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REGULARITY CONDITIONS FOR THE 3D NAVIER-STOKES EQUATIONS

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Abstract. We obtain logarithmic improvements for conditions of regularity in the 3D Navier-Stokes equations.

1. Introduction. In this paper we consider the 3D Navier-Stokes system:

$$u_t + u \cdot \nabla u + \nabla \pi - \Delta u = 0 \qquad in \qquad \mathbb{R}^3 \times (0, T), \tag{1.1}$$

$$\operatorname{div} u = 0 \qquad in \qquad \mathbb{R}^3 \times (0, T), \tag{1.2}$$

$$u|_{t=0} = u_0(x) \qquad in \qquad \mathbb{R}^3, \tag{1.3}$$

where $T \in (0, +\infty)$. The vector field u (the velocity) and the scalar field π (the pressure) are the unknowns of the problem. Taking the curl of (1.1), we obtain the following vorticity equation:

$$\omega_t + (u \cdot \nabla)\omega - \Delta\omega = (\omega \cdot \nabla)u, \tag{1.4}$$

where the vorticity ω is defined by

$$\omega = \operatorname{curl} u. \tag{1.5}$$

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The incompressibility condition (1.2) combined with (1.5) implies Biot Savart's law,

$$u(x,t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{y \times \omega(x+y,t)}{|y|^3} dy$$

for sufficiently rapidly decaying vorticity near infinity. After the pioneering work by J. Leray [1] there are many comprehensive literatures on the existence theory of the weak solutions of the NS equations ([2], [3]). The regularity of this weak solution is known as one of the most challenging problems in mathematical fluid mechanics. The first result in this direction is the one by Prodi [4], which states that if a weak solution $u(x,t)$ satisfies

$$u \in L^s(0,T;L^r(\mathbb{R}^3)), \qquad \frac{3}{r}+\frac{2}{s}=1, \tag{1.6}$$

for $3 < r \leq \infty$, then $u(x,t)$ is smooth. After that there are further developments and refinements by Serrin [5], Fabes-Jones-Riviere [6], Kozono-Taniuchi [7], and Escauriaza-Seregin-Sverak [8]. In particular, Beirão da Veiga [9] obtained a regularity condition in terms of ∇u, which is equivalent to the one in terms of the vorticity due to the Calderon-Zygmund inequality. This states that if the vorticity ω satisfies

$$\omega \in L^s(0,T;L^r(\mathbb{R}^3)), \qquad \frac{3}{r}+\frac{2}{s}=2, \tag{1.7}$$

for $\frac{3}{2} < r \leq \infty$, then u remains regular. Montgomery-Smith [10] proved that if

$$\int_0^T \frac{\|u(t)\|_{L^r}^s}{1+\log^+\|u(t)\|_{L^r}} dt < \infty, \frac{3}{r}+\frac{2}{s}=1, \tag{1.8}$$

for $3 < r < \infty$, then u is regular.

Now we are in a position to state the main result in this paper.

THEOREM 1.1. If u is a solution to (1.1)-(1.3) satisfying one of the following two conditions:

$$(i) \quad \int_0^T \frac{\|\omega(t)\|_{L^p}^s}{1+\log^+\|\omega(t)\|_{L^p}} dt < \infty, \qquad \frac{3}{p}+\frac{2}{s}=2, \quad 2 \leq p < \infty, \tag{1.9}$$

$$(ii) \quad \int_0^T \frac{\|\nabla\omega(t)\|_{L^p}^s}{1+\log^+\|\nabla\omega(t)\|_{L^p}} dt < \infty, \qquad \frac{3}{p}+\frac{2}{s}=3, \quad 2 \leq p \leq 3, \tag{1.10}$$

then u is regular.

REMARK 1.2. Note that this proof can easily be adapted to show that a sufficient condition for regularity is that

$$\int_0^T \frac{\|\omega(t)\|_{L^p}^s}{\theta(\|\omega(t)\|_{L^p})} dt < \infty$$

or

$$\int_0^T \frac{\|\nabla\omega(t)\|_{L^p}^s}{\theta(\|\nabla\omega(t)\|_{L^p})} dt < \infty,$$

where θ is any increasing function for which

$$\int_1^\infty \frac{dx}{x\theta(x)} = \infty.$$

2. Proof of Theorem 1.1. First, we assume that (1.9) holds true. We multiply (1.4) by $|\omega|^{p-2}\omega$ and we perform suitable integration by parts to obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|\omega\|_{L^p}^p + \frac{1}{2}\int |\nabla\omega|^2|\omega|^{p-2}dx + \frac{4(p-2)}{p^2}\int \left|\nabla|\omega|^{\frac{p}{2}}\right|^2 dx \\
= &\int (\omega\cdot\nabla)u\cdot\omega|\omega|^{p-2}dx \\
\le &\|\omega\|_{L^{p+1}}^p\|\nabla u\|_{L^{p+1}} \quad \text{(by Hölder's inequality)} \\
\le &C\|\omega\|_{L^{p+1}}^{p+1} \qquad (2.1)
\end{aligned}$$

by the Calderon-Zygmund's inequality

$$\|\nabla u\|_{L^{p+1}} \le C\|\omega\|_{L^{p+1}}. \qquad (2.2)$$

Since $\|\omega\|_{L^{p+1}}^{\frac{p}{2}} = \|\ |\omega|^{\frac{p}{2}}\|_{L^{\frac{2(p+1)}{p}}}$ we apply the Gagliardo-Nirenberg inequality

$$\|f\|_{L^{\frac{2(p+1)}{p}}} \le C\|f\|_{L^2}^{1-\theta}\|\nabla f\|_{L^2}^{\theta}, \quad \text{with} \quad \theta = \frac{3}{2(p+1)},$$

to the function $|\omega|^{\frac{p}{2}}$. We obtain

$$\|\omega\|_{L^{p+1}}^{p+1} \le C\|\omega\|_{L^p}^{\frac{2p-1}{2}}\|\nabla|\omega|^{\frac{p}{2}}\|_{L^2}^{\frac{3}{p}}.$$

Then the Young's inequality with exponents $\frac{2p}{2p-3}$ and $\frac{2p}{3}$ finally gives

$$\frac{1}{p}\frac{d}{dt}\|\omega\|_{L^p}^p + \frac{1}{4}\int |\nabla\omega|^2|\omega|^{p-2}dx + \frac{2(p-2)}{p^2}\int \left|\nabla|\omega|^{\frac{p}{2}}\right|^2 dx \le C\|\omega\|_{L^p}^{p+\frac{2p}{2p-3}}$$

which gives

$$\frac{d}{dt}\|\omega\|_{L^p} \le C\|\omega\|_{L^p}\log^+\|\omega\|_{L^p}\cdot\frac{\|\omega\|_{L^p}^{\frac{2p}{2p-3}}}{1+\log^+\|\omega\|_{L^p}}$$

and hence

$$\sup_{0\le t\le T}\|\omega(t)\|_{L^p} \le C$$

by (1.9) with $s = \frac{2p}{2p-3}$.

Next we assume that (1.10) holds true. We take $D = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ to (1.4), and then take the inner product of it with $D\omega|D\omega|^{p-2}$. After integration by parts we have

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|D\omega\|_{L^p}^p + \frac{1}{2}\int |\nabla D\omega|^2|D\omega|^{p-2}dx + \frac{4(p-2)}{p^2}\int \left|\nabla|D\omega|^{\frac{p}{2}}\right|^2 dx \\
= &-\int D[(u\cdot\nabla)\omega]\cdot D\omega|D\omega|^{p-2}dx + \int D[(\omega\cdot\nabla)u]\cdot D\omega|D\omega|^{p-2}dx \\
= &: I + J. \qquad (2.3)
\end{aligned}$$

We estimate I, J below.

$$\begin{aligned}
I &= -\int Du\cdot\nabla\omega\cdot D\omega|D\omega|^{p-2}dx - \int (u\cdot\nabla)D\omega\cdot D\omega|D\omega|^{p-2}dx \\
&= : I_1 + I_2.
\end{aligned}$$

Integrating by parts, and using the fact that $\operatorname{div} u = 0$, we get

$$I_2 = -\frac{1}{p}\int u\cdot\nabla|D\omega|^p dx = \frac{1}{p}\int |D\omega|^p \operatorname{div} u dx = 0.$$

By Hölder's inequality and Sobolev inequality together with the Calderon-Zygmund inequality we estimate,

$$\begin{aligned} I_1 &\le \int |Du|\cdot|D\omega|^p dx \le \|Du\|_{L^{p_1}}\|D\omega\|^p_{L^{p_2}} \quad \left(\frac{1}{p_1}+\frac{p}{p_2}=1\right)\\ &\le C\|\omega\|_{L^{p_1}}\|D\omega\|^p_{L^{p_2}}\\ &\le C\|D\omega\|^{p+1}_{L^{p_2}} \quad \left(\frac{1}{p_1}=\frac{1}{p_2}-\frac{1}{3}, p_2=\frac{3}{4}(p+1)\right)\\ &= C\|\ |D\omega|^{\frac{p}{2}}\|^{\frac{2(p+1)}{p}}_{L^{\frac{2}{p}p_2}}. \end{aligned}$$

We apply the Gagliardo-Nirenberg inequality

$$\|f\|_{L^q} \le C\|f\|^{\theta}_{L^2}\|\nabla f\|^{1-\theta}_{L^2}, \qquad \theta = \frac{3}{q}-\frac{1}{2}, \tag{2.4}$$

to the function $|D\omega|^{\frac{p}{2}}$. We obtain

$$I_1 \le \epsilon\int \left|\nabla|D\omega|^{\frac{p}{2}}\right|^2 dx + C\|D\omega\|^{p+\frac{2p}{3p-3}}_{L^p}$$

for any $\epsilon > 0$ by Young's inequality.

In order to estimate J we first decompose it into two terms as follows:

$$\begin{aligned} J &= \int D\omega\cdot\nabla u\cdot D\omega|D\omega|^{p-2}dx + \int(\omega\cdot\nabla)Du\cdot D\omega|D\omega|^{p-2}dx\\ &= : J_1 + J_2. \end{aligned}$$

Since

$$\begin{aligned} J_1 &\le \|\nabla u\|_{L^{p_1}}\|D\omega\|^p_{L^{p_2}} \quad \left(\text{by Hölder's inequality}, \ \frac{1}{p_1}+\frac{p}{p_2}=1\right)\\ &\le C\|\omega\|_{L^{p_1}}\|D\omega\|^p_{L^{p_2}} \quad (\text{by Calderon}-\text{Zygmund inequality})\\ &\le C\|D\omega\|^{p+1}_{L^{p_2}} \quad \left(\frac{1}{p_1}=\frac{1}{p_2}-\frac{1}{3}, p_2=\frac{3}{4}(p+1)\right), \end{aligned}$$

the estimate of J_1 is the same as that of I_1. On the other hand, by the Hölder, Sobolev and Calderon-Zygmund inequalities,

$$\begin{aligned} J_2 &\le \|\omega\|_{L^{p_1}}\|\nabla Du\|_{L^{p_2}}\|D\omega\|^{p-1}_{L^{p_2}} \quad \left(\frac{1}{p_1}+\frac{p}{p_2}=1\right)\\ &\le C\|\omega\|_{L^{p_1}}\|\Delta u\|_{L^{p_2}}\|D\omega\|^{p-1}_{L^{p_2}}\\ &\le C\|\omega\|_{L^{p_1}}\|D\omega\|^p_{L^{p_2}} \quad \left(\frac{1}{p_1}=\frac{1}{p_2}-\frac{1}{3}, p_2=\frac{3}{4}(p+1)\right)\\ &\le C\|D\omega\|^{p+1}_{L^{p_2}}. \end{aligned}$$

Hence, the estimate of J_2 is also the same as that of I_1. Combining the above estimates I and J and taking ϵ small enough, we have

$$\frac{d}{dt}\|D\omega\|_{L^p}^p + C\int \left|\nabla|D\omega|^{\frac{p}{2}}\right|^2 dx \leq C\|D\omega\|_{L^p}^{p+\frac{2p}{3p-3}}$$

which implies

$$\sup_{0\leq t\leq T}\|D\omega\|_{L^p} \leq C$$

by (1.10) with $s = \frac{2p}{3p-3}$.

This completes the proof. □

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