QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXVII, NUMBER 2
JUNE 2009, PAGES 311–326
S 0033-569X(09)01112-8
Article electronically published on March 20, 2009

ASYMPTOTICS OF RESONANCES IN A THERMOELASTIC MODEL WITH LIGHT LOCAL MASS PERTURBATIONS

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Abstract. The limit behaviour of a linear one-dimensional thermoelastic system with local mass perturbations is studied. The mass density is supposed to be nearly homogeneous everywhere except in an ε-vicinity of a given point, where it is of order ε^{-m}, with $m \in \mathbb{R}$. The resonance vibrations of the string are investigated as $\varepsilon \to 0$. An important ingredient of the analysis is the construction of an operator in a space of higher regularity such that its spectrum coincides with that of the classical operator in linearised thermoelasticity, with a correspondence of generalised eigenspaces. The convergence of eigenvalues and eigenprojectors is established along with error bounds for two classes of relatively light mass perturbations, $m < 1$ and $m = 1$, which exhibit contrasting limit behaviour.

1. Statement of the problem. We consider resonance vibrations of a finite string modelled in the framework of linearised, one-dimensional thermo-elasticity. The evolution of the displacement $u = u(x,t)$ and the relative temperature $\theta = \theta(x,t)$ is governed by the system of differential equations [9, 8, 5]

$$\rho_\varepsilon(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\beta(x)\theta = f(x,t), \tag{1}$$

$$k(x)\frac{\partial \theta}{\partial t} + \beta(x)\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial}{\partial x}\left(\varkappa(x)\frac{\partial \theta}{\partial x}\right) = \phi(x,t), \tag{2}$$

where ρ_ε is the mass density of the string, α is a stiffness coefficient, k is the specific heat, $\varkappa$ denotes the thermal conductivity coefficient, and β is a coupling coefficient; f and ϕ represent an external force and a heat source, respectively.

Received November 3, 2007.
2000 *Mathematics Subject Classification.* Primary 35P15; Secondary 34E10, 74F05.

For a finite string, it is not restrictive to assume that the reference configuration is the interval (a, b) with $a < 0 < b$. We are interested in local perturbations of the mass density,

$$\rho_\varepsilon(x) = \begin{cases} p(x) & \text{if } x \in (a, -\varepsilon) \cup (\varepsilon, b), \\ \varepsilon^{-m} q\left(\frac{x}{\varepsilon}\right) & \text{if } x \in (-\varepsilon, \varepsilon), \end{cases}$$

represented by two parameters $\varepsilon \to 0$ and $m \in \mathbb{R}$. The mass density functions $p\colon [a, b] \to \mathbb{R}$ and $q\colon [-1, 1] \to \mathbb{R}$ are bounded and strictly positive. We assume that p is continuous on $[a, 0)$ and $(0, b]$, while q is continuous on $[-1, 1]$. We also suppose that all other parameters are strictly positive in $\bar{\Omega}$ and smooth enough, namely $\alpha, \beta, \varkappa \in C^1(a, b)$, and $k \in C^0(a, b)$. In this note, we analyse the asymptotic behaviour as $\varepsilon \to 0$ of the eigenvalues λ_ε and eigenvectors $(u_\varepsilon, \theta_\varepsilon)$ of the eigenvalue problem associated with (1) and (2) on $\Omega_\varepsilon := (a, -\varepsilon) \cup (-\varepsilon, \varepsilon) \cup (\varepsilon, b)$,

$$-\left(\alpha(x)u_\varepsilon'\right)' + \left(\beta(x)\theta_\varepsilon\right)' = -\lambda_\varepsilon^2 \rho_\varepsilon(x) u_\varepsilon, \tag{3}$$

$$-\left(\varkappa(x)\theta_\varepsilon'\right)' - \lambda_\varepsilon \beta(x) u_\varepsilon' = \lambda_\varepsilon k(x)\theta_\varepsilon, \tag{4}$$

complemented with the Dirichlet boundary conditions at the outer ends

$$u_\varepsilon(a) = u_\varepsilon(b) = 0 \quad \text{and} \quad \theta_\varepsilon(a) = \theta_\varepsilon(b) = 0, \tag{5}$$

and the interfacial conditions

$$[\![u_\varepsilon]\!]_{\pm\varepsilon} = [\![u_\varepsilon']\!]_{\pm\varepsilon} = 0 \quad \text{and} \quad [\![\theta_\varepsilon]\!]_{\pm\varepsilon} = [\![\theta_\varepsilon']\!]_{\pm\varepsilon} = 0. \tag{6}$$

The prime $'$ denotes the derivative with respect to the spatial variable. We use the notation $[\![y]\!]_z := y(z+0) - y(z-0)$ for the jump of a function y at point z. The smoothness assumptions imply that u_ε and θ_ε, given as solutions of the ordinary differential equations (3) and (4), belong to $C^1(a, b)$, and that their restrictions to the intervals $(a, -\varepsilon)$, $(-\varepsilon, \varepsilon)$ and (ε, b) are twice continuously differentiable.

For $\beta = 0$, (3)–(6) splits into two independent problems, with the perturbation being present only in the elastic part. The corresponding study of the asymptotic behaviour for eigenvalues and eigenfunctions in the isothermal elastic problem is presented in [4]. We refer to [9, 10] for the first systematic study of problems with concentrated masses (see also [7, 3]). We remark that a thermoelastic setting has been investigated before [9, 8]; there, perturbations of thermal characteristics are investigated, while the present analysis focuses on the local perturbation of elastic coefficients and their influence on the thermal behaviour of the system.

Methodologically, the thermoelastic problem is different from the isothermal one. One of the differences is the non-selfadjointness of the corresponding operator, which renders the justification methods used for the isothermal problem [4] inapplicable. Even the existence of eigenvalues and eigenfunctions requires special analysis [12]. Another main difference is that the operator of classical thermo-elasticity acts in a space of low regularity, which would not allow the application of methods developed for localised perturbations in [4]. In this article, we construct the operator framework for thermoelastic problems in a space of higher regularity, namely $[H_0^1]^3$. The equality of the spectra and a correspondence of the eigenfunctions is established, which is a self-contained problem in its own right. Then the limit behaviour for the spectral properties and the rates of convergence are estimated for two classes of relatively light mass density perturbations, namely $m < 1$ and $m = 1$. The case $m = 1$ amounts to the presence of a concentrated (finite and non-zero) mass in the limit, whereas $m < 1$ corresponds to an unperturbed string in the limit. Within the scope of this article, we analyse these two cases of relatively light perturbations, rather than heavy mass perturbations $m > 1$.

2. Operator frameworks. We note that (3)–(6) is an eigenvalue problem for a quadratic operator pencil. Introducing the independent variables $u_1^\varepsilon = u_\varepsilon$, $u_2^\varepsilon = -\lambda_\varepsilon u_\varepsilon$, $u_3^\varepsilon = \theta_\varepsilon$ we linearise problem (3)–(6):

$$-u_2^\varepsilon = \lambda_\varepsilon u_1^\varepsilon, \tag{7}$$

$$-\left(\alpha\left(u_1^\varepsilon\right)'\right)' + \left(\beta u_3^\varepsilon\right)' = \lambda_\varepsilon \rho_\varepsilon u_2^\varepsilon, \tag{8}$$

$$-\left(\varkappa\left(u_3^\varepsilon\right)'\right)' + \beta\left(u_2^\varepsilon\right)' = \lambda_\varepsilon k u_3^\varepsilon \quad \text{for} \quad x \in \Omega_\varepsilon, \tag{9}$$

$$u_j^\varepsilon(a) = u_j^\varepsilon(b) = 0 \quad \text{and} \quad \left[\!\left[u_j^\varepsilon\right]\!\right]_{\pm\varepsilon} = \left[\!\left[\left(u_j^\varepsilon\right)'\right]\!\right]_{\pm\varepsilon} = 0 \quad \text{for } j = 1, 2, 3. \tag{10}$$

We use a standard notation for Lebesgue and Sobolev spaces: $L_p^2(\Omega)$ is a p-weighted L^2-space of square integrable functions in Ω, $H^n(\Omega)$ is the space of functions whose distributional derivatives up to the order n inclusive are in $L^2(\Omega)$, while $H_0^1(\Omega)$ consists of the functions in $H^1(\Omega)$ with a zero trace on the boundary $\partial\Omega$; the inner product in $H_0^1(\Omega)$ is $(u, v)_{H_0^1(\Omega)} := (\alpha u', v')_{L^2(\Omega)}$. We introduce the Hilbert spaces

$$\mathcal{L}_\varepsilon := H_0^1(a,b) \times L_{\rho_\varepsilon}^2(a,b) \times L_k^2(a,b) \quad \text{and} \quad \mathcal{H} := H_0^1(a,b) \times H_0^1(a,b) \times H_0^1(a,b),$$

equipped with inner products for $U = (u_1, u_2, u_3)^T$ and $V = (v_1, v_2, v_3)^T$ (the bar denotes complex conjugation),

$$(U, V)_{\mathcal{L}_\varepsilon} := \int_a^b \left[\alpha(x) u_1' \bar{v}_1' + \rho_\varepsilon(x) u_2 \bar{v}_2 + k(x) u_3 \bar{v}_3\right] \mathrm{d}x$$

and

$$(U, V)_{\mathcal{H}} := \int_a^b \alpha(x) \left[u_1' \bar{v}_1' + u_2' \bar{v}_2' + u_3' \bar{v}_3'\right] \mathrm{d}x.$$

We introduce an operator $\mathcal{A}_\varepsilon \colon \mathcal{L}_\varepsilon \to \mathcal{L}_\varepsilon$ with the domain $\mathfrak{D}\left(\mathcal{A}_\varepsilon\right) = \mathfrak{D}_* \times H_0^1(a,b) \times \mathfrak{D}_*$, where

$$\mathfrak{D}_* := H^2(a,b) \cap H_0^1(a,b) = \left\{ y \in H^2(a,b) \;\middle|\; y(a) = y(b) = 0 \right\},$$

and with an action given by $\mathcal{A}_\varepsilon V := A_\varepsilon\left(x, \frac{d}{dx}\right) V(x)$, where

$$A_\varepsilon\left(x, \mathcal{D}\right) = \begin{pmatrix} 0 & -I & 0 \\ -\rho_\varepsilon^{-1} \mathcal{D} \alpha \mathcal{D} & 0 & \rho_\varepsilon^{-1} \mathcal{D} \beta \\ 0 & k^{-1} \beta \mathcal{D} & -k^{-1} \mathcal{D} \varkappa \mathcal{D} \end{pmatrix}. \tag{11}$$

Then (7)–(10) become

$$\mathcal{A}_\varepsilon U_\varepsilon = \lambda_\varepsilon U_\varepsilon. \tag{12}$$

The operator $\mathcal{A}_\varepsilon$ is classical for the problems in linearised thermoelasticity [9, 8]. Nevertheless, since $U_\varepsilon \in \mathcal{H}$ and $\mathcal{H}$ is more regular than $\mathcal{L}_\varepsilon$, for our purposes it is more convenient to consider (7)–(10) in $\mathcal{H}$. The higher regularity is necessary for the justification of the convergence and the estimation of error bounds as $\varepsilon \to 0$. Thus, we further construct an operator framework for (7)–(10) in $\mathcal{H}$.

A variational formulation of (12) is

$$(\mathcal{A}_\varepsilon U_\varepsilon, \Phi)_{\mathcal{L}_\varepsilon} = \lambda_\varepsilon (U_\varepsilon, \Phi)_{\mathcal{L}_\varepsilon} \qquad \text{for every } \Phi \in \mathcal{L}_\varepsilon, \tag{13}$$

which in the component-wise representation for $U_\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ and $\Phi = (\varphi_1, \varphi_2, \varphi_3)$ reads

$$-\int_a^b \alpha(x)\,(u_2^\varepsilon)'\,\bar\varphi_1'\,\mathrm{d}x = \lambda_\varepsilon \int_a^b \alpha(x)\,(u_1^\varepsilon)'\,\bar\varphi_1'\,\mathrm{d}x, \tag{14}$$

$$\int_a^b \left[\alpha(x)\,(u_1^\varepsilon)'\,\bar\varphi_2' + (\beta(x) u_3^\varepsilon)'\,\bar\varphi_2\right]\,\mathrm{d}x = \lambda_\varepsilon \int_a^b \rho_\varepsilon(x) u_2^\varepsilon \bar\varphi_2\,\mathrm{d}x, \tag{15}$$

$$\int_a^b \left[\varkappa(x)\,(u_3^\varepsilon)'\,\bar\varphi_3' + \beta(x)\,(u_2^\varepsilon)'\,\bar\varphi_3\right]\,\mathrm{d}x = \lambda_\varepsilon \int_a^b k(x) u_3^\varepsilon \bar\varphi_3\,\mathrm{d}x. \tag{16}$$

Note that the left-hand side of (14)–(16) is independent of ε. We introduce a sesquilinear form τ on $\mathcal{H}$,

$$\tau(U, \Phi) := \int_a^b \left[-\alpha u_2' \bar\varphi_1' + \alpha u_1' \bar\varphi_2' + (\beta u_3)'\,\bar\varphi_2 + \beta u_2' \bar\varphi_3 + \varkappa\, u_3' \bar\varphi_3'\right]\,\mathrm{d}x, \tag{17}$$

which represents the sum of the left-hand sides in (14)–(16). Since $\tau(\cdot,\cdot)$ is obviously continuous on $\mathcal{H}$, there exists by the Lax-Milgram theorem [11] a bounded operator $\mathcal{B}\colon \mathcal{H} \to \mathcal{H}$ such that $(\mathcal{B}U, \Phi)_\mathcal{H} = \tau(U, \Phi)$ for every $\Phi \in \mathcal{H}$. Thus the eigenvalue problem (14)–(16) can be written in the form

$$(\mathcal{B}U_\varepsilon, \Phi)_\mathcal{H} = \lambda_\varepsilon (U_\varepsilon, \Phi)_{\mathcal{L}_\varepsilon} \text{ for every } \Phi \in \mathcal{H}. \tag{18}$$

Likewise, since the sesquilinear form $(\cdot,\cdot)_{\mathcal{L}_\varepsilon}$ is continuous on $\mathcal{H}$, there exists, again by the Lax-Milgram theorem, a bounded operator $\mathcal{Q}_\varepsilon\colon \mathcal{H} \to \mathcal{H}$ such that

$$(\mathcal{Q}_\varepsilon U, \Phi)_\mathcal{H} = (U, \Phi)_{\mathcal{L}_\varepsilon} \text{ for every } \Phi \in \mathcal{H}. \tag{19}$$

Hence, (18) yields that the eigenvalue problem can be represented as

$$\mathcal{B}U_\varepsilon = \lambda_\varepsilon \mathcal{Q}_\varepsilon U_\varepsilon \text{ in } \mathcal{H}. \tag{20}$$

The following sections analyse the equivalence of (20) and (12).

3. Existence of inverse operators.

LEMMA 1. The inverse $\mathcal{B}^{-1}$ exists and is a bounded operator in $\mathcal{H}$.

Proof. We need to prove that for every $F \in \mathcal{H}$, there exists a unique solution $U \in \mathcal{H}$ to the problem $\mathcal{B}U = F$. The weak formulation of this problem is to find, for arbitrary $F \in \mathcal{H}$, $U \in \mathcal{H}$ such that

$$\tau(U, \Phi) = (F, \Phi)_\mathcal{H} \text{ for every } \Phi \in \mathcal{H}. \tag{21}$$

A consecutive substitution of the test functions $\Phi = (\phi_1, 0, 0)^T$, $\Phi = (0, \phi_2, 0)^T$ and $\Phi = (0, 0, \phi_3)^T$ into (21) yields

$$-\int_a^b \alpha(x) u_2' \bar{\varphi}_1' \, \mathrm{d}x = \int_a^b \alpha(x) f_1' \bar{\varphi}_1' \, \mathrm{d}x \quad \text{for every } \varphi_1 \in H_0^1(a,b), \tag{22}$$

$$\int_a^b \left[\alpha(x) u_1' \bar{\varphi}_2' + (\beta(x) u_3)' \, \bar{\varphi}_2 \right] \mathrm{d}x = \int_a^b \alpha(x) f_2' \bar{\varphi}_2' \, \mathrm{d}x \quad \text{for every } \varphi_2 \in H_0^1(a,b), \tag{23}$$

$$\int_a^b \left[\varkappa(x) u_3' \bar{\varphi}_3' + \beta(x) u_2' \bar{\varphi}_3 \right] \mathrm{d}x = \int_a^b \alpha(x) f_3' \bar{\varphi}_3' \, \mathrm{d}x \quad \text{for every } \varphi_3 \in H_0^1(a,b). \tag{24}$$

Then (22) implies $u_2 = -f_1 \in H_0^1(a,b)$. Obviously,

$$\|u_2\|_{H_0^1(a,b)} = \|f_1\|_{H_0^1(a,b)} \leq \|F\|_{\mathcal{H}} \, . \tag{25}$$

Hence, (24) becomes

$$\int_a^b \varkappa(x) u_3' \bar{\varphi}_3' \, \mathrm{d}x = \int_a^b \left[\beta(x) f_1' \bar{\varphi}_3 + \alpha(x) f_3' \bar{\varphi}_3' \right] \mathrm{d}x \quad \text{for every } \phi_3 \in H_0^1(a,b). \tag{26}$$

Since for each fixed pair (f_1, f_3) the right-hand side of (26) is a semilinear bounded functional on $H_0^1(a,b)$, the Lax-Milgram theorem provides the existence of a unique $u_3 \in H_0^1(a,b)$ for any such pair. Moreover, by substituting $\varphi_3 = u_3$ we immediately find

$$\|u_3\|^2_{H_0^1(a,b)} \leq C \left| (\beta f_1', u_3)_{L^2(a,b)} \right| + \left| (\alpha f_3', u_3')_{L^2(a,b)} \right| \leq C \|F\|_{\mathcal{H}} \|u_3\|_{H_0^1(a,b)} \, , \tag{27}$$

with a positive constant C independent of F. We follow the convention of writing C for a generic constant whose value may change from line to line. Estimate (27) implies

$$\|u_3\|_{H_0^1(a,b)} \leq C \|F\|_{\mathcal{H}} \, . \tag{28}$$

Proceeding in a similar fashion for u_1 in (23), we find after a straightforward calculation that

$$\|u_1\|_{H_0^1(a,b)} \leq C \|F\|_{\mathcal{H}} \, . \tag{29}$$

Thus, for arbitrary $F = (f_1, f_2, f_3)^T$, there is a unique $U = (u_1, u_2, u_3)^T$ such that $\mathcal{B}U = F$. Moreover, the estimates (25), (28) and (29) imply that $\|U\|_{\mathcal{H}} \leq C \|F\|_{\mathcal{H}}$. Therefore, the operator $\mathcal{B}^{-1}$ exists and is bounded. □

We note that, as a result of Lemma 1, the eigenvalue problem (20) can be formulated equivalently as

$$U_\varepsilon = \lambda_\varepsilon \mathcal{M}_\varepsilon U_\varepsilon \text{ in } \mathcal{H} \tag{30}$$

with the operator $\mathcal{M}_\varepsilon := \mathcal{B}^{-1} \mathcal{Q}_\varepsilon$. It follows that $\lambda_\varepsilon \neq 0$, since $\lambda_\varepsilon = 0$ immediately implies $U_\varepsilon = 0$ in contradiction to the definition of an eigenvector. Thus, λ_ε^{-1} exists, and (30) allows the representation $\mathcal{M}_\varepsilon U_\varepsilon = \lambda_\varepsilon^{-1} U_\varepsilon$ in $\mathcal{H}$.

Lemma 2. The inverse operator $\mathcal{A}_\varepsilon^{-1} \colon \mathcal{L}_\varepsilon \to \mathcal{L}_\varepsilon$ exists and is compact.

Proof. The existence of $\mathcal{A}_\varepsilon^{-1}$ is equivalent to the existence of a solution U_ε to the problem $\mathcal{A}_\varepsilon U_\varepsilon = F$ for any $F \in \mathcal{L}_\varepsilon$, which reads component-wise as

$$-u_2^\varepsilon = f_1 \quad \text{in } \Omega_\varepsilon, \qquad u_2^\varepsilon \in H_0^1(a,b), \tag{31}$$

$$-\left(\alpha\,(u_1^\varepsilon)'\right)' + (\beta u_3^\varepsilon)' = \rho_\varepsilon f_2 \quad \text{in } \Omega_\varepsilon, \qquad u_1^\varepsilon \in \mathfrak{D}_*, \tag{32}$$

$$-\left(\varkappa\,(u_3^\varepsilon)'\right)' + \beta\,(u_2^\varepsilon)' = k f_3 \quad \text{in } \Omega_\varepsilon, \qquad u_3^\varepsilon \in \mathfrak{D}_*. \tag{33}$$

The argument is similar to that proposed for the proof of Lemma 1. We first find $u_2^\varepsilon = -f_1 \in H_0^1(a,b)$ from (31) such that

$$\|u_2^\varepsilon\|_{H_0^1(a,b)} = \|f_1\|_{H_0^1(a,b)} \le \|F\|_{\mathcal{L}_\varepsilon}.$$

Next, from (33) we find $u_3^\varepsilon \in H^2(a,b) \cap H_0^1(a,b)$ such that $\left(\varkappa\,(u_3^\varepsilon)'\right)' = \beta\,(u_2^\varepsilon)' - k f_3 \in L^2(a,b)$, which yields

$$\|u_3^\varepsilon\|_{H_0^1(a,b)} = C\left(\|u_2^\varepsilon\|_{H_0^1(a,b)} + \|f_3\|_{L_k^2(a,b)}\right) \le C\,\|F\|_{\mathcal{L}_\varepsilon}. \tag{34}$$

Finally, we find from (32), $u_1^\varepsilon \in H^2(a,b) \cap H_0^1(a,b)$ such that $\left(\alpha\,(u_1^\varepsilon)'\right)' = (\beta u_3^\varepsilon)' - \rho_\varepsilon f_2 \in L^2(a,b)$, which yields $\|u_1^\varepsilon\|_{H_0^1(a,b)} \le C\,\|F\|_{\mathcal{L}_\varepsilon}$ and, moreover,

$$\|u_1^\varepsilon\|_{H^2(a,b)} = C\left(\|u_1^\varepsilon\|_{H_0^1(a,b)} + \|u_3^\varepsilon\|_{H_0^1(a,b)} + \|f_2\|_{L_{\rho_\varepsilon}^2(a,b)}\right) \le C\,\|F\|_{\mathcal{L}_\varepsilon}. \tag{35}$$

Thus, the solution $U_\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)^T$ of (31)–(33) can be represented as a bounded operator $\hat{A}_\varepsilon^{-1} \colon \mathcal{L}_\varepsilon \to \mathcal{H}_2$, where $\mathcal{H}_2 = H^2(a,b) \times H_0^1(a,b) \times H_0^1(a,b)$. Then the operator $\mathcal{A}_\varepsilon^{-1} \colon \mathcal{L}_\varepsilon \to \mathcal{L}_\varepsilon$ exists and can be represented as $\mathcal{A}_\varepsilon^{-1} = J_* \hat{A}_\varepsilon^{-1}$, where $J_* \colon \mathcal{H}_2 \to \mathcal{L}_\varepsilon$ is a compact embedding operator

$$J_* = \begin{pmatrix} J & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_1 \end{pmatrix}$$

with compact embedding components $J \colon H^2(a,b) \to H^1(a,b)$, $J_1 \colon H_0^1(a,b) \to L^2(a,b)$. Therefore, $\mathcal{A}_\varepsilon^{-1}$ is compact as a product of a bounded and a compact operator. □

The compactness of $\mathcal{A}_\varepsilon^{-1}$ yields the following result (see [6, Theorem III.6.29]).

Corollary 1. The spectrum $\sigma(\mathcal{A}_\varepsilon)$ of $\mathcal{A}_\varepsilon$ is no more than a countable set of isolated eigenvalues with finite multiplicity. There is no accumulation point of the spectrum other than infinity. The eigenprojector $\mathcal{P}_\varepsilon$ to the generalised eigenspace corresponding to each $\lambda_\varepsilon \in \sigma(\mathcal{A}_\varepsilon)$ is identical with the eigenprojector corresponding to the eigenvalue λ_ε^{-1} of $\mathcal{A}_\varepsilon^{-1}$. Moreover, if $\lambda_\varepsilon \in \sigma(\mathcal{A}_\varepsilon)$, then $\bar{\lambda}_\varepsilon \in \sigma(\mathcal{A}_\varepsilon^*)$ has the same multiplicity, both in the geometric and the algebraic sense by the Riesz-Schauder Theorem [6, Remark 6.27] (the star * denotes the adjoint operator).

4. Equivalence of eigenvalue problems. The aim of this section is to establish the equivalence between the spectral properties of the operators $\mathcal{M}_\varepsilon := \mathcal{B}^{-1}\mathcal{Q}_\varepsilon$ and $\mathcal{A}_\varepsilon$ in the sense of Lemma 6. Since the operators are defined on two different Hilbert spaces, we first formulate auxiliary statements in Lemmas 3–5.

Let the operators $\mathcal{P}_j \colon \mathcal{H} \to H_0^1(a,b)$ be given by $\mathcal{P}_j U = u_j$ for $U = (u_1, u_2, u_3)^T$, $j = 1, 2, 3$.

LEMMA 3. For every $V \in \mathcal{H}$, the images $\mathcal{P}_1\mathcal{M}_\varepsilon V$ and $\mathcal{P}_3\mathcal{M}_\varepsilon V$ are functions in $H^2(a,b)$.

Proof. First, we show that the operator $\mathcal{Q}_\varepsilon$ defined by (19) admits the representation

$$\mathcal{Q}_\varepsilon = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & JQ_1^\varepsilon & 0 \\ 0 & 0 & JQ_2, \end{pmatrix}, \tag{36}$$

where $I_1\colon H_0^1(a,b) \to H_0^1(a,b)$ is the identity operator, $J\colon H^2(a,b) \to H^1(a,b)$ is a compact embedding operator and the operators Q_1^ε and Q_2 are as follows. The operator $Q_1^\varepsilon\colon H_0^1(a,b) \to H^2(a,b)$ gives a solution $w_1 = Q_1^\varepsilon u$ to the problem

$$-\left(\alpha w_1'\right)' = \rho_\varepsilon u \quad \text{in } (a,b), \qquad [\![w_1]\!]_{\pm\varepsilon} = [\![w_1']\!]_{\pm\varepsilon} = 0, \qquad w_1(a) = w_1(b) = 0. \tag{37}$$

The operator $Q_2\colon H_0^1(a,b) \to H^2(a,b)$ gives a solution $w_2 = Q_2 u$ to the problem

$$-\left(\alpha w_2'\right)' = ku \text{ in } (a,b), \qquad w_2(a) = w_2(b) = 0. \tag{38}$$

Since the weak formulations of (37) and (38) are such that for any $\varphi_1,\ \varphi_2 \in H_0^1(a,b)$,

$$\int_a^b \alpha w_1'\varphi_1'\,\mathrm{d}x = \int_a^b \rho_\varepsilon w_1\varphi_1\,\mathrm{d}x \quad \text{and} \quad \int_a^b \alpha w_2'\varphi_2'\,\mathrm{d}x = \int_a^b k w_2\varphi_2\,\mathrm{d}x,$$

the operator given by the representation (36) coincides with (19). Clearly, Q_1^ε and Q_2 are bounded, $\|Q_1^\varepsilon\| \le C_\varepsilon$, $\|Q_2\| \le C$, with positive constants C_ε and C. Therefore JQ_1^ε and JQ_2 are compact for any fixed $\varepsilon > 0$.

Second, we show that in any vector of the form $U_\varepsilon = \mathcal{M}_\varepsilon V = \mathcal{B}^{-1}\mathcal{Q}_\varepsilon V$ the first and the third components belong to $H^2(a,b)$. Note that $U_\varepsilon \in \mathcal{H}$ solves (22)–(24) with $F = \mathcal{Q}_\varepsilon V = \left(v_1, JQ_1^\varepsilon v_2, JQ_2 v_3\right)^T$. Then (22) yields

$$u_2 = -v_1 \in H_0^1(a,b). \tag{39}$$

Since $Q_1^\varepsilon v_2$ and $Q_2 v_3$ belong to $H^2(a,b)$ and (24) implies

$$\int_a^b \varkappa\, u_3'\bar{\varphi}_3'\,\mathrm{d}x = \int_a^b \left[-\left(\alpha\left(Q_2 v_3\right)'\right)' + \beta v_1'\right]\bar{\varphi}_3\,\mathrm{d}x \quad \text{for every } \phi_3 \in H_0^1(a,b), \tag{40}$$

we obtain for the distributional derivatives

$$\left(\varkappa\, u_3'\right)' = \left(\alpha\left(Q_2 v_3\right)'\right)' + \beta u_2' \in L^2(a,b). \tag{41}$$

Since $\varkappa \in C^1(a,b)$ is strictly positive, $u_3 = \mathcal{P}_3 U_\varepsilon \in H_0^1(a,b)$ along with (41) provides $u_3 \in H^2(a,b)$. Then identity (23) yields $\int_a^b \alpha u_1'\bar{\varphi}_2'\,\mathrm{d}x = \int_a^b \left[-\left(\alpha\left(Q_1^\varepsilon v_2\right)'\right)' - (\beta u_3)'\right]\bar{\varphi}_2\,\mathrm{d}x$ for every $\phi_2 \in H_0^1(a,b)$. Thus

$$\left(\alpha u_1'\right)' = \left(\alpha\left(Q_1^\varepsilon v_2\right)'\right)' + (\beta u_3)' \in L^2(a,b). \tag{42}$$

This in turn implies with $u_1 = \mathcal{P}_1 U_\varepsilon \in H_0^1(a,b)$ and the strict positivity of $\alpha \in C^1(a,b)$ that $u_1 \in H^2(a,b)$. □

Let $\mathfrak{R}\left(\mathcal{M}_\varepsilon\right)$ be the range of $\mathcal{M}_\varepsilon$, and let $J_{**}\colon \mathcal{H} \to \mathcal{L}_\varepsilon$ be an embedding operator

$$J_{**} = \begin{pmatrix} I & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_1 \end{pmatrix}$$

with the identity operator I and J_1 as above.

LEMMA 4. The following embedding of the functional sets:

$$\mathfrak{R}\left(\mathcal{M}_\varepsilon\right) \subset \mathfrak{D}\left(\mathcal{A}_\varepsilon\right) = \mathfrak{R}\left(\mathcal{A}_\varepsilon^{-1}\right) \subset \mathcal{H} = \mathfrak{D}\left(\mathcal{M}_\varepsilon\right) \subset \mathcal{L}_\varepsilon = \mathfrak{D}\left(\mathcal{A}_\varepsilon^{-1}\right) \tag{43}$$

holds, along with the operator equality $\mathcal{A}_\varepsilon^{-1} J_{**} = J_{**}\mathcal{M}_\varepsilon$.

Proof. The embedding (43) is a corollary to Lemma 3 and the definitions of the involved operators.

Let us prove the operator equality. Note that the domains of the operators are equal, $\mathfrak{D}\left(\mathcal{A}_\varepsilon^{-1} J_{**}\right) = \mathcal{H} = \mathfrak{D}\left(J_{**}\mathcal{M}_\varepsilon\right)$. We now prove that for every $V \in \mathcal{H}$, the functions $U_\varepsilon^{\text{left}} := \mathcal{A}_\varepsilon^{-1} J_{**} V$ and $U_\varepsilon^{\text{right}} := \mathcal{M}_\varepsilon V$ coincide up to the embedding operator J_{**}. We set $F := J_{**}V$. Then $U_\varepsilon^{\text{left}}$ is a unique solution to problem (31)–(33). On the other hand, $U_\varepsilon^{\text{right}}$ is a unique solution to (39), (41), (42) along with Dirichlet boundary conditions. The idea is to show that the problems coincide for the given data. Note that (31) and (39) are equivalent. The equivalence of (33) and (41) with Dirichlet conditions follows from the definition of the operator Q_2 in (38). Similarly, the equivalence of (32) and (42) is guaranteed by the definition of Q_1^ε in (37). By the uniqueness of the solution, $U_\varepsilon^{\text{left}}$ and $U_\varepsilon^{\text{right}}$ coincide. Of course, they are equal as elements of $\mathcal{L}_\varepsilon$ after an application of the embedding operator J_{**}. □

LEMMA 5. $\mathfrak{R}\left(\mathcal{M}_\varepsilon\right)$ is dense in $\mathcal{H}$.

Proof. We argue by contradiction. Let us suppose there is a non-zero $Y \in \mathcal{H}$ such that $(Y, \Phi)_{\mathcal{H}} = 0$ for all $\Phi \in \mathfrak{R}\left(\mathcal{M}_\varepsilon\right)$. For an arbitrary $U \in \mathcal{H}$, consider

$$0 = \left(\mathcal{M}_\varepsilon U, Y\right)_{\mathcal{H}} = \left(\mathcal{B}^{-1}\mathcal{Q}_\varepsilon U, Y\right)_{\mathcal{H}} = \left(\mathcal{Q}_\varepsilon U, \left(\mathcal{B}^{-1}\right)^* Y\right)_{\mathcal{H}} = \left(U, \left(\mathcal{B}^{-1}\right)^* Y\right)_{\mathcal{L}_\varepsilon}.$$

Since the latter is zero for any $U \in \mathcal{H}$, which is dense in $\mathcal{L}_\varepsilon$, we obtain $\left(\mathcal{B}^{-1}\right)^* Y = 0$. Since both $\mathcal{B}$ and $\mathcal{B}^{-1}$ are bounded operators on $\mathcal{H}$, the kernel of $\left(\mathcal{B}^{-1}\right)^*$ is trivial. Therefore, $\left(\mathcal{B}^{-1}\right)^* Y = 0$ implies $Y = 0$ in contradiction to our assumption. □

Lemma 5 allows us to introduce an operator $\mathcal{G}_\varepsilon = \mathcal{M}_\varepsilon^{-1} \colon \mathcal{H} \to \mathcal{H}$ with dense domain $\mathfrak{D}\left(\mathcal{G}_\varepsilon\right) = \mathfrak{R}\left(\mathcal{M}_\varepsilon\right)$. Note that according to (43), all eigenvectors U_ε of $\mathcal{A}_\varepsilon$ belong to $\mathcal{L}_\varepsilon \cap \mathcal{H}$. In the next lemma, in order to emphasise the different functional classes, we use different notation for an eigenvector U_ε as an element of $\mathcal{L}_\varepsilon$ and for its equivalent V_ε in $\mathcal{H}$; the correspondence is then $U_\varepsilon = J_{**}V_\varepsilon$. A pair consisting of an eigenvalue λ_ε and a corresponding eigenfunction U_ε of a given operator is referred to as an *eigenmode* of the operator.

LEMMA 6. The following statements are equivalent, with $U_\varepsilon = J_{**}V_\varepsilon$.

i) $(\lambda_\varepsilon, V_\varepsilon)$ is an eigenmode of $\mathcal{G}_\varepsilon = \mathcal{M}_\varepsilon^{-1}$;
ii) $\left(\lambda_\varepsilon^{-1}, V_\varepsilon\right)$ is an eigenmode of $\mathcal{M}_\varepsilon$;
iii) $\left(\lambda_\varepsilon^{-1}, U_\varepsilon\right)$ is an eigenmode of $\mathcal{A}_\varepsilon^{-1}$;
iv) $(\lambda_\varepsilon, U_\varepsilon)$ is an eigenmode of $\mathcal{A}_\varepsilon$.

Proof. The trivial equivalence *iii*) ⇔ *iv*) is already stated in Corollary 1. The statement *i*) ⇔ *ii*) is also immediate. We now prove *ii*) ⇒ *iii*). Let $\mu_\varepsilon = \lambda_\varepsilon^{-1}$. We apply J_{**} to both sides of $\mathcal{M}_\varepsilon V_\varepsilon = \mu_\varepsilon V_\varepsilon$. Then, by Lemma 4, $\mathcal{A}_\varepsilon^{-1} J_{**} V_\varepsilon = \mu_\varepsilon J_{**} V_\varepsilon$. Obviously if $V_\varepsilon \neq 0$, then $J_{**}V_\varepsilon \neq 0$. We finally show *iii*) ⇒ *ii*). Note that if U_ε is an eigenvector of

$\mathcal{A}_\varepsilon^{-1}$,

$$\mathcal{A}_\varepsilon^{-1} U_\varepsilon = \mu_\varepsilon U_\varepsilon, \tag{44}$$

then $U_\varepsilon \in \mathfrak{R}\left(\mathcal{A}_\varepsilon^{-1}\right)$. Moreover, (43) provides $\mathfrak{R}\left(\mathcal{A}_\varepsilon^{-1}\right) \subset \mathcal{H}$. Then $U_\varepsilon \in \mathcal{L}_\varepsilon \cap \mathcal{H}$ can be presented in the form $U_\varepsilon = J_{**} V_\varepsilon$ with $V_\varepsilon \in \mathcal{H}$. Again applying Lemma 4 to (44), we obtain $J_{**} \mathcal{M}_\varepsilon V_\varepsilon = J_{**} \mu_\varepsilon V_\varepsilon$. Since both $\mu_\varepsilon V_\varepsilon$ and $\mathcal{M}_\varepsilon V_\varepsilon$ belong to $\mathcal{H} \subset \mathcal{L}_\varepsilon$ and their traces in $\mathcal{L}_\varepsilon$ coincide, they are equal also as elements of $\mathcal{H}$, namely $\mathcal{M}_\varepsilon V_\varepsilon = \mu_\varepsilon V_\varepsilon$. □

5. Spectral properties. The discreteness of the spectra of $\mathcal{A}_\varepsilon$ and $\mathcal{G}_\varepsilon$ is guaranteed by Corollary 1 and Lemma 6. In this section, we discuss some additional properties of the spectra of the operator $\mathcal{A}_\varepsilon$ (and thus $\mathcal{G}_\varepsilon$) for fixed $\varepsilon > 0$.

The completeness of generalised eigenfunctions for linear thermoelasticity has been proved by Yakubov [12] in the three-dimensional case. The proof can be easily adapted for one space dimension. In this case, a crucial ingredient, that is, a decay rate for approximation numbers, holds in an even stronger form [1].

LEMMA 7. Each eigenvalue λ_ε has a positive real part, $\operatorname{Re} \lambda_\varepsilon > 0$.

Proof. We find from (18) that for a non-zero eigenvector $U_\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ corresponding to λ_ε,

$$\lambda_\varepsilon = \frac{(\mathcal{B} U_\varepsilon, U_\varepsilon)_{\mathcal{H}}}{(U_\varepsilon, U_\varepsilon)_{\mathcal{L}_\varepsilon}} = \frac{\tau(U_\varepsilon, U_\varepsilon)}{(U_\varepsilon, U_\varepsilon)_{\mathcal{L}_\varepsilon}} = \frac{\tau(U_\varepsilon, U_\varepsilon)}{\|U_\varepsilon\|_{\mathcal{L}_\varepsilon}^2}.$$

Since for any $U \in \mathcal{H}$, by the definition (17) of τ,

$$\tau(U, U) = \int_a^b \left[\varkappa \left|u_3'\right|^2 + 2i \operatorname{Im}\left(\alpha u_1' \bar{u}_2' + \beta u_2' \bar{u}_3\right)\right] \mathrm{d}x,$$

we obtain

$$\operatorname{Re} \lambda_\varepsilon = \|U_\varepsilon\|_{\mathcal{L}_\varepsilon}^{-2} \int_a^b \varkappa \left|(u_3^\varepsilon)'\right|^2 \mathrm{d}x. \tag{45}$$

Then obviously $\operatorname{Re} \lambda_\varepsilon \geq 0$. Let us show that $\operatorname{Re} \lambda_\varepsilon \neq 0$. Suppose to the contrary $\operatorname{Re} \lambda_\varepsilon = 0$; we then find from (45) that $(u_3^\varepsilon)' = 0$ on (a, b). Therefore, from (10) we find $u_3^\varepsilon = 0$ on (a, b). Since $\beta > 0$, equation (9) yields $(u_2^\varepsilon)' = 0$ on (a, b); again (10) gives $u_2^\varepsilon = 0$ on (a, b). Similarly, (8) along with (9) yields $u_1^\varepsilon = 0$ on (a, b), and thus $U_\varepsilon = 0$ in contradiction to the assumption. □

LEMMA 8. If U_ε is an eigenvector of $\mathcal{A}_\varepsilon$, then each component of the vector U_ε is different from zero.

Proof. The proof of Lemma 7 shows that $u_3^\varepsilon \neq 0$. The remaining arguments are similar. If $u_2^\varepsilon = 0$, then (7) would yield $u_1^\varepsilon = 0$ and consequently (8) along with (10) implies $u_3^\varepsilon = 0$, a contradiction. Thus $u_2^\varepsilon \neq 0$. If $u_1^\varepsilon = 0$, then (7) immediately yields $u_2^\varepsilon = 0$. Then (8) and (10) yield $u_3^\varepsilon = 0$, again a contradiction. □

LEMMA 9. The adjoint operator admits the representation $\mathcal{A}_\varepsilon^* = \mathcal{T} \mathcal{A}_\varepsilon \mathcal{T}$ with

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathfrak{D}\left(\mathcal{A}_\varepsilon^*\right) = \mathfrak{D}\left(\mathcal{A}_\varepsilon\right).$$

Proof. Since the form

$$(\mathcal{A}U,V)_{\mathcal{H}} = -\int_a^b \alpha u_2'\bar{v}_1'\,\mathrm{d}x + \int_a^b \left[-(\alpha u_1')' + (\beta u_3)'\right]\bar{v}_2\,\mathrm{d}x + \int_a^b \left[-(\varkappa u_3')' + \beta u_2'\right]\bar{v}_3\,\mathrm{d}x \tag{46}$$

for all $U \in \mathfrak{D}(\mathcal{A})$ can be represented as

$$(U,W)_{\mathcal{H}} = \int_a^b \alpha u_1'\bar{w}_1'\,\mathrm{d}x + \int_a^b \rho_\varepsilon u_2\bar{w}_2\,\mathrm{d}x + \int_a^b k u_3\bar{w}_3\,\mathrm{d}x$$

only with $w_1 = v_2$, $w_2 = \frac{1}{\rho_\varepsilon}\left((\alpha v_1')' - (\beta v_3)'\right)$, $w_3 = -\frac{1}{k}\left((\varkappa v_3')' + \beta v_2'\right)$, for $v_2(a) = v_2(b) = 0$ and $v_3(a) = v_3(b) = 0$, then by the definition of the adjoint operator, $W = \mathcal{A}_\varepsilon^* V$ with

$$\mathcal{A}_\varepsilon^* V = A_\varepsilon^*\left(x, \frac{d}{dx}\right)V \quad \text{given by } A_\varepsilon^*(x,\mathcal{D}) = \begin{pmatrix} 0 & I & 0 \\ \rho_\varepsilon^{-1}\mathcal{D}\alpha\mathcal{D} & 0 & -\rho_\varepsilon^{-1}\mathcal{D}\beta \\ 0 & -k^{-1}\beta\mathcal{D} & -k^{-1}\mathcal{D}\varkappa\mathcal{D} \end{pmatrix}.$$

Note again that the matrix-differential expression $A_\varepsilon^*\left(x, \frac{d}{dx}\right)$ can be applied only to the functions from $\mathfrak{D}_* \times H_0^1(a,b) \times \mathfrak{D}_*$. Thus $\mathfrak{D}(\mathcal{A}_\varepsilon^*)$ coincides with $\mathfrak{D}(\mathcal{A}_\varepsilon)$. The identity $\mathcal{A}_\varepsilon^* = \mathcal{T}\mathcal{A}_\varepsilon\mathcal{T}$ is easy to check. □

REMARK 1. One can see that the operator $\mathcal{A}_\varepsilon$ is not a normal operator for $\beta > 0$, i.e., $\mathcal{A}_\varepsilon^*\mathcal{A}_\varepsilon \neq \mathcal{A}_\varepsilon\mathcal{A}_\varepsilon^*$. Since we are not using this fact, the proof is not presented here.

LEMMA 10. If $\lambda_\varepsilon \in \sigma(\mathcal{A}_\varepsilon)$, then $\bar{\lambda}_\varepsilon \in \sigma(\mathcal{A}_\varepsilon)$ as well. Thus $\sigma(\mathcal{A}_\varepsilon) = \sigma(\mathcal{A}_\varepsilon^*)$.

Proof. If $\lambda_\varepsilon \in \sigma(\mathcal{A}_\varepsilon)$, then $\bar{\lambda}_\varepsilon \in \sigma(\mathcal{A}_\varepsilon^*)$ and thus $\mathcal{A}_\varepsilon^* W = \bar{\lambda}_\varepsilon W$ for some non-zero $W \in \mathcal{L}_\varepsilon$. Then, by Lemma 9,

$$\mathcal{T}\mathcal{A}_\varepsilon\mathcal{T}W = \bar{\lambda}_\varepsilon W. \tag{47}$$

Note that $\mathcal{T}^2$ is an identity operator. We apply $\mathcal{T}$ to (47) to get $\mathcal{A}_\varepsilon\mathcal{T}W = \bar{\lambda}_\varepsilon\mathcal{T}W$. Since $W \neq 0$, it follows that $\mathcal{T}W \neq 0$ is an eigenfunction of $\mathcal{A}_\varepsilon$ corresponding to $\bar{\lambda}_\varepsilon$. □

COROLLARY 2. Let U_ε be an eigenfunction corresponding to an eigenvalue λ_ε of $\mathcal{A}_\varepsilon$ and let U_ε^* be an eigenfunction corresponding to the same eigenvalue $\lambda_\varepsilon^* = \lambda_\varepsilon$ of $\mathcal{A}_\varepsilon^*$. The following statements hold.

(i) If $\lambda_\varepsilon \notin \mathbb{R}$, then $(U_\varepsilon, U_\varepsilon^*)_{\mathcal{L}_\varepsilon} = 0$.

(ii) If $\bar{\lambda}_\varepsilon$, U_ε^* is an eigenmode of $\mathcal{A}_\varepsilon^*$, then $\bar{\lambda}_\varepsilon$, $\mathcal{T}U_\varepsilon^*$ is an eigenmode of $\mathcal{A}_\varepsilon$.

(iii) If λ_ε, U_ε is an eigenmode of $\mathcal{A}_\varepsilon$, then λ_ε, $\mathcal{T}U_\varepsilon$ is an eigenmode of $\mathcal{A}_\varepsilon^*$.

(iv) If λ_ε, U_ε is an eigenmode of $\mathcal{A}_\varepsilon$ and $\lambda_\varepsilon \notin \mathbb{R}$, then $(U_\varepsilon, \mathcal{T}U_\varepsilon)_{\mathcal{L}_\varepsilon} = 0$.

(v) Any eigenfunction U_ε corresponding to a non-real eigenvalue satisfies the equality

$$\int_a^b \alpha\left|(u_1^\varepsilon)'\right|^2\,\mathrm{d}x + \int_a^b k\,|u_3^\varepsilon|^2\,\mathrm{d}x = \int_a^b \rho_\varepsilon\,|u_2^\varepsilon|^2\,\mathrm{d}x. \tag{48}$$

Proof. Claim *(i)* is proved by noticing that for $\bar{\lambda}_\varepsilon \neq \lambda_\varepsilon^*$ the corresponding eigenspaces of $\mathcal{A}_\varepsilon$ and $\mathcal{A}_\varepsilon^*$ are orthogonal. Claim *(ii)* has been shown in the proof of Lemma 10. Thus $U_\varepsilon = \mathcal{T}U_\varepsilon^*$ is an eigenvector of $\mathcal{A}_\varepsilon$. Since $\mathcal{T}^{-1} = \mathcal{T}$ we obtain $U_\varepsilon^* = \mathcal{T}U_\varepsilon$. Note that the kernel of $\mathcal{T}$ is trivial, so $U_\varepsilon^* \neq 0$. Thus *(iii)* is proved. Claim *(iv)* follows from applying *(i)* to *(iii)*. Finally, *(v)* is a component-wise representation of *(iv)*. □

Remark 2. We can interpret the norm of eigenvibrations as the energy; namely,

$$E_{\text{total}}(U_\varepsilon) = \|U_\varepsilon\|^2_{\mathcal{L}_\varepsilon} = \int_a^b \alpha \left|(u_1^\varepsilon)'\right|^2 \,\mathrm{d}x + \int_a^b \rho_\varepsilon \left|u_2^\varepsilon\right|^2 \,\mathrm{d}x + \int_a^b k \left|u_3^\varepsilon\right|^2 \,\mathrm{d}x.$$

The terms on the right-hand side are, in that order, the elastic energy E_{elastic}, the kinetic energy E_{kinetic} and the thermal energy E_{thermal}. Then (48) shows that eigenvibrations with non-real eigenvalues have an equal distribution between kinetic and thermo-elastic energies:

$$E_{\text{kinetic}}(U_\varepsilon) = E_{\text{thermal}}(U_\varepsilon) + E_{\text{elastic}}(U_\varepsilon) = \frac{1}{2} E_{\text{total}}(U_\varepsilon).$$

Moreover, each energetic contribution E_{kinetic}, E_{thermal} and E_{elastic} is non-zero.

6. The limit behaviour, $m < 1$.

In the case $m < 1$, the limit eigenvalue problem is

$$-u_2 = \lambda_0 u_1, \quad -(\alpha u_1')' + (\beta u_3)' = \lambda_0 p u_2, \quad -(\varkappa u_3')' + \beta u_2' = \lambda_0 k u_3 \quad \text{in } \Omega_0, \tag{49}$$

where $\Omega_0 = (a,0) \cup (0,b)$, with the boundary and interface conditions

$$u_j(a) = u_j(b) = 0, \quad [\![u_j]\!]_0 = [\![u_j']\!]_0 = 0 \quad \text{for } j = 1,2,3. \tag{50}$$

Remark 3. Similarly as for problem (7)–(10), it can be proved that the spectrum of (49)–(50) is discrete. That is, it has the same properties as the spectrum of the operator $\mathcal{A}_\varepsilon$ described in Corollary 1. Likewise, the completeness of generalised eigenfunctions discussed at the beginning of Section 5 holds for (49)–(50) as well.

We introduce an operator $\mathcal{Q}_0 \colon \mathcal{H} \to \mathcal{H}$ such that $(\mathcal{Q}_0 U, V)_{\mathcal{H}} = (U,V)_{\mathcal{L}^0}$ for every $U, V \in \mathcal{H}$, where $\mathcal{L}^0 := H_0^1(a,b) \times L_p^2(a,b) \times L_k^2(a,b)$. Then the eigenvalue problem (49)–(50) can be formulated as $\mathcal{B}U = \lambda_0 \mathcal{Q}_0 U$ in $\mathcal{H}$. The latter is equivalent to $\mathcal{M}_0 U = \lambda_0^{-1} U$ in $\mathcal{H}$, with a bounded operator $\mathcal{M}_0 := \mathcal{B}^{-1}\mathcal{Q}_0$. Similarly, as it is shown in the previous sections, there exists a densely defined operator $\mathcal{G}_0 \colon \mathcal{H} \to \mathcal{H}$ with domain $\mathfrak{D}(\mathcal{G}_0) = \mathfrak{R}(\mathcal{M}_0)$ such that its inverse coincides with $\mathcal{M}_0$, $\mathcal{G}_0^{-1} = \mathcal{M}_0$. We introduce the resolvent of the operator $\mathcal{G}_0$, namely $\mathcal{R}(\zeta, \mathcal{G}_0) = (\mathcal{G}_0 - \zeta)^{-1}$. Note that we already constructed explicitly $\mathcal{R}(0, \mathcal{G}_0) = \mathcal{G}_0^{-1} = \mathcal{M}_0 = \mathcal{B}^{-1}\mathcal{Q}_0$.

We now establish the estimates between the resolvents $\mathcal{R}(\zeta, \mathcal{G}_\varepsilon) = (\mathcal{G}_\varepsilon - \zeta)^{-1}$ and $\mathcal{R}(\zeta, \mathcal{G}_0)$ as $\varepsilon \to 0$ for any $m < 1$. Again, we already constructed explicitly $\mathcal{R}(0, \mathcal{G}_\varepsilon) = \mathcal{G}_\varepsilon^{-1} = \mathcal{M}_\varepsilon = \mathcal{B}^{-1}\mathcal{Q}_\varepsilon$, which depends on ε and m only via the multiplier $\mathcal{Q}_\varepsilon$. Then the asymptotic behaviour of the resolvent as $\varepsilon \to 0$ is determined by the asymptotic behaviour of the operator $\mathcal{Q}_\varepsilon$.

Lemma 11. If $m < 1$, then for small ε we have $\|\mathcal{R}(\zeta, \mathcal{G}_\varepsilon) - \mathcal{R}(\zeta, \mathcal{G}_0)\| \le C\varepsilon^\gamma$ with $\gamma := \min\{1, 1-m\}$ and a positive constant C independent of ε and m.

Proof. By virtue of [6, IV.3.13],

$$\begin{aligned}\|\mathcal{R}(\zeta, \mathcal{G}_\varepsilon) - \mathcal{R}(\zeta, \mathcal{G}_0)\| &\le C \|\mathcal{R}(0, \mathcal{G}_\varepsilon) - \mathcal{R}(0, \mathcal{G}_0)\| = C \|\mathcal{M}_\varepsilon - \mathcal{M}_0\| \\ &= C \left\|\mathcal{B}^{-1}(\mathcal{Q}_\varepsilon - \mathcal{Q}_0)\right\| \le C \left\|\mathcal{B}^{-1}\right\| \|\mathcal{Q}_\varepsilon - \mathcal{Q}_0\| \\ &\le C \|\mathcal{Q}_\varepsilon - \mathcal{Q}_0\|. \end{aligned} \tag{51}$$

For arbitrary elements $U, V \in \mathcal{H}$ we consider

$$((\mathcal{Q}_\varepsilon - \mathcal{Q}_0)\, U, V)_{\mathcal{H}} = (\mathcal{Q}_\varepsilon U, V)_{\mathcal{H}} - (\mathcal{Q}_0 U, V)_{\mathcal{H}} = (U, V)_{\mathcal{L}_\varepsilon} - (U, V)_{\mathcal{L}^0}$$
$$= \int_a^b (\rho_\varepsilon - p)\, u_2 v_2 \,\mathrm{d}x = \int_{-\varepsilon}^{\varepsilon} \left[\varepsilon^{-m} q\left(\frac{x}{\varepsilon}\right) - p(x)\right] u_2 v_2 \,\mathrm{d}x.$$

Then by Lemma 13 in the Appendix, $|((\mathcal{Q}_\varepsilon - \mathcal{Q}_0)\, U, V)_{\mathcal{H}}| \le C\varepsilon^\gamma \|u_2\|_{H_0^1(a,b)} \|v_2\|_{H_0^1(a,b)} \le C\varepsilon^\gamma \|U\|_{\mathcal{H}} \|V\|_{\mathcal{H}}$ for any $U, V \in \mathcal{H}$. Hence, $\|\mathcal{Q}_\varepsilon - \mathcal{Q}_0\| \le C\varepsilon^\gamma$. A substitution into (51) proves the claim. □

Theorem 1. Let $m < 1$. Then the eigenvalues λ_ε of problem (7)–(10) converge to the eigenvalues λ_0 of problem (49)–(50) saving multiplicity. Let $\mathcal{P}_\varepsilon$ be the eigenprojector to the generalised eigenspace of $\mathcal{M}_\varepsilon$ corresponding to λ_ε, and let $\mathcal{P}_0$ be the eigenprojector to the generalised eigenspace of $\mathcal{M}_0$ corresponding to λ_0. Then

$$\|\mathcal{P}_\varepsilon - \mathcal{P}_0\| \le C\varepsilon^\gamma \quad \text{with } \gamma = \min\{1, 1-m\}. \tag{52}$$

Moreover, for each eigenvalue λ_0 of (49)–(50), there is a sequence of eigenvalues $\{\lambda_\varepsilon\}_{\varepsilon\to 0}$ of (7)–(10) such that $\lambda_\varepsilon \to \lambda_0$ and such that estimate (52) holds.

Proof. Lemma 11 states the resolvent convergence $\mathcal{R}(\cdot, \mathcal{G}_\varepsilon) \to \mathcal{R}(\cdot, \mathcal{G}_0)$ in norm as $\varepsilon \to 0$. Then the eigenvalue convergence saving multiplicity follows [2, p. 35]. Further, each eigenvalue λ_0 can be encircled with a contour Γ such that no other eigenvalue of $\mathcal{G}_0$ is inside or on the contour. Then the eigenprojector to the generalised eigenspace corresponding to λ_0 is given by the Riesz projector

$$\mathcal{P}_0 = \frac{1}{2\pi i} \int_\Gamma \mathcal{R}(\zeta, \mathcal{G}_0)\, \mathrm{d}\zeta. \tag{53}$$

Since $\lambda_\varepsilon \to \lambda_0$, for ε small enough, λ_ε is strictly inside Γ. Then again the eigenprojector corresponding to λ_ε is given by

$$\mathcal{P}_\varepsilon = \frac{1}{2\pi i} \int_\Gamma \mathcal{R}(\zeta, \mathcal{G}_\varepsilon)\, \mathrm{d}\zeta. \tag{54}$$

Since the contour Γ does not depend on ε and consists only of regular points of $\mathcal{G}_\varepsilon$ and $\mathcal{G}_0$, we can estimate

$$\|\mathcal{P}_\varepsilon - \mathcal{P}_0\| \le C_1 \int_\Gamma \|\mathcal{R}(\zeta, \mathcal{G}_\varepsilon) - \mathcal{R}(\zeta, \mathcal{G}_0)\|\, \mathrm{d}\zeta \le C_2 \varepsilon^\gamma \tag{55}$$

by virtue of Lemma 11.

If we assume that there exists λ_0 which cannot be approached by a sequence $\{\lambda_\varepsilon\}$, then there are no λ_ε in a suitable neighbourhood N of λ_0. Choose a contour Γ_1 inside N encircling λ_0. Then the corresponding Riesz projectors to the linear subspaces, whose eigenvalues are inside Γ_1, are given by (53) and (54) with $\Gamma = \Gamma_1$. Therefore the projectors satisfy (55). Hence, for small ε we obtain $\dim \mathcal{P}_\varepsilon = \dim \mathcal{P}_0$. This contradicts the assumption that inside Γ_1 there are no eigenvalues λ_ε of $\mathcal{G}_\varepsilon$. □

7. The limit behaviour, $m = 1$**.** For $m = 1$ the limit eigenvalue problem is

$$-u_2 = \lambda_0 u_1, \quad -(\alpha u_1')' + (\beta u_3)' = \lambda_0 p u_2, \quad -(\varkappa u_3')' + \beta u_2' = \lambda_0 k u_3 \quad \text{in } \Omega_0, \tag{56}$$

$$u_j(a) = u_j(b) = 0, \quad [\![u_j]\!]_0 = 0, \quad j = 1,2,3, \tag{57}$$

$$[\![u_1']\!]_0 = -\lambda_0 q_0 \alpha(0)^{-1} u_2(0), \quad [\![u_2']\!]_0 = -\lambda_0 [\![u_1']\!]_0, \quad [\![u_3']\!]_0 = 0 \tag{58}$$

with

$$q_0 = \int_{-1}^{1} q(\xi)\,\mathrm{d}\xi.$$

Remark 4. Similarly as for problem (7)–(10), it can be proved that the spectrum of (56)–(58) is discrete. That is, it has the same properties as the spectrum of the operator $\mathcal{A}_\varepsilon$ described in Corollary 1.

Note that the variational formulation of the problem leads to

$$\tau(U,\Phi) + \alpha(0)\bar{\varphi}_2(0)\,[\![u_1']\!]_0 = \lambda_0 (U,\Phi)_{\mathcal{L}^0} \quad \text{for every } \Phi \in \mathcal{H}.$$

The latter along with the transmission conditions (58) implies

$$\tau(U,\Phi) = \lambda_0 \left((U,\Phi)_{\mathcal{L}^0} + q_0 u_2(0)\bar{\varphi}_2(0)\right) \quad \text{for every } \Phi \in \mathcal{H}.$$

We introduce a bounded operator $\mathcal{Q}_1 \colon \mathcal{H} \to \mathcal{H}$ such that

$$(\mathcal{Q}_1 U, V)_{\mathcal{H}} = (U,V)_{\mathcal{L}^0} + q_0 u_2(0)\bar{v}_2(0) \quad \text{for every } U, V \in \mathcal{H}.$$

Then eigenvalue problem (56)–(58) can be formulated as $\mathcal{B}U = \lambda_0 \mathcal{Q}_1 U$ in $\mathcal{H}$. The latter is equivalent to $\mathcal{M}_1 U = \lambda_0^{-1} U$ in $\mathcal{H}$, with a bounded operator $\mathcal{M}_1 := \mathcal{B}^{-1}\mathcal{Q}_1$. Similarly, as is shown in the previous sections, it can be shown that there exists a densely defined operator $\mathcal{G}_1 \colon \mathcal{H} \to \mathcal{H}$ with domain $\mathfrak{D}(\mathcal{G}_1) = \mathfrak{R}(\mathcal{M}_1)$ such that its inverse coincides with $\mathcal{M}_1$, $\mathcal{G}_1^{-1} = \mathcal{M}_1$. We introduce the resolvent

$$\mathcal{R}(\zeta, \mathcal{G}_1) = (\mathcal{G}_1 - \zeta)^{-1}.$$

Note that we already constructed explicitly $\mathcal{R}(0,\mathcal{G}_1) = \mathcal{G}_1^{-1} = \mathcal{M}_1 = \mathcal{B}^{-1}\mathcal{Q}_1$.

Lemma 12. If $m = 1$, then for small ε we have $\|\mathcal{R}(\zeta,\mathcal{G}_\varepsilon) - \mathcal{R}(\zeta,\mathcal{G}_1)\| \le C\varepsilon^{1/2}$ with $C > 0$ independent of ε.

Proof. Similarly to the proof of Lemma 11, we establish

$$\|\mathcal{R}(\zeta,\mathcal{G}_\varepsilon) - \mathcal{R}(\zeta,\mathcal{G}_1)\| \le C\,\|\mathcal{Q}_\varepsilon - \mathcal{Q}_1\|. \tag{59}$$

For arbitrary elements $U, V \in \mathcal{H}$ we consider

$$\begin{aligned}((\mathcal{Q}_\varepsilon - \mathcal{Q}_1)U, V)_{\mathcal{H}} &= (U,V)_{\mathcal{L}_\varepsilon} - (U,V)_{\mathcal{L}^0} - q_0 u_2(0) v_2(0)\\ &= \varepsilon^{-1}\int_{-\varepsilon}^{\varepsilon} q\left(\frac{x}{\varepsilon}\right) u_2 v_2\,\mathrm{d}x - \int_{-\varepsilon}^{\varepsilon} p(x) u_2 v_2\,\mathrm{d}x - q_0 u_2(0) v_2(0).\end{aligned}$$

Then by Lemma 13 in the Appendix, we obtain

$$\|((\mathcal{Q}_\varepsilon - \mathcal{Q}_1)U,V)_{\mathcal{H}}\| \le C\varepsilon^{\frac{1}{2}}\,\|u_2\|_{H_0^1(a,b)}\,\|v_2\|_{H_0^1(a,b)} \le C\varepsilon^{\frac{1}{2}}\,\|U\|_{\mathcal{H}}\,\|V\|_{\mathcal{H}}$$

for any $U, V \in \mathcal{H}$. Hence, $\|\mathcal{Q}_\varepsilon - \mathcal{Q}_1\| \le C\varepsilon^{1/2}$. A substitution in (59) proves the claim. □

Theorem 2. Let $m = 1$. Then the eigenvalues λ_ε of (7)–(10) converge to the eigenvalues λ_0 of (56) saving multiplicity. Let $\mathcal{P}_\varepsilon$ be the eigenprojector to the generalised eigenspace corresponding to λ_ε, and let $\mathcal{P}_1$ be the eigenprojector to the generalised eigenspace corresponding to λ_0. Then the following estimate holds:

$$\|\mathcal{P}_\varepsilon - \mathcal{P}_1\| \leq C\varepsilon^{\frac{1}{2}}. \tag{60}$$

Moreover, for each eigenvalue λ_0 of (56)–(58) there is a sequence of eigenvalues $\{\lambda_\varepsilon\}_{\varepsilon\to 0}$ of (7)–(10) such that $\lambda_\varepsilon \to \lambda_0$ and such that estimate (60) holds.

The idea of the proof is the same as that for Theorem 1 and is thus omitted; it relies on the estimates of Lemma 12. We remark that for $m = 1$, the localised mass concentration influences the limit eigenfrequencies and eigenvibrations represented in transmission conditions (58) involving momentum of mass q_0 and the spectral parameter (see [4] for the same effect in the isothermal elastic problem).

Discussion. We want to finish with a brief comparison of the limit behaviour in the cases $m < 1$ and $m = 1$. Theorem 1 shows that there is no influence of localised perturbation in the mass density $\varepsilon^{-m}q$ on the limit eigenfrequencies and the generalised eigenspaces for $m < 1$. This makes the expectation sufficiently rigorous that for $m < 1$ the added mass is small enough to be neglected since

$$\int_{-\varepsilon}^{\varepsilon} \varepsilon^{-m} q\left(\frac{x}{\varepsilon}\right) \mathrm{d}x = \varepsilon^{1-m} \int_{-1}^{1} q(\xi)\, \mathrm{d}\xi \to 0 \quad \text{ as } \varepsilon \to 0.$$

In contrast, Theorem 2 shows for $m = 1$ a dependence of the eigenvibrations on the concentrated mass q_0; see (58). This makes the intuition profound that larger values of m have a stronger impact on the limit behaviour, leading to localisation of eigenvibrations; this has so far been proved rigorously in the isothermal case only [4].

Appendix.

Lemma 13 (Golovaty, Nazarov, Oleinik, Soboleva [4]). Let $p \in C[a, b]$, $q \in C[-1, 1]$ and $q_0 := \int_{-1}^{1} q(\xi)\, \mathrm{d}\xi$. For any $u, v \in H_0^1(a, b)$ the following estimates hold:

$$\left\| \varepsilon^{-1} \int_{-\varepsilon}^{\varepsilon} q\left(\frac{x}{\varepsilon}\right) u(x)v(x)\, \mathrm{d}x - q_0 u(0)v(0) \right\| \leq C\varepsilon^{\frac{1}{2}} \|u\|_{H_0^1(a,b)} \|v\|_{H_0^1(a,b)}, \tag{61}$$

$$\left| \int_{-\varepsilon}^{\varepsilon} p(x)u(x)v(x)\, \mathrm{d}x \right| \leq C\varepsilon \|u\|_{H_0^1(a,b)} \|v\|_{H_0^1(a,b)}. \tag{62}$$

For the reader's convenience, we give the proof here.

Proof. The second claim follows from the compact embedding $C(a, b) \subset H_0^1(a, b)$, since

$$\begin{aligned} \left| \int_{-\varepsilon}^{\varepsilon} p(x)u(x)v(x)\, \mathrm{d}x \right| &\leq \int_{-\varepsilon}^{\varepsilon} |p(x)u(x)v(x)|\, \mathrm{d}x \\ &\leq 2\varepsilon \max_{x\in[a,b]} |p(x)u(x)v(x)| \leq C\varepsilon \|u\|_{H_0^1(a,b)} \|v\|_{H_0^1(a,b)}. \end{aligned}$$

To prove the first claim, we note that

$$\left|\varepsilon^{-1}\int_{-\varepsilon}^{\varepsilon} q\left(\frac{x}{\varepsilon}\right) u(x)v(x)\,\mathrm{d}x - q_0 u(0)v(0)\right| = \left|\int_{-1}^{1} q(\xi)\left(u(\varepsilon\xi)v(\varepsilon\xi) - u(0)v(0)\right)\,\mathrm{d}\xi\right|$$
$$= \left|\int_{-1}^{1} q(\xi)\left(\int_0^{\varepsilon\xi}(u'v - uv')\,\mathrm{d}t\right)\,\mathrm{d}\xi\right|$$
$$\leq \int_{-1}^{1} q(\xi)\left|\int_0^{\varepsilon\xi} u'v\,\mathrm{d}t\right|\,\mathrm{d}\xi + \int_{-1}^{1} q(\xi)\left|\int_0^{\varepsilon\xi} uv'\,\mathrm{d}t\right|\,\mathrm{d}\xi. \tag{63}$$

For any $\xi \in (-1,1)$, we have

$$\left|\int_0^{\varepsilon\xi} u'v\,\mathrm{d}t\right| \leq \left|\int_0^{\varepsilon\xi}(u')^2\,\mathrm{d}t\right|^{\frac{1}{2}}\left|\int_0^{\varepsilon\xi} v^2\,\mathrm{d}t\right|^{\frac{1}{2}}$$
$$\leq \left|\int_a^b (u')^2\,\mathrm{d}t\right|^{\frac{1}{2}} |\varepsilon\xi|^{\frac{1}{2}} \max_{t\in[a,b]}\left|v^2(t)\right| \leq C\,|\varepsilon\xi|^{\frac{1}{2}}\,\|u\|_{H_0^1(a,b)}\,\|v\|_{H_0^1(a,b)}\,.$$

Therefore

$$\int_{-1}^{1} q(\xi)\left|\int_0^{\varepsilon\xi} u'v\,\mathrm{d}t\right|\,\mathrm{d}\xi \leq C\varepsilon^{\frac{1}{2}}\,\|u\|_{H_0^1(a,b)}\,\|v\|_{H_0^1(a,b)}\,.$$

Since the last estimate is symmetric in u and v, the other term in (63) satisfies the same estimate; this proves (61). □

Acknowledgements. The authors are grateful to V. P. Smyshlyaev and I. V. Kamotski for helpful discussions. Both authors gratefully acknowledge the financial support of the EPSRC (GR / S99037 / 1 and GR / S99044 / 1).

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