REGULARITY CONDITIONS FOR THE 3D NAVIER-STOKES EQUATIONS

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Abstract. We obtain logarithmic improvements for conditions of regularity in the 3D Navier-Stokes equations.

1. Introduction. In this paper we consider the 3D Navier-Stokes system:

\[
\begin{align*}
    u_t + u \cdot \nabla u + \nabla \pi - \Delta u &= 0 & \text{in} & \mathbb{R}^3 \times (0,T), \\
    \text{div } u &= 0 & \text{in} & \mathbb{R}^3 \times (0,T), \\
    u|_{t=0} &= u_0(x) & \text{in} & \mathbb{R}^3,
\end{align*}
\]

where \( T \in (0, +\infty) \). The vector field \( u \) (the velocity) and the scalar field \( \pi \) (the pressure) are the unknowns of the problem. Taking the curl of (1.1), we obtain the following vorticity equation:

\[
\omega_t + (u \cdot \nabla)\omega - \Delta \omega = (\omega \cdot \nabla) u,
\]

where the vorticity \( \omega \) is defined by

\[
\omega = \text{curl } u.
\]
The incompressibility condition (1.2) combined with (1.5) implies Biot-Savart’s law,
\[ u(x,t) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{y \times \omega(x+y,t)}{|y|^3} dy \]
for sufficiently rapidly decaying vorticity near infinity. After the pioneering work by J. Leray [1] there are many comprehensive literatures on the existence theory of the weak solutions of the NS equations ([2], [3]). The regularity of this weak solution is known as one of the most challenging problems in mathematical fluid mechanics. The first result in this direction is the one by Prodi [4], which states that if a weak solution \( u(x,t) \) satisfies
\[ u \in L^r(0,T; L^r(\mathbb{R}^3)), \quad \frac{3}{r} + \frac{2}{s} = 1, \] (1.6)
for \( 3 < r \leq \infty \), then \( u(x,t) \) is smooth. After that there are further developments and refinements by Serrin [5], Fabes-Jones-Riviere [6], Kozono-Taniuchi [7], and Escauriaza-Seregin-Sverak [8]. In particular, Beirão da Veiga [9] obtained a regularity condition in terms of \( \nabla u \), which is equivalent to the one in terms of the vorticity due to the Calderon-Zygmund inequality. This states that if the vorticity \( \omega \) satisfies
\[ \omega \in L^s(0,T; L^r(\mathbb{R}^3)), \quad \frac{3}{r} + \frac{2}{s} = 2, \] (1.7)
for \( \frac{3}{2} < r \leq \infty \), then \( u \) remains regular. Montgomery-Smith [10] proved that if
\[ \int_0^T \frac{\|u(t)\|_{L^r}^s}{1 + \log^+ \|u(t)\|_{L^r}} dt < \infty, \quad \frac{3}{r} + \frac{2}{s} = 1, \] (1.8)
for \( 3 < r < \infty \), then \( u \) is regular.

Now we are in a position to state the main result in this paper.

**Theorem 1.1.** If \( u \) is a solution to (1.1)-(1.3) satisfying one of the following two conditions:
\[ \begin{align*}
(i) \quad & \int_0^T \frac{\|\omega(t)\|_{L^r}^s}{1 + \log^+ \|\omega(t)\|_{L^r}} dt < \infty, \quad \frac{3}{p} + \frac{2}{s} = 2, \quad 2 \leq p < \infty, \\
(ii) \quad & \int_0^T \frac{\|\nabla \omega(t)\|_{L^r}^s}{1 + \log^+ \|\nabla \omega(t)\|_{L^r}} dt < \infty, \quad \frac{3}{p} + \frac{2}{s} = 3, \quad 2 \leq p \leq 3, 
\end{align*} \] (1.9, 1.10)
then \( u \) is regular.

**Remark 1.2.** Note that this proof can easily be adapted to show that a sufficient condition for regularity is that
\[ \int_0^T \frac{\|\omega(t)\|_{L^r}^s}{\theta(\|\omega(t)\|_{L^r})} dt < \infty \]
or
\[ \int_0^T \frac{\|\nabla \omega(t)\|_{L^r}^s}{\theta(\|\nabla \omega(t)\|_{L^r})} dt < \infty, \]
where \( \theta \) is any increasing function for which
\[ \int_1^\infty \frac{dx}{x \theta(x)} = \infty. \]
2. **Proof of Theorem 1.1**

First, we assume that (1.9) holds true. We multiply (1.4) by $|\omega|^{-2}\omega$ and perform suitable integration by parts to obtain

$$\frac{1}{p} \frac{d}{dt} \|\omega\|_{L^p}^p + \frac{1}{2} \int |\nabla \omega|^2 |\omega|^{p-2} \, dx + \frac{4(p - 2)}{p^2} \int |\nabla |\omega|^\frac{p}{2}|^2 \, dx$$

$$= \int (\omega \cdot \nabla) u \cdot \omega |\omega|^{p-2} \, dx$$

$$\leq \|\omega\|_{L^{p+1}}^p \|\nabla u\|_{L^{p+1}} \text{ (by Hölder's inequality)}$$

$$\leq C \|\omega\|_{L^{p+1}}^p$$

(2.1)

by the Calderon-Zygmund's inequality

$$\|\nabla u\|_{L^{p+1}} \leq C \|\omega\|_{L^{p+1}}^2.$$  

(2.2)

Since $\|\omega\|_{L^{p+1}}^p = \| |\omega|^\frac{p}{2} \|_{L^{2(p+1)}}$ we apply the Gagliardo-Nirenberg inequality

$$\|f\|_{L^{2(p+1)}} \leq C \|f\|_{L^2}^{1-\theta} \|\nabla f\|_{L^2}^\theta, \quad \text{with} \quad \theta = \frac{3}{2(p+1)},$$

to the function $|\omega|^\frac{p}{2}$. We obtain

$$\|\omega\|_{L^{p+1}}^p \leq C \||\omega|^\frac{2p-3}{2p-2}\|_{L^2}^\frac{p}{2}.$$  

Then the Young's inequality with exponents $\frac{2p-3}{2p-2}$ and $\frac{2p}{3}$ finally gives

$$\frac{1}{p} \frac{d}{dt} \|\omega\|_{L^p}^p + \frac{1}{4} \int |\nabla \omega|^2 |\omega|^{p-2} \, dx + \frac{2(p - 2)}{p^2} \int |\nabla |\omega|^\frac{p}{2}|^2 \, dx \leq C \|\omega\|_{L^p}^{p + \frac{2p-3}{2p-2}}$$

which gives

$$\frac{d}{dt} \|\omega\|_{L^p}^p \leq C \|\omega\|_{L^p} \log^+ \|\omega\|_{L^p} \cdot \frac{\|\omega\|_{L^p}^{\frac{2p-3}{2p-2}}}{1 + \log^+ \|\omega\|_{L^p}}$$

and hence

$$\sup_{0 \leq t \leq T} \|\omega(t)\|_{L^p} \leq C$$

by (1.9) with $s = \frac{2p}{2p-3}$.

Next we assume that (1.10) holds true. We take $D = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ to (1.4), and then take the inner product of it with $D\omega|D\omega|^{p-2}$. After integration by parts we have

$$\frac{1}{p} \frac{d}{dt} \|D\omega\|_{L^p}^p + \frac{1}{2} \int |\nabla D\omega|^2 |D\omega|^{p-2} \, dx + \frac{4(p - 2)}{p^2} \int |\nabla |D\omega|^\frac{p}{2}|^2 \, dx$$

$$= -\int D[(u \cdot \nabla)\omega] \cdot D\omega |D\omega|^{p-2} \, dx + \int D[(\omega \cdot \nabla)u] \cdot D\omega |D\omega|^{p-2} \, dx$$

$$= : I + J.$$  

(2.3)

We estimate $I, J$ below.

$$I = -\int D u \cdot \nabla \omega \cdot D\omega |D\omega|^{p-2} \, dx - \int (u \cdot \nabla)D\omega \cdot D\omega |D\omega|^{p-2} \, dx$$

$$= : I_1 + I_2.$$
Integrating by parts, and using the fact that \( \text{div} \, u = 0 \), we get
\[
I_2 = -\frac{1}{p} \int u \cdot \nabla |D\omega|^p dx = \frac{1}{p} \int |D\omega|^p \text{div} \, u \, dx = 0.
\]

By Hölder’s inequality and Sobolev inequality together with the Calderon-Zygmund inequality we estimate,
\[
I_1 \leq \int |Du| \cdot |D\omega|^p dx \leq \|Du\|_{L^{p_1}} \|D\omega\|_{L^{p_2}}^p \left( \frac{1}{p_1} + \frac{p}{p_2} = 1 \right)
\]
\[
\leq C \|\omega\|_{L^{p_1}} \|D\omega\|_{L^{p_2}}^p
\]
\[
\leq C \|Du\|_{L^{p_1}}^p \|D\omega\|_{L^{p_2}}^p \left( \frac{1}{p_1} = \frac{1}{p_2} = \frac{1}{3}, p_2 = \frac{3}{4}(p + 1) \right)
\]
\[
= C \|D\omega\|_{L^2}^\frac{2(p+1)}{p}.
\]

We apply the Gagliardo-Nirenberg inequality
\[
\|f\|_{L^q} \leq C \|f\|_{L^p}^\theta \|\nabla f\|_{L^2}^{1-\theta}, \quad \theta = \frac{3}{q} - \frac{1}{2},
\]
(2.4)
to the function \( |D\omega|^{\frac{p}{2}} \). We obtain
\[
I_1 \leq \epsilon \int |\nabla |D\omega|^{\frac{p}{2}}|^2 dx + C \|D\omega\|_{L^p}^\frac{p+2}{2p}
\]
for any \( \epsilon > 0 \) by Young’s inequality.

In order to estimate \( J \) we first decompose it into two terms as follows:
\[
J = \int D\omega \cdot \nabla u \cdot D\omega |D\omega|^{p-2} \, dx + \int (\omega \cdot \nabla)Du \cdot D\omega |D\omega|^{p-2} \, dx
\]
\[
= : J_1 + J_2.
\]

Since
\[
J_1 \leq \|\nabla u\|_{L^{p_1}} \|D\omega\|_{L^{p_2}}^p \left( \text{by Hölder’s inequality}, \frac{1}{p_1} + \frac{p}{p_2} = 1 \right)
\]
\[
\leq C \|\omega\|_{L^{p_1}} \|D\omega\|_{L^{p_2}}^p \left( \text{by Calderon – Zygmund inequality} \right)
\]
\[
\leq C \|\omega\|_{L^{p_1}} \|D\omega\|_{L^{p_2}}^p \left( \frac{1}{p_1} = \frac{1}{p_2} = \frac{1}{3}, p_2 = \frac{3}{4}(p + 1) \right)
\]
the estimate of \( J_1 \) is the same as that of \( I_1 \). On the other hand, by the Hölder, Sobolev
and Calderon-Zygmund inequalities,
\[
J_2 \leq \|\omega\|_{L^{p_1}} \|\nabla Du\|_{L^{p_2}} \|D\omega\|_{L^{p_2}}^{p-1} \left( \frac{1}{p_1} + \frac{p}{p_2} = 1 \right)
\]
\[
\leq C \|\omega\|_{L^{p_1}} \|\Delta u\|_{L^{p_2}} \|D\omega\|_{L^{p_2}}^{p-1}
\]
\[
\leq C \|\omega\|_{L^{p_1}} \|D\omega\|_{L^{p_2}}^p \left( \frac{1}{p_1} = \frac{1}{p_2} = \frac{1}{3}, p_2 = \frac{3}{4}(p + 1) \right)
\]
\[
\leq C \|D\omega\|_{L^{p_2}}^{p+1}.
\]
Hence, the estimate of $J_2$ is also the same as that of $I_1$. Combining the above estimates $I$ and $J$ and taking $\epsilon$ small enough, we have
\[\frac{d}{dt}\|D\omega\|^p_{L^p} + C \int |\nabla |D\omega |^2|^p dx \leq C\|D\omega\|^{p+\frac{2p}{3p-3}}_{L^p}\]
which implies
\[\sup_{0 \leq t \leq T} \|D\omega\|_{L^p} \leq C\]
by (1.10) with $s = \frac{2p}{3p-3}$.
This completes the proof. \qed

References