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BOUNDED SOLUTIONS FOR THE BOLTZMANN EQUATION

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Dedicated to Professor W. A. Strauss on the occasion of his 70th birthday

Abstract. In either a periodic box $\mathbf{T}^d$ or $\mathbf{R}^d$ $(1 \leq d \leq 3)$, we establish a unified L^∞ estimate for solutions near Maxwellians for the Boltzmann equation, in terms of natural mass, momentum, energy conservation and the entropy inequality.

We study L^∞ estimates for the Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = \frac{1}{\kappa} Q(F, F), \qquad F(0, x, v) = F_0(x, v), \tag{1}$$

where $F(t, x, v) \geq 0$ is the density of particles of velocity $v \in \mathbf{R}^3$, and position $x \in \Omega = \mathbf{R}^d$ or $\mathbf{T}^d$, a periodic box, for $1 \leq d \leq 3$. The Knudsen number κ is a bounded constant. For simplicity we assume a hard-sphere interaction for the collision kernel Q. We define a global Maxwellian given by

$$\mu = \frac{\rho}{\{2\pi T\}^{3/2}} \exp\left\{-\frac{[v-u]^2}{2T}\right\}, \tag{2}$$

where ρ, u, T are independent of t and x. Our main result is

Theorem 1. Assume that the excess conservations of mass, momentum and energy,

$$\begin{aligned} \int\int \{F(t,x,v) - \mu\} dv dx &= \int\int \{F_0(x,v) - \mu\} dv dx \equiv M_0, \\ \int\int v\{F(t,x,v) - \mu\} dv dx &= \int\int v F_0(x,v) dv dx \equiv J_0, \\ \int\int |v|^2 \{F(t,x,v) - \mu\} dv dx &= \int\int |v|^2 \{F_0(x,v) - \mu\} dv dx \equiv E_0, \end{aligned} \tag{3}$$

as well as the excess entropy inequality, hold:

$$\mathcal{H}(F(t)) - \mathcal{H}(\mu) \leq \mathcal{H}(F_0) - \mathcal{H}(\mu), \tag{4}$$

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where $\mathcal{H}(g) \equiv \int\int g \ln g dv dx$. Then for any $\beta \geq 0$, there exists $C > 0$ such that

$$\sup_{0\leq t\leq\infty}\left\|\frac{\{1+|v|^2\}^\beta(F(t)-\mu)}{\sqrt{\mu}}\right\|_\infty \leq C\{\left\|\frac{\{1+|v|^2\}^\beta(F_0-\mu)}{\sqrt{\mu}}\right\|_\infty + \frac{1}{\kappa^{d/2}}\sqrt{\mathcal{H}(F_0)-\mathcal{H}(\mu)+|E_0|+|J_0|+|E_0|}\} \tag{5}$$

provided the right-hand side is sufficiently small.

Both the excess conservation laws (3) and the excess entropy inequality (4) are clearly valid when Ω is a periodic box. We remark that in the case $\Omega = \mathbf{R}^d$, local-in-time solutions satisfying both (3) and (4) can be constructed via the following approximate Boltzmann equation with finite propagation of speed in the physical space:

$$\partial_t F_n + v\mathbf{1}_{\{|v|\leq n\}}\cdot\nabla_x F_n = \frac{1}{\kappa}Q(F_n, F_n)$$

as $n \to \infty$.

It is well known that the pointwise control of F is crucial for uniqueness when β is large. The new L^∞ estimate (5) is solely based on the most natural a priori estimates in the Boltzmann theory. Even though no time decay rate is obtained, the proof is direct and robust.

Proof. Denote the weight function by $w(v) = \{1+|v|^2\}^\beta$ and define

$$h = w(v) \times \frac{F-\mu}{\sqrt{\mu}}.$$

Recall the standard linearized Boltzmann operator as

$$Lg = -\frac{1}{\sqrt{\mu}}\{Q(\mu, \sqrt{\mu}g) + Q(\sqrt{\mu}g, \mu)\} = \{\nu(v) + K\}g,$$

where the collision frequency $\nu(v) = c\int|v-u|\mu dv \backsim |v|+1$. Letting $K_w g \equiv wK(\frac{g}{w})$, we obtain

$$\partial_t h + v\cdot\nabla_x h + \frac{\nu}{\kappa}h + \frac{1}{\kappa}K_w h = \frac{1}{\kappa}w\Gamma(\frac{h}{w}, \frac{h}{w}),$$

where $\Gamma(g_1, g_2) = \frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}g_1, \sqrt{\mu}g_2)$.

For any (t, x, v), we denote $\bar{v} = (v_1, ..., v_d)$. Integrating along its backward trajectory $\frac{dX(s)}{ds} = V(s)$, $\frac{dV(s)}{ds} = 0$, we express $h(t, x, v)$ as

$$\exp\{-\frac{\nu t}{\kappa}\}h(0, x-\bar{v}t, v) + \int_0^t \exp\{-\frac{\nu(t-s)}{\kappa}\}\left(\frac{1}{\kappa}K_w h\right)(s, x-\bar{v}(t-s), v)ds + \int_0^t \exp\{-\frac{\nu(t-s)}{\kappa}\}\frac{w}{\kappa}\Gamma(\frac{h}{w}, \frac{h}{w})(s, x-\bar{v}(t-s), v)ds. \tag{6}$$

Since $|\frac{w}{\kappa}\Gamma(\frac{h}{w}, \frac{h}{w})(v)| \leq C\nu(v)||h||_\infty^2$ from Lemma 10 of [2], and since

$$\int_0^t \exp\{-\frac{\nu(t-s)}{\kappa}\}\nu ds \leq O(\kappa),$$

the last term in (6) is bounded by ($\nu(v)$ is bounded from below)

$$\frac{C}{\kappa}\int_0^t \exp\{-\frac{\nu(t-s)}{\kappa}\}\{\nu(v)|h(s,x-\bar{v}(t-s),v)|+||h(s)||_\infty\}||h(s)||_\infty ds \leq C\sup_{0\leq s\leq t}||h(s)||_\infty^2. \tag{7}$$

We shall mainly concentrate on the second term in (6). Let $\mathbf{k}(v,v')$ be the corresponding kernel accociated with K. We now use (6) again to evaluate $\{K_w h\}(s,x-(t-s)\bar{v}) = \int \mathbf{k}_w(v,v')h(s,x-\bar{v}(t-s),v')dv'$. By (7), we can bound the above by

$$\begin{aligned}
&\frac{1}{\kappa}\int_0^t \exp\{-\frac{\nu(t-s)}{\kappa}\}\int_{\mathbf{R}^3}|\mathbf{k}_w(v,v')\exp\{-\frac{\nu s}{\kappa}\}h(0,x-\bar{v}(t-s)-\bar{v}'s,v')|dv'ds\\
&+\frac{1}{\kappa^2}\int_0^t \exp\{-\frac{\nu(t-s)}{\kappa}\}\int_{\mathbf{R}^3\times\mathbf{R}^3}|\mathbf{k}_w(v,v')\mathbf{k}_w(v',v'')|\\
&\times|\int_0^s \exp\{-\frac{\nu(v')(s-s_1)}{\kappa}\}h(s_1,x-\bar{v}(t-s)-\bar{v}'(s-s_1),v'')|dv'dv''ds_1ds\\
&+\frac{C}{\kappa}\int_0^t \exp\{-\frac{\nu(t-s)}{\kappa}\}\times\sup_v\int_{\mathbf{R}^3}|\mathbf{k}_w(v,v')|dv'\times\{\sup_{0\leq s\leq t}||h(s)||_\infty^2\},
\end{aligned} \tag{8}$$

where $\mathbf{k}_w(\cdot) = w\mathbf{k}(\frac{\cdot}{w})$ and $\bar{v}' = (v_1',...,v_d')$. Since $\sup_v\int_{\mathbf{R}^3}\mathbf{k}_w|(v,v')|dv' < +\infty$ from Lemma 7 of [2], the first and the third terms above are bounded by $C||h(0)||_\infty + C\{\sup_{0\leq s\leq t}||h(s)||_\infty\}^2$.

We now concentrate on the second term in (8), which will be estimated as in the proof of Theorem 21 in [2].

Case 1. For $|v|\geq N$. By Lemma 7 in [2],

$$\int\int \mathbf{k}_w(v,v')\mathbf{k}_w(v',v'')dv'dv'' \leq \frac{C}{1+|v|}\leq\frac{C}{N}.$$

We therefore can find an upper bound for the second term in this case as

$$\begin{aligned}
\frac{C}{\kappa^2 N}\int_0^t \exp\{-\frac{\nu(v)(t-s)}{\kappa}\} \quad &\times\int_0^s \exp\{-\frac{\nu(v')(s-s_1)}{\kappa}\}||h(s_1)||_\infty ds_1 ds\\
&\leq \frac{C}{N}\sup_{0\leq s\leq t}||h(s)||_\infty.
\end{aligned}$$

Case 2. For $|v|\leq N$, $|v'|\geq 2N$, or $|v'|\leq 2N$, $|v''|\geq 3N$. Notice that we have either $|v'-v|\geq N$ or $|v'-v''|\geq N$, and either one of the following are valid correspondingly for some $\eta>0$:

$$|\mathbf{k}_w(v,v')|\leq e^{-\frac{\eta}{8}N^2}|\mathbf{k}_w(v,v')e^{\frac{\eta}{8}|v-v'|^2}|, \qquad |\mathbf{k}_w(v',v'')|\leq e^{-\frac{\eta}{8}N^2}|\mathbf{k}_w(v',v'')e^{\frac{\eta}{8}|v'-v''|^2}|. \tag{9}$$

From Lemma 8 in [2], both $\int |\mathbf{k}_w(v,v')e^{\frac{\eta}{8}|v-v'|^2}|$ and $\int |\mathbf{k}_w(v',v'')e^{\frac{\eta}{8}|v'-v''|^2}|$ are still finite. We use (9) to combine the cases of $|v'-v|\geq N$ or $|v'-v''|\geq N$ as:

$$\begin{aligned}
&\int_0^t\int_0^{s_1}\cdots\left\{\int_{|v|\leq N,|v'|\geq 2N} + \int_{|v'|\leq 2N,|v''|\geq 3N}\right\} \\
\leq\ & C\int_0^t\int_0^{s_1}\cdots\left\{\int_{|v|\leq N,|v'|\geq 2N}|\mathbf{k}_w(v,v')|dv' + \sup_{v'}\int_{|v'|\leq 2N,|v''|\geq 3N}|\mathbf{k}_w(v',v'')|dv''\right\} \\
\leq\ & \frac{C_\eta}{\kappa^2}e^{-\frac{\eta}{8}N^2}\int_0^t\int_0^{s_1}\exp\{-\frac{\nu(t-s)}{\kappa}\}\exp\{-\frac{v(s-s_1)}{\kappa}\}||h(s_1)||_\infty ds_1 ds \\
\leq\ & C_\eta e^{-\frac{\eta}{8}N^2}\sup_{0\leq s\leq t}\{||h(s)||_\infty\}. \qquad (10)
\end{aligned}$$

CASE 3. $s-s_1\leq\varepsilon\kappa$, for $\varepsilon>0$ small. We now can simply bound the second term in (8) by

$$\begin{aligned}
&\frac{1}{\kappa^2}\int_0^t\int_{s-\varepsilon\kappa}^s C\exp\{-\frac{\nu(t-s)}{\kappa}\}\exp\{-\frac{v(s-s_1)}{\kappa}\}||h(s_1)||_\infty ds_1 ds \\
\leq\ & C\sup_{0\leq s\leq t}\{||h(s)||_\infty\}\times\frac{1}{\kappa}\int_0^t\exp\{-\frac{\nu(t-s)}{\kappa}\}ds\times\int_{s-\varepsilon\kappa}^s\frac{1}{\kappa}ds_1 \\
\leq\ & \varepsilon C\sup_{0\leq s\leq t}\{||h(s)||_\infty\}. \qquad (11)
\end{aligned}$$

CASE 4. $s-s_1\geq\varepsilon\kappa$, and $|v|\leq N$, $|v'|\leq 2N, |v''|\leq 3N$. This is the last remaining case because if $|v'|>2N$, it is included in Case 2; while if $|v''|>3N$, either $|v'|\leq 2N$ or $|v'|\geq 2N$ are also included in Case 2. We now can bound the second term in (8) by

$$C\int_0^t\int_B\int_0^{s-\varepsilon\kappa}e^{-\frac{\nu(v)(t-s)}{\kappa}}e^{-\frac{v(v')(s-s_1)}{\kappa}}|\mathbf{k}_w(v,v')\mathbf{k}_w(v',v'')h(s_1,x_1-(s-s_1)\bar{v}',v'')|,$$

where $B=\{|v'|\leq 2N,\ |v''|\leq 3N\}$ and $x_1=x-(t-s)\bar{v}$. Notice that $\mathbf{k}_w(v,v')$ has a possible integrable singularity of $\frac{1}{|v-v'|}$. We can choose $\mathbf{k}_N(v,v')$ smooth with compact support such that

$$\sup_{|p|\leq 3N}\int_{|v'|\leq 3N}|\mathbf{k}_N(p,v')-\mathbf{k}_w(p,v')|dv'\leq\frac{1}{N}. \qquad (12)$$

Splitting

$$\begin{aligned}
\mathbf{k}_w(v,v')\mathbf{k}_w(v',v'') \quad = \quad & \{\mathbf{k}_w(v,v')-\mathbf{k}_N(v,v')\}\mathbf{k}_w(v',v'') \\
& +\{\mathbf{k}_w(v',v'')-\mathbf{k}_N(v',v'')\}\mathbf{k}_N(v,v')+\mathbf{k}_N(v,v')\mathbf{k}_N(v',v''),
\end{aligned}$$

we can use such an approximation (12) to bound the above s_1, s integration by

$$\begin{aligned}
&\frac{C}{N}\sup_{0\leq s\leq t}\{||h(s)||_\infty\}\times\left\{\sup_{|v'|\leq 2N}\int|\mathbf{k}_w(v',v'')|dv''+\sup_{|v|\leq 2N}\int|\mathbf{k}_w(v,v')|dv'\right\} \qquad (13) \\
&+C\int_0^t\int_B\int_0^{s-\varepsilon\kappa}e^{-\frac{\nu(v)(t-s)}{\kappa}}e^{-\frac{v(v')(s-s_1)}{\kappa}}|\mathbf{k}_N(v,v')\mathbf{k}_N(v',v'')|h(s,x_1-(s-s_1)\bar{v}',v'')|.
\end{aligned}$$

We now make use of the conservation laws (3) and the entropy inequality (4) to estimate the last term. Recall from the Taylor expansion,

$$\begin{aligned}\mathcal{H}(F(t))-\mathcal{H}(\mu) &= \int\int\{\ln\mu+1\}\{F-\mu\}+\int\int\frac{\{F(t)-\mu\}^2}{2\tilde{F}}\\ &\leq \mathcal{H}(F_0)-\mathcal{H}(\mu),\end{aligned}$$

where $\tilde{F}$ is between $F(t)$ and μ. Since $\mu=\frac{\rho}{\{2\pi T\}^{3/2}}\exp\left\{-\frac{[v-u]^2}{2T}\right\}, \ln\mu=\ln\frac{\rho}{\{2\pi T\}^{3/2}}-\frac{[v-u]^2}{2T}$. Therefore, from the conservations of mass, momentum and energy (3), we get

$$\int\int\frac{\{F(t)-\mu\}^2}{2\tilde{F}}\leq\mathcal{H}(F_0)-\mathcal{H}(\mu)+C_{\rho,u,T}\{|M_0|+|J_0|+|E_0|\}.$$

The key is to estimate $\frac{\{F(t)-\mu\}^2}{2\tilde{F}}$ in the case of $|F(t)-\mu|\geq\delta\mu$ for a small parameter δ. Notice that either $F(t)\leq\{1-\delta\}\mu$ or $F(t)-\mu\geq\delta\mu$ in this case. If $F(t)\leq\{1-\delta\}\mu$,

$$\frac{|F(t)-\mu|}{\tilde{F}(t)}\geq\frac{|F(t)-\mu|}{\mu}\geq 1-\frac{F(t)}{\mu}\geq 1-(1-\delta)=\delta.$$

On the other hand, if $F(t)\geq\{1+\delta\}\mu$,

$$\frac{|F(t)-\mu|}{\tilde{F}(t)}\geq\frac{|F(t)-\mu|}{F(t)}\geq 1-\frac{\mu}{F(t)}\geq 1-\frac{1}{1+\delta}=\frac{\delta}{1+\delta}.$$

In summary, we have

$$\begin{aligned}&\mathcal{H}(F_0)-\mathcal{H}(\mu)+C\{|M_0|+|J_0|+|E_0|\} \qquad (14)\\ \geq& \int\int\frac{\{F(t)-\mu\}^2}{2\tilde{F}}\mathbf{1}_{|F(t)-\mu|\leq\delta\mu}+\int\int\frac{\{F(t)-\mu\}^2}{2\tilde{F}}\mathbf{1}_{|F(t)-\mu|\geq\delta\mu}\\ \geq& \int\int\frac{\{F(t)-\mu\}^2}{2(1+\delta)\mu}\mathbf{1}_{|F(t)-\mu|\leq\delta\mu}+\frac{1}{2}\frac{\delta}{1+\delta}\int|F(t)-\mu|\mathbf{1}_{|F(t)-\mu|\geq\delta\mu}.\end{aligned}$$

Since $\mathbf{k}_N(v,v')\mathbf{k}_N(v',v'')$ is bounded, we first integrate over v' (bounded) to get

$$\begin{aligned}& C_N\int_{|v'|\leq 2N}|h(s_1,x_1-(s-s_1)\bar{v}',v'')|\mathbf{1}_{|F(s_1,x_1-(s-s_1)\bar{v}',v'')-\mu|\leq\delta\mu}dv'\\ &+C_N\int_{|v'|\leq 2N}|h(s_1,x_1-(s-s_1)\bar{v}',v'')|\mathbf{1}_{|F(s_1,x_1-(s-s_1)\bar{v}',v'')-\mu|\geq\delta\mu}dv'\\ \leq\;& C_N\delta+C_N\left\{\int_{|\bar{v}'|\leq 2N}\mathbf{1}_\Omega(x_1-(s-s_1)\bar{v}')|h(s_1,x_1-(s-s_1)\bar{v}',v'')|\right.\\ &\left.\times\,\mathbf{1}_{|F(s_1,x_1-(s-s_1)\bar{v}',v'')-\mu|\geq\delta\mu}d\bar{v}'\right\}\\ \leq\;& C_N\delta+\frac{C_N}{\kappa^d\varepsilon^d}\left\{\int_{|y-x_1|\leq(s-s_1)3N}|h(s_1,y,v'')|\mathbf{1}_{|F(s_1,y,v'')-\mu|\geq\delta\mu}dy\right\}\\ \leq\;& C_N\delta+\frac{C_N\{(s-s_1)^d+1\}}{\kappa^d\varepsilon^d}\left\{\int_{\mathbf{\Omega}}|\{F-\mu\}(s_1,y,v'')|\mathbf{1}_{|F(s_1,y,v'')-\mu|\geq\delta\mu}dy\right\}.\end{aligned}$$

Here we have made a change of variable $y=x_1-(s-s_1)\bar{v}'$, and for $s-s_1\geq\varepsilon\kappa$, $\frac{dy}{dv'}\geq\frac{1}{\kappa^d\varepsilon^d}$. In the case of $\Omega=\mathbf{R}^d$, the factor $\{(s_1-s)^d+1\}$ is not needed. By further integrating over v'' (bounded), we then control the last term in (13) by (14):

$$
\begin{aligned}
& \frac{C_{N,\varepsilon}}{\kappa^2}\int_0^t\int_0^s e^{-\frac{\nu(v)(t-s)}{\kappa}} e^{-\frac{v(v')(s-s_1)}{\kappa}} \\
& \times(\delta+\frac{\{(s-s_1)^d+1\}}{\kappa^d}\int_{|v''|\leq 3N}\int_{\mathbf{\Omega}}|\{F-\mu\}(s_1,y,v'')|\mathbf{1}_{|F(s_1,y,v'')-\mu|\geq\delta\mu}dydv'')ds_1ds \\
\leq\ & \frac{C_{N,\varepsilon}}{\kappa^2}\int_0^t\int_0^s e^{-\frac{\nu(v)(t-s)}{\kappa}} e^{-\frac{v(v')(s-s_1)}{\kappa}}\{(s-s_1)^d+1\}ds_1ds \\
& \times\left[\delta+\frac{1}{\kappa^d\delta}\{|\mathcal{H}(F_0)-\mathcal{H}(\mu)|+|M_0|+|J_0|+|E_0|\}\right] \\
\leq\ & C_{N,\varepsilon}\left[\delta+\frac{1}{\kappa^d\delta}\{|\mathcal{H}(F_0)-\mathcal{H}(\mu)|+|M_0|+|J_0|+|E_0|\}\right] \\
\leq\ & \frac{C_{N,\varepsilon}}{\kappa^{d/2}}\sqrt{|\mathcal{H}(F_0)-\mathcal{H}(\mu)|+|M_0|+|J_0|+|E_0|\}}.
\end{aligned}
$$

We have optimized δ such that (for sufficiently small $|\mathcal{H}(F_0)-\mathcal{H}(\mu)|+|M_0|+|J_0|+|E_0|$),

$$
\delta=\frac{1}{\kappa^d\delta}\{|\mathcal{H}(F_0)-\mathcal{H}(\mu)|+|M_0|+|J_0|+|E_0|\}.
$$

In summary, we have established, for any $\varepsilon>0$ and large $N>0$,

$$
\begin{aligned}
\sup_{0\leq s\leq\infty}||h(s)||_\infty \quad\leq\quad & \{\varepsilon+\frac{C_\varepsilon}{N}\}\sup_{0\leq s\leq t}||h(s)||_\infty+||h(0)||_\infty+C\{\sup_{0\leq s\leq t}||h(s)||_\infty\}^2 \\
& +\frac{C_{N,\varepsilon}}{\kappa^{d/2}}\sqrt{|\mathcal{H}(F_0)-\mathcal{H}(\mu)|+|M_0|+|J_0|+|E_0|\}}.
\end{aligned}
$$

First choosing ε small, then N sufficiently large so that $\{\varepsilon+\frac{C_\varepsilon}{N}\}<\frac{1}{2}$,

$$
\sup_{0\leq s\leq\tau}||h(s)||_\infty\leq C\{||h(0)||_\infty+\frac{1}{\kappa^{d/2}}\sqrt{|\mathcal{H}(F_0)-\mathcal{H}(\mu)|+|M_0|+|J_0|+|E_0|}\},
$$

and we conclude our proof provided the right-hand side is sufficiently small. □

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