

Since $u\in W^{1,q}(\omega;\mathbb{R}^3)$, for $\mathcal{L}^2$ a.e. $x^0_\alpha\in A$ we have

$$\lim_{\delta\to0^+}\frac{1}{\delta^{2+q}}\int_{Q'(x_\alpha,\delta)}|u(x_\alpha)-u(x^0_\alpha)-D_pu(x^0_\alpha)(x_\alpha-x^0_\alpha)|^q\,dx_\alpha = 0. \tag{1.18}$$

Moreover, viewing b as a Bochner integrable function, that is, an element of

$$L^q\left(A;L^q\left(I;\mathbb{R}^3\right)\right),$$

for $\mathcal{L}^2$ a.e. $x^0_\alpha\in A$ we have

$$\lim_{\delta\to0^+}\int_{-\frac12}^{\frac12}\left|\frac{1}{\delta^2}\int_{Q'(x^0_\alpha,\delta)}b(x_\alpha,x_3)\,dx_\alpha - b(x^0_\alpha,x_3)\right|^q dx_3 = 0. \tag{1.19}$$

Fix a point $x_\alpha^0 \in A$ that satisfies (1.18), (1.19), and such that

$$b(x_\alpha^0, \cdot) \in W^{1,2}\left(I; \mathbb{R}^3\right) \tag{1.20}$$

and

$$\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_\alpha^0) \text{ exists and is finite.}$$

We claim that (1.15) holds at x_α^0.

Consider a sequence $\{\delta_i\}$, with $\delta_i \to 0^+$ such that

$$\lim_{i\to\infty} \frac{1}{(\delta_i)^2} \int_{Q'(x_\alpha^0,\delta_i)} b(x_\alpha, x_3)\, dx_\alpha = b(x_\alpha^0, x_3) \tag{1.21}$$

for $\mathcal{L}^1$ a.e. $x_3 \in I$, and

$$\mu\left(\partial\left(Q'(x_\alpha^0, \delta_i) \times I\right)\right) = 0.$$

From the definition of $\hat{\mu}$ together with that of the Radon-Nikodým derivative, we obtain

$$\begin{aligned}
\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_\alpha^0) &= \lim_{i\to\infty} \frac{\hat{\mu}(Q'(x_\alpha^0, \delta_i))}{(\delta_i)^2} = \lim_{i\to\infty} \frac{\mu\left(Q'(x_\alpha^0, \delta_i) \times I\right)}{(\delta_i)^2} \\
&= \lim_{i\to\infty} \lim_{k\to\infty} \frac{1}{(\delta_i)^2} \left(\int_{Q'(x_\alpha^0,\delta)\times I} W\left(D_p u_k(x) \middle| \frac{1}{\varepsilon_k} D_3 u_k(x)\right) dx \right. \\
&\quad \left. + \int_{Q'(x_\alpha^0,\delta)\times I} \left(\varepsilon_k^2 \left|D_p^2 u_k(x)\right|^2 + |D_{p3} u_k(x)|^2 + \frac{1}{\varepsilon_k^2} |D_{33} u_k(x)|^2 \right) dx \right) \\
&= \lim_{i\to\infty} \lim_{k\to\infty} \left(\int_Q W\left(D_p v_{k,i}(y) \middle| \frac{\delta_i}{\varepsilon_k} D_3 v_{k,i}(y)\right) dy \right. \\
&\quad \left. + \int_Q \left(\left(\frac{\varepsilon_k}{\delta_i}\right)^2 \left|D_p^2 v_{k,i}(y)\right|^2 + |D_{p3} v_{k,i}(y)|^2 + \left(\frac{\delta_i}{\varepsilon_k}\right)^2 |D_{33} v_{k,i}(y)|^2 \right) dy \right),
\end{aligned}$$

where for $y \in Q$,

$$\begin{aligned}
v_{k,i}(y) &:= \frac{u_k(x_\alpha^0 + \delta_i y_\alpha, y_3) - u(x_\alpha^0)}{\delta_i}, \\
u_0(y_\alpha) &:= D_p u(x_\alpha^0) \cdot y_\alpha, \quad b_0(y_3) := b(x_\alpha^0, y_3).
\end{aligned}$$

Note that, since $u_k \to u$ in $L^q(\Omega; \mathbb{R}^3)$ and by (1.18), we have

$$\begin{aligned}
&\lim_{i\to\infty} \lim_{k\to\infty} \int_Q |v_{k,i}(y) - u_0(y_\alpha)|^q \, dy \\
&= \lim_{i\to\infty} \lim_{k\to\infty} \frac{1}{(\delta_i)^{2+q}} \int_{Q'(x_\alpha^0,\delta_i)\times I} |u_k(x) - u(x_\alpha^0) - D_p u(x_\alpha^0) \cdot (x_\alpha - x_\alpha^0)|^q \, dx \\
&= \lim_{i\to\infty} \frac{1}{(\delta_i)^{2+q}} \int_{Q'(x_\alpha^0,\delta_i)} |u(x_\alpha) - u(x_\alpha^0) - D_p u(x_\alpha^0) \cdot (x_\alpha - x_\alpha^0)|^q \, dx_\alpha = 0.
\end{aligned}$$

On the other hand, in view of (1.16), for all $y_3 \in I$,

$$\begin{aligned}
\lim_{k\to\infty} \int_{Q'} \frac{\delta_i}{\varepsilon_k} D_3 v_{k,i}(y_\alpha, y_3)\, dy_\alpha &= \lim_{k\to\infty} \frac{1}{(\delta_i)^2} \int_{Q'(x_\alpha^0,\delta_i)} \frac{1}{\varepsilon_k} D_3 u_k(x_\alpha, y_3)\, dx_\alpha \\
&= \frac{1}{(\delta_i)^2} \int_{Q'(x_\alpha^0,\delta_i)} b(x_\alpha, y_3)\, dx_\alpha,
\end{aligned}$$

and so by (1.17) it follows from Lebesgue's Dominated Convergence Theorem that

$$\lim_{k\to\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\int_{Q'}\frac{\delta_i}{\varepsilon_k}D_3v_{k,i}(y_\alpha,y_3)\,dy_\alpha-b_0(y_3)\right|^q dy_3$$
$$=\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{1}{(\delta_i)^2}\int_{Q'(x_\alpha^0,\delta_i)}b(x_\alpha,x_3)\,dx_\alpha-b(x_\alpha^0,x_3)\right|^q dx_3.$$

By (1.19) we have

$$\lim_{i\to\infty}\lim_{k\to\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\int_{Q'}\frac{\delta_i}{\varepsilon_k}D_3v_{k,i}(y_\alpha,y_3)\,dy_\alpha-b_0(y_3)\right|^q dy_3=0.$$

By a standard diagonalization argument, we may extract subsequences $v_i := v_{k_i,i}$ and $\varepsilon_i := \frac{\varepsilon_{k_i}}{\delta_i} \to 0^+$ such that

$$\infty > \frac{d\hat{\mu}}{d\mathcal{L}^2}(x_\alpha^0) = \lim_{i\to\infty}\left(\int_Q W\left(D_pv_i\left|\frac{1}{\varepsilon_i}D_3v_i\right.\right)\right. \tag{1.22}$$
$$\left.+\int_Q\left(\varepsilon_i^2\left|D_p^2v_i\right|^2+\left|D_{p3}v_i\right|^2+\frac{1}{\varepsilon_i^2}\left|D_{33}v_i\right|^2\right)dy\right),$$
$$\lim_{i\to\infty}\int_Q|v_i-u_0|^q\,dy=0, \tag{1.23}$$
$$\lim_{i\to\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\int_{Q'}\frac{1}{\varepsilon_i}D_3v_i(y_\alpha,y_3)\,dy_\alpha-b_0(y_3)\right|^q dy_3=0. \tag{1.24}$$

Reasoning as in Theorem D, we can assume, without loss of generality, that $v_i = u_0$ in a neighborhood of $\partial Q' \times I$.

For $y \in Q$ define

$$\varphi_i(y) := v_i(y) - u_0(y_\alpha) - \int_{Q'}(v_i-u_0)(w_\alpha,y_3)\,dw_\alpha.$$

Then

$$D_p\varphi_i(y) = D_pv_i(y) - D_pu(x_\alpha^0), \tag{1.25}$$
$$\frac{1}{\varepsilon_i}D_3\varphi_i(y) = \frac{1}{\varepsilon_i}D_3v_i(y) - \int_{Q'}\frac{1}{\varepsilon_i}D_3v_i(w_\alpha,y_3)\,dw_\alpha, \tag{1.26}$$
$$\frac{1}{\varepsilon_i}D_{33}\varphi_i(y) = \frac{1}{\varepsilon_i}D_{33}v_i(y) - \int_{Q'}\frac{1}{\varepsilon_i}D_{33}v_i(w_\alpha,y_3)\,dw_\alpha, \tag{1.27}$$

and note that

$$\int_{Q'}D_3\varphi_i(y_\alpha,y_3)\,dy_\alpha = 0 \text{ for all } y_3\in I. \tag{1.28}$$

Since $\varphi_i(\cdot, y_3)$ is Q'-periodic for $\mathcal{L}^1$ a.e. y_3, it follows that φ_i is admissible in the definition of $\overline{\mathcal{W}}_2(D_p u(x_\alpha^0)|b(x_\alpha^0, \cdot))$. Moreover,

$$\begin{aligned}
&\int_Q W\left(D_p v_i\left(y\right) \middle| \frac{1}{\varepsilon_i} D_3 v_i\left(y\right)\right)\, dy \\
&+ \int_Q \left(\varepsilon_i^2 \left|D_p^2 v_i\left(y\right)\right|^2 + \left|D_{p3} v_i\left(y\right)\right|^2 + \frac{1}{\varepsilon_i^2}\left|D_{33} v_i\left(y\right)\right|^2\right) dy \\
&= \int_Q W\left(D_p u(x_\alpha^0) + D_p \varphi_i\left(y\right) \middle| b_0(y_3) + \frac{1}{\varepsilon_i} D_3 \varphi_i\left(y\right) + z_i\left(y_3\right)\right) \\
&+ \int_Q \left(\varepsilon_i^2 \left|D_p^2 \varphi_i\left(y\right)\right|^2 + \left|D_{p3}\varphi_i\left(y\right)\right|^2 + \left|b_0'(y_3) + \frac{1}{\varepsilon_i} D_{33}\varphi_i\left(y\right) + z_i'\left(y_3\right)\right|^2\right) dy,
\end{aligned} \tag{1.29}$$

where

$$z_i\left(y_3\right) := \int_{Q'} \frac{1}{\varepsilon_i} D_3 v_i(w_\alpha, y_3)\, dw_\alpha - b_0(y_3). \tag{1.30}$$

Since by (1.28),

$$\int_{Q'} D_{33}\varphi_i(y_\alpha, y_3)\, dy_\alpha = D_3\left(\int_{Q'} D_3\varphi_i(y_\alpha, y_3)\, dy_\alpha\right) = 0 \text{ for all } y_3 \in I,$$

it follows that

$$\begin{aligned}
\int_Q \left|b_0' + \frac{1}{\varepsilon_i} D_{33}\varphi_i + z_i'\right|^2 dy &\geq \int_Q \left|b_0' + \frac{1}{\varepsilon_i} D_{33}\varphi_i\right|^2 dy + 2\int_Q \left(b_0' + \frac{1}{\varepsilon_i} D_{33}\varphi_i\right)\cdot z_i'\, dy \\
&= \int_Q \left|b_0' + \frac{1}{\varepsilon_i} D_{33}\varphi_i\right|^2 dy + 2\int_Q b_0' \cdot z_i'\, dy.
\end{aligned} \tag{1.31}$$

We claim that

$$z_i \rightharpoonup 0 \text{ in } W^{1,2}\left(I; \mathbb{R}^3\right). \tag{1.32}$$

If the claim holds, then letting $i \to \infty$ in the previous inequality yields

$$\limsup_{i\to\infty} \int_Q \left|b_0' + \frac{1}{\varepsilon_i} D_{33}\varphi_i + z_i'\right|^2 dy \geq \limsup_{i\to\infty} \int_Q \left|b_0' + \frac{1}{\varepsilon_i} D_{33}\varphi_i\right|^2 dy. \tag{1.33}$$

To prove (1.32) note that, up to a subsequence, from (1.22) and (1.24) we may assume that $z_i\left(y_3\right) \to 0$ for $\mathcal{L}^1$ a.e. $y_3 \in I$ and that

$$\sup_i \int_Q \left|\frac{1}{\varepsilon_i} D_{33} v_i\right|^2 dy < \infty.$$

Hence by Hölder's Inequality,

$$\begin{aligned}
\int_{-\frac12}^{\frac12} \left|z_i'\left(y_3\right)\right|^2 dy_3 &\leq 2\int_{-\frac12}^{\frac12} \left[\left|\int_{Q'} \frac{1}{\varepsilon_i} D_{33} v_i(w_\alpha, y_3)\, dw_\alpha\right|^2 + \left|b_0'(y_3)\right|^2\right] dy_3 \\
&\leq 2\int_{-\frac12}^{\frac12} \left[\int_{Q'} \left|\frac{1}{\varepsilon_i} D_{33} v_i(w_\alpha, y_3)\right|^2 dw_\alpha + \left|b_0'(y_3)\right|^2\right] dy_3,
\end{aligned}$$

and so also by (1.20),

$$\sup_i \int_{-\frac{1}{2}}^{\frac{1}{2}} |z_i'(y_3)|^2 \, dy_3 < \infty.$$

By extracting a further subsequence, if necessary, and appealing to (1.24) we have shown (1.32). In particular, $z_i \to 0$ uniformly. Moreover, by the coercivity hypothesis $(H_1)'$ and (1.22),

$$\sup_i \int_Q \left(|D_p \varphi_i(y)|^q + \left| \frac{1}{\varepsilon_i} D_3 \varphi_i(y) \right|^q \right) dy < \infty.$$

Hence, using the q-Lipschitz condition (1.3) we obtain

$$\begin{aligned} &\int_Q W\left(D_p u(x_\alpha^0) + D_p \varphi_i \left| b_0 + \frac{1}{\varepsilon_i} D_3 \varphi_i + z_i \right) dy \right. \\ &\geq \int_Q W\left(D_p u(x_\alpha^0) + D_p \varphi_i \left| b_0 + \frac{1}{\varepsilon_i} D_3 \varphi_i \right) dy + o(1). \right. \end{aligned}$$

In turn, using also (1.22), (1.33), we have that

$$\begin{aligned} \frac{d\hat{\mu}}{d\mathcal{L}^2}(x_\alpha^0) \geq \limsup_{i \to \infty} &\left(\int_Q \left[W\left(D_p u(x_\alpha^0) + D_p \varphi_i \left| b_0 + \frac{1}{\varepsilon_i} D_3 \varphi_i \right) \right] dy \right. \right. \\ &\left. + \int_Q \left(\varepsilon_i^2 \left| D_p^2 \varphi_i \right|^2 + |D_{p3} \varphi_i|^2 + \left| b_0' + \frac{1}{\varepsilon_i} D_{33} \varphi_i \right|^2 \right) dy \right). \end{aligned}$$

Since by construction φ_i are admissible functions in the definition of $\overline{\mathcal{W}}(D_p u(x_\alpha^0)|b(x_\alpha^0, \cdot))$ (see (1.28)), it follows that

$$\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_\alpha^0) \geq \overline{\mathcal{W}}_2(D_p u(x_\alpha^0)|b(x_\alpha^0, \cdot)),$$

and the proof of (1.15) is complete.

Upper bound. We first prove the upper bound

$$E^-_{\{\varepsilon_k\}}(u, b; A) \leq \int_A \overline{\mathcal{W}}_2(D_p u(x_\alpha)|b(x_\alpha, \cdot)) \, dx_\alpha \tag{1.34}$$

when $u(x_\alpha) = \overline{F} x_\alpha + c$ for all $x_\alpha \in Q'$ and for some $\overline{F} \in \mathbb{R}^{3\times 2}$, $c \in \mathbb{R}^3$, and $b \in W^{1,2}(I; \mathbb{R}^3)$. For $\eta > 0$ fixed, choose $\varphi \in W^{2,\infty}(Q; \mathbb{R}^3)$, and $L > 0$, with $\varphi(\cdot, x_3)$ Q'-periodic and $\int_{Q'} D_3 \varphi(x_\alpha, x_3) \, dx_\alpha = 0$ for $\mathcal{L}^1$ a.e. $x_3 \in I$, such that

$$\begin{aligned} \int_Q \Bigg[W\left(\overline{F} + D_p \varphi(x) | b(x_3) + \left| \frac{1}{L} D_p^2 \varphi(x) \right|^2 + |D_{p3} \varphi(x)|^2 + L D_3 \varphi(x) \right) & \\ + |b'(x_3) + L D_{33} \varphi(x)|^2 \Bigg] \, dx & \\ \leq \overline{\mathcal{W}}_2(\overline{F}|b) + \eta \, . & \end{aligned} \tag{1.35}$$

Extend $\varphi(\cdot, x_3)$ periodically with period Q' and for $x \in \Omega$ define

$$u_k(x_\alpha, x_3) := \overline{F} x_\alpha + c + \varepsilon_k \int_0^{x_3} b(s) \, ds + L \varepsilon_k \varphi\left(\frac{x_\alpha}{L \varepsilon_k}, x_3 \right). \tag{1.36}$$

Since b and φ are bounded, it follows that $\{u_k\}$ converges uniformly to u. In addition, for $x \in \Omega$ we have that

$$D_p u_k(x) = \overline{F} + D_p\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right),$$
$$\frac{1}{\varepsilon_k} D_3 u_k(x) = b(x_3) + L D_3\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right).$$

Hence $\{D_p u_k\}$ is bounded in L^∞ while $\{D_3 u_k\}$ goes to 0 in L^q, and so $u_k \rightharpoonup u$ in $W^{1,q}(\Omega;\mathbb{R}^3)$, while by the Riemann-Lebesgue Lemma, Fubini's Theorem, and the fact that

$$\int_{Q'} D_3\varphi(x_\alpha, x_3)\, dx_\alpha = 0$$

for $\mathcal{L}^1$ a.e. $x_3 \in I$, we have that $\frac{1}{\varepsilon_k} D_3 u_k \rightharpoonup b$ in $L^q(\Omega;\mathbb{R}^3)$. This proves that the sequence $\{u_k\}$ is admissible for $E^-_{\{\varepsilon_k\}}(u, b; A)$, and we have

$$\begin{aligned} E^-_{\{\varepsilon_k\}}(u,b;A) &\le \liminf_{k\to\infty} E^2_{\varepsilon_k}(u_k;A) \\ &= \liminf_{k\to\infty} \left(\int_{A\times I} W\left(D_p u_k \middle| \frac{1}{\varepsilon_k} D_3 u_k\right) dx \right. \\ &\quad \left. + \int_{A\times I} \varepsilon_k^2 \left(\left|D_p^2 u_k\right|^2 + \frac{1}{\varepsilon_k^2}\left|D_{p3}u_k\right|^2 + \frac{1}{\varepsilon_k^4}\left|D_{33}u_k\right|^2 \right) dx \right). \end{aligned} \tag{1.37}$$

We have that

$$\begin{aligned} &\int_{A\times I} W\left(D_p u_k(x) \middle| \frac{1}{\varepsilon_k} D_3 u_k(x)\right) dx \\ &= \int_{A\times I} W\left(\overline{F} + D_p\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right) \middle| b(x_3) + L D_3\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right)\right) dx. \end{aligned} \tag{1.38}$$

On the other hand, for $x \in \Omega$ we have that

$$\begin{aligned} D_p^2 u_k(x) &= \frac{1}{L\varepsilon_k} D_p^2\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right), \\ D_{p3} u_k(x) &= D_{p3}\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right), \\ D_{33} u_k(x) &= \varepsilon_k b'(x_3) + L\varepsilon_k D_{33}\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right), \end{aligned}$$

and so

$$\int_{A\times I} \varepsilon_k^2 \left(\left|D_p^2 u_k\right|^2 + \frac{1}{\varepsilon_k^2}\left|D_{p3}u_k\right|^2 + \frac{1}{\varepsilon_k^4}\left|D_{33}u_{n,k}\right|^2 \right) dx \tag{1.39}$$

$$= \int_{A\times I} \left(\frac{1}{L^2}\left|D_p^2\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right)\right|^2 + \left|D_{p3}\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right)\right|^2 \right.$$

$$\left. + \left|b'(x_3) + L D_{33}\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right)\right|^2 \right) dx. \tag{1.40}$$

Since, for $\mathcal{L}^1$ a.e. $x_3 \in I$ the function

$$W\left(\overline{F} + D_p\varphi(\cdot, x_3)\,|b(x_3) + LD_3\varphi(\cdot, x_3)\right) + \frac{1}{L^2}\left|D_p^2\varphi(\cdot, x_3)\right|^2 + |D_{p3}\varphi(\cdot, x_3)|^2 + |b'(x_3) + LD_{33}\varphi(\cdot, x_3)|^2$$

is Q'-periodic, it converges weakly in $L^1(A)$ to its mean, that is, to

$$\int_{Q'}\left(W\left(\overline{F} + D_p\varphi(x_\alpha, x_3)\,|b(x_3) + LD_3\varphi(x_\alpha, x_3)\right) + \frac{1}{L^2}\left|D_p^2\varphi\left(\frac{x_\alpha}{L}, x_3\right)\right|^2 + \left|D_{p3}\varphi\left(\frac{x_\alpha}{L}, x_3\right)\right|^2 + \left|b'(x_3) + LD_{33}\varphi\left(\frac{x_\alpha}{L}, x_3\right)\right|^2\right) dx_\alpha.$$

Lebesgue's Dominated Convergence Theorem and Fubini's Theorem imply that

$$\begin{aligned}&\lim_{k\to\infty}\int_{A\times I} W\left(\overline{F} + D_p\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right)\middle|\, b(x_3) + LD_3\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right)\right) dx\\ &= \mathcal{L}^2(A)\int_Q W\left(\overline{F} + D_p\varphi(x_\alpha, x_3)\,|b(x_3) + LD_3\varphi(x_\alpha, x_3)\right) dx\end{aligned}$$

and

$$\begin{aligned}&\lim_{k\to\infty}\int_{A\times I}\left(L^2\left|D_p^2\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right)\right|^2 + \left|D_{p3}\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right)\right|^2\right.\\ &\left.+\left|b'(x_3) + LD_{33}\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right)\right|^2\right) dx \qquad (1.41)\\ &= \mathcal{L}^2(A)\int_Q\left(\frac{1}{L^2}\left|D_p^2\varphi(x_\alpha, x_3)\right|^2 + |D_{p3}\varphi(x_\alpha, x_3)|^2 + |b'(x_3) + LD_{33}\varphi(x_\alpha, x_3)|^2\right) dx,\end{aligned}$$

which, in view of (1.35), (1.37), (1.38), and (1.39), finally yields

$$E^-_{\{\varepsilon_{n_k}\}}(u, b; A) \le \mathcal{L}^2(A)[\overline{\mathcal{W}}_2(\overline{F}|b) + \eta].$$

Letting η tend to 0, we conclude

$$E^-_{\{\varepsilon_{n_k}\}}(u, b; A) \le \mathcal{L}^2(A)\overline{\mathcal{W}}_2(\overline{F}|b). \qquad (1.42)$$

This proves (1.34) when $u(x_\alpha) = \overline{F}x_\alpha + c$ for all $x_\alpha \in Q'$ and $b \in W^{1,2}\left(I; \mathbb{R}^3\right)$. The general case follows as in Steps 2 and 3 of Theorem 4.4 in [2]. We omit the details. □

In order to address the case $\gamma > 2$ we first recall the result obtained in [1], where

$$\mathcal{I}_\varepsilon(u; A) := \int_{A\times I} W\left(D_p u\,\middle|\,\frac{1}{\varepsilon}D_3 u\right) dx$$

if $u \in W^{1,q}(\Omega; \mathbb{R}^3)$, and $\mathcal{I}_\varepsilon(u; A) := \infty$ otherwise. Fix a countable dense family $\{\theta_i\}_{i\in\mathbb{N}}$ in $L^{q'}(I; \mathbb{R}^3)$, where q' is the conjugate exponent of q. For $\overline{F} \in \mathbb{R}^{2\times 3}$, $b \in L^q(I; \mathbb{R}^3)$ we define

$$\mathcal{Q}_\infty W(\overline{F}|b) := \sup_n \mathcal{Q}_n W(\overline{F}|b) = \lim_{nj\to\infty}\mathcal{Q}_n W(\overline{F}|b),$$

where

$$\mathcal{Q}_n W(\overline{F}|b) := \inf_{\varphi,L}\left\{\int_Q W\left(\overline{F}+D_p\varphi(x)|b\left(x_3\right)+LD_3\varphi\left(x\right)\right)\,dx\,:\right.$$
$$L>0,\ \varphi\in W^{1,q}(Q;\mathbb{R}^3),\ \varphi(\cdot,x_3)\ Q'\text{-periodic for } \mathcal{L}^1 \text{ a.e. } x_3, \tag{1.43}$$
$$\left.\left|\int_Q LD_3\varphi(x_\alpha,x_3)\theta_i\left(x_3\right)\,dx\right|\le\frac{1}{n},\ i=1,\ldots,n\right\}.$$

REMARK F. If W satisfies $(H_1)'$, then a density argument shows that in the definition above it is possible to restrict admissible functions $\varphi\in W^{1,q}(Q;\mathbb{R}^3)$ to functions $\varphi\in C^\infty(\overline{Q};\mathbb{R}^3)$.

The main result in [1] is that under condition $(H_1)'$,

$$\mathcal{I}(u,b;A) := \inf\left\{\liminf_{\varepsilon\to0^+}\mathcal{I}_\varepsilon(u_\varepsilon;A):\ u_\varepsilon\in\ W^{1,q}\left(A\times I;\mathbb{R}^3\right),\right.$$
$$=\ u_\varepsilon\rightharpoonup u \text{ in } W^{1,q}\left(A\times I;\mathbb{R}^3\right),\quad \left.\frac{1}{\varepsilon}D_3u_\varepsilon\rightharpoonup b \text{ in } L^q\left(A\times I;\mathbb{R}^3\right)\right\} \tag{1.44}$$
$$=\int_A \mathcal{Q}_\infty W(D_pu(x_\alpha)|b(x_\alpha,\cdot))\,dx_\alpha$$

for all $(u,b)\in W^{1,q}\left(A;\mathbb{R}^3\right)\times L^q\left(A\times I;\mathbb{R}^3\right)$ and $A\in\mathcal{A}(\omega)$.

Next we show that the $\Gamma-\liminf E_-^\gamma(u,b;A)$ coincides with $\mathcal{I}(u,b;A)$. This establishes that in the case $\gamma>2$ the second-order perturbation plays no role.

THEOREM G. Assume that $\gamma>2$ and that condition $(H_1)'$ is satisfied. Then for all $(u,b)\in\mathcal{V}^\gamma$ and $A\in\mathcal{A}(\omega)$,

$$E_-^\gamma(u,b;A)=\int_A \mathcal{Q}_\infty W(D_pu(x_\alpha)|b(x_\alpha,\cdot))\,dx_\alpha.$$

Proof. For any given sequence $\{\varepsilon_n\}$, with $\varepsilon_n\to0^+$, we extract a subsequence $\{\varepsilon_{n_k}\}$ such that

$$\Gamma\text{-}\lim_{k\to\infty}E^\gamma_{\varepsilon_{n_k}}(u,b;A) \text{ exists for all } (u,b)\in\mathcal{V}^\gamma \text{ and for all } A'\in\mathcal{R}(\omega),$$

and we define the Γ-lower limit

$$E^-_{\{\varepsilon_{n_k}\}}(u,b;A) := \inf\left\{\liminf_{k\to\infty}E^\gamma_{\varepsilon_{n_k}}(u_k;A):\ u_k\in W^{2,2}\left(\Omega;\mathbb{R}^3\right),\right.$$
$$\left. u_k\rightharpoonup u \text{ in } W^{1,q}\left(\Omega;\mathbb{R}^3\right),\quad \frac{1}{\varepsilon_{n_k}}D_3u_{n_k}\rightharpoonup b \text{ in } L^q\left(\Omega;\mathbb{R}^3\right)\right\}.$$

As in the case $\gamma=2$, it suffices to show that

$$E^-_{\{\varepsilon_{n_k}\}}(u,b;A)=\int_A \mathcal{Q}_\infty W(D_pu(x_\alpha)|b(x_\alpha,\cdot))\,dx_\alpha.$$

In the sequel, to simplify the notation, we write ε_k in place of ε_{n_k}.
Lower bound. In view of (1.44) we deduce immediately that

$$E^-_{\{\varepsilon_k\}}(u,b;A)\ge\mathcal{I}(u,b;A)=\int_A \mathcal{Q}_\infty W(D_pu(x_\alpha)|b(x_\alpha,\cdot))\,dx_\alpha.$$

Upper bound. We first prove the upper bound

$$E^-_{\{\varepsilon_k\}}(u,b;A) \le \int_A \mathcal{Q}_\infty W(D_p u(x_\alpha)|b(x_\alpha,\cdot))\,dx_\alpha \tag{1.45}$$

when $u\,(x_\alpha) = \overline{F}x_\alpha + c$ for all $x_\alpha \in Q'$ and for some $\overline{F} \in \mathbb{R}^{3\times 2}$, $c \in \mathbb{R}^3$, and $b \in L^q\left(I;\mathbb{R}^3\right)$. In this case the right-hand side reduces to $\mathcal{Q}_\infty W(\overline{F}|b(\cdot))\mathcal{L}^2\,(A)$. In view of $(H_1)'$ we have that

$$\mathcal{Q}_\infty W(\overline{F}|b(\cdot)) < \infty. \tag{1.46}$$

By definition of $\mathcal{Q}_n W(\overline{F}|b(\cdot))$ (see (1.43) and Remark F) we may find admissible $\varphi_n \in C^\infty(\overline{Q};\mathbb{R}^3)$ and $L_n > 0$ such that

$$\int_Q W\left(\overline{F} + D_p\varphi_n(x)|b\,(x_3) + L_n D_3\varphi_n\,(x)\right)\,dx \le \mathcal{Q}_n W(\overline{F}|b(\cdot)) + \frac{1}{n}. \tag{1.47}$$

Note that in view of (1.46) and $(H_1)'$ we have that

$$\sup_n \|(D_p\varphi_n|L_n D_3\varphi_n)\|_{L^q(Q;\mathbb{R}^{3\times 3})} < \infty. \tag{1.48}$$

Extend $\varphi(\cdot,x_3)$ to $\mathbb{R}^2$ periodically with period Q' and extend b to $\mathbb{R}$ by zero, and for $x \in \Omega$ define

$$u_{k,n}(x_\alpha,x_3) := \overline{F}x_\alpha + c + \varepsilon_k \int_0^{x_3} (b*\rho_k)\,(s)\;ds + L_n\varepsilon_k\varphi_n\left(\frac{x_\alpha}{L_n\varepsilon_k},x_3\right),$$

where

$$\rho_k\,(t) := \frac{1}{\delta_k}\rho\left(\frac{t}{\delta_k}\right), \quad \delta_k := \varepsilon_k^{\frac{\gamma-2}{4}},$$

and $\rho \in C_c^\infty\,(\mathbb{R})$, $\rho \ge 0$, and $\int_{\mathbb{R}} \rho\,(t)\;dt = 1$. Note that

$$\lim_{n\to\infty}\lim_{k\to\infty} \|u_{k,n} - u\|_{L^q(\Omega;\mathbb{R}^3)} = 0. \tag{1.49}$$

Moreover, for $x \in \Omega$ we have that

$$D_p u_{k,n}\,(x) = \overline{F} + D_p\varphi_n\left(\frac{x_\alpha}{L_n\varepsilon_k},x_3\right),$$

$$\frac{1}{\varepsilon_k}D_3 u_{k,n}\,(x) = (b*\rho_k)\,(x_3) + L_n D_3\varphi_n\left(\frac{x_\alpha}{L_n\varepsilon_k},x_3\right).$$

Since $\varphi_n(\cdot,x_3)$ is Q'-periodic for $\mathcal{L}^1$ a.e. x_3, by (1.48) it follows that

$$\sup_{k,n\in\mathbb{N}} \left\|\left(D_p u_{k,n}\left|\frac{1}{\varepsilon_k}D_3 u_{k,n}\right.\right)\right\|_{L^q(\Omega;\mathbb{R}^{3\times 3})} < \infty. \tag{1.50}$$

For every $i \in \mathbb{N}$, by (1.43),

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{k\to\infty}\left|\int_\Omega \left(\frac{1}{\varepsilon_k}D_3 u_{k,n}\,(x) - b\,(x_3)\right)\theta_i\,(x_3)\;dx\right| \\
&= \lim_{n\to\infty}\lim_{k\to\infty}\left|\int_\Omega \left((b*\rho_k)\,(x_3) - b\,(x_3) + L_n D_3\varphi_n\left(\frac{x_\alpha}{L_n\varepsilon_k},x_3\right)\right)\theta_i\,(x_3)\;dx\right| \qquad (1.51)\\
&= \lim_{n\to\infty}\left|\int_{\omega\times I}\int_{Q'} L_n D_3\varphi_n\,(y_\alpha,x_3)\;dy_\alpha\theta_i\,(x_3)\;dx\right| = 0,
\end{aligned}$$

where we have used the Riemann-Lebesgue Lemma and the fact that for $n \geq i$,

$$\left| \int_Q L_n D_3 \varphi_n(x_\alpha, x_3) \theta_i(x_3)\, dx \right| \leq \frac{1}{n}.$$

Also by (1.47), $(H_1)'$, the Lebesgue Dominated Convergence Theorem, and the Riemann-Lebesgue Lemma we have that

$$\begin{aligned}
&\lim_{n\to\infty} \lim_{k\to\infty} \int_{A\times I} W\left(D_p u_{k,n}(x) \left| \frac{1}{\varepsilon_k} D_3 u_{k,n}(x) \right.\right) dx \\
&= \lim_{n\to\infty} \lim_{k\to\infty} \int_{A\times I} W\left(\overline{F} + D_p \varphi_n\left(\frac{x_\alpha}{L_n \varepsilon_k}, x_3 \right) \left| (b * \rho_k)(x_3) + L_n D_3 \varphi_n\left(\frac{x_\alpha}{L_n \varepsilon_k}, x_3 \right) \right.\right) dx \qquad (1.52)\\
&= \lim_{n\to\infty} \int_{A\times I} \int_{Q'} W\left(\overline{F} + D_p \varphi_n(y_\alpha, x_3) \,|\, b(x_3) + L_n D_3 \varphi_n(y_\alpha, x_3) \right) dy_\alpha\, dx \\
&\leq \mathcal{L}^2(A) \lim_{n\to\infty} \mathcal{Q}_n W(\overline{F}|b(\cdot)) = \mathcal{L}^2(A)\, \mathcal{Q}_\infty W(\overline{F}|b(\cdot)).
\end{aligned}$$

Finally,

$$\begin{aligned}
&\lim_{n\to\infty} \lim_{k\to\infty} \int_{A\times I} \varepsilon_k^\gamma \left(\left| D_p^2 u_{k,n} \right|^2 + \frac{1}{\varepsilon_k^2} \left| D_{p3} u_{k,n} \right|^2 + \frac{1}{\varepsilon_k^4} \left| D_{33} u_{k,n} \right|^2 \right) dx \\
&= \lim_{n\to\infty} \lim_{k\to\infty} \int_{A\times I} \varepsilon_k^\gamma \left(\left| \frac{1}{L_n \varepsilon_k} D_p^2 \varphi_n\left(\frac{x_\alpha}{L_n \varepsilon_k}, x_3 \right) \right|^2 + \frac{1}{\varepsilon_k^2} \left| D_{p3} \varphi_n\left(\frac{x_\alpha}{L_n \varepsilon_k}, x_3 \right) \right|^2 \right. \qquad (1.53)\\
&\left. + \frac{1}{\varepsilon_k^4} \left| \varepsilon_k (b * \rho_k')(x_3) + L_n \varepsilon_k D_{33} \varphi_n\left(\frac{x_\alpha}{L_n \varepsilon_k}, x_3 \right) \right|^2 \right) dx = 0,
\end{aligned}$$

where we have used the facts that $\gamma > 2$,

$$\lim_{k\to\infty} \int_{A\times I} \left| D^2 \varphi_n\left(\frac{x_\alpha}{L_n \varepsilon_k}, x_3 \right) \right|^2 dx = \mathcal{L}^2(A) \int_{Q'\times I} \left| D^2 \varphi_n(y_\alpha, x_3) \right|^2 dy_\alpha\, dx_3,$$

and

$$\lim_{k\to\infty} \int_I \varepsilon_k^{\gamma-2} \left| b * \rho_k'(x_3) \right|^2 dx_3 = 0,$$

because

$$\begin{aligned}
\|b * \rho_k'\|_{L^\infty(I;\mathbb{R}^3)} &\leq C \|\rho_k'\|_{L^\infty(I)} \|b\|_{L^q(I;\mathbb{R}^3)} \\
&\leq \frac{C}{\delta_k^2} \|b\|_{L^q(I;\mathbb{R}^3)} = \frac{C}{\varepsilon_k^{\frac{\gamma-2}{2}}} \|b\|_{L^q(I;\mathbb{R}^3)}.
\end{aligned}$$

Hence, recalling (1.49), (1.50), (1.51), (1.52), and (1.53), we may find a diagonal sequence

$$u_n := u_{k_n,n}$$

such that $u_n \rightharpoonup u$ in $W^{1,q}(\Omega;\mathbb{R}^3)$ and $\frac{1}{\varepsilon_{k_n}} D_3 u_n \rightharpoonup b$ in $L^q(\Omega;\mathbb{R}^3)$ and

$$\limsup_{n\to\infty} E^\gamma_{\varepsilon_{k_n}}(u_n; A) \leq \mathcal{L}^2(A)\, \mathcal{Q}_\infty W(\overline{F}|b(\cdot)).$$

Define

$$\tilde{u}_k(x) := \begin{cases} u_{k_n,n}(x) & \text{if } k = k_n \text{ for some } n \in \mathbb{N}, \\ \overline{F} x_\alpha + c + \varepsilon_k \int_0^{x_3} (b * \rho_k)(s)\, ds & \text{otherwise.} \end{cases}$$

Since the sequence $\{\tilde{u}_k\}$ is admissible for $E^-_{\{\varepsilon_k\}}(u,b;A)$, we have

$$E^-_{\{\varepsilon_k\}}(u,b;A) \leq \liminf_{n\to\infty} E^\gamma_{\varepsilon_{k_n}}(u_n;A) \leq \mathcal{L}^2\,(A)\,\mathcal{Q}_\infty W(\overline{F}|b(\cdot)).$$

Just as in the proof of Theorem D, it can be shown that for every $(u,b) \in \mathcal{V}^\gamma$, $E^-_{\{\varepsilon_k\}}(u,b;\cdot)$ is the trace of a Radon measure absolutely continuous with respect to $\mathcal{L}^2 \lfloor \omega$. Therefore, to establish the inequality (1.45) for arbitrary $(u,b) \in \mathcal{V}^\gamma$ we may proceed as in the proof of Steps 2 and 3 of Theorem 4.4 in [2]. We omit the details. □

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