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KERNEL SECTIONS OF MULTI-VALUED PROCESSES WITH APPLICATION TO THE NONLINEAR REACTION-DIFFUSION EQUATIONS IN UNBOUNDED DOMAINS

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Abstract. First, we introduce the concept of pullback ω-limit compactness for multi-valued processes, as an extension of the similar concept in the autonomous and nonautonomous framework. Next, we present the necessary and sufficient conditions (pullback dissipativeness and pullback ω-limit compactness) for the existence of a nonempty local bounded kernel (kernel sections are all compact, invariant and pullback attracting) of an infinite dimensional multi-valued process. In addition, we prove a result ensuring the existence of a uniform attractor and the uniform forward attraction of the inflated kernel sections of a family of multi-valued processes under the general assumptions of point dissipativeness and uniform ω-limit compactness. Finally, we illustrate the abstract theory with a nonlinear reaction-diffusion model in an unbounded domain.

1. Introduction. Set-valued dynamical systems have received extensive investigations (see, e.g., [1, 2, 3, 4, 5, 6, 7, 13, 15, 19, 26] and the references cited therein). Recently, under the assumptions that a family of multi-valued semiprocesses $\{U_\sigma(t,\tau) \mid$

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$t \geqslant \tau,\ \tau \in \mathbb{R}^+\}$, $\sigma \in \Sigma$, is uniformly dissipative and uniformly ω-limit compact, and that $\{T(t)\}$ is a continuous invariant semigroup on a compact subset of Σ, we [29] proved the existence of kernel sections of an infinite dimensional general multi-valued process constructed by the set-valued backward extension of multi-valued semiprocesses, and studied the structure of the uniform attractors of a family of multi-valued semiprocesses and the uniform forward attraction of the inflated kernel sections of a family of general multi-valued processes. Our main purpose here is to generalize the existing results of kernel sections to the case where the multi-valued process is pullback dissipative and pullback ω-limit compact, and to obtain the existence of the uniform attractors and the uniform forward attraction of the inflated kernel sections of a family of multi-valued processes in the case where the family of multi-valued processes is point dissipative and uniformly ω-limit compact.

First, we give the necessary and sufficient conditions for the existence of a local bounded kernel and compact kernel sections of an infinite dimensional multi-valued process. Let $\{U(t,\tau) \mid t \geqslant \tau,\ \tau \in \mathbb{R}\}$ be a multi-valued process (MVP) on a state space X. In particular, we suppose that X is a Banach space with norm $\|\cdot\|_X$ and $U(t,\tau)x$ is norm-to-weak upper-semicontinuous in x for any fixed $t \geqslant \tau$, $\tau \in \mathbb{R}$; i.e., if $x_n \to x$, then for any $y_n \in U(t,\tau)x_n$, there exists a $y \in U(t,\tau)x$, such that $y_n \rightharpoonup y$ (weak convergence). Let $\mathcal{K}_l$ be the local bounded kernel of the MVP $\{U(t,\tau)\}$. The local bounded kernel $\mathcal{K}_l$ defined by

$$\mathcal{K}_l = \left\{ u(\cdot) \mid \sup_{t\in[a,b]} \|u(t)\|_X \leqslant C_{a,b},\ \forall a,\, b \in \mathbb{R},\ u(t) \in U(t,\tau)u(\tau),\ \forall t \geqslant \tau,\, \tau \in \mathbb{R} \right\}.$$

$\mathcal{K}_l(s)$ denotes the kernel section at a time moment $s \in \mathbb{R}$:

$$\mathcal{K}_l(s) = \{u(s) \mid u(\cdot) \in \mathcal{K}_l\},\, \mathcal{K}_l(s) \subseteq X.$$

Here, using the technique of Kuratowski measure of noncompactness, we will show that the MVP $\{U(t,\tau)\}$ has a nonempty local bounded kernel; moreover, the kernel sections $\mathcal{K}_l(t)$ are all compact, invariant ($U(t,\tau)\mathcal{K}_l(\tau) = \mathcal{K}_l(t)$ for all $t \geqslant \tau$ and all $\tau \in \mathbb{R}$) and pullback attracting if and only if $\{U(t,\tau)\}$ is pullback dissipative and pullback ω-limit compact. Furthermore, we give simple and sufficient methods for verifying the pullback ω-limit compactness. These results also generalize some of the previous ones in [8, 28] to multi-valued systems.

Then we are interested in the existence of uniform attractors and the uniform forward attraction of the inflated kernel sections. Let $\{T(h) \mid h \in \mathbb{R}^+\}$ be a continuous invariant semigroup on a compact Banach space Σ with norm $\|\cdot\|_\Sigma$, and let $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, be a family of multi-valued processes on X. Let $\mathcal{K}_\sigma$ be the kernel of the MVP $\{U_\sigma(t,\tau)\}$ with $\sigma \in \Sigma$. The kernel $\mathcal{K}_\sigma$ consists of all bounded complete trajectories of the multi-valued process, i.e.,

$$\mathcal{K}_\sigma = \left\{ u(\cdot) \mid \sup_{t\in(-\infty,+\infty)} \|u(t)\|_X \leqslant C_u,\ u(t) \in U_\sigma(t,\tau)u(\tau),\ \forall t \geqslant \tau,\, \tau \in \mathbb{R} \right\}.$$

Here we consider the family of inflated kernel sections $\{\mathcal{K}_\sigma^{[\varepsilon_0]}(\tau)\}$, $\sigma \in \Sigma$, for any fixed $\varepsilon_0 > 0$, with component sets defined by

$$\mathcal{K}_\sigma^{[\varepsilon_0]}(\tau) = \bigcup_{\|\sigma' - \sigma\|_\Sigma \leqslant \varepsilon_0} \mathcal{K}_{\sigma'}(\tau). \tag{1.1}$$

We will prove that the family of MVPs $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, has a unique uniform attractor, and that the family of inflated kernel sections $\{\mathcal{K}_\sigma^{[\varepsilon_0]}(\tau)\}$, $\sigma \in \Sigma$, is uniformly (w.r.t. $\sigma \in \Sigma$) forward attracting if and only if $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is point dissipative and uniformly ω-limit compact. Some of the ideas used here come from [3].

Finally, we illustrate our abstract results by studying the nonautonomous reaction-diffusion equations in unbounded domains without uniqueness. For reaction-diffusion equations possessing a unique solution, we refer the reader to [13, 24, 25, 30], whereas, there is little reference on the kernel sections of nonautonomous reaction-diffusion equations without uniqueness. As an application of our abstract results as above, we discuss the nonautonomous reaction-diffusion equations in unbounded domains without uniqueness and obtain the existence of a nonempty local bounded kernel (kernel sections are compact, invariant in $L^2(\mathbb{R}^n)$ and pullback attracts every family of bounded subsets of $L^2(\mathbb{R}^n)$). It is worth mentioning that here we consider the case without uniform ω-limit compactness, which is different from [29]. In addition, we get that the family of inflated kernel sections in $L^p(\mathbb{R}^n)$ uniformly forward attracts every bounded subset of $L^2(\mathbb{R}^n)$ in the topology of $L^p(\mathbb{R}^n)$ with $p \geqslant 2$.

This paper is organized as follows. In Section 2 we present some preliminary results and definitions and then in Sections 3-4 we state and prove our main results. Finally, in Section 5 we deal with nonautonomous reaction-diffusion equations in unbounded domains without uniqueness.

2. Preliminaries. Let X be a Banach space with norm $\|\cdot\|_X$, and let 2^X be the set of all subsets of X. Denote by $H_X^*(\cdot,\cdot)$ and $H_X(\cdot,\cdot)$, respectively, the Hausdorff semidistance and Hausdorff distance between two nonempty subsets of a Banach space $(X, \|\cdot\|_X)$, which are defined by

$$H_X^*(A,\, B) = \sup_{a \in A} \operatorname{dist}_X(a,\, B),$$

where $\operatorname{dist}_X(a,\, B) = \inf_{b \in B} \|a - b\|_X$, and

$$H_X(A,\, B) = \max\{H_X^*(A,\, B), H_X^*(B,\, A)\}.$$

Finally, denote by $\mathcal{N}(A, r)$ the open neighborhood $\{y \in X \mid \operatorname{dist}_X(y, A) < r\}$ of radius $r > 0$ of a subset A of a Banach space X.

Definition 2.1. A family of mappings $F(t) : X \to 2^X$, $t \in \mathbb{R}^+$ is said to be a (autonomous) multi-valued semidynamical system (MVSS in short) if the following axioms hold:

(1) $F(0)x = \{x\},\ \forall x \in X$;
(2) $F(s)F(t)x = F(s+t)x,\ \forall s,\, t \in \mathbb{R}^+,\, x \in X$;

(3) $F(t)x$ is norm-to-weak upper-semicontinuous in x for fixed $t \in \mathbb{R}^+$ (i.e., if $x_n \to x$ in X, then for any $y_n \in F(t)x_n$, there exists a $y \in F(t)x$ such that $y_n \rightharpoonup y$ (weak convergence)).

It should be pointed out that the multi-valued semidynamical system defined here is indeed a strict multi-valued semidynamical system in [5, 22].

Let X, Y be two Banach spaces, and X^*, Y^* be their dual spaces, respectively. We also assume that X is a dense subspace of Y, the injection $i : X \hookrightarrow Y$ is continuous and its adjoint $i^* : Y^* \hookrightarrow X^*$ is densely injective.

Theorem 2.2 ([29]). Let X, Y be two Banach spaces satisfying the assumptions just above, $\{F(t)\}$ be an MVSS on X and Y, respectively. Assume that $\{F(t)\}$ is upper-semicontinuous or weak upper-semicontinuous on Y. If for fixed $t \in \mathbb{R}^+$, $F(t)$ maps compact subsets of X into bounded subsets of 2^X, then $F(t)$ is norm-to-weak upper-semicontinuous on X.

Remark 2.3. In concrete problems, we can choose Y to be a larger and weaker topology space, in which the upper semicontinuity of the MVSS can be obtained easily.

Definition 2.4. Let $\{F(t)\}$ be a multi-valued semidynamical system on X. We say that $\{F(t)\}$ is

(1) dissipative, if there exists a bounded subset $\mathcal{V}$ of X such that for any bounded set $B \subset X$, there exists a $T_0 = T_0(B) \in \mathbb{R}^+$, such that

$$F(t)B \subset \mathcal{V}, \ \forall t \geqslant T_0;$$

(2) ω-limit compact, if for any bounded subset B of X and $\varepsilon > 0$, there exists a $T_1 = T_1(B, \varepsilon) \in \mathbb{R}^+$, such that

$$k\left(\bigcup_{t \geqslant T_1} F(t)B\right) \leqslant \varepsilon,$$

where k is the Kuratowski measure of noncompactness.

Definition 2.5. A nonempty compact subset $\mathcal{A}$ of X is called to be a global attractor for the multi-valued semidynamical system $\{F(t)\}$ if it satisfies

(1) $\mathcal{A}$ is an invariant set, i.e.,

$$F(t)\mathcal{A} = \mathcal{A}, \ \forall t \in \mathbb{R}^+;$$

(2) $\mathcal{A}$ attracts each bounded subset B of X, i.e.,

$$\lim_{t \to +\infty} H_X^*(F(t)B, \mathcal{A}) = 0.$$

Definition 2.6. Let A be a subset of a Banach space X. The weakly sequential closure $\bar{A}^{WS}$ of A is defined by

$$\bar{A}^{WS} = \{x \in X \mid \exists \{x_n\} \subset A, \text{ s. t. } x_n \rightharpoonup x, \text{ that is, } x_n \text{ converges weakly to } x\}.$$

In general topology space, $\bar{A}^{WS}$ is different from $\bar{A}$ or the weak closure $\bar{A}^W$ of A. But if A is a convex subset of some Banach space, then we know that $\bar{A} = \bar{A}^W = \bar{A}^{WS}$.

Lemma 2.7 ([30]). Let X be a Banach space and k be the Kuratowski measure of noncompactness. Then for any subset A of X, we have

$$k(A) = k(\bar{A}^{WS}).$$

Let $\mathcal{K}$ be the kernel of the MVSS $\{F(t)\}$. The kernel $\mathcal{K}$ consists of all bounded complete trajectories of the MVSS $\{F(t)\}$, i.e.,

$$\mathcal{K} = \left\{ u(\cdot) \ \mid \sup_{t\in(-\infty,+\infty)} \|u(t)\|_X \leqslant C_u,\ u(t+\tau) \in F(\tau)u(t),\ \forall t \in \mathbb{R},\, \tau \in \mathbb{R}^+ \right\}.$$

As usual, $\mathcal{K}(s)$ denotes the kernel section at a time moment $s \in \mathbb{R}$:

$$\mathcal{K}(s) = \{u(s) \ \mid \ u(\cdot) \in \mathcal{K}\},\ \mathcal{K}(s) \subset X.$$

Obviously,

$$\mathcal{K}(t+\tau) \subseteq F(\tau)\mathcal{K}(t),\ \forall t \in \mathbb{R},\, \tau \in \mathbb{R}^+.$$

Definition 2.8. Let $\{F(t)\}$ be a (autonomous) multi-valued semidynamical system on X. For any subset B of X, the ω-limit set $\omega(B)$ is defined by

$$\omega(B) = \bigcap_{s\in\mathbb{R}^+} \overline{\bigcup_{t\geqslant s} F(t)B}^{WS}.$$

Theorem 2.9 ([29]). Let $\{F(t)\}$ be a (autonomous) multi-valued semidynamical system on X. Then $\{F(t)\}$ has a unique global attractor $\mathcal{A} = \omega(\mathcal{V})$; moreover, $\mathcal{A}$ coincides with the kernel section at time τ; i.e., $\mathcal{A} = \mathcal{K}(\tau)$ for any $\tau \in \mathbb{R}$ if and only if $\{F(t)\}$ is dissipative and ω-limit compact.

Definition 2.10. A family of mappings $U(t,\tau) : X \to 2^X, t \geqslant \tau, \tau \in \mathbb{R}$, is said to be a multi-valued process (MVP in short) if it satisfies:

(1) $U(\tau,\tau)x = \{x\}$, $\forall \tau \in \mathbb{R}$, $x \in X$;
(2) $U(t,s)U(s,\tau)x = U(t,\tau)x$, $\forall t \geqslant s \geqslant \tau$, $\tau \in \mathbb{R}$, $x \in X$;
(3) $U(t,\tau)x$ is norm-to-weak upper-semicontinuous in x for fixed $t \geqslant \tau, \tau \in \mathbb{R}$ (i.e., if $x_n \to x$ in X, then for any $y_n \in U(t,\tau)x_n$, there exists a $y \in U(t,\tau)x$ such that $y_n \rightharpoonup y$ (weak convergence)).

Theorem 2.11 ([29]). Let X, Y be two Banach spaces satisfying the assumptions just above, and let $\{U(t,\tau)\}$ be an MVP on X and Y, respectively. Assume that $\{U(t,\tau)\}$ is upper-semicontinuous or weak upper-semicontinuous on Y. If for fixed $t \geqslant \tau, \tau \in \mathbb{R}$, $U(t,\tau)$ maps compact subsets of X into bounded subsets of 2^X, then $U(t,\tau)$ is norm-to-weak upper-semicontinuous on X.

Definition 2.12. Let $\{U(t,\tau)\}$ be a multi-valued process on X. We say that $\{U(t,\tau)\}$ is

(1) pullback dissipative if there exists a family of bounded sets $\mathcal{Q} = \{Q(t)\}_{t\in\mathbb{R}}$ in X so that for any family of bounded subsets $\mathcal{B} = \{B(t)\}_{t\in\mathbb{R}}$ of X and any $t \in \mathbb{R}$, there exists a $t_0 = t_0(\mathcal{B}, t) \in \mathbb{R}^+$, such that

$$U(t, t-s)B(t-s) \subset Q(t),\ \forall s \geqslant t_0;$$

(2) pullback ω-limit compact with respect to each $t \in \mathbb{R}$ if for any family of bounded subsets $\mathcal{B} = \{B(t)\}_{t\in\mathbb{R}}$ of X and $\varepsilon > 0$, there exists a $t_1 = t_1(\mathcal{B}, t, \varepsilon) \in \mathbb{R}^+$, such that

$$k\left(\bigcup_{s\geqslant t_1} U(t, t-s)B(t-s)\right) \leqslant \varepsilon.$$

Now we recall briefly the basic concept of the Kuratowski measure of noncompactness and recapitulate its basic properties; see [14] etc. for more details.

Definition 2.13. Let M be a complete metric space and A be a bounded subset of M. The Kuratowski measure of noncompactness $k(A)$ of A is defined by

$$k(A) = \inf\{\delta > 0 \mid \ A \text{ admits a finite cover by sets whose diameter } \ \leqslant \delta\}.$$

If A is a nonempty, unbounded set in M, then we define $k(A) = \infty$.

The properties of $k(A)$, which we will use in this paper, are given in the following lemmas:

Lemma 2.14. The Kuratowski measure of noncompactness $k(A)$ on a complete metric space M satisfies the following properties:

(1) $k(A) = 0$ if and only if $\bar{A}$ is compact, where $\bar{A}$ is the closure of A.
(2) If $A_1 \subset A_2$, then $k(A_1) \leqslant k(A_2)$.
(3) $k(A_1 \cup A_2) \leqslant \max\{k(A_1), k(A_2)\}$.
(4) $k(\bar{A}) = k(A)$.
(5) If A_t is a family of nonempty, closed, bounded sets defined for $t > r$ that satisfy $A_t \subset A_s$, whenever $s \leqslant t$, and $k(A_t) \to 0$, as $t \to \infty$, then $\bigcap_{t>r} A_t$ is a nonempty, compact set in M.

If, in addition, M is a Banach space, then the following statements are valid:

(6) $k(A_1 + A_2) \leqslant k(A_1) + k(A_2)$.
(7) $k(\overline{coA}) = k(A)$, where $\overline{coA}$ is the closed convex hull of A.
(8) Let M have the following decomposition:

$$M = M_1 \oplus M_2, \quad \text{with } \dim M_1 < \infty,$$

and $P : M \to M_1$, $Q : M \to M_2$ be the canonical projectors, and A be a bounded subset of M. If the diameter of QA is less than ε, then $k(A) < \varepsilon$.

3. Necessary and sufficient conditions for the existence of a local bounded kernel and compact kernel sections of multi-valued processes. In this section, we give the necessary and sufficient conditions for the existence of a local bounded kernel (kernel sections are all compact, invariant and pullback attracting) of an infinite dimensional multi-valued process by using the Kuratowski measure of noncompactness.

Let $\{U(t, \tau) \mid t \geqslant \tau, \tau \in \mathbb{R}\}$ be a multi-valued process on X. The local bounded kernel $\mathcal{K}_l$ of the MVP $\{U(t, \tau)\}$ is defined by

$$\mathcal{K}_l = \left\{u(\cdot) \mid \sup_{t\in[a,b]} \|u(t)\|_X \leqslant C_{a,b}, \ \forall a,\, b \in \mathbb{R}, \ u(t) \in U(t, \tau)u(\tau), \ \forall t \geqslant \tau, \, \tau \in \mathbb{R}\right\}.$$

As usual, $\mathcal{K}_l(s)$ denotes the kernel section at a time moment $s \in \mathbb{R}$:

$$\mathcal{K}_l(s) = \{u(s) \mid u(\cdot) \in \mathcal{K}_l\},\ \mathcal{K}_l(s) \subset X.$$

Evidently, the following assertion holds:

LEMMA 3.1. Let $\mathcal{K}_l$ be the local bounded kernel of the multi-valued process $\{U(t,\tau)\}$. Then

$$\mathcal{K}_l(t) \subseteq U(t,\tau)\mathcal{K}_l(\tau),\ \forall t \geqslant \tau,\ \tau \in \mathbb{R}. \tag{3.1}$$

REMARK 3.2. In the single-valued case (see [13]), the kernel sections of the process $\{U(t,\tau)\}$ are always invariant, whereas, the kernel sections of the multi-valued process $\{U(t,\tau)\}$ defined as above only possess negative invariance, i.e., (3.1) holds true.

DEFINITION 3.3. Let $\{U(t,\tau)\}$ be a multi-valued process on X. For every family of nonempty subsets $\mathcal{B} = \{B(t)\}_{t\in\mathbb{R}}$ of X and any $t \in \mathbb{R}$, the pullback ω-limit set $\omega_t(\mathcal{B})$ is defined by

$$\omega_t(\mathcal{B}) = \bigcap_{\tau\in\mathbb{R}^+} \overline{\bigcup_{s\geqslant\tau} U(t,t-s)B(t-s)}^{WS}.$$

THEOREM 3.4. Let X be a Banach space and let $\{U(t,\tau)\}$ be a multi-valued process on X. Then the local bounded kernel $\mathcal{K}_l$ of the MVP $\{U(t,\tau)\}$ is nonempty, and the kernel sections

$$\mathcal{K}_l(t) = \omega_t(\mathcal{Q}) = \bigcap_{T\in\mathbb{R}^+} \overline{\bigcup_{s\geqslant T} U(t,t-s)Q(t-s)}^{WS} \subset Q(t),\ \forall t \in \mathbb{R}$$

are all compact, invariant ($U(t,\tau)\mathcal{K}_l(\tau) = \mathcal{K}_l(t)$ for all $t \geqslant \tau$ and all $\tau \in \mathbb{R}$) and pullback attract every family of bounded subsets $\mathcal{B} = \{B(t)\}_{t\in\mathbb{R}}$ of X; i.e., for any fixed $t \in \mathbb{R}$,

$$\lim_{s\to+\infty} H_X^*(U(t,t-s)B(t-s),\ \mathcal{K}_l(t)) = 0$$

if and only if $\{U(t,\tau)\}$ is

(1) pullback dissipative, i.e., there exists a family of bounded subsets $\mathcal{Q} = \{Q(t)\}_{t\in\mathbb{R}}$ of X so that for any family of bounded subsets $\mathcal{B} = \{B(t)\}_{t\in\mathbb{R}}$ of X and each $t \in \mathbb{R}$, there exists a $t_0 = t_0(\mathcal{B},t) \in \mathbb{R}^+$, such that

$$U(t,t-s)B(t-s) \subset Q(t),\ \forall s \geqslant t_0; \tag{3.2}$$

(2) pullback ω-limit compact with respect to each $t \in \mathbb{R}$.

Proof. " $\Rightarrow$ " Let $Q(t) = \mathcal{N}(\mathcal{K}_l(t),1)$. By the pullback attraction of the kernel sections, it is easy to see that $\{U(t,\tau)\}$ is pullback dissipative.

Note that for every family of bounded subsets $\mathcal{B} = \{B(t)\}_{t\in\mathbb{R}}$ of X and any fixed $t \in \mathbb{R}$,

$$\lim_{s\to+\infty} H_X^*(U(t,t-s)B(t-s),\ \mathcal{K}_l(t)) = 0.$$

Therefore for any $\varepsilon > 0$, there exists a $\tilde{t}_\varepsilon = \tilde{t}_\varepsilon(\mathcal{B},t,\varepsilon) > 0$ such that

$$\bigcup_{s\geqslant\tilde{t}_\varepsilon} U(t,t-s)B(t-s) \subset \mathcal{N}\left(\mathcal{K}_l(t),\frac{\varepsilon}{4}\right) = \left\{x \in X \mid \operatorname{dist}_X(x,\mathcal{K}_l(t)) < \frac{\varepsilon}{4}\right\}. \tag{3.3}$$

On the other hand, since $\mathcal{K}_l(t)$ is compact, there exists a finite number of elements $x_1, x_2, \ldots, x_n \in X$ such that

$$\mathcal{K}_l(t) \subset \bigcup_{i=1}^{n} \mathcal{N}\left(x_i, \frac{\varepsilon}{4}\right).$$

Hence,

$$\mathcal{N}\left(\mathcal{K}_l(t), \frac{\varepsilon}{4}\right) \subset \bigcup_{i=1}^{n} \mathcal{N}\left(x_i, \frac{\varepsilon}{2}\right). \tag{3.4}$$

Combining (3.3) and (3.4) together, we can conclude that

$$k\left(\bigcup_{s \geqslant \tilde{t}_\varepsilon} U(t, t-s)B(t-s)\right) \leqslant k\left(\mathcal{N}\left(\mathcal{K}_l(t), \frac{\varepsilon}{4}\right)\right) \leqslant \varepsilon,$$

which implies that $\{U(t,\tau)\}$ is pullback ω-limit compact.

" $\Leftarrow$ " Since the MVP $\{U(t,\tau)\}$ is pullback ω-limit compact and $\mathcal{Q}$ is a family of bounded sets, for each fixed $t \in \mathbb{R}$ and any $\varepsilon > 0$, there exists a $t_\varepsilon = t_\varepsilon(t, \mathcal{Q}, \varepsilon) > 0$ such that

$$k\left(\bigcup_{s \geqslant t_\varepsilon} U(t, t-s)Q(t-s)\right) \leqslant \varepsilon.$$

Take $\varepsilon = \frac{1}{n}$, $n = 1, 2, \ldots$. We can find a sequence $\{t_n\}$, $0 < t_1 < t_2 \cdots < t_n < \cdots$ such that

$$k\left(\bigcup_{s \geqslant t_n} U(t, t-s)Q(t-s)\right) \leqslant \frac{1}{n},\ n = 1, 2, \ldots.$$

By Lemma 2.7, we have

$$k\left(\overline{\bigcup_{s \geqslant t_n} U(t, t-s)Q(t-s)}^{WS}\right) = k\left(\bigcup_{s \geqslant t_n} U(t, t-s)Q(t-s)\right).$$

Thanks to the property (5) in Lemma 2.14, noticing that the set

$$\overline{\bigcup_{s \geqslant t_n} U(t, t-s)Q(t-s)}^{WS}$$

is closed in X, we know that $\bigcap_{n=1}^{\infty} \overline{\bigcup_{s \geqslant t_n} U(t, t-s)Q(t-s)}^{WS}$ is a nonempty compact set, and also the pullback ω-limit set with respect to $t \in \mathbb{R}$. That is,

$$\begin{aligned} A(t) = \omega_t(\mathcal{Q}) &= \bigcap_{\tau \in \mathbb{R}^+} \overline{\bigcup_{s \geqslant \tau} U(t, t-s)Q(t-s)}^{WS} \\ &= \bigcap_{n=1}^{\infty} \overline{\bigcup_{s \geqslant t_n} U(t, t-s)Q(t-s)}^{WS}. \end{aligned} \tag{3.5}$$

Observe that $\mathcal{Q}$ is a pullback absorbing set, therefore there exists a $\tau_0 = \tau_0(\mathcal{Q}, t) \in \mathbb{R}^+$, such that

$$U(t, t-s)Q(t-s) \subset Q(t),\ \forall s \geqslant \tau_0. \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$A(t) = \omega_t(\mathcal{Q}) \subset Q(t),\ \forall t \in \mathbb{R}. \tag{3.7}$$

Let us show that

$$A(t) = \mathcal{K}_l(t),\ \forall t \in \mathbb{R}, \tag{3.8}$$

where $\mathcal{K}_l(t)$ is the section of the local bounded kernel $\mathcal{K}_l$ of the MVP $\{U(t,\tau)\}$ at time t. We consider an arbitrary complete trajectory $u(s)$ of the MVP $\{U(t,\tau)\}$ with

$$\sup_{s\in[a,b]} \|u(s)\|_X \leqslant C_{a,b},\ \forall a, b \in \mathbb{R}.$$

Then, according to (3.2), $u(t) \in Q(t)$ for all $t \in \mathbb{R}$. Indeed, $u(t) \in U(t, t-s)u(t-s)$ for all $s \geqslant 0$. Clearly, $\mathcal{B} = \{u(t)\}_{t\in\mathbb{R}}$ is a family of bounded sets in X. Then (3.2) implies that for s sufficiently large,

$$U(t, t-s)u(t-s) \subset Q(t).$$

Hence

$$u(t) \in U(t, t-s)u(t-s) \subset Q(t).$$

On the other hand, it follows from $u(t-T) \in Q(t-T)$ for all $t \in \mathbb{R}$ and all $T \in \mathbb{R}^+$ that

$$u(t) \in U(t, t-T)u(t-T) \subset U(t, t-T)Q(t-T) \subset \overline{\bigcup_{s\geqslant T} U(t, t-s)Q(t-s)}^{WS}.$$

Therefore, $u(t) \in \overline{\bigcup_{s\geqslant T} U(t,t-s)Q(t-s)}^{WS}$ for all $T \geqslant 0$. So,

$$u(t) \in A(t) = \bigcap_{T\geqslant 0} \overline{\bigcup_{s\geqslant T} U(t, t-s)Q(t-s)}^{WS}.$$

Thus we have established that

$$\mathcal{K}_l(t) \subset A(t),\ \forall t \in \mathbb{R}. \tag{3.9}$$

To prove the reverse inclusion we need the following lemmas.

LEMMA 3.5. For a family of bounded sets $\mathcal{B} = \{B(t)\}_{t\in\mathbb{R}}$ in X, $y \in \omega_t(\mathcal{B}) \Leftrightarrow$ there exist sequences $s_n \in \mathbb{R}^+$, $s_n \to +\infty (n \to \infty)$, $x_n \in B(t-s_n)$, $y_n \in U(t, t-s_n)x_n$, such that $y_n \rightharpoonup y$ as $n \to \infty$.

Proof. " $\Leftarrow$ " According to the definition of weakly sequential closure, we know that if there exist sequences $s_n \in \mathbb{R}^+$, $s_n \to +\infty (n \to \infty)$, $x_n \in B(t-s_n)$, $y_n \in U(t, t-s_n)x_n$ such that $y_n \rightharpoonup y$ as $n \to \infty$, then we can conclude that $y \in \overline{\bigcup_{s\geqslant T} U(t,t-s)B(t-s)}^{WS}$ for any $T \geqslant 0$, which means that $y \in \omega_t(\mathcal{B}) = \bigcap_{T\geqslant 0} \overline{\bigcup_{s\geqslant T} U(t,t-s)B(t-s)}^{WS}$.

" $\Rightarrow$ " Assume that $y \in \omega_t(\mathcal{B})$. Then for any $n \in \mathbb{N}$, we have

$$y \in \overline{\bigcup_{s\geqslant n} U(t, t-s)B(t-s)}^{WS}.$$

It follows from the definition of weakly sequential closure that there exist sequences $s_n^k \geqslant n$, $x_n^k \in B(t - s_n^k)$, and $y_n^k \in U(t, t - s_n^k)x_n^k$ such that

$$y_n^k \rightharpoonup y \quad \text{as } k \to \infty.$$

Since the MVP $\{U(t,\tau)\}$ is pullback ω-limit compact, we know that the weakly sequential closure $\overline{K}^{WS}$ of $K = \{y_n^k \mid k, n = 1, 2, \dots\}$ is weakly compact. So $\overline{K}^{WS}$ is metrizable, and the metric generates the weak topology of $\overline{K}^{WS}$. Suppose that the equivalent metric is d; i.e., for $\{y_n\} \subset \overline{K}^{WS}$ and $y \in \overline{K}^{WS}$, $y_n \rightharpoonup y$ if and only if $d(y_n, y) \to 0$ as $n \to \infty$. Using this equivalent metric of $\overline{K}^{WS}$, for each $n \in \mathbb{N}$, we can extract an element $y_n^{k_n}$ which belongs to $\{y_n^k\}_{k=1}^{\infty}$ such that

$$d(y_n^{k_n}, y) < \frac{1}{n}.$$

Then $d(y_n^{k_n}, y) \to 0$ as $n \to \infty$, which implies that $y_n^{k_n} \rightharpoonup y$ as $n \to \infty$; i.e., we can find sequences $s_n^{k_n} \geqslant n$, $s_n^{k_n} \to +\infty (n \to \infty)$, $x_n^{k_n} \in B(t - s_n^{k_n})$, $y_n^{k_n} \in U(t, t - s_n^{k_n})x_n^{k_n}$ such that $y_n^{k_n} \rightharpoonup y$ as $n \to \infty$. The proof of this lemma is completed. □

LEMMA 3.6. $A(t) \subset U(t,\tau)A(\tau)$, $\forall t \geqslant \tau$, $\tau \in \mathbb{R}$.

Proof. Let $y \in A(t)$. By Lemma 3.5, we can find sequences $s_n \in \mathbb{R}^+$, $s_n \to +\infty (n \to \infty)$, $x_n \in Q(t - s_n)$, and $y_n \in U(t, t - s_n)x_n$ such that $y_n \rightharpoonup y$ as $n \to \infty$. Clearly for n sufficiently large,

$$y_n \in U(t, t - s_n)x_n = U(t,\tau)U(\tau, t - s_n)x_n.$$

Thus there exists a sequence $\tilde{x}_n \in U(\tau, t - s_n)x_n = U(\tau, \tau - (\tau + s_n - t))x_n$ such that $y_n \in U(t,\tau)\tilde{x}_n$.

We need to prove that $\{\tilde{x}_n\}$ has a subsequence which converges in X. Noticing that for any $\varepsilon > 0$, there exists a $\tau_\varepsilon > 0$ such that

$$k\left(\bigcup_{s \geqslant \tau_\varepsilon} U(\tau, \tau - s)Q(\tau - s)\right) \leqslant \varepsilon,$$

and that there exists an N_0 such that $\tau + s_n - t \geqslant \tau_\varepsilon$ for all $n \geqslant N_0$ and

$$\bigcup_{n \geqslant N_0} \tilde{x}_n \subset \bigcup_{n \geqslant N_0} U(\tau, \tau - (\tau + s_n - t))x_n \subset \bigcup_{s \geqslant \tau_\varepsilon} U(\tau, \tau - s)Q(\tau - s).$$

Hence

$$k\left(\bigcup_{n \geqslant N_0} \tilde{x}_n\right) \leqslant \varepsilon.$$

On the other hand, $\bigcup_{n=N_0'}^{N_0} \tilde{x}_n$ contains only a finite number of elements, where N_0' is fixed such that $\tau + s_n - t \geqslant 0$ as $n \geqslant N_0'$. Using the property (3) for the measure of noncompactness in Lemma 2.14, we have

$$k\left(\bigcup_{n \geqslant N_0'} \tilde{x}_n\right) = k\left(\bigcup_{n \geqslant N_0} \tilde{x}_n\right) \leqslant \varepsilon.$$

Let $\varepsilon \to 0$. We then derive that

$$k\left(\bigcup_{n \geqslant N_0'} \tilde{x}_n\right) = 0.$$

This means that $\{\tilde{x}_n\}$ is relatively compact. So, there is a subsequence of $\{\tilde{x}_n\}$ such that $\tilde{x}_n \to x$ as $n \to \infty$ and by Lemma 3.5, we see that $x \in \omega_\tau(\mathcal{Q}) = A(\tau)$.

Finally, by the norm-to-weak upper semicontinuity of the MVP $\{U(t,\tau)\}$, we can conclude that there exist a subsequence y_{n_k} of y_n and a $y' \in U(t,\tau)x$ such that

$$y_{n_k} \rightharpoonup y' \text{ as } k \to \infty.$$

Observe that

$$y_n \rightharpoonup y \text{ as } n \to \infty.$$

Hence $y = y' \in U(t,\tau)x \subset U(t,\tau)A(\tau)$ and $A(t) \subset U(t,\tau)A(\tau)$.

LEMMA 3.7. Assume that $\{V(t)\}_{t\in\mathbb{R}}$ is negatively invariant, i.e., $V(t) \subset U(t,\tau)V(\tau)$ for all $t \geqslant \tau$ and all $\tau \in \mathbb{R}$. Let $W(t) = \bigcup_{s\geqslant 0} U(t,t-s)V(t-s)$. Then $\{W(t)\}_{t\in\mathbb{R}}$ is invariant, i.e.,

$$U(t,\tau)W(\tau) = W(t),\ \forall t \geqslant \tau,\ \tau \in \mathbb{R}.$$

□

Proof. For any $t \geqslant \tau$ and $\tau \in \mathbb{R}$, observe that

$$\begin{aligned} U(t,\tau)W(\tau) &= U(t,\tau)\bigcup_{s\geqslant 0} U(\tau,\tau-s)V(\tau-s) = \bigcup_{s\geqslant 0} U(t,\tau-s)V(\tau-s) \\ &= \bigcup_{s'\geqslant t-\tau} U(t,t-s')V(t-s')\ (s' = t+s-\tau) \\ &\subset \bigcup_{s'\geqslant 0} U(t,t-s')V(t-s') = W(t). \end{aligned}$$

Clearly, $\{W(t)\}_{t\in\mathbb{R}}$ is positively invariant. In the following, we prove that $\{W(t)\}_{t\in\mathbb{R}}$ is also negatively invariant, i.e., $W(t) \subset U(t,\tau)W(\tau)$ for all $t \geqslant \tau$ and all $\tau \in \mathbb{R}$. Assume that $y \in W(t)$. Then there exist an $s \in \mathbb{R}^+$ and $x \in V(t-s)$ such that $y \in U(t,t-s)x$. Two cases may occur.

CASE 1. $t - s \geqslant \tau$. Since $\{V(t)\}_{t\in\mathbb{R}}$ is negatively invariant, we can find an $x_0 \in V(\tau) \subset W(\tau)$ such that $x \in U(t-s,\tau)x_0$. Hence

$$y \in U(t,t-s)x \subset U(t,t-s)U(t-s,\tau)x_0 = U(t,\tau)x_0 \subset U(t,\tau)W(\tau).$$

CASE 2. $t - s < \tau$. In this case,

$$y \in U(t,t-s)x = U(t,\tau)U(\tau,t-s)x.$$

Therefore, there exists an $x_0 \in U(\tau,t-s)x = U(\tau,\tau-(s+\tau-t))x \subset W(\tau)$, such that $y \in U(t,\tau)x_0 \subset U(t,\tau)W(\tau)$.

In conclusion, $W(t) \subset U(t,\tau)W(\tau)$ for all $t \geqslant \tau$ and all $\tau \in \mathbb{R}$. □

LEMMA 3.8. $U(t,\tau)A(\tau) = A(t),\ \forall t \geqslant \tau,\ \tau \in \mathbb{R}$.

Proof. By Lemma 3.6, only the positive invariance of $\{A(t)\}_{t\in\mathbb{R}}$ needs to be checked. Let $W(t) = \bigcup_{s\geqslant 0} U(t, t-s)A(t-s)$. By Lemmas 3.6 and 3.7, we know that $\{W(t)\}_{t\in\mathbb{R}}$ is invariant. In view of $A(t) = \omega_t(\mathcal{Q}) \subset Q(t)$ for all $t \in \mathbb{R}$, so for any $t \geqslant \tau$ and $\tau \in \mathbb{R}$,

$$\begin{aligned} W(t) &= U(t,\tau)W(\tau) = U(t,\tau)\bigcup_{s\geqslant 0} U(\tau,\tau-s)A(\tau-s) \\ &= \bigcup_{s\geqslant 0} U(t,\tau-s)A(\tau-s) \subset \bigcup_{s'\geqslant t-\tau} U(t,t-s')Q(t-s')\ (s'=s-\tau+t), \end{aligned}$$

which implies that

$$\bigcup_{s\geqslant 0} U(t,t-s)A(t-s) = W(t) \subset \omega_t(\mathcal{Q}) = A(t),$$

i.e.,

$$U(t,t-s)A(t-s) \subset A(t),\ \forall t\in\mathbb{R},\ s\in\mathbb{R}^+.$$

Thus writing $\tau = t-s$, we can conclude that

$$U(t,\tau)A(\tau) \subset A(t),\ \forall t\geqslant\tau,\ \tau\in\mathbb{R}.$$

The proof of this lemma is finished. □

Now we are ready to complete the proof of Theorem 3.4.

Using Lemma 3.8, let us show that $A(t) \subset \mathcal{K}_l(t)$ for all $t \in \mathbb{R}$. Indeed, let u_t be an arbitrary element of $A(t)$. We shall construct a complete trajectory $u(s), s \in \mathbb{R}$ of the MVP $\{U(s,t)\}$ such that $\sup_{s\in[a,b]} \|u(s)\|_X \leqslant C_{a,b}$ for all $a, b \in \mathbb{R}$ and $u(t) = u_t$. We take $u(s) \in U(s,t)u_t$, where $s \geqslant t$. Let us extend $u(s)$ to $s \leqslant t$. Lemma 3.8 implies that there exists $u_{t-1} \in A(t-1)$ such that $u_t \in U(t,t-1)u_{t-1}$. If we now take $u(s) \in U(s,t-1)u_{t-1}$ for all $s \in [t-1,t]$, then we have $u(s) \in A(s) \subset Q(s)$ for all $s \geqslant t-1$. Applying the above procedure several times we can construct $u(s) \in A(s) \subset Q(s)$ for all $s \geqslant t-n, n \in \mathbb{N}$. Letting $n \to \infty$, we obtain a complete trajectory $u(s), s \in \mathbb{R}$, of the MVP $\{U(s,t)\}$ such that $u(s) \in A(s) \subset Q(s)$ for all $s \in \mathbb{R}$ and $u(t) = u_t$. Recall that $A(t)$ is compact for each $t \in \mathbb{R}$, so we can see that for any fixed $a, b \in \mathbb{R}$, $\|u(s)\|_X$ is bounded for all $s \in [a,b]$. Therefore, $u_t = u(t) \in \mathcal{K}_l(t)$ and $A(t) \subset \mathcal{K}_l(t)$. Taking into account (3.9), we obtain the identity (3.8).

Finally, it suffices to prove that for every family of bounded subsets $\mathcal{B} = \{B(t)\}_{t\in\mathbb{R}}$ of X and for each fixed $t \in \mathbb{R}$,

$$\lim_{s\to+\infty} H_X^*(U(t,t-s)B(t-s),\, \mathcal{K}_l(t)) = 0.$$

Assume, otherwise, that there exist a family of bounded subsets $\mathcal{B}_0 = \{B_0(t)\}_{t\in\mathbb{R}}$ of X and $\tilde{t}_0 \in \mathbb{R}$, such that

$$H_X^*(U(\tilde{t}_0,\tilde{t}_0-s)B_0(\tilde{t}_0-s),\, \mathcal{K}_l(\tilde{t}_0)) \nrightarrow 0\ (s\to+\infty). \tag{3.10}$$

Thus there exist $\varepsilon' > 0$ and sequences $\{s_n\} \subset \mathbb{R}^+$, $s_n \to +\infty\,(n\to\infty)$, $\{x_n\} \subset B_0(\tilde{t}_0-s_n)$, and $y_n \in U(\tilde{t}_0,\tilde{t}_0-s_n)x_n$, such that

$$\mathrm{dist}_X(y_n,\, \mathcal{K}_l(\tilde{t}_0)) \geqslant \varepsilon' > 0,\ \forall n\in\mathbb{N}. \tag{3.11}$$

Since $\mathcal{Q}$ is a pullback absorbing set, for each integer $k \geqslant 1$, there exists an $s_{n_k} \in \{s_n\}$, such that $s_{n_k} \geqslant k$ and

$$U(\tilde{t}_0-k,\tilde{t}_0-s_{n_k})B_0(\tilde{t}_0-s_{n_k}) \subset Q(\tilde{t}_0-k).$$

In particular, as $x_{n_k} \in B_0(\tilde{t}_0 - s_{n_k})$, so

$$y_{n_k} \in U(\tilde{t}_0, \tilde{t}_0 - s_{n_k})x_{n_k} = U(\tilde{t}_0, \tilde{t}_0 - k)U(\tilde{t}_0 - k, \tilde{t}_0 - s_{n_k})x_{n_k} \subset U(\tilde{t}_0, \tilde{t}_0 - k)Q(\tilde{t}_0 - k).$$

As in the proof above, due to the pullback ω-limit compactness, we can verify that y_{n_k} is relatively compact and possesses at least one cluster point y_0. Hence y_0 belongs to $A(\tilde{t}_0) = \mathcal{K}_l(\tilde{t}_0) = \omega_{\tilde{t}_0}(\mathcal{Q})$ and this contradicts (3.11). Thus we have completed the proof of Theorem 3.4. □

Remark 3.9. From Theorem 3.4, it seems that the kernel sections

$$\mathcal{K}_l(t) = \bigcap_{T \geqslant 0} \overline{\bigcup_{s \geqslant T} U(t, t-s)Q(t-s)}^{WS}$$

are larger than $\bigcap_{T\geqslant 0} \overline{\bigcup_{s\geqslant T} U(t,t-s)Q(t-s)}$, which is obtained by the usual method (see [8]). However, under the assumption that the MVP $\{U(t,\tau)\}$ is pullback ω-limit compact with respect to each $t \in \mathbb{R}$, we can show that

$$\mathcal{K}_l(t) = \bigcap_{T \geqslant 0} \overline{\bigcup_{s \geqslant T} U(t, t-s)Q(t-s)}^{WS} = \bigcap_{T \geqslant 0} \overline{\bigcup_{s \geqslant T} U(t, t-s)Q(t-s)}.$$

Remark 3.10. In particular, under the assumption that the MVP $\{U(t,\tau)\}$ is uniformly dissipative, i.e., there exists a bounded subset $\tilde{Q}$ of X so that for any bounded subset B of X, there exists a $\tilde{t}_0 = \tilde{t}_0(B) \in \mathbb{R}^+$ independent of $\tau \in \mathbb{R}$, such that

$$U(t+\tau, \tau)B \subset \tilde{Q}, \ \forall t \geqslant \tilde{t}_0, \, \tau \in \mathbb{R}.$$

Similar to the arguments in Theorem 3.4, we can show that the kernel $\mathcal{K}$ of the MVP $\{U(t,\tau)\}$ is nonempty, and the kernel sections

$$\mathcal{K}(t) = \omega_t(\tilde{Q}) = \bigcap_{T \in \mathbb{R}^+} \overline{\bigcup_{s \geqslant T} U(t, t-s)\tilde{Q}}^{WS} \subset \tilde{Q}, \ \forall t \in \mathbb{R}$$

are all compact, invariant and pullback attract any bounded subset of X if and only if $\{U(t,\tau)\}$ is pullback ω-limit compact with respect to each $t \in \mathbb{R}$; i.e., for every bounded set B in X and any $\varepsilon > 0$, there exists a $\tilde{t}_1 = \tilde{t}_1(B, t, \varepsilon) \in \mathbb{R}^+$, such that

$$k\left(\bigcup_{s \geqslant \tilde{t}_1} U(t, t-s)B\right) \leqslant \varepsilon.$$

It is worth mentioning that the kernel $\mathcal{K}$ here consists of all bounded complete trajectories of the multi-valued process, i.e.,

$$\mathcal{K} = \left\{ u(\cdot) \mid \sup_{t \in (-\infty, +\infty)} \|u(t)\|_X \leqslant C_u, \ u(t) \in U(t,\tau)u(\tau), \ \forall t \geqslant \tau, \, \tau \in \mathbb{R} \right\}.$$

Now we give the relation between the pullback flattening and the pullback ω-limit compactness of the MVP $\{U(t,\tau)\}$. First, we need the following definition:

Definition 3.11. A multi-valued process $\{U(t,\tau)\}$ on a Banach space X is said to be pullback flattening if for each $t \in \mathbb{R}$, any family of bounded subsets $\mathcal{B} = \{B(t)\}_{t\in\mathbb{R}}$ of X and $\varepsilon > 0$, there exist $\tau_0 = \tau_0(t, \mathcal{B}, \varepsilon) > 0$ and a finite dimensional subspace X_1 of X such that

(1) $\left\{ P\left(\bigcup_{s \geqslant \tau_0} U(t,t-s)B(t-s) \right) \right\}$ is bounded,

(2) $\left\| (I-P)\left(\bigcup_{s \geqslant \tau_0} U(t,t-s)B(t-s) \right) \right\|_X < \varepsilon,$

where $P : X \to X_1$ is the canonical projector.

Analogous to the proofs of Theorems 4.10–4.11 in [29], we can get the following results.

Theorem 3.12. Let X be a Banach space and let $\{U(t,\tau)\}$ be a multi-valued process on X.

(1) If the MVP $\{U(t,\tau)\}$ is pullback flattening, then $\{U(t,\tau)\}$ is pullback ω-limit compact with respect to each $t \in \mathbb{R}$.
(2) Let X be a uniformly convex Banach space; in particular, let X be a Hilbert space. Then $\{U(t,\tau)\}$ is pullback ω-limit compact with respect to each $t \in \mathbb{R}$ if and only if $\{U(t,\tau)\}$ is pullback flattening.

Theorem 3.13. Let X be a uniformly convex Banach space; in particular, let X be a Hilbert space. Then the MVP $\{U(t,\tau)\}$ possesses a nonempty local bounded kernel $\mathcal{K}_l$; moreover, the kernel sections $\mathcal{K}_l(t)$ are all compact, invariant and pullback attract every family of bounded subsets $\mathcal{B} = \{B(t)\}_{t\in\mathbb{R}}$ of X if and only if $\{U(t,\tau)\}$ is pullback dissipative and pullback flattening.

An MVP $\{U(t,\tau)\}$ is said to be asymptotically upper-semicompact in X if for each fixed $t \in \mathbb{R}$, and any family of bounded subsets $\mathcal{B} = \{B(t)\}_{t\in\mathbb{R}}$ of X, any sequence $T_n \to +\infty (n \to \infty)$, $y_n \in U(t,t-T_n)B(t-T_n)$ is precompact in X. The following theorem shows that pullback ω-limit compactness equals asymptotically upper-semicompactness.

By slightly modifying the arguments in [28], Theorem 3.9, we have

Theorem 3.14. Let $\{U(t,\tau)\}$ be a multi-valued process on X. Then $\{U(t,\tau)\}$ is asymptotically upper-semicompact if and only if $\{U(t,\tau)\}$ is pullback ω-limit compact.

Remark 3.15. Let X be a uniformly convex Banach space; in particular, let X be a Hilbert space, and $\{U(t,\tau)\}$ be an MVP on X. We can deduce by Theorems 3.12 and 3.14 that $\{U(t,\tau)\}$ is asymptotically upper-semicompact if and only if $\{U(t,\tau)\}$ is pullback flattening.

4. Uniform attractors and uniform forward attraction of kernel sections. In this section, we present the necessary and sufficient conditions for the existence of uniform attractors and the uniform forward attraction of the inflated kernel sections of a family of multi-valued processes. Let Σ be a compact Banach space with norm $\|\cdot\|_\Sigma$ and let $\{T(h) \mid h \in \mathbb{R}^+\}$ be a continuous invariant ($T(h)\Sigma = \Sigma$) semigroup on Σ. Let $\{U_\sigma(t,\tau) \mid t \geqslant \tau,\, \tau \in \mathbb{R}\}$, $\sigma \in \Sigma$, be a family of multi-valued processes on X satisfying the following translation identity:

$$U_{T(h)\sigma}(t,\tau) = U_\sigma(t+h,\tau+h),\ \forall h \geqslant 0,\, t \geqslant \tau,\, \tau \in \mathbb{R}. \tag{4.1}$$

In particular, we recall the concept of the multi-valued skew product flow, which will be used in the proof of Theorem 4.9. For any fixed $\tau \in \mathbb{R}$, the family of multi-valued mappings $\{F_\tau(t)\}_{t\geqslant 0}$ acting on the extended space $\mathcal{Y} = X \times \Sigma$ defined by

$$F_\tau(t)(x,\sigma) = (U_\sigma(t+\tau,\tau)x,\, T(t)\sigma),\ \forall t \geqslant 0,\, (x,\sigma) \in X \times \Sigma \tag{4.2}$$

forms an autonomous multi-valued semidynamical system on $\mathcal{Y}$ over $\mathbb{R}^+$, which is called the multi-valued skew product flow associated with the family of multi-valued processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, and the semigroup $\{T(t)\}$.

We also need the following definitions and results.

Definition 4.1. Let $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, be a family of multi-valued processes on X. We say that the family of MVPs $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, is

(1) σ-uniformly dissipative, if for any fixed $\tau \in \mathbb{R}$, there exists a bounded subset Θ of X so that for any bounded set $B \subset X$, there exists a $\tilde{\tau}_1 = \tilde{\tau}_1(B) \in \mathbb{R}^+$ independent of $\sigma \in \Sigma$, such that

$$U_\sigma(t+\tau,\tau)B \subset \Theta,\ \forall \sigma \in \Sigma,\, t \geqslant \tilde{\tau}_1;$$

(2) σ-uniformly ω-limit compact, if for any fixed $\tau \in \mathbb{R}$, every bounded subset B of X and any $\varepsilon > 0$, there exists a $\tau_1' = \tau_1'(B,\varepsilon,\tau) \in \mathbb{R}^+$ which is independent of $\sigma \in \Sigma$, such that

$$k\left(\bigcup_{t\geqslant \tau_1'}\bigcup_{\sigma\in\Sigma} U_\sigma(t+\tau,\tau)B\right) < \varepsilon;$$

(3) point dissipative, if for any fixed $\tau \in \mathbb{R}$, there is a bounded set $\mathcal{M}_0$ in X, such that $\mathcal{M}_0$ attracts each trajectory starting from any point in X with the initial time τ;

(4) σ-uniformly eventually bounded, if for any fixed $\tau \in \mathbb{R}$ and for every bounded set $B \subset X$, there exists a $T_3 \geqslant \tau$ independent of $\sigma \in \Sigma$, such that $\bigcup_{t\geqslant T_3}\bigcup_{\sigma\in\Sigma} U_\sigma(t,\tau)B$ is bounded.

Definition 4.2. A compact set $\mu \subset X$ is said to be a uniform attractor of the family of multi-valued processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, if it satisfies

(1) μ σ-uniformly attracts every bounded subset B of X; i.e., for any fixed $\tau \in \mathbb{R}$,

$$\lim_{t\to+\infty}\sup_{\sigma\in\Sigma} H_X^*(U_\sigma(t+\tau,\tau)B,\, \mu) = 0.$$

(2) If there is another compact set A' satisfying (1), then $\mu \subset A'$.

Theorem 4.3 ([29]). Let X, Y be two Banach spaces satisfying the assumptions in the preliminary, and let $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, be a family of MVPs on X and Y, respectively. Assume that $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, is jointly upper-semicontinuous or weak upper-semicontinuous on $Y \times \Sigma$. If for fixed $t \geqslant \tau, \tau \in \mathbb{R}$, the family of MVPs $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, maps compact subsets of $X \times \Sigma$ into bounded subsets of 2^X, then $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, is jointly norm-to-weak upper-semicontinuous on $X \times \Sigma$.

Lemma 4.4. Let the family of multi-valued processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, be σ-uniformly ω-limit compact. Then $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is σ-uniformly eventually bounded.

Proof. Let $a \in X$, bounded set $B \subset X$ and $\tau \in \mathbb{R}$ be given arbitrarily, and suppose for a contradiction that $\bigcup_{t\geqslant T}\bigcup_{\sigma\in\Sigma} U_\sigma(t,\tau)B$ is unbounded for all $T \geqslant \tau$. Then there exist $x_n \in B$, $\sigma_n \in \Sigma$, $t_n \to +\infty$, and $y_n \in U_{\sigma_n}(t_n,\tau)x_n$ with $\|y_n - a\|_X \to \infty$. By the uniform ω-limit compactness and the property (1) in Lemma 2.14, we can deduce that y_n has a convergent subsequence. This leads to a contradiction and thus the proof of this lemma is completed. □

LEMMA 4.5. Let the family of multi-valued processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, satisfying the translation identity (4.1) be point dissipative, σ-uniformly ω-limit compact, and jointly norm-to-weak upper-semicontinuous in each $(x,\sigma) \in X \times \Sigma$ uniformly on the interval $[\tau, T]$ for any fixed $T \geqslant \tau$, where Σ is a compact Banach space. Then there exists a bounded set $\mathcal{M} \subset X$, so that for any compact set $D \subset X$, there exist $\varepsilon = \varepsilon(D) > 0$ and $t_2 = t_2(D) > \tau$ independent of $\sigma \in \Sigma$, such that $U_\sigma(t,\tau)\mathcal{N}(D,\varepsilon) \subset \mathcal{M}$ for all $t \geqslant t_2$ and all $\sigma \in \Sigma$.

Proof. Let $\eta > 0$. By Lemma 4.4, we know that $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is σ-uniformly eventually bounded. Hence for any fixed $\tau \in \mathbb{R}$, there exists a $\tau_3 \geqslant \tau$, such that

$$\mathcal{M} := \bigcup_{t \geqslant \tau_3} \bigcup_{\sigma \in \Sigma} U_\sigma(t,\tau)\mathcal{N}(\mathcal{M}_0, \eta)$$

is bounded, where $\mathcal{M}_0$ is given in Definition 4.1 (3). Assume on the contrary that there exist a compact set $D \subset X$ and sequences $\varepsilon_n \to 0$, $t_n > \tau_3$ ($t_n \to \infty$), $\sigma_n \in \Sigma$, $x_n \in \mathcal{N}(D,\varepsilon_n)$ and $y_n \in U_{\sigma_n}(t_n,\tau)x_n$ such that $y_n \notin \mathcal{M}$. Noticing that there exists a sequence $\tilde{x}_n \in D$ such that $\|x_n - \tilde{x}_n\| \leqslant \varepsilon_n$, we can assume that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \tilde{x}_n = x_0 \in D$ and $\lim_{n\to\infty} \sigma_n = \sigma_0 \in \Sigma$ (recall that Σ is compact). In view of (4.1), $U_{\sigma_n}(t_n,\tau)x_n = U_{\sigma_n}(t_n,t)U_{\sigma_n}(t,\tau)x_n = U_{T(t-\tau)\sigma_n}(t_n+\tau-t,\tau)U_{\sigma_n}(t,\tau)x_n$, so there exists $\tilde{y}_n(t) \in U_{\sigma_n}(t,\tau)x_n$, such that $y_n \in U_{T(t-\tau)\sigma_n}(t_n+\tau-t,\tau)\tilde{y}_n(t)$ and $\tilde{y}_n(t) \notin \mathcal{N}(\mathcal{M}_0,\eta)$ for all $\tau \leqslant t \leqslant t_n + \tau - \tau_3$. Thanks to the jointly norm-to-weak upper semicontinuity of the family of MVPs $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, in each $(x,\sigma) \in X \times \Sigma$ uniformly on any interval $[\tau,T]$, by diagonal procedure, we can deduce that there exists $y_0(t) \in U_{\sigma_0}(t,\tau)x_0 \notin \mathcal{M}_0$ for all $t \geqslant \tau$. This contradicts the point dissipativeness of $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$. We have finished the proof of Lemma 4.5. □

REMARK 4.6. Similar to Theorem 4.3, we have the following result:
Let X, Y be two Banach spaces satisfying the assumptions in the preliminary, and let $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, be a family of MVPs on X and Y, respectively. Assume that $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, is jointly upper-semicontinuous or weak upper-semicontinuous on $Y \times \Sigma$ uniformly on the interval $[\tau,T]$ for any fixed $T \geqslant \tau$. If for all $t \in [\tau,T]$, the family of MVPs $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, maps compact subsets of $X \times \Sigma$ into bounded subsets of 2^X, then $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, is jointly norm-to-weak upper-semicontinuous on $X \times \Sigma$ uniformly on the interval $[\tau,T]$.

The following theorem gives the necessary and sufficient conditions for the existence of uniform attractors.

THEOREM 4.7. Let the family of multi-valued processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, on X satisfying the translation identity (4.1) be jointly norm-to-weak upper-semicontinuous in each $(x,\sigma) \in X \times \Sigma$ uniformly on the interval $[\tau,T]$ for any fixed $T \geqslant \tau$, where Σ is a compact Banach space. Then the following statements are equivalent:

(1) There is a compact uniformly attracting set for the family of MVPs $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$.
(2) $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is point dissipative and σ-uniformly ω-limit compact.
(3) $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, has a unique uniform attractor μ.
(4) $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is σ-uniformly dissipative and σ-uniformly ω-limit compact.

Proof. Invoking Theorem 5.7 in [29], we only need to prove (2) $\Longrightarrow$ (3). For any $\tau \in \mathbb{R}$, let

$$\mu = \bigcap_{s \geqslant \tau} \overline{\bigcup_{t \geqslant s} \bigcup_{\sigma \in \Sigma} U_\sigma(t,\tau)\mathcal{M}}^{WS},$$

where $\mathcal{M}$ is given in Lemma 4.5. The compactness of μ follows immediately from the uniform ω-limit compactness.

Let us show that μ uniformly attracts every bounded subset B of X. Let $K = \bigcap_{s\geqslant\tau} \overline{\bigcup_{t\geqslant s}\bigcup_{\sigma\in\Sigma} U_\sigma(t,\tau)B}^{WS}$. By making use of uniform ω-limit compactness, it is easy to see that K is compact and uniformly attracts B. Let $\varepsilon(K), t_2 = t_2(K)$ be given as in Lemma 4.5, and let $0 < \varepsilon < \varepsilon(K)$. Since K uniformly attracts B, $U_\sigma(t_3,\tau)B \subset \mathcal{N}(K,\varepsilon)$ for all $\sigma \in \Sigma$ and some $t_3 > \tau$. By Lemma 4.5, hence $U_\sigma(t_2+t_3-\tau,\tau)B = U_\sigma(t_2 + t_3 - \tau, t_3)U_\sigma(t_3,\tau)B \subset U_\sigma(t_2+t_3-\tau,t_3)\mathcal{N}(K,\varepsilon) = U_{T(t_3-\tau)\sigma}(t_2,\tau)\mathcal{N}(K,\varepsilon) \subset \mathcal{M}$ for all $\sigma \in \Sigma$. Thus $U_\sigma(t+t_2+t_3-\tau,\tau)B = U_\sigma(t+t_2+t_3-\tau,t_2+t_3-\tau)U_\sigma(t_2+t_3-\tau,\tau)B \subset U_\sigma(t+t_2+t_3-\tau,t_2+t_3-\tau)\mathcal{M} = U_{T(t_2+t_3-2\tau)\sigma}(t+\tau,\tau)\mathcal{M}$ for all $t \geqslant 0$ and all $\sigma \in \Sigma$. Observing that $\mathcal{M}$ is σ-uniformly attracted to μ under the family of MVPs $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ (recall the definition of μ), so is B.

It suffices to check that μ is the minimal compact uniformly attracting set; i.e., if ν is another compact uniformly attracting set, then we need to show that $\mu := \bigcap_{s\geqslant\tau} \overline{\bigcup_{t\geqslant s}\bigcup_{\sigma\in\Sigma} U_\sigma(t,\tau)\mathcal{M}}^{WS} \subset \nu$. Let $y \in \mu$. Then there exist $t_n \to +\infty$, $\sigma_n \in \Sigma$, $x_n \in \mathcal{M}$, and $y_n \in U_{\sigma_n}(t_n,\tau)x_n$ such that $y_n \rightharpoonup y$. Note that ν uniformly attracts $\mathcal{M}$. Hence $\mathrm{dist}_X(y_n,\nu) \to 0$ as $n \to \infty$. Clearly, we have $y \in \nu$ and $\mu \subset \nu$.

The proof is complete. □

Analogous to the proof of Theorem 4.7, in view of Theorem 5.2 and Corollary 5.3 in [29], we have

THEOREM 4.8. Let $\{F(t)\}$ be a (autonomous) multi-valued semidynamical system on X, and let $F(t)x$ be norm-to-weak upper-semicontinuous in each $x \in X$ uniformly on interval $[0,T]$ for any fixed $T \geqslant 0$. Then the following statements are equivalent:
(1) There is a compact attracting set for $\{F(t)\}$.
(2) $\{F(t)\}$ is point dissipative and ω-limit compact.
(3) $\{F(t)\}$ has a unique global attractor $\mathcal{A}$; moreover, $\mathcal{A}$ coincides with the kernel section at time τ, i.e., $\mathcal{A} = \mathcal{K}(\tau)$ for any $\tau \in \mathbb{R}$.
(4) $\{F(t)\}$ is dissipative and ω-limit compact.

Now we generalize Theorem IV. 5.1 in [13] to the multi-valued case and show the uniform forward attraction of the inflated kernel sections.

THEOREM 4.9. Let a family of multi-valued processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, acting on X be point dissipative, σ-uniformly ω-limit compact and jointly norm-to-weak upper-semicontinuous in each $(x,\sigma) \in X \times \Sigma$ uniformly on interval $[\tau, T]$ for any fixed $T \geqslant \tau$. Also let Σ be a compact Banach space and let $\{T(t)\}$ be a continuous (uniformly on the interval $[0,T]$ for any fixed $T \geqslant 0$) invariant semigroup on Σ satisfying the translation identity (4.1). Then the multi-valued semidynamical system $\{F_\tau(t)\}$ corresponding to the family of multi-valued processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, and acting on $X \times \Sigma$ (see Theorem 4.8) possesses a unique compact attractor $\mathcal{A}$ which is strictly invariant with respect to $\{F_\tau(t)\} : F_\tau(t)\mathcal{A} = \mathcal{A}$ for all $t \geqslant 0$. Furthermore,

(1) $\Pi_1\mathcal{A} = \mu$ is the uniform (w.r.t. $\sigma \in \Sigma$) attractor of the family of multi-valued processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$;
(2) $\Pi_2\mathcal{A} = \Sigma$;
(3) the global attractor satisfies

$$\mathcal{A} = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(\tau) \times \{\sigma\};$$

(4) the uniform attractor satisfies

$$\Pi_1\mathcal{A} = \mu = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(\tau);$$

(5) for any fixed $\varepsilon_0 > 0$, the family of inflated kernel sections $\{\mathcal{K}_\sigma^{[\varepsilon_0]}(\tau)\}$, $\sigma \in \Sigma$, defined in (1.1) σ-uniformly pullback (respectively forward) attracts each bounded subset B of X; i.e., for any $\varepsilon > 0$, there is a $T_1 = T_1(B,\varepsilon) > 0$ independent of $\sigma \in \Sigma$, such that

$$H_X^*(U_\sigma(\tau, \tau - t)B,\, \mathcal{K}_\sigma^{[\varepsilon_0]}(\tau)) < \varepsilon,\ \forall \sigma \in \Sigma,\, t \geqslant T_1$$
$$\left(\text{respectively } H_X^*(U_\sigma(t+\tau, \tau)B,\, \mathcal{K}_{T(t)\sigma}^{[\varepsilon_0]}(\tau)) < \varepsilon,\ \forall \sigma \in \Sigma,\, t \geqslant T_1\right).$$

Here the kernel $\mathcal{K}_\sigma$ of the MVP $\{U_\sigma(t,\tau)\}$ with $\sigma \in \Sigma$ be given as in Remark 3.10, $\mathcal{K}_\sigma(\tau)$ is the section at $t = \tau$ of the kernel $\mathcal{K}_\sigma$.

Proof. Write $\mathcal{Y} = X \times \Sigma$ and endow $\mathcal{Y}$ with the norm $\|\cdot\|_{\mathcal{Y}}$ defined by

$$\|(x,\sigma_1) - (y,\sigma_2)\|_{\mathcal{Y}} = \|x - y\|_X + \|\sigma_1 - \sigma_2\|_\Sigma,\ \forall (x,\sigma_1), (y,\sigma_2) \in \mathcal{Y}. \tag{4.3}$$

Clearly, $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a Banach space.

Let $\tau \in \mathbb{R}$ be given arbitrarily. Now we consider the multi-valued semidynamical system $\{F_\tau(t)\}$ on $\mathcal{Y}$ over $\mathbb{R}^+$ defined by

$$F_\tau(t)(x,\sigma) = (U_\sigma(t+\tau,\tau)x, T(t)\sigma),\ \forall (x,\sigma) \in \mathcal{Y}.$$

Then $F_\tau(t)$ is well defined on $\mathcal{Y}$ and due to the jointly norm-to-weak upper semicontinuity of $U_\sigma(t,\tau)x$ in (x,σ) uniformly on the interval $[\tau,T]$ for any fixed $T \geqslant \tau$ and the continuity of $T(t)\sigma$ in σ uniformly on the interval $[0,T]$ for any fixed $T \geqslant 0$, we see that $F_\tau(t)(x,\sigma)$ is norm-to-weak upper-semicontinuous in (x,σ) uniformly on the interval $[0,T]$ for any fixed $T \geqslant 0$. Since the family of MVPs $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, is point dissipative, there exists a bounded subset $\mathcal{M}_0$ of X, such that $\mathcal{M}_0$ attracts each trajectory starting from any point in X with the initial time τ. It follows that $\Omega := \mathcal{M}_0 \times \Sigma$ is a bounded set in $X \times \Sigma$ and attracts each trajectory starting from any point in $X \times \Sigma$, i.e., $\{F_\tau(t)\}$ is point dissipative. The σ-uniform ω-limit compactness of the family of MVPs $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, implies that $\{F_\tau(t)\}$ is ω-limit compact (recall that Σ is compact). Thus, according to Theorem 4.8, the multi-valued semidynamical system $\{F_\tau(t)\}$ has a unique global attractor $\mathcal{A}$.

Now we will show that

$$\mathcal{A} = \omega(\mathcal{M} \times \Sigma) = \bigcap_{s \in \mathbb{R}^+} \overline{\bigcup_{t \geqslant s} F_\tau(t)(\mathcal{M} \times \Sigma)}^{WS},$$

where $\mathcal{M}$ is defined in the proof of Lemma 4.5.

Similar to the proof of Theorem 3.4, in view of the ω-limit compactness of $\{F_\tau(t)\}$ and the boundedness of the set $\mathcal{M} \times \Sigma$, we can verify that $\omega(\mathcal{M} \times \Sigma)$ is compact. By making use of some techniques from the proof of Theorem 4.7 and [3], we can deduce that $\omega(\mathcal{M} \times \Sigma)$ attracts every bounded subset of $X \times \Sigma$. Thanks to the ω-limit compactness of $\{F_\tau(t)\}$, by the similar arguments of Proposition 2.13 in [19], we can show that for any $x \in \omega(\mathcal{M} \times \Sigma)$, there is a trajectory γ of $\{F_\tau(t)\}$ on $\mathbb{R}$ which lies in $\omega(\mathcal{M} \times \Sigma)$ with $\gamma(0) = x$. On the other hand, by Lemma 4.5, it is easy to see that there is a $t_2' > 0$ such that $F(t_2')\omega(\mathcal{M} \times \Sigma) \subset \mathcal{M} \times \Sigma$. We then derive that $\omega(\mathcal{M} \times \Sigma) \subset \mathcal{M} \times \Sigma$. Let $W = \bigcup_{t \geqslant 0} F_\tau(t)\omega(\mathcal{M} \times \Sigma)$. Analogous to the proofs of Proposition 4.1 in [19] and Lemma 3.8, we can deduce that $\omega(\mathcal{M} \times \Sigma)$ is invariant. Therefore, the uniqueness of the global attractor of $\{F_\tau(t)\}$ implies that $\mathcal{A} = \omega(\mathcal{M} \times \Sigma)$.

Since $\{T(t)\}$ is an invariant semigroup, i.e., $T(t)\Sigma = \Sigma$ for all $t \in \mathbb{R}^+$, we can deduce that

$$\Pi_2 \mathcal{A} = \Pi_2 \omega(\mathcal{M} \times \Sigma) = \Pi_2 \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} F_\tau(t)(\mathcal{M} \times \Sigma)}^{WS} = \Sigma,$$

and (2) is proved.

Now let us show that $\mu = \Pi_1 \mathcal{A}$ is the uniform attractor of the family of MVPs $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$. Because of the compactness of $\mathcal{A}$, we see that μ is compact. Let us verify that μ is a σ-uniformly attracting set. Let B be a bounded subset of X. Since $\mathcal{A}$ attracts $B \times \Sigma$ under $\{F_\tau(t)\}$,

$$H^*_{\mathcal{Y}}(F_\tau(t)(B \times \Sigma), \mathcal{A}) \to 0 \quad \text{as } t \to +\infty. \tag{4.4}$$

By (4.2), we have

$$\sup_{\sigma \in \Sigma} H^*_X(U_\sigma(t+\tau,\tau)B, \mu) \to 0 \quad \text{as } t \to +\infty. \tag{4.5}$$

In fact, we can replace τ in (4.5) by any $s \in \mathbb{R}$. Indeed, by Theorem 4.7, we see that the family of MVPs $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, is σ-uniformly dissipative, and the set $B_1 = \bigcup_{\sigma \in \Sigma} U_\sigma(\tilde{\tau}_1 + s, s)B$ is bounded in X for some positive constant $\tilde{\tau}_1 > \tau - s$. Let $h = \tilde{\tau}_1 + s - \tau$. Therefore if $t \geqslant \tilde{\tau}_1$, then

$$\begin{aligned} U_\sigma(t+s,s)B &= U_\sigma(t+s, \tilde{\tau}_1 + s)U_\sigma(\tilde{\tau}_1 + s, s)B \\ &\subset U_\sigma(t+s, \tilde{\tau}_1 + s)B_1 = U_{T(h)\sigma}(t - \tilde{\tau}_1 + \tau, \tau)B_1. \end{aligned}$$

By (4.5), we find that

$$\sup_{\sigma \in \Sigma} H^*_X(U_\sigma(t+s,s)B,\, \mu) \leqslant \sup_{\sigma \in \Sigma} H^*_X(U_\sigma(t - \tilde{\tau}_1 + \tau, \tau)B_1,\, \mu) \to 0 \ (t \to +\infty).$$

To verify the minimality property, let us show that

$$\mu = \Pi_1\mathcal{A} = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma(\tau). \tag{4.6}$$

Indeed, let a pair $(u_\tau, \sigma_\tau) \in \mathcal{A}$ be given, when $u_\tau \in \Pi_1\mathcal{A}$ and $\sigma_\tau \in \Sigma$. It follows from Theorem 4.8 that there exists a bounded complete trajectory $\varphi(s) = (u(s), \sigma(s))$ of the multi-valued semidynamical system $\{F_\tau(t)\}$ ($(u(t+s), \sigma(t+s)) \in F_\tau(t)(u(s), \sigma(s))$ for all $s \in \mathbb{R}$ and all $t \in \mathbb{R}^+$) such that $\varphi(\tau) = (u(\tau), \sigma(\tau)) = (u_\tau, \sigma_\tau)$. Hence

$$u(t+s) \in U_{\sigma(s)}(t+\tau, \tau)u(s),\ \forall s \in \mathbb{R},\ t \geqslant 0. \tag{4.7}$$

Noticing that $T(t)$ is a semigroup, therefore

$$\sigma(t+s) = T(t)\sigma(s),\ \forall s \in \mathbb{R},\ t \geqslant 0. \tag{4.8}$$

Let us show that $u(s)$ is a bounded complete trajectory of the MVP $\{U_{\sigma_\tau}(t,\tau)\}$ with $\sigma_\tau = \sigma(\tau)$. If $s \geqslant \tau$, then by (4.1) and (4.7)–(4.8), we have

$$u(t+s) \in U_{\sigma(s)}(t+\tau, \tau)u(s) = U_{T(s-\tau)\sigma(\tau)}(t+\tau, \tau)u(s) = U_{\sigma(\tau)}(t+s, s)u(s).$$

If $s < \tau$, then by $T(\tau - s)\sigma(s) = \sigma(\tau)$, and again by (4.1) and (4.7),

$$u(t+s) \in U_{\sigma(s)}(t+\tau, \tau)u(s) = U_{T(\tau-s)\sigma(s)}(t+s, s)u(s) = U_{\sigma(\tau)}(t+s, s)u(s).$$

Thus, $u(s)$ is a bounded complete trajectory of the MVP $\{U_{\sigma_\tau}(t,\tau)\}$ with $\sigma_\tau = \sigma(\tau) \in \Sigma$ or, in other words, $u_\tau = u(\tau) \in \mathcal{K}_{\sigma(\tau)}(\tau)$. We have established that

$$\Pi_1\mathcal{A} \subset \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma(\tau). \tag{4.9}$$

Let us verify the converse inclusion. Let $u_\tau \in \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma(\tau)$ and $u_\tau = u(\tau)$, where $u(s), s \in \mathbb{R}$ is the corresponding bounded complete trajectory of the MVP $\{U_{\sigma_\tau}(t,\tau)\}$ with some $\sigma_\tau \in \Sigma$. Since Σ is strictly invariant with respect to $\{T(t)\}$, there exists a bounded complete trajectory $\sigma(s), s \in \mathbb{R}$ of $\{T(t)\}$ such that $\sigma(\tau) = \sigma_\tau$. Let us show that $(u(s), \sigma(s))$ is a complete bounded trajectory of the multi-valued semidynamical system $\{F_\tau(t)\}$. If $s \geqslant \tau$, then by (4.1) and (4.2), we have

$$\begin{gathered}(u(t+s), \sigma(t+s)) \in (U_{\sigma(\tau)}(t+s, s)u(s), \sigma(t+s)) \\ = (U_{T(s-\tau)\sigma(\tau)}(t+\tau, \tau)u(s), \sigma(t+s)) = (U_{\sigma(s)}(t+\tau, \tau)u(s), T(t)\sigma(s)) \\ = F_\tau(t)(u(s), \sigma(s)).\end{gathered}$$

Now if $s < \tau$, then $T(\tau - s)\sigma(s) = \sigma(\tau)$, that is,

$$u(t+s) \in U_{\sigma(\tau)}(t+s, s)u(s) = U_{T(\tau-s)\sigma(s)}(t+s, s)u(s) = U_{\sigma(s)}(t+\tau, \tau)u(s).$$

So,

$$(u(t+s), \sigma(t+s)) \in (U_{\sigma(s)}(t+\tau, \tau)u(s), T(t)\sigma(s)) = F_\tau(t)(u(s), \sigma(s)),$$

that is, $(u(t+s), \sigma(t+s)) \in F_\tau(t)(u(s), \sigma(s))$ for all $t \geqslant 0$ and all $s \in \mathbb{R}$. By Theorem 4.8, $(u_\tau, \sigma_\tau) = (u(\tau), \sigma(\tau)) \in \mathcal{A}$ and $u_\tau = u(\tau) \in \Pi_1\mathcal{A}$. Hence,

$$\Pi_1\mathcal{A} \supset \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma(\tau). \tag{4.10}$$

Combining (4.9) and (4.10) together, we have

$$\Pi_1\mathcal{A} = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma(\tau). \tag{4.11}$$

We have proved (3).

To show that $\Pi_1\mathcal{A} = \bigcup_{\sigma\in\Sigma}\mathcal{K}_\sigma(\tau) = \mu$ is the uniform attractor of the family of MVPs $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, we have to check the minimality property. Let us show that $\Pi_1\mathcal{A} = \mu \subset A'$ where A' is an arbitrary compact σ-uniformly attracting set for the family of MVPs $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$. By (4.11), it is sufficient to verify that $u(\tau) \in A'$ for an arbitrary bounded complete trajectory $u(s)$, $s \in \mathbb{R}$ of the MVP $\{U_\sigma(t,\tau)\}$ with $\sigma \in \Sigma$. As we already know there exists a bounded complete trajectory $\sigma(s)$ of $\{T(t)\}$ with $\sigma(\tau) = \sigma$. Consider the set $B_2 = \{u(-n+\tau),\, n \in \mathbb{N}\}$. Evidently, B_2 is bounded in X. We have

$$\begin{aligned} u(\tau) &\in U_{\sigma(\tau)}(\tau,-n+\tau)u(-n+\tau) = U_{T(n)\sigma(-n+\tau)}(\tau,-n+\tau)u(-n+\tau)\\ &= U_{\sigma(-n+\tau)}(\tau+n,\tau)u(-n+\tau). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{dist}_X(u(\tau),\, A') &\leqslant H_X^*(U_{\sigma(-n+\tau)}(\tau+n,\tau)u(-n+\tau),\, A')\\ &\leqslant \sup_{\sigma\in\Sigma} H_X^*(U_\sigma(\tau+n,\tau)B_2,\, A') \to 0\ (n\to+\infty), \end{aligned}$$

i.e., $\mathrm{dist}_X(u(\tau), A') = 0$. The set A' is compact; hence $u(\tau) \in A'$ and $\mu = \Pi_1\mathcal{A} \subset A'$. Thus $\Pi_1\mathcal{A} = \bigcup_{\sigma\in\Sigma}\mathcal{K}_\sigma(\tau) = \mu$ is the uniform attractor of the family of MVPs $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, and the conclusions (1) and (4) hold true.

Let $\varepsilon_0 > 0$. Now it remains to show that for any bounded subset B of X and $\varepsilon > 0$ (we can assume that $\varepsilon < \varepsilon_0$), there is a $T_1 = T_1(B,\varepsilon) > 0$ independent of $\sigma \in \Sigma$ such that

$$H_X^*(U_\sigma(\tau,\tau-t)B,\, \mathcal{K}_\sigma^{[\varepsilon_0]}(\tau)) < \varepsilon,\ \forall\sigma\in\Sigma,\, t \geqslant T_1,$$

and

$$H_X^*(U_\sigma(t+\tau,\tau)B,\, \mathcal{K}_{T(t)\sigma}^{[\varepsilon_0]}(\tau)) < \varepsilon,\ \forall\sigma\in\Sigma,\, t \geqslant T_1.$$

Let $\mathcal{U} = B \times \Sigma$. By (4.4), there is $T_1 = T_1(\mathcal{U},\varepsilon) > 0$ such that when $t \geqslant T_1$, we have

$$\inf_{v\in\mathcal{A}} H_{\mathcal{Y}}^*(F_\tau(t)u,\, v) < \frac{\varepsilon}{2},\ \forall u := (u',\sigma) \in \mathcal{U},$$

i.e.,

$$\inf_{v\in\mathcal{A}} H_{\mathcal{Y}}^*((U_\sigma(t+\tau,\tau)u', T(t)\sigma),\, v) < \frac{\varepsilon}{2},\ \forall u' \in B,\, \sigma\in\Sigma,\, t \geqslant T_1.$$

Let $\sigma' = T(t)\sigma$. Then

$$\begin{aligned} &\inf_{v\in\mathcal{A}} H_{\mathcal{Y}}^*((U_{\sigma'}(\tau,\tau-t)u',\sigma'),\, v)\\ =\ &\inf_{v\in\mathcal{A}} H_{\mathcal{Y}}^*((U_{T(t)\sigma}(\tau,\tau-t)u', T(t)\sigma),\, v)\\ =\ &\inf_{v\in\mathcal{A}} H_{\mathcal{Y}}^*((U_\sigma(t+\tau,\tau)u', T(t)\sigma),\, v)\\ <\ &\frac{\varepsilon}{2},\ \forall u' \in B,\, \sigma' \in \Sigma,\, t \geqslant T_1. \end{aligned} \tag{4.12}$$

Now for each σ' we divide $\mathcal{A}$ into two parts,

$$\mathcal{A} = \mathcal{A}_{\sigma'}[\varepsilon_0] \cup \mathcal{A}_{\sigma'}^C[\varepsilon_0],$$

where

$$\mathcal{A}_{\sigma'}[\varepsilon_0] = \bigcup_{\sigma\in\Sigma, \|\sigma-\sigma'\|_\Sigma \leqslant \varepsilon_0} (\mathcal{K}_\sigma(\tau)\times\{\sigma\}),\ \mathcal{A}^C_{\sigma'}[\varepsilon_0] = \bigcup_{\sigma\in\Sigma, \|\sigma-\sigma'\|_\Sigma > \varepsilon_0} (\mathcal{K}_\sigma(\tau)\times\{\sigma\}).$$

Let $(u',\sigma')\in B\times\Sigma$, $t\geqslant T_1$, and $x\in U_{\sigma'}(\tau,\tau-t)u'$. Note that, if $v:=(y,\sigma)\in\mathcal{A}^C_{\sigma'}[\varepsilon_0]$, then by the definition of $\|\cdot\|_{\mathcal{Y}}$,

$$\|(x,\sigma')-(y,\sigma)\|_{\mathcal{Y}} = \|x-y\|_X + \|\sigma'-\sigma\|_\Sigma \geqslant \|\sigma'-\sigma\|_\Sigma > \varepsilon_0 > \varepsilon.$$

So, by (4.12), we necessarily have

$$\inf_{v\in\mathcal{A}_{\sigma'}[\varepsilon_0]} \|(x,\sigma')-v\|_{\mathcal{Y}} < \frac{\varepsilon}{2}.$$

Thus, in particular, there exists a point $v':=(y',\sigma'')\in\mathcal{A}_{\sigma'}[\varepsilon_0]$ such that

$$\|(x,\sigma')-(y',\sigma'')\|_{\mathcal{Y}} \leqslant \frac{2\varepsilon}{3}. \tag{4.13}$$

Since $\|\sigma''-\sigma'\|_\Sigma\leqslant\varepsilon_0$, we conclude that $y'\in\mathcal{K}_{\sigma''}(\tau)\subset\mathcal{K}^{[\varepsilon_0]}_{\sigma'}(\tau)$. From this and (4.13) it follows that

$$\begin{aligned}\operatorname{dist}_X(x,\,\mathcal{K}_{\sigma''}(\tau)) &\leqslant \|x-y'\|_X \\ &\leqslant \|(x,\sigma')-(y',\sigma'')\|_{\mathcal{Y}} \leqslant \tfrac{2\varepsilon}{3}\end{aligned}$$

and hence that

$$\operatorname{dist}_X(x,\,\mathcal{K}^{[\varepsilon_0]}_{\sigma'}(\tau)) \leqslant \operatorname{dist}_X(x,\,\mathcal{K}_{\sigma''}(\tau)) \leqslant \frac{2\varepsilon}{3} < \varepsilon.$$

Observing that $u'\in B$, $\sigma'\in\Sigma$, $t\geqslant T_1$ and $x\in U_{\sigma'}(\tau,\tau-t)u'$ are otherwise arbitrary, we obtain

$$H^*_X(U_{\sigma'}(\tau,\tau-t)B,\,\mathcal{K}^{[\varepsilon_0]}_{\sigma'}(\tau)) < \varepsilon,\ \forall t\geqslant T_1,\ \sigma'\in\Sigma. \tag{4.14}$$

In view of (4.1), (4.14) and $\sigma'=T(t)\sigma$, we have

$$\begin{aligned}H^*_X(U_\sigma(t+\tau,\tau)B,\,\mathcal{K}^{[\varepsilon_0]}_{T(t)\sigma}(\tau)) &= H^*_X(U_{T(t)\sigma}(\tau,\tau-t)B,\,\mathcal{K}^{[\varepsilon_0]}_{T(t)\sigma}(\tau)) \\ &= H^*_X(U_{\sigma'}(\tau,\tau-t)B,\,\mathcal{K}^{[\varepsilon_0]}_{\sigma'}(\tau)) \\ &< \varepsilon,\ \forall t\geqslant T_1,\ \sigma\in\Sigma.\end{aligned}$$

The proof of Theorem 4.9 is finished. □

5. Nonautonomous reaction-diffusion equations in unbounded domains without uniqueness. The main purpose of this section is to apply our abstract theory developed in Sections 3 and 4 to consider nonautonomous reaction-diffusion equations in unbounded domains without uniqueness.

5.1. *Kernel sections for nonautonomous reaction-diffusion equations in* $L^2(\mathbb{R}^n)$. In this subsection, we will prove the existence of local bounded kernel and compact kernel sections for nonautonomous reaction-diffusion equations in $L^2(\mathbb{R}^n)$.

We investigate the system

$$\frac{\partial u}{\partial t} - \triangle u + \lambda u + f(u) = g(x,t),\quad \text{in } \mathbb{R}^n\times[\tau,+\infty), \tag{5.1}$$

$$u(x,\tau) = u_0(x),\ x\in\mathbb{R}^n,\ \tau\in\mathbb{R}. \tag{5.2}$$

We impose the following conditions on $f\in C(\mathbb{R},\mathbb{R})$ and $g\in L^2_{loc}(\mathbb{R};L^2(\mathbb{R}^n))$:

$$\alpha_2|v|^p - k_2|v|^2 \leqslant f(v)v \leqslant \alpha_1|v|^p + k_1|v|^2,\ \forall v\in\mathbb{R}, \tag{5.3}$$

where $2 < p < \infty$, $k_2 < \lambda$, α_1, α_2, k_1, k_2 are positive constants, and

$$\int_{-\infty}^{t} e^s |g(s)|_2^2 ds < +\infty, \ \forall t \in \mathbb{R}. \tag{5.4}$$

Without loss of generality, we assume that

$$\lambda - k_2 \geqslant 1. \tag{5.5}$$

For convenience, hereafter let $|\cdot|_q$ be the norm of $L^q(\mathbb{R}^n)$ $(q \geqslant 1)$, $|u|$ the modular (or absolute value) of u, and C the arbitrary positive constants, which may be different from line to line and even in the same line.

We start with the following general existence of solutions, which can be obtained by the normal Faedo-Galerkin methods. Here we only state the results, and the interested readers are referred to [13] for details.

LEMMA 5.1. Assume that $g \in L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^n))$ and f satisfies (5.3). Then for any initial data $u_0 \in L^2(\mathbb{R}^n)$ and any $\tau \in \mathbb{R}$, there exists a weak solution u for system (5.1)–(5.2) which satisfies

$$u \in L^2(\tau, T; H^1(\mathbb{R}^n)) \cap L^\infty(\tau, T; L^2(\mathbb{R}^n)), \ \forall T \geqslant \tau.$$

In addition, by the similar arguments in [13], we can define a family of multi-valued mappings $U(t,\tau) : L^2(\mathbb{R}^n) \to 2^{L^2(\mathbb{R}^n)}$, $t \geqslant \tau$, $\tau \in \mathbb{R}$ by setting

$$U(t,\tau)u_0 = \{u(t) \ | \ u(\cdot) \text{ is a weak solution of system (5.1)–(5.2)}\}.$$

It is easy to verify that the properties (1), (2) in Definition 2.10 hold true. Let $u_{n0} \to u_0$ in $L^2(\mathbb{R}^n)$. Similar to the proof of the existence of weak solutions (see [13] for details), in view of (5.3), we can show that for any fixed $t \geqslant \tau$, $\tau \in \mathbb{R}$ and any $u_n(t) \in U(t,\tau)u_{n0}$, there exists a $u(t) \in U(t,\tau)u_0$ such that $u_n(t) \rightharpoonup u(t)$ in $L^2(\mathbb{R}^n)$. Thus, the family of multi-valued mappings $\{U(t,\tau)\}$ forms a multi-valued process on $L^2(\mathbb{R}^n)$.

From now on, we denote by $\mathfrak{R}$ the set of all functions $r : \mathbb{R} \to (0, +\infty)$ such that

$$\lim_{t \to -\infty} e^t r^2(t) = 0, \tag{5.6}$$

and denote by $\mathfrak{B}$ the class of all families $\mathcal{B} = \{B(t)\}_{t\in\mathbb{R}} \subset \mathfrak{F}(L^2(\mathbb{R}^n))$ such that $B(t) \subset \overline{\mathcal{N}}(0, r_{\mathcal{B}}(t))$, for some $r_{\mathcal{B}} \in \mathfrak{R}$, where $\mathfrak{F}(L^2(\mathbb{R}^n))$ denotes the family of all nonempty subsets of $L^2(\mathbb{R}^n)$ and $\overline{\mathcal{N}}(0, r_{\mathcal{B}}(t))$ denotes the closed ball in $L^2(\mathbb{R}^n)$ centered at zero with radius $r_{\mathcal{B}}(t)$.

LEMMA 5.2. Suppose that f and $g \in L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^n))$ satisfy (5.3)–(5.5). Then the multi-valued process $\{U(t,\tau)\}$ is pullback $\mathfrak{B}$-dissipative in $L^2(\mathbb{R}^n)$; i.e., there exists a family of bounded sets $\mathcal{Q} = \{Q(t)\}_{t\in\mathbb{R}} \in \mathfrak{B}$ so that for any family of bounded subsets $\mathcal{B} := \{B(t)\}_{t\in\mathbb{R}} \in \mathfrak{B}$ and any $t \in \mathbb{R}$, there exists a $t_0 = t_0(\mathcal{B}, t) \in \mathbb{R}^+$ such that

$$U(t, t-s)B(t-s) \subset Q(t), \ \forall s \geqslant t_0.$$

Proof. Let $t \in \mathbb{R}$ and $u_0 \in L^2(\mathbb{R}^n)$ be fixed. We observe that for any $T \geqslant t - s$ with $s \geqslant 0$,

$$U(T, t-s)u_0 = \{u(T) \ | \ u(\cdot) \text{ is a weak solution of system (5.1)–(5.2) and } u(t-s) = u_0 \in B(t-s) \text{ for all } s \geqslant 0\}.$$

Multiplying (5.1) by u, after the standard integration by parts and using assumptions (5.3), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dT}|u(T)|_2^2+|\nabla u(T)|_2^2+\lambda|u(T)|_2^2\\&+\alpha_2\int_{\mathbb{R}^n}|u(T)|^p-k_2|u(T)|_2^2=\int_{\mathbb{R}^n}g(T)u(T).\end{aligned}\tag{5.7}$$

Using Young's inequality,

$$|(g(T),u(T))|\leqslant\frac{1}{2}|u(T)|_2^2+\frac{1}{2}|g(T)|_2^2.$$

By (5.5),

$$\frac{d}{dT}|u(T)|_2^2+|u(T)|_2^2\leqslant|g(T)|_2^2.\tag{5.8}$$

Let $\mathcal{B}\in\mathfrak{B}$ be given. From (5.8), we easily obtain

$$\begin{aligned}|u(t)|_2^2&\leqslant e^{-s}|u(t-s)|_2^2+e^{-t}\int_{t-s}^{t}e^{s'}|g(s')|_2^2ds'\\&\leqslant e^{-s}|u_0|_2^2+e^{-t}\int_{-\infty}^{t}e^{s'}|g(s')|_2^2ds',\end{aligned}\tag{5.9}$$

for all $u_0\in B(t-s)$, $t\in\mathbb{R}$, $s\geqslant 0$.

Denote by $R(t)$ the nonnegative number given for each $t\in\mathbb{R}$ by

$$(R(t))^2=2e^{-t}\int_{-\infty}^{t}e^{s'}|g(s')|_2^2ds',\tag{5.10}$$

and consider the family of closed balls $\mathcal{Q}=\{Q(t)\}_{t\in\mathbb{R}}$ in $L^2(\mathbb{R}^n)$ defined by

$$Q(t)=\{v\in L^2(\mathbb{R}^n):|v|_2\leqslant R(t)\}.$$

It is straightforward to check that $\mathcal{Q}\in\mathfrak{B}$, and moreover, by (5.6) and (5.9), the family of $\mathcal{Q}$ is pullback $\mathfrak{B}$-absorbing for the MVP $\{U(t,\tau)\}$ and thus the proof is completed. □

Lemma 5.3. Assume that the hypotheses in Lemma 5.2 hold. Then for each $t\in\mathbb{R}$ and any $\varepsilon>0$, there exist $k>0$ and $T_4>0$, such that

$$\int_{\Omega_k^C}|u|_2^2dx<\varepsilon,\ \forall s\geqslant T_4,\ u_0\in Q(t-s),\ u\in U(t,t-s)u_0,$$

where $\mathcal{Q}=\{Q(t)\}_{t\in\mathbb{R}}\in\mathfrak{B}$ is a pullback absorbing set in $L^2(\mathbb{R}^n)$ and $\Omega_k^C=\{x\in\mathbb{R}^n|\ |x|\geqslant k\}$.

Proof. Choose a smooth function θ such that $0\leqslant\theta(s)\leqslant 1$ for any $s\in\mathbb{R}^+$, and

$$\theta(s)=0\ \text{ for } 0\leqslant s\leqslant 1\ \text{ and }\ \theta(s)=1\ \text{ for } s\geqslant 2.$$

Then there exists a constant C such that $|\theta'(s)|\leqslant C$ for any $s\in\mathbb{R}^+$. Let $t\in\mathbb{R}$ and $\varepsilon>0$ be fixed. As above, we define $U(T,t-s)u_0$ for all $T\geqslant t-s$ and $s\geqslant 0$ with $u_0\in Q(t-s)$.

Multiplying (5.1) by $\theta^2(|x|^2/k^2)\cdot u$, and integrating in $\mathbb{R}^n$, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dT}\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)|u|^2-\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)u\triangle u+\lambda\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)|u|^2\\ =&-\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)f(u)u+\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)g(T)u.\end{aligned}\tag{5.11}$$

Now we estimate each term in (5.11) as follows.

First we have

$$-\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)u\triangle u=\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2+\int_{\mathbb{R}^n}\frac{4x}{k^2}\theta'\left(\frac{|x|^2}{k^2}\right)\theta\left(\frac{|x|^2}{k^2}\right)u\nabla u\tag{5.12}$$

and

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\frac{4x}{k^2}\theta'\left(\frac{|x|^2}{k^2}\right)\theta\left(\frac{|x|^2}{k^2}\right)u\nabla u\right|\\ \leqslant&\frac{C}{k}\int_{k\leqslant|x|\leqslant\sqrt{2}k}\theta\left(\frac{|x|^2}{k^2}\right)|u||\nabla u|\\ \leqslant&\frac{C}{2k}\left(\int_{\mathbb{R}^n}|u|^2+\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2\right).\end{aligned}\tag{5.13}$$

For the first term on the right-hand side of (5.11), by (5.3), we have

$$-\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)f(u)u\leqslant k_2\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)|u|^2.\tag{5.14}$$

Moreover, we also have

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)g(T)u\right|\leqslant&\frac{1}{2}\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)|g(T)|^2+\frac{1}{2}\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)|u|^2\\ \leqslant&\frac{1}{2}\int_{|x|\geqslant k}|g(T)|^2+\frac{1}{2}\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)|u|^2.\end{aligned}\tag{5.15}$$

Combining (5.11)–(5.15) together, in view of (5.5), we can deduce that for k sufficiently large,

$$\frac{d}{dT}\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)|u|^2+\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)|u|^2\leqslant\int_{|x|\geqslant k}|g(T)|^2+\frac{C}{2k}\int_{\mathbb{R}^n}|u|^2.$$

By the Gronwall lemma, we get

$$\begin{aligned}&\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)|u(t)|^2\leqslant e^{-s}\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)|u(t-s)|^2\\ &+\frac{C}{2k}e^{-t}\int_{t-s}^{t}e^{s'}|u(s')|_2^2+e^{-t}\int_{t-s}^{t}e^{s'}\int_{|x|\geqslant k}|g(s')|^2\\ \leqslant&e^{-s}R^2(t-s)+\frac{C}{2k}e^{-t}\int_{-\infty}^{t}e^{s'}R^2(s')+e^{-t}\int_{-\infty}^{t}e^{s'}\int_{|x|\geqslant k}|g(s')|^2.\end{aligned}\tag{5.16}$$

By making use of (5.10), we can conclude that when s and k are sufficiently large,

$$e^{-s}R^2(t-s)+\frac{C}{2k}e^{-t}\int_{-\infty}^{t}e^{s'}R^2(s')<\frac{\varepsilon}{2}.\tag{5.17}$$

Similarly, we can choose a $t_4 > 0$ sufficiently large, such that

$$e^{-t}\int_{-\infty}^{-t_4} e^{s'}|g(s')|_2^2 < \frac{\varepsilon}{4}, \tag{5.18}$$

and when k is sufficiently large,

$$e^{-t}\int_{-t_4}^{t} e^{s'}\int_{|x|\geqslant k}|g(s')|^2 < \frac{\varepsilon}{4}. \tag{5.19}$$

Hence, (5.16)–(5.19) imply that

$$\int_{\Omega_{2k}^C}|u(t)|^2dx \leqslant \int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)|u(t)|^2 < \varepsilon,\ u_0\in Q(t-s),\ u(t)\in U(t,t-s)u_0,$$

provided that k and s are large enough. Thus the conclusion follows immediately. □

LEMMA 5.4. Let $t\in\mathbb{R}$ be any given. Let h, y, be two positive locally integrable functions on $(-\infty,t]$ such that y' is locally integrable on $(-\infty,t]$, and which satisfy

$$\frac{dy}{ds}\leqslant ry+h \quad \text{for } s\leqslant t, \tag{5.20}$$

$$\int_{-\infty}^{t}h(s)ds\leqslant a_1,\ \int_{-\infty}^{t}y(s)ds\leqslant a_2, \tag{5.21}$$

where r, a_1, a_2, are positive constants. Then

$$y(t)\leqslant \exp(r)(a_1+a_2).$$

Proof. By (5.20)–(5.21) and the Gronwall inequality, we know that

$$y(t)\leqslant\left(y(s')+\int_{s'}^{t}h(s)ds\right)\exp\int_{s'}^{t}rds\leqslant (y(s')+a_1)\exp(rt-rs').$$

Integrating this inequality with respect to s' between $t-1$ and t, we have

$$\begin{aligned}\int_{t-1}^{t}y(t)ds' &\leqslant \int_{t-1}^{t}y(s')\exp(rt-rs')ds'+a_1\int_{t-1}^{t}\exp(rt-rs')ds'\\ &\leqslant \exp(r)\left(\int_{t-1}^{t}y(s')ds'+a_1\right).\end{aligned}$$

In view of (5.21),

$$y(t)\leqslant\exp(r)\left(\int_{-\infty}^{t}y(s')ds'+a_1\right)\leqslant\exp(r)(a_1+a_2),$$

which completes the proof of this lemma. □

THEOREM 5.5. Suppose that the hypotheses in Lemma 5.2 hold. Then the MVP $\{U(t,\tau)\}$ generated by (5.1)–(5.2) possesses a nonempty local bounded kernel $\mathcal{K}_l$ in $L^2(\mathbb{R}^n)$; moreover, the kernel sections $\{\mathcal{K}_l(t)\}_{t\in\mathbb{R}}\in\mathfrak{B}$ are all compact, invariant in $L^2(\mathbb{R}^n)$ and pullback attract every family of bounded subsets $\mathcal{B}=\{B(t)\}_{t\in\mathbb{R}}\in\mathfrak{B}$.

Proof. Let $\Omega_k=\{x\in\mathbb{R}^n \mid |x|\leqslant k\}$, where k is sufficiently large such that Lemma 5.3 holds. Thanks to Lemmas 5.2 and 5.3, by Theorem 3.13, it suffices to show that for each $t\in\mathbb{R}$, $U(t,t-s)Q(t-s)$ is precompact in $L^2(\Omega_k)$ for s sufficiently large.

Let $t \in \mathbb{R}$ be fixed. As above, we define $U(T, t-s)u_0$ for all $T \geqslant t-s$ and $s \geqslant 0$ with $u_0 \in Q(t-s)$. Similar to the arguments in Lemma 5.2, we can show that

$$e^T \frac{d}{dT}|u(T)|_2^2 + 2\alpha_2 e^T \int_{\Omega_k} |u(T)|^p \leqslant e^T |g(T)|_2^2. \tag{5.22}$$

Integrating this inequality from $t-s$ to t, by (5.6) and (5.10), we obtain when s is sufficiently large,

$$\begin{aligned} 2\alpha_2 \int_{t-s}^t e^T \int_{\Omega_k} |u(T)|^p &\leqslant \int_{t-s}^t e^T |g(T)|_2^2 - \int_{t-s}^t e^T \frac{d}{dT}|u(T)|_2^2 \\ &\leqslant \frac{(R(t))^2 e^t}{2} + e^{t-s}|u(t-s)|_2^2 + \int_{-\infty}^t e^T |u(T)|_2^2 \\ &\leqslant 2\alpha_2 C(t). \end{aligned} \tag{5.23}$$

Similarly, we can also show that when s is sufficiently large,

$$\int_{t-s}^t e^T |\nabla u(T)|_2^2 \leqslant C(t). \tag{5.24}$$

Meanwhile, let $F(v) = \int_0^v f(\tau) d\tau$; then by (5.3), we can deduce that

$$\tilde{\alpha}_2 |v|^p - \tilde{k}_2 |v|^2 \leqslant F(v) \leqslant \tilde{\alpha}_1 |v|^p + \tilde{k}_1 |v|^2, \ \forall v \in \mathbb{R}. \tag{5.25}$$

Therefore, in view of (5.10), when s is sufficiently large,

$$\int_{t-s}^t e^T \int_{\Omega_k} F(u(T)) \leqslant \int_{t-s}^t e^T \left(\tilde{\alpha}_1 \int_{\Omega_k} |u(T)|^p + \tilde{k}_1 \int_{\Omega_k} |u(T)|^2 \right) \leqslant C(t). \tag{5.26}$$

By (5.10), (5.24) and (5.26), we obtain

$$\int_{-\infty}^t e^T \left(|\nabla u(T)|_2^2 + |u(T)|_2^2 + 2\int_{\Omega_k} F(u(T)) \right) \leqslant C(t). \tag{5.27}$$

On the other hand, multiplying (5.1) by $u_T(T)$, we get

$$|u_T(T)|_2^2 + \frac{1}{2}\frac{d}{dT}(|\nabla u(T)|_2^2 + \lambda |u(T)|_2^2) + \frac{d}{dT}\int_{\Omega_k} F(u(T)) = (g(T), u_T(T)). \tag{5.28}$$

By the Hölder inequality and the Cauchy inequality, it follows from (5.28) that

$$\frac{d}{dT}\left(|\nabla u(T)|_2^2 + \lambda |u(T)|_2^2 + 2\int_{\Omega_k} F(u(T)) \right) \leqslant |g(T)|_2^2.$$

Thus,

$$\begin{aligned} &\frac{d}{dT} e^T \left(|\nabla u(T)|_2^2 + \lambda |u(T)|_2^2 + 2\int_{\Omega_k} F(u(T)) \right) \\ \leqslant &e^T \left(|\nabla u(T)|_2^2 + \lambda |u(T)|_2^2 + 2\int_{\Omega_k} F(u(T)) \right) + e^T |g(T)|_2^2. \end{aligned} \tag{5.29}$$

Due to Lemma 5.4, we can deduce from (5.10), (5.27) and (5.29) that when s is sufficiently large,

$$|\nabla u(t)|_2^2 + \lambda |u(t)|_2^2 + 2\int_{\Omega_k} F(u(t)) \leqslant C(t).$$

Note that $u(t) \in Q(t)$ and $H^1(\Omega_k)$ is compactly embedded in $L^2(\Omega_k)$, in view of (5.25); hence $U(t, t-s)Q(t-s)$ is precompact in $L^2(\Omega_k)$ for s sufficiently large.

We have finished the proof of Theorem 5.5. □

5.2. *Uniform attractors and uniform forward attraction of the inflated kernel sections.* Let $L_b^2(\mathbb{R}; L^2(\mathbb{R}^n))$ be a space of functions g from $L_{loc}^2(\mathbb{R}; L^2(\mathbb{R}^n))$ such that

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \mid g(s) \mid_2^2 ds < \infty.$$

Evidently, $L_b^2(\mathbb{R}; L^2(\mathbb{R}^n))$ is a Banach space endowed with the norm

$$\|g\|_{L_b^2(\mathbb{R};L^2(\mathbb{R}^n))} = \left(\sup_{t\in\mathbb{R}} \int_t^{t+1} \mid g(s) \mid_2^2 ds\right)^{1/2}.$$

Fully analogous to Lemma 5.1, we have

LEMMA 5.6. Assume that for $g \in L_b^2(\mathbb{R}; L^2(\mathbb{R}^n))$, (5.3) holds true. Then for any initial data $u_0 \in L^2(\mathbb{R}^n)$ and any $\tau \in \mathbb{R}$, there exists a weak solution u for system (5.1)–(5.2) which satisfies

$$u \in L^2(\tau, T; H^1(\mathbb{R}^n)) \cap L^p(\tau, T; L^p(\mathbb{R}^n)) \cap L^\infty(\tau, T; L^2(\mathbb{R}^n)),\ \forall T > \tau.$$

LEMMA 5.7 (Chepyzhov and Vishik [13]). Let $y(t)$ be uniformly continuous on $[t_0, +\infty)$ and satisfy

$$y'(t) + \gamma y(t) \leqslant h(t),$$

where $\gamma \geqslant 0$ and $h(t) \geqslant 0$ for all $t \geqslant t_0$. Suppose also that

$$\int_t^{t+1} h(s)ds \leqslant L,\ \forall t \geqslant t_0.$$

Then

$$y(t) \leqslant y(t_0)e^{-\gamma(t-t_0)} + L(1 - e^{-\gamma})^{-1} \leqslant y(t_0)e^{-\gamma(t-t_0)} + L(1 + \gamma^{-1}),\ \forall t \geqslant t_0.$$

LEMMA 5.8 (The Uniform Gronwall Lemma [25]). Let g, h, y, be three positive locally integrable functions on $]t_0, +\infty[$ such that y' is locally integrable on $]t_0, +\infty[$, and which satisfy

$$\frac{dy}{dt} \leqslant gy + h \quad \text{for } t \geqslant t_0,$$
$$\int_t^{t+r} g(s)ds \leqslant a_1,\ \int_t^{t+r} h(s)ds \leqslant a_2,\ \int_t^{t+r} y(s)ds \leqslant a_3, \quad \text{for } t \geqslant t_0,$$

where r, a_1, a_2, a_3 are positive constants. Then

$$y(t + r) \leqslant \left(\frac{a_3}{r} + a_2\right) e^{a_1},\ \forall t \geqslant t_0.$$

LEMMA 5.9. We assume that for $g \in L_b^2(\mathbb{R}; L^2(\mathbb{R}^n))$, (5.3) and (5.5) hold true. Let u be a solution of (5.1)–(5.2). Then for any $t \geqslant \tau$,

(1) $|u(t)|_2^2 \leqslant |u(\tau)|_2^2 e^{-(t-\tau)} + 2\|g\|_{L_b^2}^2$;

(2) $|\nabla u(t+1)|_2^2 + \lambda |u(t+1)|_2^2 + 2\int_{\mathbb{R}^n} F(u(t+1)) \leqslant C|u(\tau)|_2^2 + C\|g\|_{L_b^2}^2$.

Proof. Multiplying (5.1) by u, in view of Young's inequality and Lemma 5.7, we obtain

$$|u(t)|_2^2 \leqslant |u(\tau)|_2^2 e^{-(t-\tau)} + 2\|g\|_{L_b^2}^2, \ \forall t \geqslant \tau. \tag{5.30}$$

Multiplying (5.1) by u, by (5.5) and Young's inequality,

$$\frac{d}{dt}|u(t)|_2^2 + 2|\nabla u(t)|_2^2 + |u(t)|_2^2 + 2\alpha_2 \int_{\mathbb{R}^n} |u(t)|^p \leqslant |g(t)|_2^2. \tag{5.31}$$

Integrating (5.31) from s to $s+1$ and by (5.30), we have

$$\begin{aligned} &2\int_s^{s+1} |\nabla u(t)|_2^2 + \int_s^{s+1} |u(t)|_2^2 + 2\alpha_2 \int_s^{s+1} \int_{\mathbb{R}^n} |u(t)|^p \\ \leqslant &\|g\|_{L_b^2}^2 + |u(s)|_2^2 \leqslant |u(\tau)|_2^2 + 3\|g\|_{L_b^2}^2, \ \forall s \geqslant \tau. \end{aligned} \tag{5.32}$$

On the other hand, multiplying (5.1) by u_t, we obtain

$$\frac{d}{dt}\left(|\nabla u(t)|_2^2 + \lambda |u(t)|_2^2 + 2\int_{\mathbb{R}^n} F(u(t)) \right) \leqslant |g(t)|_2^2, \tag{5.33}$$

and it follows from (5.25) and (5.32) that

$$\begin{aligned} &\int_s^{s+1} \left(|\nabla u(t)|_2^2 + \lambda |u(t)|_2^2 + 2\int_{\mathbb{R}^n} F(u(t)) \right) \\ \leqslant &\int_s^{s+1} \left(|\nabla u(t)|_2^2 + (\lambda + 2\tilde{k}_1)|u(t)|_2^2 + 2\tilde{\alpha}_1 \int_{\mathbb{R}^n} |u(t)|^p \right) \\ \leqslant &\, C|u(\tau)|_2^2 + C||g||_{L_b^2}^2, \ \forall s \geqslant \tau. \end{aligned} \tag{5.34}$$

Invoking Lemma 5.8, we have

$$|\nabla u(t+1)|_2^2 + \lambda |u(t+1)|_2^2 + 2\int_{\mathbb{R}^n} F(u(t+1)) \leqslant C|u(\tau)|_2^2 + C\|g\|_{L_b^2}^2, \ \forall t \geqslant \tau. \tag{5.35}$$

We have completed the proof of this lemma. □

Suppose that a function $g_0(s)$ is translation compact in $L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^n))$. In particular, g_0 is translation bounded in $L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^n))$, i.e., $g_0 \in L^2_b(\mathbb{R}; L^2(\mathbb{R}^n))$. Let $\mathcal{H}(g_0)$ be the closure of the $\{g_0(\tau + \cdot); \tau \in \mathbb{R}\}$ in $L^2_b(\mathbb{R}; L^2(\mathbb{R}^n))$. The hull $\mathcal{H}(g_0)$ is a compact set in $L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^n))$.

Lemma 5.10. For all $g(x,s) \in \mathcal{H}(g_0)$, we have

$$\|g\|_{L_b^2}^2 = \sup_{t\in\mathbb{R}} \int_t^{t+1} |g|_2^2 ds \leqslant \|g_0\|_{L_b^2}^2.$$

Consider the family of systems:

$$\frac{\partial u}{\partial t} - \triangle u + \lambda u + f(u) = g(x,t), \ g(x,t) \in \mathcal{H}(g_0), \tag{5.36}$$

$$u|_{t=\tau} = u_0, \ \tau \in \mathbb{R}. \tag{5.37}$$

As above, in view of Lemma 5.10, we can see that Lemmas 5.6 and 5.9 hold for any $g(x,t) \in \mathcal{H}(g_0)$. Thus we can define a family of multi-valued processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$, on $L^2(\mathbb{R}^n)$ corresponding to system (5.36)–(5.37), and $\{U_g(t,\tau)\}$ is jointly norm-to-weak upper-semicontinuous in $L^p(\mathbb{R}^n)$; i.e., for any fixed $t \geqslant \tau$, $\tau \in \mathbb{R}$, if $u_{n0} \to u_0$ in

$L^2(\mathbb{R}^n)$ and $g_n \to g$ in $L^2_b(\mathbb{R}; L^2(\mathbb{R}^n))$, then for any $u_n(t) \in U_{g_n}(t,\tau)u_{n0}$, there exists a $u(t) \in U_g(t,\tau)u_0$ such that $u_n(t) \rightharpoonup u(t)$ in $L^p(\mathbb{R}^n)$.

LEMMA 5.11 ([24, 30]). Let $B \subset L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ $(p \geqslant 2)$ be bounded in both $L^2(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$. Then for any $\varepsilon > 0$, B has a finite ε-net in $L^p(\mathbb{R}^n)$ if there exists a positive constant $M = M(\varepsilon)$ which depends on ε, such that

(1) B has a finite $(3M)^{(2-p)/2}(\varepsilon/2)^{p/2}$-net in $L^2(\mathbb{R}^n)$;

(2) $\left(\int_{\mathbb{R}^n(|u|\geqslant M)} |u|^p\right)^{1/p} < 2^{-(2p+2)/p}\varepsilon$ for any $u \in B$, where $\mathbb{R}^n(|u| \geqslant M) = \{x \in \mathbb{R}^n \mid |u(x)| \geqslant M\}$.

From the definition of uniform ω-limit compactness of the multi-valued process and the lemma above, we have

LEMMA 5.12. Let $\{U_g(t,\tau)\}, g \in \mathcal{H}(g_0)$, be a family of MVPs on $L^p(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$, respectively, where $2 \leqslant p < \infty$. Then $\{U_g(t,\tau)\}, g \in \mathcal{H}(g_0)$, is uniformly ω-limit compact in $L^p(\mathbb{R}^n)$ provided that the following conditions hold:

(1) $\{U_g(t,\tau)\}, g \in \mathcal{H}(g_0)$, is uniformly ω-limit compact in $L^2(\mathbb{R}^n)$;

(2) for any fixed $\tau \in \mathbb{R}$, any $\varepsilon > 0$ and every bounded subset $B \subset L^2(\mathbb{R}^n)$, there exist positive constants $M = M(B,\varepsilon)$ and $T_5 = T_5(B,\varepsilon)$ which are all independent of $g \in \mathcal{H}(g_0)$, such that

$$\int_{\mathbb{R}^n(|u|\geqslant M)} |u|^p < \varepsilon,\ \forall u_0 \in B,\ t \geqslant T_5,\ g \in \mathcal{H}(g_0),\ u \in U_g(t,\tau)u_0.$$

LEMMA 5.13 ([24, 30]). Let B be a bounded subset in $L^q(\mathbb{R}^n)$ $(q \geqslant 1)$. If B has a finite ε-net in $L^q(\mathbb{R}^n)$, then there exists an $M = M(B,\varepsilon)$, such that for any $u \in B$, the following estimate is valid:

$$\int_{\mathbb{R}^n(|u|\geqslant M)} |u|^q \leqslant 2^{q+1}\varepsilon^q.$$

Now let us recall the concept of the multi-valued skew product flow, which will be used in the proof of Theorem 5.14. The mapping $F_\tau(t) : \mathbb{R}^+ \times Y \to Y$ defined by

$$F_\tau(t)(u_0, g) = (U_g(t+\tau,\tau)u_0, g(\cdot + t))$$

forms an autonomous multi-valued semidynamical system on $Y = L^2(\mathbb{R}^n) \times \mathcal{H}(g_0)$ over $\mathbb{R}^+$, which is called the multi-valued skew product flow associated with the family of multi-valued processes $\{U_g(t,\tau)\}, g \in \mathcal{H}(g_0)$.

THEOREM 5.14. Let $g_0(x,s)$ be translation compact in $L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^n))$ and let (5.3) and (5.5) hold true. Then the multi-valued semidynamical system $\{F_\tau(t)\}$ corresponding to the family of multi-valued processes $\{U_g(t,\tau)\}, g \in \mathcal{H}(g_0)$, and acting on $L^2(\mathbb{R}^n) \times \mathcal{H}(g_0)$ possesses a unique compact attractor $\mathcal{A}$ in $L^p(\mathbb{R}^n) \times \mathcal{H}(g_0)$ which is strictly invariant with respect to $\{F_\tau(t)\} : F_\tau(t)\mathcal{A} = \mathcal{A}$ for all $t \geqslant 0$. Furthermore,

(1) $\Pi_1\mathcal{A} = \mu$ is the uniform (w.r.t. $g \in \mathcal{H}(g_0)$) attractor of the family of multi-valued processes $\{U_g(t,\tau)\}, g \in \mathcal{H}(g_0)$, in $L^p(\mathbb{R}^n)$;

(2) $\Pi_2\mathcal{A} = \mathcal{H}(g_0)$;

(3) the global attractor satisfies

$$\mathcal{A} = \bigcup_{g \in \mathcal{H}(g_0)} \mathcal{K}_g(\tau) \times \{g\};$$

(4) the uniform attractor satisfies

$$\Pi_1 \mathcal{A} = \mu = \bigcup_{g \in \mathcal{H}(g_0)} \mathcal{K}_g(\tau);$$

(5) for any fixed $\varepsilon_0 > 0$, the family of inflated kernel sections $\{\mathcal{K}_g^{[\varepsilon_0]}(\tau)\}$, $g \in \mathcal{H}(g_0)$, uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) pullback (respectively forward) attracts each bounded subset B of $L^2(\mathbb{R}^n)$ in the topology of $L^p(\mathbb{R}^n)$.
Here the kernel $\mathcal{K}_g$ of the MVP $\{U_g(t,\tau)\}$ with $g \in \mathcal{H}(g_0)$ is given as in Remark 3.10, and $\mathcal{K}_g(\tau)$ is the section at $t = \tau$ of the kernel $\mathcal{K}_g$.

Proof. Thanks to Theorems 4.7 and 4.9, we only need to show that the family of MVPs $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$, corresponding to system (5.36)–(5.37) is uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) dissipative and uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) ω-limit compact.

It follows from Lemmas 5.9 and 5.10 that the family of MVPs $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$, has a uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) absorbing set $\tilde{\mathcal{B}}$ in $L^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$,

$$\tilde{\mathcal{B}} = \{u \in L^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^p(\mathbb{R}^n) \mid |u|_2^2 + |u|_p^p + |\nabla u|_2^2 \leqslant \rho_1^p\};$$

i.e., for any fixed $\tau \in \mathbb{R}$ and every bounded set $B \subset L^2(\mathbb{R}^n)$, there is a $\tilde{\tau}_1 > 0$ independent of $g \in \mathcal{H}(g_0)$, such that

$$U_g(t+\tau,\tau)B \subset \tilde{\mathcal{B}},\ \forall g \in \mathcal{H}(g_0),\ t \geqslant \tilde{\tau}_1. \tag{5.38}$$

Let us show that $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$, is uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) ω-limit compact in $L^2(\mathbb{R}^n)$.

Observe that for any $\varepsilon' > 0$ and $g \in \mathcal{H}(g_0)$, there exists a sequence $g_0(\cdot, \cdot + h_n)$ with $h_n \in \mathbb{R}$, such that for n sufficiently large,

$$\|g(s) - g_0(s+h_n)\|_{L_2^b}^2 = \left(\sup_{t \in \mathbb{R}} \int_t^{t+1} \mid g(s) - g_0(s+h_n) \mid_2^2\right) < \varepsilon',\ \forall s \in \mathbb{R}. \tag{5.39}$$

Since g_0 is translation compact in $L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^n))$, invoking Theorem 4.2 in [20], for any $\varepsilon > 0$, there exists $\delta_5 > 0$, such that if $e_1 \subset \mathbb{R}^n$ and $m(e_1) \leqslant \delta_5$, then

$$\int_t^{t+1} \int_{e_1} |g_0(x,s)|^2 < \frac{\varepsilon}{2},\ \forall t \in \mathbb{R}, \tag{5.40}$$

where $m(e_1)$ denotes the Lebesgue measure of $e_1 \subset \mathbb{R}^n$. By (5.39) and (5.40), we can conclude that

$$\begin{aligned}\sup_{t\in\mathbb{R}}\int_t^{t+1}\int_{e_1}|g(x,s)|^2 &\leqslant 2\sup_{t\in\mathbb{R}}\int_t^{t+1}\int_{e_1}|g(x,s)-g_0(x,s+h_n)|^2\\ &\quad +2\sup_{t\in\mathbb{R}}\int_t^{t+1}\int_{e_1}|g_0(x,s+h_n)|^2\\ &\leqslant 2\sup_{t\in\mathbb{R}}\int_t^{t+1}\int_{\mathbb{R}^n}|g(x,s)-g_0(x,s+h_n)|^2\\ &\quad +2\sup_{t\in\mathbb{R}}\int_{t+h_n}^{t+h_n+1}\int_{e_1}|g_0(x,T)|^2\\ &\leqslant 2\varepsilon'+2\sup_{t\in\mathbb{R}}\int_t^{t+1}\int_{e_1}|g_0(x,T)|^2\\ &\leqslant 2\varepsilon'+\varepsilon.\end{aligned}$$

Let $\varepsilon' \to 0$. Thus,

$$\sup_{t\in\mathbb{R}}\int_t^{t+1}\int_{e_1}|g(x,s)|^2 \leqslant \varepsilon. \tag{5.41}$$

Let $\Omega_{k_1}^C = \{x \in \mathbb{R}^n \mid |x| \geqslant k_1\}$. We can choose k_1 sufficiently large, such that

$$\sup_{t\in\mathbb{R}}\int_t^{t+1}\int_{\Omega_{k_1}^C}|g(x,s)|^2 \leqslant \varepsilon. \tag{5.42}$$

Observe that for any $g \in \mathcal{H}(g_0)$ and any $s \geqslant \tau$,

$$U_g(s,\tau)u_0 = \{u(s) \mid u(\cdot) \text{ is a weak solution of system (5.36)–(5.37)}\}.$$

Similar to the proof in Lemma 5.3, we have

$$\frac{d}{ds}\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k_1^2}\right)|u|^2+\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k_1^2}\right)|u|^2 \leqslant \int_{|x|\geqslant k_1}|g(s)|^2+\frac{C}{2k_1}\int_{\mathbb{R}^n}|u|^2, \tag{5.43}$$

where the function θ is defined as in Lemma 5.3. By (5.38), we can take k_1 and s sufficiently large, such that

$$\frac{C}{2k_1}\int_{\mathbb{R}^n}|u(s)|^2 \leqslant \varepsilon.$$

In view of (5.42), we obtain that when t and k_1 are sufficiently large,

$$\int_t^{t+1}\int_{\Omega_{k_1}^C}|g(x,s)|^2+\frac{C}{2k_1}\int_t^{t+1}\int_{\mathbb{R}^n}|u(s)|^2 \leqslant 2\varepsilon. \tag{5.44}$$

Due to Lemma 5.7 and (5.38), we can deduce that when k_1 and s are sufficiently large,

$$\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k_1^2}\right)|u(s)|^2 \leqslant e^{-(s-\tau)}\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k_1^2}\right)|u(\tau)|^2+4\varepsilon \leqslant 5\varepsilon,$$

which implies that

$$\int_{\Omega_{2k_1}^C}|u(s)|^2 \leqslant \int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k_1^2}\right)|u(s)|^2 \leqslant 5\varepsilon. \tag{5.45}$$

On the other hand, note that $H^1(\Omega_{2k_1})$ is compactly embedded in $L^2(\Omega_{2k_1})$. Thus we can see from (5.38) and (5.45) that $\{U_g(t,\tau)\}$, $g\in\mathcal{H}(g_0)$, is uniformly (w.r.t. $g\in\mathcal{H}(g_0)$) ω-limit compact in $L^2(\mathbb{R}^n)$.

Taking into account Lemma 5.12, it only remains to show that for any fixed $\tau\in\mathbb{R}$, any $\varepsilon>0$ and any bounded subset $B\subset L^2(\mathbb{R}^n)$, there exist two positive constants $M_3=M_3(B,\varepsilon)$ and $\tau_5=\tau_5(B,\varepsilon)$ which are all independent of $g\in\mathcal{H}(g_0)$, such that

$$\int_{\mathbb{R}^n(|u(s)|\geqslant M_3)}|u(s)|^p<\varepsilon, \tag{5.46}$$

for any $g\in\mathcal{H}(g_0)$, any $s\geqslant\tau_5$ and every $u(s)\in U_g(s+\tau,\tau)B$.

Let $\tau\in\mathbb{R}$ and let $B\subset L^2(\mathbb{R}^n)$ be any given bounded set. It follows from (5.38) that

$$|\ u\ |_p^p\leqslant\rho_1^p,\ \forall g\in\mathcal{H}(g_0),\ t\geqslant\tilde\tau_1,\ u\in U_g(t+\tau,\tau)B.$$

Therefore,

$$\rho_1^p\geqslant\int_{\mathbb{R}^n}|u|^p\geqslant\int_{\mathbb{R}^n(|u|\geqslant M_1)}|u|^p\geqslant\int_{\mathbb{R}^n(|u|\geqslant M_1)}M_1^p=M_1^p m(\mathbb{R}^n(|u|\geqslant M_1)),$$

where $\mathbb{R}^n(|u|\geqslant M_1)=\{x\in\mathbb{R}^n\mid|u(x)|\geqslant M_1\}$. By Lemma 5.13 and the uniform ω-limit compactness of the family of MVPs $\{U_g(t,\tau)\}$, $g\in\mathcal{H}(g_0)$, there exist $\tau_5'>0$ and $M_1>0$ large enough, such that

$$m(\mathbb{R}^n(|u|\geqslant M_1))\leqslant\min\{\varepsilon,\delta_5\},\ \forall g\in\mathcal{H}(g_0),\ t\geqslant\tau_5',\ u\in U_g(t+\tau,\tau)B \tag{5.47}$$

and

$$\int_{\mathbb{R}^n(|u|\geqslant M_1)}|u|^2\leqslant 8\varepsilon,\ \forall g\in\mathcal{H}(g_0),\ t\geqslant\tau_5',\ u\in U_g(t+\tau,\tau)B. \tag{5.48}$$

In addition, it follows from (5.3) that $f(v)\geqslant 0$ provided that $v\geqslant(k_2/\alpha_2)^{1/(p-2)}$. Let $M_2=\max\{M_1,(k_2/\alpha_2)^{1/(p-2)}\}$. We consider $u(s)\in U_g(s,\tau)u_0$ with $g\in\mathcal{H}(g_0)$, $s\geqslant\tau_5:=\tilde\tau_1+\tau_5'+\tau$ and $u(\tau)=u_0\in B$. Multiplying (5.36) by $(u-M_2)_+$ and integrating over $\mathbb{R}^n$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{ds}|(u-M_2)_+|_2^2+\int_{\mathbb{R}^n(u\geqslant M_2)}|\nabla(u-M_2)_+|^2\\&+\lambda\int_{\mathbb{R}^n}u\cdot(u-M_2)_++\int_{\mathbb{R}^n}f(u)(u-M_2)_+=\int_{\mathbb{R}^n}g(x,s)(u-M_2)_+,\end{aligned} \tag{5.49}$$

where $(u-M_2)_+$ denotes the positive part of $u-M_2$, that is,

$$(u-M_2)_+=\begin{cases}u-M_2, & u\geqslant M_2,\\ 0, & u\leqslant M_2.\end{cases}$$

Let $\mathbb{R}_1^n=\mathbb{R}^n(u(s)\geqslant M_2)$. Then we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{ds}|(u-M_2)_+|_2^2+\int_{\mathbb{R}_1^n}|\nabla u|^2+\lambda\int_{\mathbb{R}^n}u\cdot(u-M_2)_+\\&+\int_{\mathbb{R}_1^n}f(u)(u-M_2)=\int_{\mathbb{R}_1^n}g(x,s)(u-M_2).\end{aligned}$$

By the Cauchy inequality and the Hölder inequality, we know that

$$\begin{aligned}\frac{d}{ds}|(u-M_2)_+|_2^2 + C\left(\int_{\mathbb{R}_1^n}|\nabla u|^2 + \int_{\mathbb{R}_1^n} u\cdot(u-M_2)_+\right.\\ \left.+\int_{\mathbb{R}_1^n} f(u)(u-M_2)\right) \leqslant C\left(\int_{\mathbb{R}_1^n}|g(x,s)|^2\right).\end{aligned} \tag{5.50}$$

Combining (5.41), (5.48) and (5.50) together and integrating from t to $t+1$, we obtain

$$\int_t^{t+1}\left(\int_{\mathbb{R}_1^n}|\nabla u|^2 + \int_{\mathbb{R}_1^n} u\cdot(u-M_2)_+ + \int_{\mathbb{R}_1^n} f(u)(u-M_2)\right) \leqslant C\varepsilon,\ \forall t\geqslant \tau_5.$$

Let $M_3 = 2M_2$. Hence,

$$\begin{aligned}&\int_t^{t+1}\left(\int_{\mathbb{R}^n(u\geqslant M_3)}|\nabla u|^2 + \int_{\mathbb{R}^n(u\geqslant M_3)}|u|^2 + \int_{\mathbb{R}^n(u\geqslant M_3)} f(u)u\right)\\ &\leqslant C\varepsilon,\ \forall t\geqslant \tau_5.\end{aligned} \tag{5.51}$$

On the other hand, we multiply (5.36) by $(u-M_3)_{+s}$ and denote $\mathbb{R}^n(u\geqslant M_3)$ by $\mathbb{R}_2^n$. Then we get

$$\frac{d}{ds}\left(\int_{\mathbb{R}_2^n}|\nabla u|^2 + \int_{\mathbb{R}_2^n}|u|^2 + \int_{\mathbb{R}_2^n} F(u)\right) \leqslant C\int_{\mathbb{R}_2^n}|g(x,s)|^2, \tag{5.52}$$

in the same fashion as in proving (5.33).

From (5.41), (5.51) and (5.52), using the uniform Gronwall lemma, we obtain that

$$\int_{\mathbb{R}_2^n}|\nabla u|^2 + \int_{\mathbb{R}_2^n}|u|^2 + \int_{\mathbb{R}_2^n} F(u) \leqslant C\varepsilon.$$

So, we have

$$\int_{\mathbb{R}_2^n}|\nabla u|^2 \leqslant C\varepsilon \tag{5.53}$$

and

$$\int_{\mathbb{R}_2^n} F(u) \leqslant C\varepsilon. \tag{5.54}$$

Repeating the same steps above, just taking $(u+M_3)_-$ and $(u+M_3)_{-s}$ instead of $(u-M_3)_+$ and $(u-M_3)_{+s}$, respectively, we deduce that

$$\int_{\mathbb{R}^n(u\leqslant -M_3)}|\nabla u|^2 \leqslant C\varepsilon \tag{5.55}$$

and

$$\int_{\mathbb{R}^n(u\leqslant -M_3)} F(u) \leqslant C\varepsilon. \tag{5.56}$$

Then (5.53)–(5.56) mean that

$$\int_{\mathbb{R}^n(|u(s)|\geqslant M_3)}|\nabla u(s)|^2 \leqslant C\varepsilon \tag{5.57}$$

and

$$\int_{\mathbb{R}^n(|u(s)|\geqslant M_3)} F(u(s)) \leqslant C\varepsilon. \tag{5.58}$$

Thus, thanks to (5.25) and (5.48), (5.46) follows from (5.58) and the proof of Theorem 5.14 is completed. □

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References

[1] J. P. Aubin and A. Cellina, *Differential Inclusions.* Springer, Berlin, 1984. MR755330 (85j:49010)

[2] J. P. Aubin and H. Frankowska, *Set-Valued Analysis.* Birkhäuser, 1990. MR1048347 (91d:49001)

[3] J. M. Ball, *Global attractors for damped semilinear wave equations,* Discrete Continuous Dynamical Systems, **10** (2004), 31-52. MR2026182 (2005a:37149)

[4] T. Caraballo and J. A. Langa, *Global attractors for multivalued random dynamical systems generated by random differential inclusions with multiplicative noise,* J. Math. Anal. Appl., **260** (2001), 602-622. MR1845571 (2002d:34095)

[5] T. Caraballo, J. A. Langa, V. S. Melnik and J. Valero, *Pullback attractors of nonautonomous and stochastic multivalued dynamical systems,* Set-Valued Analysis, **11** (2003), 153-201. MR1966698 (2004b:37101)

[6] T. Caraballo, P. Marin-Rubio and J. C. Robinson, *A comparison between two theories for multi-valued semiflows and their asymptotic behavior,* Set-Valued Analysis, **11** (2003), 297-322. MR1992066 (2004d:37132)

[7] T. Caraballo, P. Marin-Rubio and J. Valero, *Autonomous and non-autonomous attractors for differential equations with delays,* J. Differential Equations, **208** (2005), no. 1, 9-41. MR2107292 (2005i:37095)

[8] T. Caraballo, G. Lukaszewicz and J. Real, *Pullback attractors for asymptotically compact non-autonomous dynamical systems,* Nonlinear Analysis TMA, **64** (2006), 484-498. MR2191992 (2006h:37028)

[9] D. N. Cheban, P. E. Kloeden and B. Schmalfuss, *The relationship between pullback, forward and global attractors of nonautonomous dynamical systems,* Nonlinear Dynamical Systems Theory, **2** (2002), 9-28. MR1989935 (2004e:34090)

[10] V. V. Chepyzhov and M. I. Vishik, *A Hausdorff dimension estimate for kernel sections of nonautonomous evolution equations,* Indiana Univ. Math. J., **42** (1993), no. 3, 1057-1076. MR1254132 (94m:34157)

[11] Vladimir V. Chepyzhov and M. I. Vishik, *Dimension estimates for attractors and for kernel sections of nonautonomous evolution equations,* C. R. Acad. Sci. Paris Sér. I Math., **317** (1993), 365-370. MR1235450 (94h:35116)

[12] V. V. Chepyzhov and M. I. Vishik, *Attractors of nonautonomous dynamical systems and their dimension,* J. Math. Pures Appl., **73** (1994), 279-333. MR1273705 (95c:34106)

[13] V. V. Chepyzhov and M. I. Vishik, *Attractors for equations of mathematical physics.* American Mathematical Society, Providence, RI, 2002. MR1868930 (2003f:37001c)

[14] K. Deimling, *Nonlinear Functional Analysis.* Springer-Verlag, 1985. MR787404 (86j:47001)

[15] K. Deimling, *Multivalued Differential Equations.* de Gruyter, Berlin, 1992. MR1189795 (94b:34026)

[16] X. M. Fan and S. F. Zhou, *Kernel sections for non-autonomous strongly damped wave equations of non-degenerate Kirchhoff-type,* Applied Mathematics and Computation, **158** (2004), 253-266. MR2091247

[17] J. K. Hale, *Asymptotic Behavior of Dissipative Systems.* American Mathematical Society, Providence, 1988. MR941371 (89g:58059)

[18] P. E. Kloeden and J. A. Langa, *Flattening, squeezing and the existence of random attractors,* Proceedings of the Royal Society of London, **463** (2007), 163-181. MR2281716 (2008a:37056)

[19] D. S. Li, *On dynamical stability in general dynamical systems,* J. Math. Anal. Appl., **263** (2001), 455-478. MR1866058 (2002i:37019)

[20] S. S. Lu, H. Q. Wu and C. K. Zhong, *Attractors for nonautonomous 2D Navier-Stokes equations with normal external forces,* Discrete Continuous Dynamical Systems, **13** (2005), 701-719. MR2153139 (2006c:37091)

[21] Q. F. Ma, S. H. Wang and C. K. Zhong, *Necessary and sufficient conditions for the existence of global attractors for semigroups and applications,* Indiana Univ. Math. J., **51** (2002), 1541-1559. MR1948459 (2003j:37137)

[22] V. S. Melnik and J. Valero, *On attractors of multivalued semi-flows and differential inclusions,* Set-Valued Analysis, **6** (1998), 83-111. MR1631081 (99e:34087)

[23] G. R. Sell and Y. C. You, *Dynamics of Evolutionary Equations.* Springer-Verlag, New York, 2002. MR1873467 (2003f:37001b)

[24] C. Y. Sun and C. K. Zhong, *Attractors for the semilinear reaction-difussion equation with distribution derivatives in unbounded domains,* Nonlinear Analysis TMA, **63** (2005), 49-65. MR2167314 (2006g:35152)

[25] R. Temam, *Infinite Dimensional Dynamical System in Mechanics and Physics.* 2nd Edition, Springer-Verlag, New York, 1997. MR1441312 (98b:58056)

[26] M. I. Vishik and V. V. Chepyzhov, *Trajectory and global attractors of the three-dimensional Navier-Stokes systems,* Mathematical Notes, **71** (2002), 177-193. Translated from Matematicheskie Zametki, **71** (2002), 194-213. MR1900793 (2003a:37114)

[27] Y. J. Wang, D. S. Li and P. E. Kloeden, *On the asymptotical behavior of nonautonomous dynamical systems,* Nonlinear Analysis TMA, **59** (2004), 35-53. MR2092077 (2005j:34079)

[28] Y. J. Wang, C. K. Zhong and S. F. Zhou, *Pullback attractors of nonautonomous dynamical systems,* Discrete Continuous Dynamical Systems, **16** (2006), 587-614. MR2257151 (2007f:37131)

[29] Y. J. Wang and S. F. Zhou, *Kernel sections and uniform attractors of multi-valued semiprocesses,* J. Differential Equations, **232** (2007), 573-622. MR2286392 (2008b:37134)

[30] C. K. Zhong, M. H. Yang and C. Y. Sun, *The existence of global attractors for the norm-to-weak continuous semigroup and application to the nonlinear reaction-diffusion equations,* J. Differential Equations, **223** (2006), 367-399. MR2214940 (2006k:37221)

[31] S. F. Zhou, *Kernel sections for damped non-autonomous wave equations with linear memory and critical exponent,* Quart. Appl. Math., **61** (2003), 731-757. MR2019621 (2006b:37169)

[32] S. F. Zhou and L. S. Wang, *Kernel sections for damped non-autonomous wave equations with critical exponent,* Discrete Continuous Dynamical Systems, **9** (2003), 399-412. MR1952382 (2003k:35172)