

CRITICAL RAYLEIGH NUMBER IN RAYLEIGH-BÉNARD CONVECTION

BY

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Dedicated to Prof. W. A. Strauss on the occasion of his 70th birthday

Abstract. The Rayleigh-Bénard convection is a classical problem in fluid dynamics. In the presence of rigid boundary condition, we identify the critical Rayleigh number R_a^* by a reduced variational problem. We prove nonlinear asymptotic stability for motionless steady states for $R_a < R_a^*$, and their nonlinear instability for $R_a > R_a^*$. The dynamic of such instability is determined by the leading growing mode(s) for the corresponding linearized system within the time interval of instability.

1. Introduction. Rayleigh-Bénard convection in a shallow horizontal layer of a fluid heated from below has been widely studied in [2, 3, 4, 6, 10, 11, 12, 13, 16, 17]. Assuming so-called Boussinesq approximation [3], we obtain the basic hydrodynamic equations governing Rayleigh-Bénard convection as

$$\begin{aligned}\partial_t v + (v \cdot \nabla)v + \frac{1}{\rho_0} \nabla p &= \nu \Delta v + g[\alpha(T - T_0) - 1] \mathbf{e}_z, \\ \partial_t T + (v \cdot \nabla)T &= \kappa \Delta T, \quad u|_{z=0,h} = 0, \\ T|_{z=0} &= T_1, \quad T|_{z=h} = T_2, \quad T_1 > T_2, \\ v|_{t=0} &= u_0(x, y, z), \quad T|_{t=0} = T_0(x, y, z).\end{aligned}$$

Here $v = (v_1, v_2, v_3)$ is the velocity field of the fluid satisfying $\nabla \cdot v = 0$, p the pressure, ν the kinematic viscosity, α the thermal expansion coefficient, $\mathbf{e}_z = (0, 0, 1)$ the unit upward vector, T the temperature field of the fluid, κ the thermal diffusivity coefficient, T_0 the properly chosen mean temperature, and ρ_0 the density at the temperature T_0 . We

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also impose periodic boundary conditions in the horizontal directions with period $2\pi h$. There is a motionless steady state

$$\begin{aligned} v_s &\equiv 0, \\ p_s &= -g\rho_0 + g\alpha[(T_s - T_0)z + \frac{T_2 - T_1}{2h}z^2], \\ T_s &= T_1 + \frac{T_2 - T_1}{h}z. \end{aligned} \quad (1.1)$$

We denote the perturbation of such a steady state (1.1) as:

$$v = v_s + u, \quad p = p_s + P, \quad T = T_s + \theta.$$

By using the units of the layer depth h as the typical length scale, $(gh)^{1/2}$ as the typical velocity, $(h/g)^{1/2}$ as the typical time, $\rho_0 gh$ as the typical pressure, and $T_1 - T_2$ as the typical temperature, we derive the nondimensional form of the Boussinesq system for the perturbation as:

$$\partial_t u + (u \cdot \nabla)u + \nabla P = \mu_1 \Delta u + \mu_1 \mu_1 R_a \theta \mathbf{e}_z, \quad \nabla \cdot u = 0, \quad (1.2)$$

$$\partial_t \theta + (u \cdot \nabla)\theta + u \cdot \mathbf{e}_z = \mu_2 \Delta \theta, \quad (1.3)$$

with the following initial conditions:

$$u|_{t=0} = u_0(x, y, z), \quad \theta|_{t=0} = \theta_0(x, y, z), \quad (1.4)$$

and the boundary conditions

$$u|_{z=0, h} = 0, \quad u(x + 2\pi, y, z, t) = u(x, y + 2\pi, z, t) = u(x, y, z, t), \quad (1.5)$$

$$\theta|_{z=0, h} = 0, \quad \theta(x + 2\pi, y, z, t) = \theta(x, y + 2\pi, z, t) = \theta(x, y, z, t).$$

Here $\mu_1 = \frac{\nu}{g^{1/2}h^{3/2}}$ and $\mu_2 = \frac{\kappa}{g^{1/2}h^{3/2}}$, and the Rayleigh number is given by:

$$R_a \equiv \frac{\alpha(T_1 - T_2)}{\mu_1 \mu_2} > 0. \quad (1.6)$$

In order to characterize the stability and instability of the motionless steady state (1.1), we introduce the critical Rayleigh number R_a^* as follows. For any integer $k \geq 1$, let $\Theta_k(z)$ be the minimizer of the following variational problem:

$$R(k) = \min_{\Theta \in B, k^2 \int_0^1 (|\partial_z \Theta|^2 + k^2 |\Theta|^2) dz = 1} \int_0^1 |(\partial_z^2 - k^2)^2 \Theta|^2 dz, \quad (1.7)$$

where the function space B is defined by

$$\left\{ \Theta_k \in H^4, \quad \Theta_k \Big|_{z=0,1} = (\partial_z^2 - k^2)\Theta_k \Big|_{z=0,1} = \partial_z(\partial_z^2 - k^2)\Theta_k \Big|_{z=0,1} = 0 \right\}.$$

We define

$$R_a^* = \min_{k \neq 0} \{R(k)\}. \quad (1.8)$$

The stability problem in the Rayleigh-Bénard convection has been investigated in [3, 4, 6, 10, 12, 13]. The purpose of this paper is to identify the sharp Rayleigh number R_a^* for stability.

In section 2, we first establish that the motionless steady state (1.1) is linearly stable for $R_a < R_a^*$ while linearly unstable for $R_a > R_a^*$. We carefully study appropriate variational problems and obtain a complete set of eigenfunctions for the linearized Boussinesq system, which leads to a precise formula for the linear solutions.

In section 3, we prove the nonlinear stability for $R_a < R_a^*$ by a standard semigroup approach (Theorem 3). In section 4, we prove that the motionless steady state (1.1) is nonlinear unstable for $R_a > R_a^*$ (Theorem 8). Moreover, the dynamics of the nonlinear instability is characterized by the fastest exponential growing mode(s) constructed in section 2 for the linearized Boussinesq system, within the instability regime $0 \leq t \leq \frac{1}{-\lambda_1} \ln \frac{1}{\delta}$, where δ is the strength of the initial perturbation and $-\lambda_1 > 0$ is the largest eigenvalue (Theorem 8). These growing mode(s) can exhibit interesting circular and role structures observed in experiments. The proof for the nonlinear instability is based on a general framework given in [8, 9]. The crucial step is to establish a bootstrap energy estimate (Lemma 6) in which we employ higher order anisotropic Sobolev norms in the presence of rigid boundary conditions.

We introduce the following notation: Let $(0, 2\pi)^2 = (0, 2\pi) \times (0, 2\pi)$ and $(E)^3 = E \times E \times E$, where E is any Banach space. The Hilbert space H denotes the completion of

$$\{(u_1, u_2, u_3) \mid u_1, u_2, u_3 \in C_{per}^\infty((0, 2\pi)^2; C_0^\infty(0, 1)); \partial_x u_1 + \partial_y u_2 + \partial_z u_3 = 0\} \quad (1.9)$$

with respect to the norm of $(L^2(Q))^3$, and endowed with the scalar product of $(L^2(Q))^3$, where $Q = (0, 2\pi)^2 \times (0, 1)$. We denote

$$V = \{(u_1, u_2, u_3) \mid (u_1, u_2, u_3) \in H^1(Q) \cap L_{per}^2((0, 2\pi)^2; H_0^1(0, 1)); \partial_x u_1 + \partial_y u_2 + \partial_z u_3 = 0\} \quad (1.10)$$

endowed with the scalar product and the norm of $(H^1(Q))^3$.

2. Linear stability and instability. We study the linearized Boussinesq system around the steady state (1.1):

$$\begin{aligned} \partial_t u + \nabla P &= \mu_1 \Delta u + R_a \mu_1 \mu_2 \theta \mathbf{e}_z, & \nabla \cdot u &= 0, \\ \partial_t \theta - u_3 &= \mu_2 \Delta \theta \end{aligned} \quad (2.1)$$

with the initial condition (1.4) and boundary conditions (1.5). We rewrite the equations (1.20)–(1.22) as

$$\partial_t(u, \theta) = L(u, \theta). \quad (2.2)$$

LEMMA 1. There exist countable eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ for the eigenvalue problem:

$$\begin{aligned} -\lambda u + \nabla P &= \mu_1 \Delta u + R_a \mu_1 \mu_2 \theta \mathbf{e}_z, & \nabla \cdot u &= 0, \\ -\lambda \theta - u_3 &= \mu_2 \Delta \theta. \end{aligned} \quad (2.3)$$

The corresponding eigenfunctions $[u_k, \theta_k]_{k=1}^\infty$ form an orthonormal basis with respect to

$$\langle [u, \theta], [\tilde{u}, \tilde{\theta}] \rangle = (u, \tilde{u}) + R_a \mu_1 \mu_2 (\theta, \tilde{\theta}), \quad (2.4)$$

with $[u_1, P_1, \theta_1]$ smooth. Moreover, for any initial condition $[u^0, \theta^0] \in L^2$, if

$$[u^0, \theta^0] = \sum_k \gamma_k [u_k, \theta_k],$$

then the solution to the linearized Boussinesq system (2.1) is given by

$$e^{Lt}[u^0, \theta^0] = \sum_k \gamma_k e^{-\lambda_k t} [u_k, \theta_k]. \quad (2.5)$$

In particular, there exists a constant $C > 0$ such that

$$\|e^{Lt}[u^0, \theta^0]\| \leq C e^{-\lambda_1 t} \|[u^0, \theta^0]\|. \quad (2.6)$$

Proof. Recall the inner product of (2.4) with the corresponding Hilbert space $L^2_{R_a}$, and recall H and V in (1.9) and (1.10). We consider an equivalent eigenvalue problem as

$$\begin{aligned} -\lambda u &= \mu_1 \mathcal{P} \Delta u + R_a \mu_1 \mu_2 \mathcal{P} \{\theta \mathbf{e}_z\}, \\ -\lambda \theta &= \mu_2 \Delta \theta + u_3, \end{aligned} \quad (2.7)$$

where \mathcal{P} denotes the projection $\{L^2(Q)\}^3 \rightarrow H$. Clearly, by the definition of (2.4), the operator

$$\begin{pmatrix} (\mu_1 \Delta - \lambda_0) I & R_a \mu_1 \mu_2 \mathbf{e}_z^t \\ \mathbf{e}_z & (\mu_2 \Delta - \lambda_0) \end{pmatrix}^{-1}$$

is a bounded, linear, compact, symmetric operator mapping $L^2_{R_a}(Q) \cap \{H \times L^2\}$ into itself for λ_0 large. The theory of compact, symmetric operators implies that all the eigenvalues of (2.7) are real with finite multiplicity, $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots$. There are corresponding eigenfunctions $\{(u_k, \theta_k)\}_{k=1}^\infty$ which make up an orthonormal basis of $L^2_{R_a}(Q)$. The minimizer $(u_{\lambda_1}, \theta_{\lambda_1})$ of the following variational problem

$$\min_{(U, \Theta) \in A} F(U, \Theta) = \min_{(U, \Theta) \in A} \int_Q \{\mu_1 |\nabla U|^2 + R_a \mu_1 \mu_2^2 |\nabla \Theta|^2 - 2R_a \mu_1 \mu_2 U_3 \Theta\} dx dy dz$$

is a weak solution of (2.7), where the function space A is given by

$$\left\{ U \in V, \Theta \in H^1(Q) \cap L^2_{per}((0, 2\pi)^2; H_0^1(0, 1)), \text{ and } \|U\|^2 + R_a \mu_1 \mu_2 \|\Theta\|^2 = 1 \right\}.$$

Since $F(U, \Theta)$ is coercive and convex, there exists at least one $(u_1, \theta_1) \in A$ solving (2.7). By Lemma 1.1 in [5, Page 180], there exists a pressure field $P_1 \in L^2_{per}((0, 2\pi)^2; L^2(0, 1))$ such that (u_1, P_1, θ_1) is the weak solution of the original (2.3) with $\lambda = \lambda_1$. Thanks to the periodic boundary condition, (u_1, P_1, θ_1) satisfies (2.3) with $\lambda = \lambda_1$ in the domain $\Omega = \{(x, y, z) \mid -2\pi < x, y < 4\pi, 0 < z < 1\}$. Let Ω_0 be any bounded subdomain of Ω with $\partial\Omega_0 \in C^\infty$ and $\partial\Omega_0 \cap \partial\Omega = \{(x, y, z) \mid -\pi \leq x, y \leq 3\pi; z = 0, 1\}$. By Theorem 5.1 in [5, Page 218], we have $u_1 \in (H^3_{per}((0, 2\pi)^2; H^3(0, 1)))^3 \cap V$ and $P_1 \in H^2_{per}((0, 2\pi)^2; H^2(0, 1))$. By the regular theory of weak solutions to elliptic equations [7], we deduce $\theta_1 \in H^3_{per}((0, 2\pi)^2; H^3(0, 1) \cap H_0^1(0, 1))$. By a bootstrap method, we have $u_1 \in (H^{m+1}_{per}((0, 2\pi)^2; H^{m+1}(0, 1)))^3 \cap V$, $P_1 \in H^m_{per}((0, 2\pi)^2; H^m(0, 1))$, $\theta_1 \in H^{m+1}_{per}((0, 2\pi)^2; H^{m+1}(0, 1) \cap H_0^1(0, 1))$, $\forall m \geq 2$. Both (2.5) and (2.6) then follow.

LEMMA 2. Recall the critical Rayleigh number R_a^* defined in (1.8). If $R_a < R_a^*$, then $\lambda_1 > 0$. If $R_a > R_a^*$, then $\lambda_1 < 0$.

Proof. In order to construct an eigenfunction to (2.3), we first notice that it suffices to find the third component of u and θ . In fact, taking the curl of equation (2.1) and letting $\omega = (\omega_1, \omega_2, \omega_3) = \text{curl } u = \nabla \times u$, we have

$$\partial_t \omega = \mu_1 \Delta \omega + R_a \mu_1 \mu_2 (\nabla \times \mathbf{e}_z) \theta. \quad (2.8)$$

Taking the curl of equation (2.1) once again, we have

$$\partial_t (\nabla \times \omega) = -\partial_t \Delta u = -\mu_1 \Delta^2 u + R_a \mu_1 \mu_2 (\nabla \times (\nabla \times \mathbf{e}_z)) \theta. \quad (2.9)$$

Since the horizontal components u_1 and u_2 of the velocity can be determined by u_3 and ω_3 , see [3], equations (2.3) are equivalent to the following equations:

$$\begin{aligned} -\lambda \omega_3 &= \mu_1 \Delta \omega_3, \\ -\lambda \Delta u_3 &= \mu_1 \Delta^2 u_3 + R_a \mu_1 \mu_2 (\partial_x^2 + \partial_y^2) \theta, \\ -\lambda \theta &= \mu_2 \Delta \theta + u_3. \end{aligned} \quad (2.10)$$

We now construct an eigenfunction to (2.3) by first studying the following reduced variational problem for the third component function $U_3(z)$ and $\Theta(z)$. For any $k \geq 1$, define

$$\begin{aligned} F_3(U_3, \Theta, R_a) \equiv \int_0^1 \{ \mu_1 |\partial_z^2 U_3 - k^2 U_3|^2 + R_a \mu_1 \mu_2^2 k^2 (|\partial_z \Theta|^2 + k^2 |\Theta|^2) \\ - 2R_a \mu_1 \mu_2 k^2 U_3 \Theta \} dz. \end{aligned} \quad (2.11)$$

We consider

$$\lambda(R_a) = \min_{(U, \Theta) \in A_3} F_3(U_3, \Theta, R_a), \quad (2.12)$$

where the function space is

$$A_3 = \left\{ U_3 \in H_0^2, \Theta \in H_0^1, \int_0^1 (|\partial_z U_3|^2 + k^2 |U_3|^2 + R_a \mu_1 \mu_2 k^2 |\Theta|^2) dz = 1 \right\}. \quad (2.13)$$

It is standard to show that there exists a minimizer $[U_3, \Theta]$ for such a variational problem, which satisfies the Euler-Lagrange equations

$$\begin{aligned} -\lambda(R_a) (\partial_z^2 - k_0^2) U_3 &= \mu_1 (\partial_z^2 - k^2)^2 U_3 - R_a \mu_1 \mu_2 k^2 \Theta, \\ -\lambda(R_a) \Theta &= \mu_2 (\partial_z^2 - k^2) \Theta + U_3, \end{aligned} \quad (2.14)$$

with boundary conditions: $U_3|_{z=0,1} = \partial_z U_3|_{z=0,1} = 0$, $\Theta|_{z=0,1} = 0$. A direct computation shows that

$$\left[-\frac{1}{k_0} \partial_z U_3(z) \sin\{k_0 x\}, 0, U_3(z) \cos\{k_0 x\}, \Theta(z) \cos\{k_0 x\} \right] \quad (2.15)$$

is an eigenfunction with eigenvalue $\lambda(R_a)$ for the original equations (2.3).

We first prove that if $R_a > R_a^*$, then $\lambda_1 \leq \lambda(R_a) < 0$. By (1.8), there exists $k_0 \geq 1$ such that

$$R(k_0) = \min_{\Theta \in B, k_0^2 \int_0^1 (|\partial_z \Theta|^2 + k_0^2 |\Theta|^2) dz = 1} \int_0^1 |(\partial_z^2 - k_0^2)^2 \Theta|^2 dz < R_a.$$

Since the linear operator

$$(\partial_z^2 - k_0^2)^{-1} : L^2(0, 1) \rightarrow L^2(0, 1)$$

is bounded, compact and symmetric, the eigenvalue $0 < R(k_0)$ is real with finite multiplicity and there exists a minimizer $\Theta_{k_0} \in B$ of $R(k_0)$. Letting

$$U_{3,k_0} \equiv -\mu_2(\partial_z^2 - k_0^2)\Theta_{k_0},$$

we plug such a pair $[U_{3,k_0}, \Theta_{k_0}]$ into (2.11) to get

$$\begin{aligned} & \lambda(R_a) \int_0^1 (|\partial_z U_{3,k_0}|^2 + k^2|U_{3,k_0}|^2 + R_a \mu_1 \mu_2 k_0^2 |\Theta|^2) dz \\ & \leq \int_0^1 \{ \mu_1 |\partial_z^2 U_{3,k_0} - k_0^2 U_{3,k_0}|^2 + R_a \mu_1 \mu_2^2 k_0^2 (|\partial_z \Theta_{k_0}|^2 + k_0^2 |\Theta_{k_0}|^2) \\ & \quad - 2R_a \mu_1 \mu_2 k_0^2 U_{3,k_0} \Theta_{k_0} \} dz \\ & = \mu_1 \mu_2^2 \int_0^1 \{ (|\partial_z^2 - k_0^2|^2 \Theta_{k_0}|^2 - R_a k_0^2 (|\partial_z \Theta_{k_0}|^2 + k_0^2 |\Theta_{k_0}|^2)) \} dz \\ & < \mu_1 \mu_2^2 \int_0^1 \{ (|\partial_z^2 - k_0^2|^2 \Theta_{k_0}|^2 - R(k_0) k_0^2 (|\partial_z \Theta_{k_0}|^2 + k_0^2 |\Theta_{k_0}|^2)) \} dz = 0. \end{aligned}$$

Hence $\lambda(R_a) < 0$.

We now assume that $R_a < R_a^*$ and prove $\lambda_1 > 0$ by contradiction. If not, $\lambda_1 \leq 0$, then the first equation in (2.10) implies $\omega = 0$, so that the corresponding eigenfunction $[u_3, \theta]$ satisfies the last two equations in (2.10), or the Euler-Lagrange equations (2.14). This implies that, for all $k \geq 1$,

$$\lambda(R_a) \leq \lambda_1 \leq 0.$$

By a change of $\tilde{\Theta} = \sqrt{R_a} \Theta$, we deduce that $\lambda(R_a)$ takes the form

$$\min_{\int_0^1 (|\partial_z U_3|^2 + k^2 |U_3|^2 + \mu_1 \mu_2 k^2 |\tilde{\Theta}|^2) dz = 1} F_3(U_3, \tilde{\Theta}, R_a), \quad (2.16)$$

where

$$\begin{aligned} F_3(U_3, \tilde{\Theta}, R_a) = \int_0^1 \{ \mu_1 |\partial_z^2 U_3 - k^2 U_3|^2 + \mu_1 \mu_2^2 k_0^2 (|\partial_z \tilde{\Theta}|^2 + k^2 |\tilde{\Theta}|^2) \\ - 2\sqrt{R_a} \mu_1 \mu_2 k^2 U_3 \tilde{\Theta} \} dz. \end{aligned} \quad (2.17)$$

We claim that as a function of the Rayleigh number R_a , $\lambda(R_a)$ is continuous in R_a . In fact, for any two Rayleigh numbers R_{a_1} and R_{a_2} , we choose corresponding minimizers $[U_1, \Theta_1]$ and $[U_2, \Theta_2]$. Clearly, by (2.17), we have

$$|F_3(U_1, \tilde{\Theta}, R_{a_1}) - F_3(U_1, \tilde{\Theta}, R_{a_2})| \leq C|R_{a_1} - R_{a_2}|.$$

Letting $\tilde{\Theta} = \Theta_2$, we deduce that

$$\lambda(R_{a_1}) \leq F_3(U_1, \Theta_2, R_{a_1}) < \lambda(R_{a_2}) + C|R_{a_1} - R_{a_2}|.$$

Similarly, letting $\tilde{\Theta} = \Theta_1$, we have

$$\lambda(R_{a_2}) \leq F_3(U_2, \Theta_1, R_{a_2}) < \lambda(R_{a_1}) + C|R_{a_1} - R_{a_2}|.$$

Hence, the continuity of $\lambda(R_a)$ follows. Moreover, by letting $U_3 = \tilde{\Theta}$ in (2.17), we have

$$\lim_{R_a \rightarrow \infty} \lambda(R_a) = -\infty.$$

This implies that for any $k \geq 1$, there exists $R_a^0(k) \leq R_a$ such that

$$\lambda(R_a^0(k)) = 0.$$

We now denote the corresponding minimizer of $\lambda(R_a^0(k))$ in (2.16) and (2.11) by $[U_3^0, \Theta^0]$, which satisfies (2.14) with $\lambda(R_a^0(k)) = 0$:

$$\begin{aligned} 0 &= \mu_1(\partial_z^2 - k^2)^2 U_3^0 - R_a^0 \mu_1 \mu_2 k^2 \Theta^0, \\ 0 &= \mu_2(\partial_z^2 - k^2) \Theta^0 + U_3^0. \end{aligned} \quad (2.18)$$

Equivalently, we have

$$-(\partial_z^2 - k^2)^4 \Theta^0 - R_a^0(k) k^2 (\partial_z^2 - k^2) \Theta^0 = 0,$$

which implies that $R_a^0(k) \geq R(k)$. Therefore, by (1.8),

$$R_a \geq R_a^0(k) \geq R_a^*,$$

a contradiction. We therefore conclude the lemma.

3. Nonlinear stability for $Ra < Ra^*$.

THEOREM 3. If the Rayleigh number $Ra < Ra^*$, then the motionless solution (u_s, P_s, T_s) in (1.1) of system (1.2), (1.3) is nonlinear stable with respect to the norm of $C(0, \infty; H^2(Q)) \cap W_\infty^1(0, \infty; L^2(Q))$.

Proof. The unconditional nonlinear stability with respect to the norm of $C(0, \infty; L^2(Q))$ can be found in [6].

Now we only give the proof of nonlinear stability with respect to the norm of $C(0, \infty; H^2(Q)) \cap W_\infty^1(0, \infty; L^2(Q))$. Let

$$\mathcal{A} = \begin{pmatrix} (\mu_1 \mathcal{P} \Delta) I & 0 \\ 0 & \mu_2 \Delta \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & \sqrt{R_a \mu_1 \mu_2} \mathbf{e}_z^t \\ \sqrt{R_a \mu_1 \mu_2} \mathbf{e}_z & 0 \end{pmatrix},$$

where vector $\mathbf{e}_z = (0, 0, 1)$ and I is the 3×3 identity matrix. If $\tilde{\lambda}$ is an eigenvalue and $[\tilde{u}, \tilde{\theta}]$ is the corresponding eigenfunction of the following eigenvalue problem,

$$(\mathcal{A} + \mathcal{B})[\tilde{u}, \tilde{\theta}] = -\tilde{\lambda}[\tilde{u}, \tilde{\theta}], \quad (3.1)$$

then $\lambda = \tilde{\lambda}$ is the eigenvalue and $[u, \theta] = [\tilde{u}, \tilde{\theta} / \sqrt{R_a \mu_1 \mu_2}]$ is the corresponding eigenfunction of the eigenvalue problem (2.3). Similarly, for the eigenvalue λ and the corresponding eigenfunction $[u, \theta]$ of (2.3), then $\tilde{\lambda} = \lambda$ is the eigenvalue and $[\tilde{u}, \tilde{\theta}] = [u, \sqrt{R_a \mu_1 \mu_2} \theta]$ is the corresponding eigenfunction of the eigenvalue problem (3.1). Thus $\tilde{\lambda}_1 = \lambda_1 > 0$.

It is well known [14, 15] that $\mathcal{A} + \mathcal{B}$ is the infinitesimal generator of the analytic semigroup $e^{t(\mathcal{A} + \mathcal{B})} : H \times L^2(Q) \rightarrow H \times L^2(Q)$ with

$$\|e^{t(\mathcal{A} + \mathcal{B})}\| \leq C e^{-\lambda_1 t}, \quad \forall t > 0, \quad (3.2)$$

$$\|(-\mathcal{A} - \mathcal{B})^{1/2} e^{t(\mathcal{A} + \mathcal{B})}\| \leq C t^{-1/2} e^{-\lambda_1 t}, \quad \forall t > 0, \quad (3.3)$$

where $\lambda_1 > 0$ is defined in Lemma 1. Since $(-\mathcal{A} - \mathcal{B})^{-1} : H \times L^2(Q) \rightarrow H \times L^2(Q)$ and $\mathcal{A}(-\mathcal{A} - \mathcal{B})^{-1} : H \times L^2(Q) \rightarrow H \times L^2(Q)$ are both selfadjoint linear bounded operators, we have

$$\begin{aligned} & \|(-\mathcal{A})^{1/2} e^{t(\mathcal{A}+\mathcal{B})}[u, \theta]\| \\ &= \left((-\mathcal{A})(-\mathcal{A} - \mathcal{B})^{-1}(-\mathcal{A} - \mathcal{B})^{1/2} e^{t(\mathcal{A}+\mathcal{B})}[u, \theta], (-\mathcal{A} - \mathcal{B})^{1/2} e^{t(\mathcal{A}+\mathcal{B})}[u, \theta] \right)^{1/2} \\ &\leq \|\mathcal{A}(-\mathcal{A} - \mathcal{B})^{-1}\|^{1/2} \|(-\mathcal{A} - \mathcal{B})^{1/2} e^{t(\mathcal{A}+\mathcal{B})}[u, \theta]\| \\ &\leq Ct^{-1/2} e^{-\lambda_1 t} \|[u, \theta]\|, \quad \forall t > 0, [u, \theta] \in H \times L^2(Q). \end{aligned} \quad (3.4)$$

Equations (1.2) and (1.3) can be rewritten in the following equivalent form:

$$\begin{aligned} & [u(t), \sqrt{R_a \mu_1 \mu_2} \theta(t)] \\ &= e^{t(\mathcal{A}+\mathcal{B})}(u_0, \sqrt{R_a \mu_1 \mu_2} \theta_0) - \int_0^t e^{(t-s)(\mathcal{A}+\mathcal{B})} \left((u \cdot \nabla)u(s), (u \cdot \nabla)(\sqrt{R_a \mu_1 \mu_2} \theta(s)) \right) ds. \end{aligned} \quad (3.5)$$

Applying (3.2), (3.4) and (3.5), we have

$$\begin{aligned} & \|u(t)\|_{L^2} + \|\theta(t)\|_{L^2} \\ &\leq C_1 e^{-\lambda_1 t} \left(\|u_0\|_{L^2} + \|\theta_0\|_{L^2} \right) \\ &\quad + C_2 \int_0^t e^{-\lambda_1(t-s)} \left(\|(u \cdot \nabla)u(s)\|_{L^2} + \|(u \cdot \nabla)\theta(s)\|_{L^2} \right) ds \\ &\leq C_1 \left(\|u_0\|_{L^2} + \|\theta_0\|_{L^2} \right) + C_2 \sup_{0 \leq s \leq T} \left(\|u(s)\|_{H^2}^2 + \|\theta(s)\|_{H^2}^2 \right), \quad \forall T \geq t, \\ & \|\Delta u(t)\|_{L^2} + \|\Delta \theta(t)\|_{L^2} \leq C_1 e^{-\lambda_1 t} \left(\|\Delta u_0\|_{L^2} + \|\Delta \theta_0\|_{L^2} \right) \\ &\quad + C_2 \int_0^t (t-s)^{-1/2} e^{-\lambda_1(t-s)} \left(\|\nabla\{(u \cdot \nabla)u(s)\}\|_{L^2} + \|\nabla\{(u \cdot \nabla)\theta(s)\}\|_{L^2} \right) ds \\ &\leq C_1 \left(\|\Delta u_0\|_{L^2} + \|\Delta \theta_0\|_{L^2} \right) + C_2 \sup_{0 \leq s \leq T} \left(\|u(s)\|_{H^2}^2 + \|\theta(s)\|_{H^2}^2 \right), \quad \forall T \geq t. \end{aligned} \quad (3.7)$$

Let

$$E(T) = \sup_{0 \leq s \leq T} \left(\|u(s)\|_{H^2} + \|\theta(s)\|_{H^2} \right).$$

Then we have

$$E(T) \leq C_1 \left(\|u_0\|_{H^2} + \|\theta_0\|_{H^2} \right) + C_2 E^2(T), \quad \forall T \geq 0, \quad (3.8)$$

where the positive constants C_1 and C_2 are independent of T . Therefore, if $\|u_0\|_{H^2} + \|\theta_0\|_{H^2}$ is small enough, then there exists a unique solution $(u, \theta) \in C([0, \infty); (H^2(Q))^4)$ of (1.2) and (1.3) such that

$$\|u(t)\|_{H^2} + \|\theta(t)\|_{H^2} \leq 2C_1 \left(\|u_0\|_{H^2} + \|\theta_0\|_{H^2} \right). \quad \forall t \geq 0, \quad (3.9)$$

and our theorem follows. \square

4. Instability of motionless state. We now turn to the nonlinear problem (1.2) and (1.3). We first recall a general framework for nonlinear instability as:

LEMMA 4 (Bootstrap Instability [8]). Assume that L is a linear operator on a Banach space X with norm $\|\cdot\|$, and e^{tL} generates a strongly continuous semigroup on X such that

$$\|e^{tL}\|_{(X,X)} \leq C_L e^{t\lambda} \quad (4.1)$$

for some C_L and $\lambda > 0$. Assume a nonlinear operator $N(y)$ on X and another norm $\|\cdot\|$, and constant C_N , such that

$$\|N(y)\| \leq C_N \|y\|^2 \quad (4.2)$$

for all $y \in X$ and $\|y\| < \infty$. Assume for any solution $y(t)$ to the equation

$$y' = Ly + N(y) \quad (4.3)$$

with $\|y(t)\|^2 \leq \sigma$ that there exists $C_\sigma > 0$ such that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that the following sharp energy estimate holds:

$$\frac{d}{dt} \|y(t)\| \leq \epsilon \|y(t)\| + C_\sigma \|y(t)\| + C_\epsilon \|y(t)\|. \quad (4.4)$$

Consider a family of initial data $y^\delta(0) = \delta y_0$ with $\|y_0\| = 1$ and $\|y_0\| < \infty$ and let β_0 be a sufficiently small (fixed) number. Then there exists some constant $C > 0$ such that if

$$0 \leq t \leq T^\delta \equiv \frac{1}{\lambda} \log \frac{\beta_0}{\delta},$$

then we have

$$\|y(t) - \delta e^{tL} y_0\| \leq C(\|y_0\|^2 + 1) \delta^2 e^{2\lambda t}. \quad (4.5)$$

In particular, if there exists a constant C_p such that $\|\delta e^{tL} y_0\| \geq C_p \delta e^{\lambda t}$, then there exists an escape time $T^{esc} \leq T^\delta$ such that

$$\|y(T^{esc})\| \geq \tau_0 > 0, \quad (4.6)$$

where τ_0 depends explicitly on $C_L, C_N, C_\sigma, C_p, \lambda, y_0, \sigma$ and is independent of δ .

To apply such a method, we need to verify (4.4) for an appropriate Sobolev norm $\|\cdot\|$. Let $\|\cdot\| = \|\cdot\|_{L^2}$, $D_{x,y}^k = \sum_{k_1+k_2=k} \partial_x^{k_1} \partial_y^{k_2}$ and

$$\begin{aligned} E_0 &= \|u(t)\|^2 + \|\theta(t)\|^2, \\ E_k &= E_0 + \|\nabla u(t)\|^2 + \|\nabla \theta(t)\|^2 + \|D_{x,y}^k \nabla u(t)\|^2 + \|D_{x,y}^k \nabla \theta(t)\|^2. \end{aligned}$$

LEMMA 5. Let $k \geq 2$. Then

$$\|(u \cdot \nabla)u\| + \|(u \cdot \nabla)\theta\| \leq C E_k. \quad (4.7)$$

Proof. Applying Sobolev imbedding theorems and Hölder's inequality, we have

$$\begin{aligned} u^2(x, y, z, t) &= 2 \int_0^z u_z u(x, y, s, t) ds \leq C \int_0^z \|u_z(\cdot, s, t)\|_{H_{x,y}^2} \|u(\cdot, s, t)\|_{H_{x,y}^2} ds, \\ \|u(t)\|_{L^\infty} &\leq C(\|u(t)\| + \|(\partial_x^2 + \partial_y^2)u(t)\| + \|\partial_z u(t)\| + \|(\partial_x^2 + \partial_y^2)\partial_z u(t)\|). \end{aligned}$$

By using the multiplicative inequality [1, p. 323], we have

$$\|(u \cdot \nabla)u\| + \|(u \cdot \nabla)\theta\| \leq \|u\|_{L^\infty}(\|\nabla u\| + \|\nabla\theta\|) \leq CE_k,$$

where we have used the following estimates:

$$\|\nabla D_{x,y}^{k-l}u\| \leq C\|\nabla u\|^{l/k}\|\nabla D_{x,y}^k u\|^{(k-l)/k}, \quad \forall 1 \leq l \leq k-1.$$

LEMMA 6. Let $k \geq 3$. Then we have

$$\frac{d}{dt}E_k \leq \epsilon E_k + CE_k^2 + C_\epsilon E_0, \quad \forall \epsilon > 0. \quad (4.8)$$

Proof. Taking scalar products of u with equation (1.2) and θ with equation (1.3), we have

$$\begin{aligned} \frac{d}{dt}\|u\|^2 + 2\mu_1\|\nabla u\|^2 &\leq 2R_a\|u_3\|\|\theta\|, \\ \frac{d}{dt}\|\theta\|^2 + 2\mu_2\|\nabla\theta\|^2 &\leq 2\|u_3\|\|\theta\|. \end{aligned}$$

Taking scalar products of u_t with equation (1.2) and $\Delta\theta$ with equation (1.3), we have

$$\begin{aligned} \frac{d}{dt}\mu_1\|\nabla u\|^2 + 2\|u_t\|^2 &\leq 2\|(u \cdot \nabla)u\|^2 + \|\partial_t u_3\|^2 + C\|\theta\|^2, \\ \frac{d}{dt}\|\nabla\theta\|^2 + 2\mu_2\|\Delta\theta\|^2 &\leq C\|(u \cdot \nabla)\theta\|^2 + \mu_2\|\Delta\theta\|^2 + C\|u_3\|^2. \end{aligned}$$

Taking scalar products of $D_{x,y}^{2k}u_t$ with equation (1.2) and $D_{x,y}^{2k}\Delta\theta$ with equation (1.3), we have

$$\begin{aligned} \frac{d}{dt}\mu_1\|\nabla D_{x,y}^k u\|^2 + 2\|D_{x,y}^k u_t\|^2 &\leq 2\|D_{x,y}^k\{(u \cdot \nabla)u\}\|^2 + \|D_{x,y}^k u_t\|^2 + C\|D_{x,y}^k\theta\|^2, \\ \frac{d}{dt}\|\nabla D_{x,y}^k\theta\|^2 + 2\mu_2\|\Delta D_{x,y}^k\theta\|^2 &\leq C\|D_{x,y}^k\{(u \cdot \nabla)\theta\}\|^2 + \mu_2\|\Delta D_{x,y}^k\theta\|^2 + C\|D_{x,y}^k u\|^2. \end{aligned}$$

By Hölder's inequality, the Sobolev imbedding theorems, the multiplicative inequality [1, p. 323] and (4.7), $\|D_{x,y}^k\{(u \cdot \nabla)u\}\|$ is bounded by

$$\begin{aligned} &C \sum_{l=0}^{k-2} \|D_{x,y}^l u\|_{L^\infty} \|D_{x,y}^{k-l} \nabla u\| + 2\|D_{x,y}^{k-1} u\|_{L_z^\infty(0,1;L_{x,y}^2)} \|D_{x,y} \nabla u\|_{L_z^2(0,1;L_{x,y}^\infty)} \\ &+ 2\|D_{x,y}^k u\|_{L_z^\infty(0,1;L_{x,y}^2)} \|\nabla u\|_{L_z^2(0,1;L_{x,y}^\infty)} \\ &\leq CE_k + C\|D_{x,y}^{k-1} u\|_{H_z^1(0,1;L_{x,y}^2)} \|D_{x,y} \nabla u\|_{L_z^2(0,1;H_{x,y}^2)} \\ &+ C\|D_{x,y}^k u\|_{H_z^1(0,1;L_{x,y}^2)} \|\nabla u\|_{L_z^2(0,1;H_{x,y}^2)} \\ &\leq CE_k, \end{aligned}$$

and $\|D_{x,y}^k \{(u \cdot \nabla)\theta\}$ is bounded by

$$\begin{aligned}
& C \sum_{l=0}^{k-2} \|D_{x,y}^l u\|_{L^\infty} \|\nabla D_{x,y}^{k-l} \theta\|_{L^2} + C \|D_{x,y}^{k-1} u\|_{L_z^\infty(0,1;L_{x,y}^2)} \|D_{x,y} \nabla \theta\|_{L_z^2(0,1;L_{x,y}^\infty)} \\
& + C \|D_{x,y}^k u\|_{L_z^\infty(0,1;L_{x,y}^2)} \|\nabla \theta\|_{L_z^2(0,1;L_{x,y}^\infty)} \\
& \leq CE_k + C \|D_{x,y}^{k-1} u\|_{H_z^1(0,1;L_{x,y}^2)} \|D_{x,y} \nabla \theta\|_{L_z^2(0,1;H_{x,y}^2)} \\
& + C \|D_{x,y}^k u\|_{H_z^1(0,1;L_{x,y}^2)} \|\nabla \theta\|_{L_z^2(0,1;H_{x,y}^2)} \\
& \leq CE_k.
\end{aligned}$$

Here we have used the following estimates:

$$\|\nabla D_{x,y}^{k-l} \theta\| \leq C \|\nabla \theta\|^{l/k} \|\nabla D_{x,y}^k \theta\|^{(k-l)/k}, \quad \forall 1 \leq l \leq k-1.$$

Since

$$\begin{aligned}
C \|D_{x,y}^k u\|^2 + C \|D_{x,y}^k \theta\|^2 & \leq C \|D_{x,y}^{k+1} u\|^{2k/(k+1)} \|u\|^{2/(k+1)} \\
& + C \|D_{x,y}^{k+1} \theta\|^{2k/(k+1)} \|\theta\|^{2/(k+1)} \\
& \leq \epsilon \{ \|D_{x,y}^{k+1} u\|^2 + \|D_{x,y}^{k+1} \theta\|^2 \} + C_\epsilon \{ \|u\|^2 + \|\theta\|^2 \},
\end{aligned}$$

by putting together all these estimates, we obtain (4.8).

Since $\lambda_1 < 0$ for $R_a > R_a^*$, there exists some corresponding wave numbers $k^* \geq 1$ such that $\lambda^* = \lambda(R_a) < 0$, where $\lambda(R_a)$ is defined in (2.12) with $k = k^*$, and the maximal growth rate is given by $-\lambda^* > 0$. Let $[U_k, \Theta_k]$ be the minimizer of (2.12).

DEFINITION 7. We define a smooth generic profile for the initial perturbation as

$$\begin{aligned}
[\tilde{u}, \tilde{\theta}] &= (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{\theta}) \\
&= \sum_{k_1^2 + k_2^2 = k^2} \left(V_{1,k_1,k_2}, V_{2,k_1,k_2}, v_{k_1,k_2} U_k(z), \vartheta_{k_1,k_2} \Theta_k(z) \right) e^{ik_1 x + ik_2 y} \quad (4.9)
\end{aligned}$$

such that if $k_1^2 + k_2^2 = (k^*)^2$, then either v_{k_1,k_2} or ϑ_{k_1,k_2} is nonzero. Moreover, $V_{1,0,0} = V_{2,0,0} = 0$,

$$\begin{aligned}
V_{1,k_1,k_2} &= \frac{ik_1}{k^2} v_{k_1,k_2} \partial_z U_k(z), \quad \forall k_1^2 + k_2^2 = k^2 \geq 1, \\
V_{2,k_1,k_2} &= \frac{ik_2}{k^2} v_{k_1,k_2} \partial_z U_k(z), \quad \forall k_1^2 + k_2^2 = k^2 \geq 1.
\end{aligned}$$

Therefore, from Lemmas 4-6, we have

THEOREM 8. Suppose that the Rayleigh number $R_a > Ra^*$. Let $[u^\delta, \theta^\delta]$ be a solution to the equations (1.2) and (1.3) with initial value

$$[u^\delta(0), \theta^\delta(0)] = \delta [\tilde{u}, \tilde{\theta}],$$

where $[\tilde{u}, \tilde{\theta}]$ is a generic profile defined in (4.9) and $\|[\tilde{u}, \tilde{\theta}]\| = 1$. Then,

$$\begin{aligned}
\| [u^\delta(t), \theta^\delta(t)] - \sum_{k_1^2 + k_2^2 = (k^*)^2} \delta e^{tL} \left(V_{1,k_1,k_2}, V_{2,k_1,k_2}, v_{k_1,k_2} U_k(z), \vartheta_{k_1,k_2} \Theta_k(z) \right) e^{ik_1 x + ik_2 y} \| \\
\leq C \{ 1 + (\|[\nabla \tilde{u}, \nabla \tilde{\theta}]\|^2 + \|[\partial_x^k \nabla \tilde{u}, \partial_x^k \nabla \tilde{\theta}]\|^2) \} \delta^2 e^{-2\lambda^* t}, \quad k \geq 3, \quad (4.10)
\end{aligned}$$

for any $\delta \leq \delta_0$ sufficiently small, and $0 \leq t \leq T^\delta$, where $\| [u, \theta] \| = \|u_1\| + \|u_2\| + \|u_3\| + \|\theta\|$.

REMARK 9. We note that $e^{tL} \left(V_{1,k_1,k_2}, V_{2,k_1,k_2}, v_{k_1,k_2} U_k(z), \vartheta_{k_1,k_2} \Theta_k(z) \right) e^{ik_1 x + ik_2 y}$ is given explicitly in (2.5). Clearly, for $k_1^2 + k_2^2 = (k^*)^2$ and $v_{k_1,k_2}, \vartheta_{k_1,k_2}$ not both zero,

$$\| e^{tL} \left(V_{1,k_1,k_2}, V_{2,k_1,k_2}, v_{k_1,k_2} U_k(z), \vartheta_{k_1,k_2} \Theta_k(z) \right) e^{ik_1 x + ik_2 y} \| \geq C e^{-\lambda^* t}$$

for some positive constant C . Then, from Lemma 4 and Theorem 8, the motionless state (u_s, P_s, T_s) of the system (1.2) and (1.3) is nonlinear unstable.

The proof of Theorem 8 follows from Lemmas 4–6 with $\| \cdot \| = E_k^{1/2}$ and $\| \cdot \| = E_0^{1/2}$. By (2.5), we split $\delta e^{tL}(\tilde{u}, \tilde{\theta})$ in Lemma 4 into maximal growing modes

$$\delta e^{tL} \left(V_{1,k_1,k_2}, V_{2,k_1,k_2}, v_{k_1,k_2} U_k(z), \vartheta_{k_1,k_2} \Theta_k(z) \right) e^{ik_1 x + ik_2 y}, \quad \forall k_1^2 + k_2^2 = (k^*)^2$$

and the remaining modes to conclude the proof.

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