QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXVIII, NUMBER 1
MARCH 2010, PAGES 161–167
S 0033-569X(09)01175-3
Article electronically published on November 9, 2009

SPACE-TIME RESONANCES

By

JALAL SHATAH

Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, New York 10012

The problem of global existence (or finite time blowup) of small solutions to nonlinear hyperbolic and dispersive equations has been studied extensively by many authors for over forty years. The earliest results depended on energy and decay estimates of the linear equations to obtain global bounds. This method of obtaining global estimates yielded many fundamental results on global existence; however, it failed to deal with many important cases where the linear decay is too weak to bound the nonlinearity. To tackle some of these problems two different techniques were introduced, namely, the vector fields method (i.e. weighted energy estimates) of S. Klainerman, and the normal form method of J. Shatah. The method of S. Klainerman proved to be very powerful in that it led to many results on global existence of small solutions based on weighted energy estimates. In particular weighted energy estimates have led to many interesting global existence results to nonlinear Schrödinger equations.

Our aim here is to introduce the concept of Space-Time resonance and use it explain and extend existing results on Schrödinger equations. This new technique bridges the gap between the vector field method and the normal forms method mentioned above. This work is part of an ongoing larger program of research conducted jointly by Pierre Germain and Nader Masmoudi at the Courant Institute.

Time resonances: Poincaré-Dulac normal forms. Time resonance is an ODE phenomenon. It explains how the behavior of solutions to a nonlinear equation that are close to a critical point differ from the behavior of the associated linear system. For example, solutions to

$$\begin{cases} \partial_t u = 2iu + v^2, \\ \partial_t v = iv \end{cases}$$

are given by $u = u_0 e^{2it} + v_0^2 t e^{2it}$. These solutions grow in time and behave very differently from solutions of the linear equation which are periodic in time. To guess which nonlinear equations exhibit the same behavior as the associated linear equation one tries to reduce the nonlinear equation to a linear one by using the Poincaré-Dulac normal forms method, which will be briefly explained below.

Received December 31, 2008.
2000 *Mathematics Subject Classification.* Primary 35B34; Secondary 35Q35.
This research is funded in part by NSF DMS 0701272.
E-mail address: shatah@cims.nyu.edu

Given an analytic ODE

$$\dot{u} = Au + f(u) = Au + M_2(u,u) + \dots, \quad u \in \mathbb{R}^d,$$

find an analytic transformation

$$v = G(u) = u + H_2(u) + H_3(u) + \dots,$$

where H_k is the k-multilinear map of u that transforms solutions of the above equation to solutions of the linear system

$$\dot{v} = Av.$$

To see if such a transformation exists, plug $v = G(u) = u + H_2 + \dots$ into the equation to get

$$G'(u)(Au + f(u)) = AG(u),$$

and expand in a power series to find the quadratic term of G,

$$H_2(Au,u) + H_2(u,Au) + M_2(u,u) = AH_2(u,u),$$

which can be solved provided $\lambda_{k_1} + \lambda_{k_2} \neq \lambda_k$. Repeating this process for the higher order terms of G leads to the following condition on the eigenvalues λ_k of the matrix A:

$$\lambda_{k_1} + \lambda_{k_2} + \dots + \lambda_{k_\ell} \neq \lambda_k.$$

Our way of computing $G(u)$ or more precisely $G^{-1}(v)$ is to look at the integral equation

$$u = e^{At}u_0 + e^{At}\int_0^t e^{-As} f(u(s))ds$$

and write it in terms of the profile of u, i.e., $w = e^{-At}u$,

$$w = u_0 + \int_0^t e^{-As} f(e^{As}w(s))ds = u_0 + \int_0^t e^{-As} M_2(e^{As}w, e^{As}w) + \dots ds.$$

Each i-multilinear term consists of matrices whose eigenvalues are given by

$$e^{(\lambda_{k_1} + \lambda_{k_2} + \dots + \lambda_{k_i} - \lambda_k)s}.$$

Since $\dot{w} = e^{-At}M_2(e^{At}w, e^{At}w) + \dots$, which is at least quadratic in w, then integrating by parts on w in

$$\int_0^t e^{-As} M_2(e^{As}w, e^{As}w) + \dots ds$$

eliminates M_2 in favor of a cubic term. If

$$\lambda_{k_1} + \lambda_{k_2} + \dots + \lambda_{k_i} - \lambda_k \neq 0,$$

then w is a function of $e^{-At}\tilde{M}_2(e^{At}w, e^{At}w) + \dots$. Otherwise we get a power of t, which makes the behavior of u and $e^{At}v_0$ quite different.

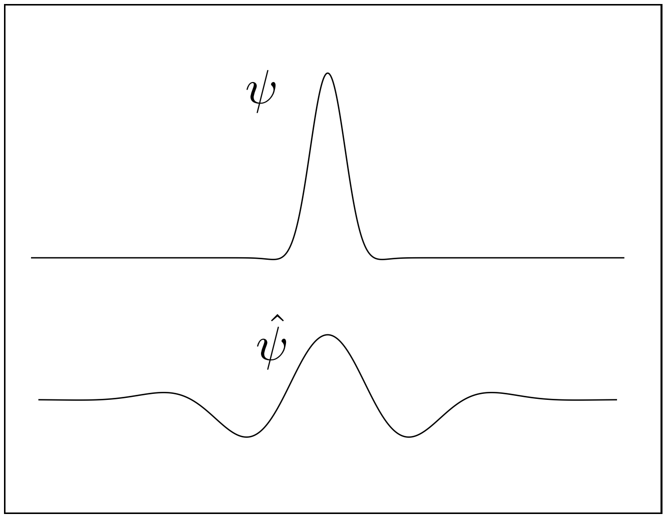

FIGURE 1. Wavepacket

Time resonance for PDEs. The same analysis can be applied to PDEs which are translation invariant with eigenvalues and eigenvectors substituted by ξ and $e^{-i\xi\cdot x}$. We will illustrate this by considering the nonlinear Schrödinger equation

$$i\partial_t u - \Delta u = u^2.$$

Writing $\hat{u}$ for the Fourier transform, we have the equation for $\hat{w} = e^{-i|\xi|^2 t}\hat{u}$,

$$\hat{w}(\xi, s) = \hat{u}_0(\xi) - i\int_{\xi_1+\xi_2=\xi}\Big(\int_0^t e^{i(-\xi^2+\xi_1^2+\xi_2^2)s}\hat{w}(\xi_1, s)\hat{w}(\xi_2, s)ds\Big).$$

If we let $\phi = -\xi^2 + \xi_1^2 + \xi_2^2$, then if $\phi \neq 0$ we can integrate by parts in s and eliminate the quadratic term in favor of a cubic term. Thus quadratic time resonances correspond to

$$\mathcal{T} = \{(\xi_1, \xi_2) \mid \phi = 0 \ \text{ whenever } \ \xi_1 + \xi_2 = \xi\}.$$

Unfortunately the set $\mathcal{T}$ is too big; it corresponds to $\xi_1 \cdot \xi_2 = 0$, which means that the quadratic term is relevant and cannot be eliminated via a transform. However, we *know* that small solutions to

$$i\partial_t u - \Delta u = u^2$$

behave like linear solutions (Hayashi, Mizumachi, and Naumkin). So the normal forms method fails to explain the behavior of small solutions for this PDE. However, this apparent failure is due to the way we computed resonances. We considered eigenvectors $e^{-i\xi\cdot x}$ which are not in the space of functions where nonlinear solutions behave like linear ones. For that to happen we need to consider functions that decay at spatial infinity.

Thus instead of $e^{-i\xi\cdot x}$ we need to consider a wave packet ψ, i.e., a smooth function on $\mathbb{R}^d$ which is strongly localized in space and in frequency.

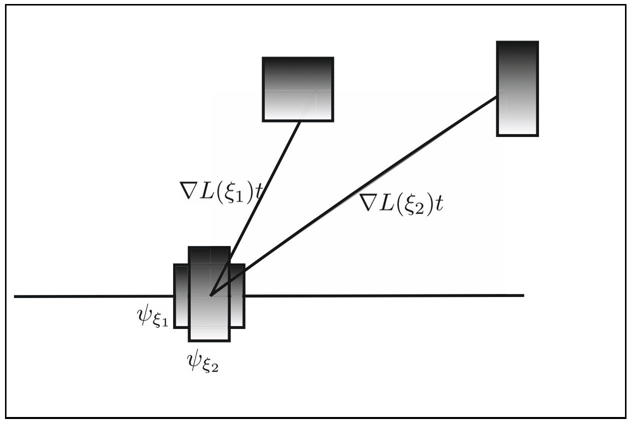

FIGURE 2. Group velocity

Thus if we propagate a wave packet, adapted to the origin with frequency ξ_1, by an equation

$$\partial_t u = iL(\frac{1}{i}\nabla)u,$$

the solution after time t is given by

$$\hat{u} = e^{iL(\xi)t}\psi \approx e^{iL(\xi_1)t}e^{i\nabla L(\xi_1)\cdot(\xi-\xi_1)t}\hat{\psi}.$$

Consequently the solution u is basically supported around $\nabla L(\xi_1)t$. Thus if we have two time-resonant frequencies ξ_1 and ξ_2, they might be located at different places in space, and if the solutions are spatially localized, these time-resonant frequencies won't even have a chance to feel the effects of one another.

Space-time resonances. We say that two wave packets adapted to a region I are spatially resonant if after time t they occupy essentially the same region in space, i.e., if they have the same group velocity. Thus for an equation of the form

$$\partial_t u = iL(\frac{1}{i}\nabla)u + u^2,$$

we define the space-resonant set $\mathcal{S} = \{(\xi_1, \xi_2) \mid \nabla L(\xi_1) = \nabla L(\xi_2)\}$, the time-resonant set $\mathcal{T} = \{(\xi_1, \xi_2) \mid L(\xi_1) + L(\xi_2) = L(\xi_1 + \xi_2)\}$, and the Space-Time resonant set $\mathcal{R} = \mathcal{T} \cap \mathcal{S}$.

To understand how space resonances impact the behavior of solutions to the nonlinear equation

$$\partial_t u = iL(\frac{1}{i}\nabla)u + u^2,$$

we consider two frequencies ξ_1 and ξ_2 that are time resonant but not space resonant. Then the two wave packets can only resonate through the tail. Thus if the tail is sufficiently small, the growth from resonance will be nullified by the smallness of the tail. The smallness of the tail can be measured by using weighted norms.

How to prove global existence and scattering? To explain our technique we consider the quadratic equation

$$\partial_t u = iL(\frac{1}{i}\nabla)u + u^2$$

and proceed as follows:

(1) Compute the space-time resonant set from $\phi = -L(\xi)+L(\eta)+L(\xi-\eta)$, the quadratic frequency interaction, and write the equation for the profile $\hat{w} = e^{-iL(\xi)t}u$ using Duhamel's formula

$$\hat{w}(\xi,s) = \hat{u}_0(\xi) - i\int_0^t \big(\int e^{i\phi(\xi,\eta)s}\hat{w}(\eta,s)\hat{w}(\xi-\eta,s)d\eta\big)ds.$$

(2) On the space-time resonant set $\mathcal{R}$, we apply the stationary phase to see if the set is small enough so that growth due to resonant behavior can be dominated by the size of the resonant set:

$$\int_0^t \int e^{i\phi(\xi,\eta)s}\hat{w}(\eta,s)\hat{w}(\xi-\eta,s)d\eta ds \approx \int^t \frac{1}{s^\alpha}ds.$$

(3) On the set $\{\phi \neq 0\}$, i.e., no time resonances, integrate by parts with respect to s. This introduces a factor of $\frac{1}{\phi}$ and changes the equation to cubic:

$$\int_0^t \int e^{i\phi(\xi,\eta)s}\hat{w}(\eta,s)\hat{w}(\xi-\eta,s)d\eta ds \approx \int_0^t \big(\int e^{i\phi(\xi,\eta)s}\frac{1}{\phi}\mathcal{O}(\hat{w}^3)d\eta ds.$$

(4) On the set $\{\phi = 0\}$ but $\{\nabla_\eta \phi \neq 0\}$, i.e., time resonances but no space resonances, integrate by parts with respect to η. This introduces an additional decay of $\frac{1}{t}$ and a factor of order $\frac{1}{|\nabla_\eta \phi|}$:

$$\int_0^t \int e^{i\phi(\xi,\eta)s}\hat{w}(\eta,s)\hat{w}(\xi-\eta,s)d\eta ds \approx \int_0^t \frac{1}{s}\int e^{i\phi(\xi,\eta)s}\frac{1}{|\nabla_\eta \phi|}\mathcal{O}(\hat{w}\nabla\hat{w})d\eta ds.$$

The factors $\frac{1}{\phi}$ and $\frac{1}{|\nabla_\eta \phi|}$ are viewed as bilinear operators. These operators are usually of Coifman-Meyer type and are harmless.

Nonlinear Schrödinger equations. We will illustrate this procedure for

$$i\partial_t u - \Delta u = u^2, \qquad x \in \mathbb{R}^3.$$

In this case the phase is given by $\phi = -|\xi|^2 + |\xi-\eta|^2 + |\eta|^2$. The time resonances $\mathcal{T}$ are given by

$$\phi = 0 \Rightarrow \eta \cdot (\xi - \eta) = 0.$$

The space resonances $\mathcal{S}$ are given by

$$\nabla_\eta \phi = 0 \Rightarrow \eta = \frac{\xi}{2},$$

and the *Space-Time* resonances $\mathcal{R}$ are given by

$$\xi = \eta = 0.$$

Using the stationary phase lemma to control the behavior near $\mathcal{R}$ ($\phi = 0$ and $\nabla_\eta \phi = 0$), we obtain:

$$\int_0^t \int e^{i\phi(\xi,\eta)s}\hat{w}(\eta,s)\hat{w}(\xi-\eta,s)d\eta ds \approx \int^t \frac{1}{s^{\frac{3}{2}}}\hat{w}(0,s)^2 ds.$$

This gives a bound on $|\hat{w}|_{L^\infty_{t,\xi}}$, which takes the place of a bound on the profile in L^1. In this case the dispersive estimate that we use for the linear equation is

$$|e^{-i\Delta t}f|_{L^\infty} \leq \frac{1}{t^{\frac{3}{2}}}|\hat{f}|_{L^\infty} + \frac{1}{t^{\frac{7}{4}}}|x^2 f|_{L^2}.$$

The energy estimates are straightforward since we are in 3 dimensions, and the weighted L^2 estimates can be achieved by computing $\partial_\xi^\ell \hat{w}$ for $\ell = 1, 2$. This procedure gives a simple proof of existence and scattering for small data in 3 dimensions.

Similarly for the nonlinear problem

$$i\partial_t u - \Delta u = \bar{u}^2, \qquad x \in \mathbb{R}^3,$$

the phase is given by $\phi = -|\xi|^2 - |\xi - \eta|^2 - |\eta|^2$, which has the same space-time resonant sets as the previous example.

For the equation

$$i\partial_t u - \Delta u = |u|^2, \qquad x \in \mathbb{R}^3,$$

the phase is given by $\phi = -|\xi|^2 + |\xi - \eta|^2 - |\eta|^2 = 2\xi \cdot \eta$. Thus the resonant set is too large: stationary phase methods imply that small data cannot behave as linear.

In two space dimensions the decay is too weak to overcome the space-time resonant set. In fact one can see that for $i\partial_t u - \Delta u = u^2, \ \ x \in \mathbb{R}^2$, the stationary phase in a neighborhood of $\mathcal{R}$ implies that solutions diverge logarithmically from the linear behavior. However, if one considers the equation

$$i\partial_t u - \Delta u = u\Lambda u, \qquad x \in \mathbb{R}^2,$$

where Λ is an operator that vanishes on $\mathcal{R}$, e.g. $\hat{\Lambda} = \xi$, then we get rid of these logarithmic divergences. However for such a Λ, we lose the energy estimate. By changing $\hat{\Lambda} = \xi$ to $\hat{\Lambda} = \frac{\xi}{\sqrt{1+|\xi|^2}}$ we can close the argument by gaining back the energy estimate. In this case the energy estimate requires the use of *space-time resonance* splitting since the decay is borderline $\emptyset(t_{-1})$. We note here that J. M. Delort has an existence proof for $\hat{\Lambda} = \xi$. His proof uses smoothing estimates, which so far have not been incorporated in our method.

References

[1] T. Cazenave, F. Weissler, *Rapidly decaying solutions of the nonlinear Schrödinger equation*, Comm. Math. Phys. **147** (1992), no. 1, 75–100. MR1171761 (93d:35150)

[2] R. Coifman, Y. Meyer, *Au delà des opérateurs pseudo-différentiels.* Astérisque, **57**. Société Mathématique de France, Paris, 1978. MR518170 (81b:47061)

[3] P. Germain, N. Masmoudi, J. Shatah, *Global Solutions for 3D Quadratic Schrödinger equations*, Internat. Math. Res. Notices 2009, no. 3, 414–432.

[4] J. Ginibre, N. Hayashi, *Almost global existence of small solutions to quadratic nonlinear Schrödinger equations in three space dimensions*, Math. Z. **219** (1995), no. 1, 119–140. MR1340853 (96e:35158)

[5] S. Gustafson, K. Nakanishi, T.-P. Tsai, *Global dispersive solutions for the Gross-Pitaevskii equation in two and three dimensions*, Ann. Henri Poincaré **8** (2007), no. 7, 1303–1331. MR2360438 (2008m:35335)

[6] S. Gustafson, K. Nakanishi, T.-P. Tsai, *Scattering theory for the Gross-Pitaevski equation in three dimensions*, preprint.

[7] N. Hayashi, T. Mizumachi, P. Naumkin, *Time decay of small solutions to quadratic nonlinear Schrödinger equations in 3D*, Differential Integral Equations **16** (2003), no. 2, 159–179. MR1947090 (2004a:35197)

[8] Y. Kawahara, *Global existence and asymptotic behavior of small solutions to nonlinear Schrödinger equations in* $3D$, Differential Integral Equations **18** (2005), no. 2, 169–194. MR2106101 (2005i:35247)

[9] S. Klainerman, G. Ponce, *Global, small amplitude solutions to nonlinear evolution equations*, Comm. Pure Appl. Math. **36** (1983), no. 1, 133–141. MR680085 (84a:35173)

[10] S. Klainerman, *Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions*, Comm. Pure Appl. Math. **38** (1985), no. 5, 631–641. MR803252 (87e:35080)

[11] K. Nakanishi, T. Ozawa, *Remarks on scattering for nonlinear Schrödinger equations*, NoDEA Nonlinear Differential Equations Appl. **9** (2002), no. 1, 45–68. MR1891695 (2003a:35177)

[12] N. Hayashi, P. Naumkin, *On the quadratic nonlinear Schrödinger equation in three space dimensions*, Internat. Math. Res. Notices 2000, no. 3, 115–132. MR1741610 (2001a:35165)

[13] J. Shatah, *Global existence of small solutions to nonlinear evolution equations*, J. Differential Equations **46** (1982), no. 3, 409–425. MR681231 (84g:35036)

[14] J. Shatah, *Normal forms and quadratic nonlinear Klein-Gordon equations*, Comm. Pure Appl. Math. **38** (1985), no. 5, 685–696. MR803256 (87b:35160)

[15] W. A. Strauss, *Nonlinear scattering theory at low energy*, J. Funct. Anal. **41** (1981), no. 1, 110–133. MR614228 (83b:47074a)