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A DERIVATION OF THE AW–RASCLE TRAFFIC MODELS FROM FOKKER–PLANCK TYPE KINETIC MODELS

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Abstract. We show how the Aw–Rascle model, a hyperbolic system of PDEs modeling traffic flow, can be derived from a simplified Fokker–Planck type kinetic equation.

1. Introduction. We are concerned with Fokker–Planck type kinetic models for multilane traffic flow on a highway as introduced in [4], [5]. The full Fokker–Planck model, homogenized over all lanes, is given by

$$\partial_t f + v\partial_x f + \partial_v \left(B[f](\rho, u, v-u)f - D_\epsilon[f](\rho, u, v-u)\partial_v f\right) \quad = \quad 0. \tag{1}$$

By $f = f(t, x, v)$ we denote the numerical density of cars which at time t are at location $x \in \mathbb{R}$ and move with speed $v \in [0, v_{max}]$. Here, v_{max} is the speed limit. $B[f]$ denotes a braking/acceleration force and D_ϵ is a nondegenerate diffusion term modeling the inability of a driver to observe speeds with accuracy. For $\epsilon = 0, D_\epsilon = D_0$ may be degenerate, as used in [5].

In the original model [5], it was further assumed that all drivers have the same constant reaction time $\tau > 0$ and observe braking and acceleration thresholds $H_B := x + H_0 + T_B v$ and $H_A := x + H_0 + T_A v$, where v is the driver's speed and T_B and T_A are reaction times (in general different from τ). This means that a driver at x moving with speed v will brake in reaction to a traffic condition observed at $x + H_0 + T_B v$, and if no condition for braking applies, he/she will accelerate (if possible) in reaction to a traffic condition observed at $x + H_0 + T_A v$. As we will see below, the nonlocalities prove to be a key for

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deriving the AR–model from a simplified version of (1). H_0 is a safety distance in the range of one to three car lengths.

To model driver behavior, the braking term B and the diffusion coefficient D_ε in [4] were chosen as follows:

$$B[f](\rho,u,v-u)=\begin{cases} -c_B\rho(v-u^B)^2(1-P(u^B,v-u^B;b)), & v1 \;>\; u^B,\\ c_A(\rho_{max}-\rho^A)(v-u^A)^2, & v \;\leq\; u^A \text{ and } v\leq u^B,\\ 0, & \text{otherwise;}\end{cases}$$

$$D_\epsilon[f](\rho,u,v-u)=\begin{cases} \sigma(\rho^B,u^B)|v-u^B|^\gamma+\epsilon, & v \;>\; u^B,\\ \sigma(\rho^A,u^A)|v-u^A|^\gamma+\epsilon, & \text{otherwise.}\end{cases}$$

Here, ρ^X and u^X for $X\in\{A,B\}$ incorporate the nonlocalities:

$$\rho^X \;:=\; \rho(t-\tau,x+H_0+T_Xv), \tag{2a}$$
$$u^X \;:=\; u(t-\tau,x+H_0+T_Xv). \tag{2b}$$

This exact form for the braking and diffusion terms was chosen to make c_A and c_B dimensionless; cf. [5]. More importantly, for $v>u^B$ we decelerate and for $v<u^B$ we accelerate (if possible). $\sigma(\rho,u)$ is chosen in such a way that realistic values emerge at the endpoints of the fundamental diagram; we refer to [5] for details. The lane-changing probability was chosen as

$$P(u,v-u;b) \;=\; \begin{cases} b\cdot\left(\frac{v-u}{v_{max}-u}\right)^\delta, & v \;>\; u,\\ 0, & v\;\leq\; u,\end{cases}$$

$\delta>0$ is a parameter and $b\in[0,1]$ can be used to model passing restrictions.

For the rest of this work we will consider a simplified Fokker–Planck type model, with Robin boundary conditions, which retains the general features from (1). In the sequel we set the reaction time $\tau=0$ and consider

$$\partial_t f+v\partial_x f+\partial_v\left(-bg(\rho^X)(v-u^X)f-d\partial_v f\right)=0, \tag{3a}$$
$$-bg(\rho^X)(v-u^X)f-d\partial_v f=0, \qquad v\in\{0,v_{max}\}. \tag{3b}$$

Here, the index X again indicates nonlocal dependence of the dependent variables ρ,u (cf. (2)) , and $g:\mathbb{R}\to\mathbb{R}^+$ is a sufficiently regular function, e.g., $g\in C^2(\mathbb{R})$. This version of the Fokker–Planck model is quite similar to the original model from [5], which we presented in (1). However, the original model includes a more complicated, nonfactorized dependence on ρ and $v-u$, and here we use $(v-u^X)$ instead of $(v-u^X)|v-u^X|$, but still obtain the (crucial) correct qualitative behavior in the braking/acceleration term: for $v>u^X$ we decelerate whereas for $v<u^X$ we accelerate. In addition we take a constant diffusion $d>0$ and set $P(\ldots)=0$. Finally, for simplicity we assume that the braking and acceleration delay is the same, i.e., we set $u^X=u^B=u^A$. Then the integrals to be evaluated during the derivation can easily be expressed in terms of the macroscopic quantities (4). In the original model we would distinguish $u^X=u^A$ and $u^X=u^B$ in the braking/acceleration terms, and that would produce more complicated expressions.

Our observation in this note is that the simple choice $u^X=u^B$ leads to the AR–model in [1].

2. From a simplified Fokker–Planck type equation to the AR–model. In this section we derive the well-known Aw–Rascle model [1] from the simplified Fokker–Planck equation presented above. As stated, some of the principal features of the more sophisticated model from [3, 4, 5] are retained.

We provide a detailed description of our procedure, including the various simplifications necessary to obtain the AR–equations from our model equation. Not surprisingly, we have to neglect a number of terms arising from the Fokker–Planck formulation. In this sense the Aw–Rascle model emerges as a simplified description of traffic flow.

There are standard procedures for deriving macroscopic (fluid dynamic) models from kinetic equations [7]; of necessity, these procedures involve closure procedures for moment equations leading to different types of fluid dynamic descriptions, and the outcome depends on the scalings of independent and dependent variables as well as on the chosen closure process. In the context of traffic flow we mention [6].

In the present context we use only the zeroth and first moments. The macroscopic density ρ and flux $j = \rho u$ are given by

$$\rho \quad := \quad \int_0^{v_{max}} f(v)\, dv, \tag{4a}$$

$$\rho u \quad := \quad \int_0^{v_{max}} v f(v)\, dv. \tag{4b}$$

In view of (4a), (4b) is equivalent to

$$\int_0^{v_{max}} (v-u) f(v)\, dv \quad = \quad 0. \tag{5}$$

Before proceeding we recall one of the several versions in which the AR–model can be stated:

$$\partial_t \rho + \partial_x(\rho u) \quad = \quad 0, \tag{6a}$$

$$\partial_t(\rho u) + (-\rho p'(\rho) + u)\partial_x(\rho u) + (\rho u)\partial_x(u + p(\rho)) \quad = \quad 0. \tag{6b}$$

The form of the momentum equation (6b) is unconventional but is easily derived from the more common formulation

$$\partial_t(\rho(u + p(\rho))) + \partial_x(\rho u(u + p(\rho))) \quad = \quad 0.$$

We will derive these equations from the simpler Fokker–Planck type kinetic equation (3)

$$\begin{aligned} \partial_t f + v\partial_x f + \partial_v \left(-bg(\rho^X)(v - u^X) f - d\partial_v f\right) = 0, \\ -bg(\rho^X)(v - u^X) f - d\partial_v f = 0, \qquad v \in \{0, v_{max}\}. \end{aligned}$$

Consistent with the notation from section 1 and from [5] we expand

$$\rho^X \quad := \quad \rho(t, x + H_X + T_X v) \approx \rho + (H_X + T_X v)\rho_x, \tag{8a}$$

$$u^X \quad := \quad u(t, x + H_X + T_X v) \approx u + (H_X + T_X v) u_x, \tag{8b}$$

where we have used a Taylor expansion up to first order. H_X is the typical car length (or minimal safety distance) and $T_X v$ is the distance a driver looks ahead in preparation for braking or other reactions to anticipated or observed traffic situations.

The no-flux boundary conditions (3b) imply the continuity equation (6a) from (3a) by a simple integration from 0 to v_{max}.

The equation for the first moment is more interesting. After multiplication of (3a) by v and integration by parts, using the boundary conditions (3b), we find

$$\partial_t j + \partial_x S + \int_0^{v_{max}} bg(\rho^X)(v-u^X)f - d\partial_v f\, dv \quad = \quad 0,$$

where S denotes the second moment of f.

For the diffusion term we immediately find

$$\int_0^{v_{max}} d\partial_v f(v)\, dv \quad = \quad df(v)\,\Big|_{v=0}^{v=v_{max}} = d(f(v_{max}) - f(0)) := D. \tag{9}$$

If the distribution function is negligibly small at 0 and v_{max}, then D is negligibly small. We defer a discussion of the significance of D until the end of the derivation.

Consider next the braking/acceleration term. Assuming that the nonlocality is of the form (8b) we have up to first order

$$(v-u^X) \quad = \quad v - u - (H_X + T_X v)u_x = -(u + H_X u_x) + (1 - T_X u_x)v. \tag{10}$$

We group the terms by powers of v, because we are interested in moments of the distribution function f to recover our macroscopic quantities (4).

Furthermore, we expand

$$\begin{aligned} g(\rho^X) \quad &= \quad g(\rho + (H_X + T_X v)\rho_x) \\ &\approx \quad g(\rho) + g'(\rho)(H_X + T_X v)\rho_x + \frac{1}{2}g''(\rho)(H_X + T_X v)^2\rho_x^2. \end{aligned} \tag{11}$$

With this Taylor approximation the integral involving the braking term can be written as

$$\begin{aligned} &\int_0^{v_{max}} bg(\rho^X)(v-u^X)f\, dv \\ \approx \quad &\int_0^{v_{max}} b(g(\rho) + g'(\rho)H_X\rho_x + \frac{1}{2}g''(\rho)H_X^2\rho_x^2)(v-u^X)f\, dv && \text{(12a)} \\ &+ \int_0^{v_{max}} b(g'(\rho)T_X\rho_x + g''(\rho)H_X T_X\rho_x^2)v(v-u^X)f\, dv && \text{(12b)} \\ &+ \frac{1}{2}\int_0^{v_{max}} bg''(\rho)T_X^2\rho_x^2 v^2(v-u^X)f\, dv. && \text{(12c)} \end{aligned}$$

In the sequel we will neglect the terms involving $g''(\rho)$ (this is, of course, exact if g is linear). To simplify (12), relation (5) plays a crucial role. Note that in general

$$0 \quad = \quad \int_0^{v_{max}} (v-u)f(v)\, dv \neq \int_0^{v_{max}} bg(\rho^X)(v-u)f(v)\, dv \tag{13}$$

because, by (11), $g(\rho^X)$ depends implicitly on v. We can incorporate this dependence by a Taylor expansion method but have to add correction terms in (13) involving higher moments of the distribution function f. However, we can use the relation

$$0 \quad = \quad \int_0^{v_{max}} (v-u)f(v)\, dv = \int_0^{v_{max}} b(g(\rho) + g'(\rho)H_X\rho_x)(v-u)f(v)\, dv. \tag{14}$$

By using (14) in (12) we are led to

$$\begin{aligned}\int_0^{v_{max}} bg(\rho^X)(v-u^X)f\,dv &= \int_0^{v_{max}} -b(g(\rho)+g'(\rho)H_X\rho_x)(u^X-u)f\,dv \\ &\quad+\int_0^{v_{max}} bg'(\rho)T_X\rho_x v(v-u^X)f\,dv \\ &= \int_0^{v_{max}} -b\big(g(\rho)+g'(\rho)H_X\rho_x\big)(H_Xu_x+T_Xu_xv)f\,dv \\ &\quad+\int_0^{v_{max}} bg'(\rho)T_X\rho_x v(v-u^X)f\,dv.\end{aligned}$$

Note that in the leading term on the right the dependence on v remains only through u^X. This is the key step in the derivation!

To proceed we consider first a simplified case: we neglect the terms involving $g'(\rho)$. Then we find the macroscopic equation

$$\partial_t(\rho u)+\partial_x S-bg(\rho)H_Xu_x\rho-bg(\rho)T_Xu_x\rho u-D = 0. \tag{15}$$

We keep the diffusion induced term (9) on the right-hand side. Furthermore, we can write

$$S = \int_0^{v_{max}} v^2 f\,dv=\int_0^{v_{max}}(v-u)^2 f\,dv-u^2\rho+2\rho u^2.$$

Equation (15) becomes

$$\partial_t(\rho u)+\partial_x\left(\int_0^{v_{max}}(v-u)^2 f\,dv+u^2\rho\right)-bg(\rho)H_Xu_x\rho = D+b\frac{g(\rho)}{\rho}T_X\rho u_x j.$$

Using $u_x\rho=(\rho u)_x-u\rho_x$ we find

$$\begin{aligned}&\partial_t(\rho u)+\partial_x\left(\int_0^{v_{max}}(v-u)^2 f\,dv+u^2\rho\right)-bg(\rho)H_Xj_x+jb\frac{g(\rho)}{\rho}H_X\rho_x\\ &=D+\frac{1}{2}b\frac{g(\rho)}{\rho}T_X(j^2)_x-jb\frac{g(\rho)}{\rho}T_Xu\rho_x.\end{aligned}$$

Performing some of the differentiations and rearranging terms we are led to

$$\partial_t j+\big(-bg(\rho)(H_X+T_Xu)+u\big)j_x+j(b\frac{g(\rho)}{\rho}(H_X+T_Xu)\rho_x+u_x) \tag{16a}$$

$$=-\partial_x\int_0^{v_{max}}(v-u)^2 f\,dv+D. \tag{16b}$$

The left-hand side here will take the same appearance as (6b) if we choose $p=p(\rho,u)$ such that

$$\frac{\partial p}{\partial\rho} = b\frac{g(\rho)}{\rho}(H_X+T_Xu).$$

Recall that up to first order the nonlocality was of the form H_X+T_Xv. The above equation for $\frac{\partial p}{\partial\rho}$ also contains this expression, but the independent speed v has been replaced with the *average* speed u.

We discuss the terms in (16). First, $p(\rho, u)$ is sometimes referred to as traffic "pressure" although it is not a pressure from a dimensional point of view—it has the dimension of speed. The terminology is therefore somewhat confusing, but we chose to follow the common notation; however, we will keep this "pressure" in quotation marks for that reason.

In contrast to this, the term $-\partial_x \int_0^{v_{max}} (v-u)^2 f\, dv$ on the right-hand side of (16) is the gradient of a pressure term, and must be expected to emerge in a derivation of fluid dynamic equations from a kinetic model; see, for example, [2] for the derivation of fluid dynamic equations from the Boltzmann equation. Of course, retaining the term introduces a second-order moment into the fluid approximation and therefore leads to a nonclosed set of equations. To close the set, one has to introduce a closure relation, such as a) simply assuming that pressure gradients are negligible, and hence deleting the term, or b) introducing a closure relation in which pressure is assumed to be given in terms of ρ and u. A third option is to introduce equations for second-order moments of f (as is common in compressible fluid dynamics), but the closure problem then arises again because third-order moments will appear. The Aw–Rascle model emerges by simply deleting this pressure term.

The other term on the right-hand side of (16) is D. This term is an artifact of the simplicity of our kinetic model. It is simply not realistic to assume that from a statistical point of view any cars should assume zero speed in equilibrated traffic; hence, while our simple model allows $f(t, 0) > 0$, it is more realistic to set this $= 0$. Similarly, the assumption of a rigidly observed speed limit v_{max} leads to the possibility that $f(t, v_{max}) > 0$. This term will readily disappear if we set $v_{max} = \infty$ and assume rapid decline of f for large v. These considerations show that it is certainly reasonable to ignore D.

If we choose $g(\rho) = \rho$, then we obtain for the traffic "pressure" $p(\rho, u) = b(H_X + T_X u)\rho \sim \rho$. We note that this satisfies the original conditions on $p(\rho)$ stated in [1] with $\gamma = 1$, provided $b, H_X, T_X > 0$.

In summary, comparing (16) to (6b) we see that the only distinctions are the source terms on the right-hand side in (16), namely the true pressure gradient term (whose emergence is inevitable), and D.

Remark 2.1. In our derivation we have neglected terms involving $g'(\rho)$. These terms can be retained, and computations similar to the ones presented above show that an Aw–Rascle model will emerge if we choose p such that $p = p(\rho, \rho_x, u)$ and

$$\frac{\partial p}{\partial \rho} \quad = \quad b\left(\frac{g(\rho)}{\rho}(H_X + T_X u) + \frac{g'(\rho)}{\rho}\,(H_X + T_X u)^2\,\rho_x\right)$$

in order to obtain an equation similar to (6b). This result indicates that a higher-order approximation in terms of the nonlocalities corresponds to a "higher-order" approximation in p.

3. Conclusion. We demonstrated how the AR–model can be derived from a Fokker–Planck type equation. We conclude with a brief review of our simplifying assumptions. We truncated the expansion of $g(\rho^X)$ after the zero-order term and ignored diffusive effects, and found a well-known macroscopic equation. Hence kinetic models may have the potential to depict traffic dynamics more accurately, as more effects are included.

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