BOUNDED SOLUTIONS FOR THE BOLTZMANN EQUATION

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Dedicated to Professor W. A. Strauss on the occasion of his 70th birthday

Abstract. In either a periodic box $T^d$ or $\mathbb{R}^d$ ($1 \leq d \leq 3$), we establish a unified $L^\infty$ estimate for solutions near Maxwellians for the Boltzmann equation, in terms of natural mass, momentum, energy conservation and the entropy inequality.

We study $L^\infty$ estimates for the Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = \frac{1}{\kappa} Q(F, F), \quad F(0, x, v) = F_0(x, v),$$

where $F(t, x, v) \geq 0$ is the density of particles of velocity $v \in \mathbb{R}^3$, and position $x \in \Omega = \mathbb{R}^d$ or $T^d$, a periodic box, for $1 \leq d \leq 3$. The Knudsen number $\kappa$ is a bounded constant. For simplicity we assume a hard-sphere interaction for the collision kernel $Q$. We define a global Maxwellian given by

$$\mu = \frac{\rho}{(2\pi T)^{3/2}} \exp \left\{ -\frac{|v - u|^2}{2T} \right\},$$

where $\rho, u, T$ are independent of $t$ and $x$. Our main result is

**Theorem 1.** Assume that the excess conservations of mass, momentum and energy,

$$\int \int \{F(t, x, v) - \mu\} dv dx = \int \int \{F_0(x, v) - \mu\} dv dx \equiv M_0,$$

$$\int \int v\{F(t, x, v) - \mu\} dv dx = \int \int vF_0(x, v) dv dx \equiv J_0,$$

$$\int \int |v|^2 \{F(t, x, v) - \mu\} dv dx = \int \int |v|^2 \{F_0(x, v) - \mu\} dv dx \equiv E_0,$$

as well as the excess entropy inequality, hold:

$$\mathcal{H}(F(t)) - \mathcal{H}(\mu) \leq \mathcal{H}(F_0) - \mathcal{H}(\mu),$$

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where $H(g) \equiv \int \int g \ln gdvdx$. Then for any $\beta \geq 0$, there exists $C > 0$ such that

\[
\sup_{0 \leq t \leq \infty} \left\| \frac{1 + |v|^2}{\sqrt{\mu}} \beta (F(t) - \mu) \right\|_\infty \leq C \left\| \frac{1 + |v|^2}{\sqrt{\mu}} \beta (F_0 - \mu) \right\|_\infty + \frac{1}{\kappa^{d/2}} \sqrt{H(F_0) - H(\mu) + |E_0| + |J_0| + |E_0|}
\]

provided the right-hand side is sufficiently small.

Both the excess conservation laws (3) and the excess entropy inequality (4) are clearly valid when $\Omega$ is a periodic box. We remark that in the case $\Omega = \mathbb{R}^d$, local-in-time solutions satisfying both (3) and (4) can be constructed via the following approximate Boltzmann equation with finite propagation of speed in the physical space:

\[
\partial_t F_n + v1_{\{|v| \leq n\}} \cdot \nabla_x F_n = \frac{1}{\kappa} Q(F_n, F_n)
\]

as $n \to \infty$.

It is well known that the pointwise control of $F$ is crucial for uniqueness when $\beta$ is large. The new $L^\infty$ estimate (5) is solely based on the most natural a priori estimates in the Boltzmann theory. Even though no time decay rate is obtained, the proof is direct and robust.

**Proof.** Denote the weight function by $w(v) = \{1 + |v|^2\}^\beta$ and define

\[
h = w(v) \times \frac{F - \mu}{\sqrt{\mu}}.
\]

Recall the standard linearized Boltzmann operator as

\[
Lg = -\frac{1}{\sqrt{\mu}} \{Q(\mu, \sqrt{\mu}g) + Q(\sqrt{\mu}g, \mu)\} = \{\nu(v) + K\}g,
\]

where the collision frequency $\nu(v) = c \int |v - u|/u dv \sim |v| + 1$. Letting $K_w g \equiv wK(\frac{g}{w})$, we obtain

\[
\partial_t h + v \cdot \nabla_x h + \frac{\nu}{\kappa} h + \frac{1}{\kappa} K_w h = \frac{1}{\kappa} w \Gamma(h, \frac{h}{w}, \frac{h}{w}),
\]

where $\Gamma(g_1, g_2) = \frac{1}{\sqrt{\mu}} \sqrt{\mu (g_1, \sqrt{\mu}g_2)}$.

For any $(t, x, v)$, we denote $\bar{v} = (v_1, ..., v_d)$. Integrating along its backward trajectory $\frac{dX(s)}{ds} = V(s)$, $\frac{dV(s)}{ds} = 0$, we express $h(t, x, v)$ as

\[
\exp\left\{-\frac{\nu}{\kappa} t\right\} h(0, x - \bar{v} t, v) + \int_0^t \exp\left\{-\frac{\nu}{\kappa} (t - s)\right\} \left(\frac{1}{\kappa} K_w h\right) (s, x - \bar{v} (t - s), v) ds \\
+ \int_0^t \exp\left\{-\frac{\nu}{\kappa} (t - s)\right\} w \frac{\kappa}{\kappa} \Gamma(h, \frac{h}{w}, \frac{h}{w}) (s, x - \bar{v} (t - s), v) ds.
\]

(6)

Since $\frac{w}{\kappa} \Gamma(h, \frac{h}{w}, \frac{h}{w}) (v) \leq C \nu(v) ||h||_\infty^2$ from Lemma 10 of [2], and since

\[
\int_0^t \exp\left\{-\frac{\nu}{\kappa} (t - s)\right\} ds \leq O(\kappa),
\]

where $\mathcal{H}(g) = \int \int g \ln gdvdx$. Then for any $\beta \geq 0$, there exists $C > 0$ such that

\[
\sup_{0 \leq t \leq \infty} \left\| \frac{1 + |v|^2}{\sqrt{\mu}} \beta (F(t) - \mu) \right\|_\infty \leq C \left\| \frac{1 + |v|^2}{\sqrt{\mu}} \beta (F_0 - \mu) \right\|_\infty + \frac{1}{\kappa^{d/2}} \sqrt{H(F_0) - H(\mu) + |E_0| + |J_0| + |E_0|}
\]
the last term in (6) is bounded by \((\nu(v)\) is bounded from below)

\[
\frac{C}{\kappa} \int_0^t \exp\left\{-\frac{\nu(t-s)}{\kappa}\right\} \left\{\nu(v)|h(s,x-\bar{v}(t-s),v)| + \|h(s)\|_\infty\right\} \|h(s)\|_\infty ds \leq C \sup_{0 \leq s \leq t} \|h(s)\|^2_\infty. \tag{7}
\]

We shall mainly concentrate on the second term in (6). Let \(k(v,v')\) be the corresponding kernel associated with \(K\). We now use (6) again to evaluate \(\{K_w, h\}(s,x-(t-s)\bar{v}) = \int k_w(v,v') h(s,x-\bar{v}(t-s),v') dv'\). By (7), we can bound the above by

\[
\frac{1}{\kappa} \int_0^t \exp\left\{-\frac{\nu(t-s)}{\kappa}\right\} |k_w(v,v')| \exp\left\{-\frac{\nu s}{\kappa}\right\} h(0,x-\bar{v}(t-s)-v',v') dv' ds
+ \frac{1}{\kappa^2} \int_0^t \exp\left\{-\frac{\nu(t-s)}{\kappa}\right\} \int \mathbb{R}^3 \times \mathbb{R}^3 |k_w(v,v') k_w(v'',v'')| \times \int_0^s \exp\left\{-\frac{\nu(v')(s-s_1)}{\kappa}\right\} |h(s_1,x-\bar{v}(t-s)-\bar{v}'(s-s_1),v'')| dv'' ds_1 ds
+ \frac{C}{\kappa} \int_0^t \exp\left\{-\frac{\nu(t-s)}{\kappa}\right\} \sup_v \int \mathbb{R}^3 |k_w(v,v')| dv' \times \left\{\sup_{0 \leq s \leq t} \|h(s)\|^2_\infty\right\}, \tag{8}
\]

where \(k_w(\cdot) = w k(\cdot)\) and \(\bar{v}' = (v'_1, \ldots, v'_d)\). Since \(\sup_v \int \mathbb{R}^3 k_w(|v,v'|) dv' < +\infty\) from Lemma 7 of [2], the first and the third terms above are bounded by \(C\|h(0)\|_\infty + C\{\sup_{0 \leq s \leq t} \|h(s)\|_\infty\}^2\).

We now concentrate on the second term in (8), which will be estimated as in the proof of Theorem 21 in [2].

**Case 1.** For \(|v| \geq N\). By Lemma 7 in [2],

\[
\int \int k_w(v,v') k_w(v',v'') dv' dv'' \leq \frac{C}{1 + |v|} \leq \frac{C}{N}.
\]

We therefore can find an upper bound for the second term in this case as

\[
\frac{C}{\kappa^2 N} \int_0^t \exp\left\{-\frac{\nu(t-s)}{\kappa}\right\} \times \int_0^s \exp\left\{-\frac{\nu(v')(s-s_1)}{\kappa}\right\} \|h(s_1)\|_\infty ds_1 ds \leq \frac{C'}{N} \sup_{0 \leq s \leq t} \|h(s)\|_\infty.
\]

**Case 2.** For \(|v| \leq N\), \(|v'| \geq 2N\), or \(|v'| \leq 2N\), \(|v''| \geq 3N\). Notice that we have either \(|v' - v| \geq N\) or \(|v' - v''| \geq N\), and either one of the following are valid correspondingly for some \(\eta > 0\):

\[
|k_w(v,v')| \leq e^{-\frac{\pi N^2}{4}} |k_w(v,v') e^{\frac{\pi}{2} |v-v'|^2}|, \quad |k_w(v',v'')| \leq e^{-\frac{\pi N^2}{4}} |k_w(v',v'') e^{\frac{\pi}{2} |v'-v''|^2}|. \tag{9}
\]
From Lemma 8 in [2], both $\int |k_w(v',v')e^{\frac{2}{N}v'v''}|$ and $\int |k_w(v',v'')e^{\frac{2}{N}v'v''}|$ are still finite. We use (9) to combine the cases of $|v' - v| \geq N$ or $|v' - v''| \geq N$ as:

$$
\int_0^t \int_0^{s_1} \cdots \left\{ \int_{|v| \leq N, |v'| \geq 2N} + \int_{|v'| \leq 2N, |v''| \geq 3N} \right\}
$$

$$\leq C \int_0^t \int_0^{s_1} \cdots \left\{ \int_{|v| \leq N, |v'| \geq 2N} |k_w(v, v')|dv' + \sup_{v'} \int_{|v'| \leq 2N, |v''| \geq 3N} |k_w(v', v'')|dv'' \right\}
$$

$$\leq C e^{-\frac{N}{k^2}} \int_0^t \int_0^{s_1} \cdots \exp\left\{ -\frac{\nu(t-s)}{k} \right\} \exp\left\{ -\frac{v(s-s_1)}{k} \right\} ||h(s)||_\infty ds_1 ds
$$

$$\leq C e^{-\frac{N}{k^2}} \sup_{0 \leq s \leq t} ||h(s)||_{\infty}. \tag{10}
$$

**Case 3.** $s - s_1 \leq \varepsilon \kappa$, for $\varepsilon > 0$ small. We now can simply bound the second term in (8) by

$$\frac{1}{k^2} \int_0^t \int_{s - \varepsilon \kappa}^s C \exp\left\{ -\frac{\nu(t-s)}{k} \right\} \exp\left\{ -\frac{v(s-s_1)}{k} \right\} ||h(s)||_\infty ds_1 ds
$$

$$\leq C \sup_{0 \leq s \leq t} \frac{1}{k} \int_0^t \exp\left\{ -\frac{\nu(t-s)}{k} \right\} ds \times \int_{s - \varepsilon \kappa}^s \frac{1}{k} ds_1
$$

$$\leq \varepsilon C \sup_{0 \leq s \leq t} ||h(s)||_{\infty}. \tag{11}
$$

**Case 4.** $s - s_1 \geq \varepsilon \kappa$, and $|v| \leq N, |v'| \leq 2N, |v''| \leq 3N$. This is the last remaining case because if $|v'| > 2N$, it is included in Case 2; while if $|v''| > 3N$, either $|v'| \leq 2N$ or $|v'| \geq 2N$ are also included in Case 2. We now can bound the second term in (8) by

$$C \int_0^t \int_0^{s - \varepsilon \kappa} B e^{-\frac{\nu(t-s)}{k}} \frac{e^{-\frac{v(s-s_1)}{k}}}{\nu} |k_w(v, v')k_w(v', v'')h(s, x_1 - (s - s_1)v', v'')|,
$$

where $B = \{|v'| \leq 2N, |v''| \leq 3N\}$ and $x_1 = x - (t-s)v$. Notice that $k_w(v, v')$ has a possible integrable singularity of $\frac{1}{|v-v'|}$. We can choose $k_N(v, v')$ smooth with compact support such that

$$\sup_{|v'| \leq 3N} \int_{|v''| \leq 3N} |k_N(p, v') - k_w(p, v')|dv' \leq \frac{1}{N}. \tag{12}
$$

Splitting

$$k_w(v, v')k_w(v', v'') = \{k_w(v, v') - k_N(v, v')\}k_w(v', v'') + \{k_w(v', v'') - k_N(v', v'')\}k_N(v, v') + k_N(v, v')k_N(v', v''),
$$

we can use such an approximation (12) to bound the above $s_1, s$ integration by

$$\frac{C}{N} \sup_{0 \leq s \leq t} ||h(s)||_\infty \times \left\{ \sup_{|v'| \leq 2N} \int |k_w(v', v'')|dv'' + \sup_{|v| \leq 2N} \int |k_w(v, v')|dv' \right\}
$$

$$+ C \int_0^t \int_0^{s - \varepsilon \kappa} B e^{-\frac{\nu(t-s)}{k}} \frac{e^{-\frac{v(s-s_1)}{k}}}{\nu} |k_N(v, v')k_N(v', v'')h(s, x_1 - (s - s_1)v', v'')|.
$$

$$\tag{13}
$$
We now make use of the conservation laws (4) and the entropy inequality (4) to estimate the last term. Recall from the Taylor expansion,

\[ \mathcal{H}(F(t)) - \mathcal{H}(\mu) = \int \int \{ \ln \mu + 1 \} \{ F - \mu \} + \int \int \frac{(F(t) - \mu)^2}{2F} \]

where \( \tilde{F} \) is between \( F(t) \) and \( \mu \). Since \( \mu = \frac{\rho}{(2\pi T)^{3/2}} \exp \left\{ -\frac{|v-u|^2}{2T} \right\}, \ln \mu = \ln \frac{\rho}{(2\pi T)^{3/2}} - \frac{|v-u|^2}{2T} \). Therefore, from the conservations of mass, momentum and energy (3), we get

\[ \int \int \frac{(F(t) - \mu)^2}{2F} \leq \mathcal{H}(F_0) - \mathcal{H}(\mu) + C_{\mu,u,T} \{ |M_0| + |J_0| + |E_0| \}. \]

The key is to estimate \( \frac{(F(t) - \mu)^2}{2F} \) in the case of \( |F(t) - \mu| \geq \delta \mu \) for a small parameter \( \delta \). Notice that either \( F(t) \leq 1 - \delta \mu \) or \( F(t) - \mu \geq \delta \mu \) in this case. If \( F(t) \leq 1 - \delta \mu \),

\[ \frac{|F(t) - \mu|}{F(t)} \geq \frac{|F(t) - \mu|}{\mu} \geq 1 - \frac{F(t)}{\mu} \geq 1 - (1 - \delta) = \delta. \]

On the other hand, if \( F(t) \geq 1 + \delta \mu \),

\[ \frac{|F(t) - \mu|}{F(t)} \geq 1 - \frac{\mu}{F(t)} \geq 1 - \frac{1}{1 + \delta} = \frac{\delta}{1 + \delta}. \]

In summary, we have

\[ \mathcal{H}(F_0) - \mathcal{H}(\mu) + C \{ |M_0| + |J_0| + |E_0| \} \]

\[ \geq \int \int \frac{(F(t) - \mu)^2}{2F} 1_{|F(t) - \mu| \leq \delta \mu} + \int \int \frac{(F(t) - \mu)^2}{2F} 1_{|F(t) - \mu| \geq \delta \mu} \]

\[ \geq \int \int \frac{(F(t) - \mu)^2}{2(1 + \delta)\mu} 1_{|F(t) - \mu| \leq \delta \mu} + \frac{\delta}{2(1 + \delta)} \int |F(t) - \mu| 1_{|F(t) - \mu| \geq \delta \mu}. \]

Since \( k_N(v, v') k_N(v', v'') \) is bounded, we first integrate over \( v' \) (bounded) to get

\[ C_N \int_{|v'| \leq 2N} |h(s_1, x_1 - (s - s_1)v', v'')| 1_{F(s_1, x_1 - (s - s_1)v', v'') - \mu| \leq \delta \mu} dv' \]

\[ + C_N \int_{|v'| \leq 2N} |h(s_1, x_1 - (s - s_1)v', v'')| 1_{F(s_1, x_1 - (s - s_1)v', v'') - \mu| \geq \delta \mu} dv' \]

\[ \leq C_N \delta + C_N \left\{ \int_{|v'| \leq 2N} \Omega(x_1 - (s - s_1)v') |h(s_1, x_1 - (s - s_1)v', v'')| \right\} \]

\[ \times 1_{|F(s_1, x_1 - (s - s_1)v', v'') - \mu| \geq \delta \mu} dv' \]

\[ \leq C_N \delta + C_N \left\{ \int_{|y| \leq 3N} |h(s_1, y, v'')| 1_{F(s_1, y, v'') - \mu| \geq \delta \mu} dy \right\} \]

\[ \leq C_N \delta + C_N \left\{ \frac{(s - s_1)^d + 1}{\kappa} \right\} \left\{ \int_{\Omega} |F - \mu| (s_1, y, v'') 1_{F(s_1, y, v'') - \mu| \geq \delta \mu} dy \right\}. \]

Here we have made a change of variable \( y = x_1 - (s - s_1)v' \), and for \( s - s_1 \geq \varepsilon \kappa, \frac{dy}{dv'} \geq \frac{1}{n_{x_1}} \). In the case of \( \Omega = \mathbb{R}^d \), the factor \( \{ (s_1 - s)^d + 1 \} \) is not needed. By further
integrating over \( v'' \) (bounded), we then control the last term in \([13]\) by \([14]\):

\[
\frac{C_{N, \varepsilon}}{\kappa^2} \int_0^t \int_0^s e^{-\frac{\psi(v)(t-s)}{\kappa}} e^{-\frac{\psi(v')s}{\kappa}} \times \left( \delta + \frac{(s-s_1)\kappa}{\kappa^d} + 1 \right) \Omega \left\{ \int_{|v''| \leq N} \varepsilon \right\} \left| \{ F - \mu \} (s_1, y, v'') \right| \int_{|v''| \leq N} \varepsilon \delta d\mu d\nu'' ds_1 ds
\]

\[
\leq \frac{C_{N, \varepsilon}}{\kappa^2} \int_0^t \int_0^s e^{-\frac{\psi(v)(t-s)}{\kappa}} e^{-\frac{\psi(v')s}{\kappa}} \times \left( (s-s_1)\kappa + 1 \right) ds_1 ds
\]

\[
\delta + \frac{1}{\kappa^d} \left( |\mathcal{H}(F_0) - \mathcal{H}(\mu)| + |M_0| + |J_0| + |E_0| \right)
\]

\[
\leq C_{N, \varepsilon} \left[ \delta + \frac{1}{\kappa^d} \left( |\mathcal{H}(F_0) - \mathcal{H}(\mu)| + |M_0| + |J_0| + |E_0| \right) \right]
\]

\[
\leq \frac{C_{N, \varepsilon}}{\kappa^{d/2}} \sqrt{|\mathcal{H}(F_0) - \mathcal{H}(\mu)| + |M_0| + |J_0| + |E_0|}.
\]

We have optimized \( \delta \) such that (for sufficiently small \( |\mathcal{H}(F_0) - \mathcal{H}(\mu)| + |M_0| + |J_0| + |E_0| \)),

\[
\delta = \frac{1}{\kappa^d} \left( |\mathcal{H}(F_0) - \mathcal{H}(\mu)| + |M_0| + |J_0| + |E_0| \right).
\]

In summary, we have established, for any \( \varepsilon > 0 \) and large \( N > 0 \),

\[
\sup_{0 \leq s \leq \tau} \| h(s) \|_{\infty} \leq \left\{ \varepsilon + \frac{C_{\varepsilon}}{N} \right\} \sup_{0 \leq s \leq \tau} \| h(s) \|_{\infty} + \| h(0) \|_{\infty} + C\left\{ \sup_{0 \leq s \leq \tau} \| h(s) \|_{\infty} \right\}^2 + \frac{C_{N, \varepsilon}}{\kappa^{d/2}} \sqrt{|\mathcal{H}(F_0) - \mathcal{H}(\mu)| + |M_0| + |J_0| + |E_0|}.
\]

First choosing \( \varepsilon \) small, then \( N \) sufficiently large so that \( \left\{ \varepsilon + \frac{C_{\varepsilon}}{N} \right\} < \frac{1}{2} \),

\[
\sup_{0 \leq s \leq \tau} \| h(s) \|_{\infty} \leq C\left\{ \| h(0) \|_{\infty} + \frac{1}{\kappa^{d/2}} \sqrt{|\mathcal{H}(F_0) - \mathcal{H}(\mu)| + |M_0| + |J_0| + |E_0|} \right\},
\]

and we conclude our proof provided the right-hand side is sufficiently small.

\[
\square
\]

\section*{References}
