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AN ENERGETIC VIEW ON THE LIMIT ANALYSIS OF NORMAL BODIES

By

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Abstract. This note presents a limit analysis for normal materials based on energy minimization. The class of normal materials includes some of those used to model masonry structures, namely, no–tension materials and materials with bounded compressive strength; it also includes the Hencky plastic materials. Considering loads $\mathfrak{L}(\lambda)$ that depend affinely on the loading multiplier $\lambda \in \mathbb{R}$, we examine the infimum $I_0(\lambda)$ of the potential energy $I(\boldsymbol{u}, \lambda)$ over the set of all admissible displacements $\boldsymbol{u}$. Since $I_0(\lambda)$ is a concave function of λ, the set Λ of all λ with $I_0(\lambda) > -\infty$ is an interval. Each finite endpoint $\lambda_c \in \mathbb{R}$ of Λ is called a collapse multiplier, and we interpret the loads corresponding to λ_c as the loads at which the collapse of the structure occurs. We show that the standard definition of collapse based on the collapse mechanism does not capture all situations: the collapse mechanism is sufficient but not necessary for the collapse. We then examine the validity of the static and kinematic theorems of limit analysis under the present definition. We show that the static theorem holds unconditionally while the kinematic theorem holds for Hencky plastic materials and materials with bounded compressive strength. For no–tension materials it generally does not hold; a weaker version is given for this class of materials.

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1. Introduction. Certain materials cannot support all stresses: in plastic materials the stresses are delimited by the yield criteria, and masonry materials are incapable of withstanding (all or large) tensile stresses. The bodies made of such materials then cannot support all loads; certain loads lead to the collapse of the body. The goal of the limit analysis is to determine the limit load, i.e., the largest possible load prior to collapse. It is customary to assume that the loads depend affinely on a scalar parameter λ, the loading multiplier, as described below, and the problem reduces to determining the collapse multiplier, i.e., the value of λ corresponding to the limit load. Limit analysis is traditionally based on the static and kinematic theorems, which determine the limit load as the supremum of statically admissible multipliers and the infimum of kinematically admissible multipliers, respectively. The traditional definition identifies the collapse multiplier as one with the collapse mechanism (as postulated in Definition 2.5(iv), below; see also Remark 2.6(iv), below). The reader is referred to [4] for the proofs of the static and kinematic theorems under this definition. Our definition of the collapse multiplier is different since (as we argue in examples below) a collapse can occur without collapse mechanisms and the collapse mechanism is only a sufficient condition for the collapse.

Our goal is to examine the validity of the static and kinematic theorems under our definition of the collapse for bodies made of normal materials, with a particular attention to masonry materials.

A normal material (called a normal linear elastic material in [3, Definition 3.3], [4, Section III]) is completely characterized by a convex set of admissible stresses K (the stress range), and by the tensor of elastic constants C. The response of a normal material is nonlinear elastic as it is governed by the projections onto K. Normal materials include Hencky plastic materials [17] (the deformation theory of plasticity) but our main interest is in no–tension materials [2], [8], [3], [5], [12] (with the stress range the set of all negative semidefinite symmetric tensors) and the materials of bounded compressive strength [11] (with the stress range as in Definition 2.3(ii), below); these are used to model masonry structures. If C possesses the major symmetry, which we assume, then the normal material is hyperelastic, with the stored energy of at most a quadratic growth but possibly with no growth at all in certain directions of the strain space.

Considering the loads $\mathfrak{L}(\lambda)$ that depend affinely on the loading multiplier $\lambda \in \mathbb{R}$, we examine the potential energy $I(\boldsymbol{u}, \lambda)$ of the displacement $\boldsymbol{u}$ under the loads $\mathfrak{L}(\lambda)$. For $I_0(\lambda)$ the infimum of $I(\boldsymbol{u}, \lambda)$ over the set of all admissible displacements $\boldsymbol{u}$, the function $\lambda \mapsto I_0(\lambda)$ is concave and thus the set Λ of all λ with $I_0(\lambda) > -\infty$ is an interval (which can be empty). We call each finite endpoint $\lambda_c \in \mathbb{R}$ of Λ a collapse multiplier, i.e., the value at which the collapse occurs in parallel with the same definition in [17, Chapter I, Section 5] in case of the Hencky plasticity. The definition has a dynamical motivation in terms of processes of masonry bodies with dissipation [16]: if $I_0(\lambda) > -\infty$ then, for large times, processes starting from arbitrary initial data stabilize and converge to the set of equilibrium states; on the other hand, if $I_0(\lambda) = -\infty$, the processes blow up in the sense of norms, i.e., the collapse occurs.

The existence of the collapse mechanism generally leads to the collapse in our sense (Theorem 2.7). In general, the collapse multiplier λ_c can be such that $I_0(\lambda_c) > -\infty$ or $I_0(\lambda_c) = -\infty$; this distinguishes our general normal material from the Hencky plasticity, for which Témam [17, Chapter I, Remark 5.1] shows that $I_0(\lambda_c) > -\infty$. We also prove this assertion for any material with a bounded stress range (in particular for materials with bounded compressive strength) but for a no–tension body it may happen that $I_0(\lambda_c) = -\infty$ as our examples show. Generalizing the argument in [17, Chapter I, Section 4] we show that λ_c, if it exists, is the supremum of all statically admissible loading multipliers, i.e., of multipliers for which the loads can be equilibrated by an admissible stressfield in the space $L^2(\Omega, \mathrm{Sym})$ of square integrable stresses; this is essentially the static theorem under the present notion of collapse. In contrast, the kinematic theorem generally does not hold for no–tension materials, as our (very singular) example shows despite the fact that it holds for the Hencky plasticity (as can be deduced from the results in [18]) and for materials with bounded compressive strength (as we show below). The reason for the failure in the case of no–tension materials is that the stress range is a cone; this makes the variational problem of the kinematic theorem degenerate in the sense that the effective domain of the involved function has empty interior. We introduce a perturbed variational problem (with an extra parameter) that formally approaches the variational problem of the kinematic theorem. The perturbed problem gives the correct value of the collapse multiplier.

In Section 2 we introduce the constitutive equations of normal materials and basic notions of the limit analysis and we summarize the general results. In Section 3 we present additional results on no–tension materials and examples without proof. The rest of the paper is devoted to a more detailed exposition and to the proofs.

Throughout we use the conventions for vectors and second-order tensors identical to those in [9]. Thus Lin denotes the set of all second-order tensors on $\mathbb{R}^n$, i.e., linear transformations from $\mathbb{R}^n$ into itself, Sym is the subspace of symmetric tensors, Sym^+ the set of all positive semidefinite elements of Sym; additionally, Sym^- is the set of all negative semidefinite elements of Sym and Sym_0 is the space of all traceless elements of Sym. The scalar product of $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{Lin}$ is defined by $\boldsymbol{A} \cdot \boldsymbol{B} = \mathrm{tr}(\boldsymbol{A}\boldsymbol{B}^{\mathsf{T}})$ and $|\cdot|$ denotes the associated euclidean norm on Lin.

2. Limit analysis for normal materials.

A normal material is completely determined by a fourth-order tensor of *elastic constants* C, interpreted as a linear transformation from Sym into itself, and by a *stress range* $K \subset \mathrm{Sym}$, such that

$$\left.\begin{aligned} \boldsymbol{E} \cdot \mathsf{C}\boldsymbol{E} > 0 \quad & \text{for all } \boldsymbol{E} \in \mathrm{Sym}, \boldsymbol{E} \neq \boldsymbol{0}, \\ \boldsymbol{E}_1 \cdot \mathsf{C}\boldsymbol{E}_2 = \boldsymbol{E}_2 \cdot \mathsf{C}\boldsymbol{E}_1 \quad & \text{for all } \boldsymbol{E}_1, \boldsymbol{E}_2 \in \mathrm{Sym}, \end{aligned}\right\} \tag{2.1}$$

and

$$K \quad \text{is a nonempty closed convex set.} \tag{2.2}$$

The elastic constants C and the stress range K determine the (nonlinear) response functions $\hat{\boldsymbol{T}}, \hat{w}$ of a normal material via the following proposition.

PROPOSITION 2.1. Assume (2.1) and (2.2). If $\boldsymbol{E} \in \mathrm{Sym}$, there exists a unique triplet $(\boldsymbol{T}, \boldsymbol{E}^{\mathrm{e}}, \boldsymbol{E}^{\mathrm{a}})$ of elements of Sym such that

$$\left.\begin{array}{c} \boldsymbol{E} = \boldsymbol{E}^{\mathrm{e}} + \boldsymbol{E}^{\mathrm{a}}, \\ \boldsymbol{T} = \mathsf{C}\boldsymbol{E}^{\mathrm{e}}, \\ \boldsymbol{T} \in K, \\ (\boldsymbol{T} - \boldsymbol{S}) \cdot \boldsymbol{E}^{\mathrm{a}} \geq 0 \quad \text{for each} \quad \boldsymbol{S} \in K. \end{array}\right\} \tag{2.3}$$

We define the stress $\hat{\boldsymbol{T}}(\boldsymbol{E})$ and the stored energy $\hat{w}(\boldsymbol{E})$ by

$$\begin{gathered} \hat{\boldsymbol{T}}(\boldsymbol{E}) = \boldsymbol{T}, \\ \hat{w}(\boldsymbol{E}) = \boldsymbol{T} \cdot \boldsymbol{E} - \tfrac{1}{2}\boldsymbol{T} \cdot \mathsf{C}^{-1}\boldsymbol{T} \equiv \tfrac{1}{2}\mathsf{C}\boldsymbol{E}^{\mathrm{e}} \cdot \boldsymbol{E}^{\mathrm{e}} + \boldsymbol{T} \cdot \boldsymbol{E}^{\mathrm{a}}; \end{gathered} \tag{2.4}$$

the map $\hat{\boldsymbol{T}} : \mathrm{Sym} \to \mathrm{Sym}$ is monotone and Lipschitz continuous. The function $\hat{w} : \mathrm{Sym} \to \mathbb{R}$ is continuously differentiable, convex and $D\hat{w} = \hat{\boldsymbol{T}}$.

DEFINITION 2.2. The functions $\hat{\boldsymbol{T}}$ and $\hat{w}$ constructed in Proposition 2.1 are called the response functions of the *normal material* determined by C and K.

The definition is identical with that of normal linear elastic materials in [3, Definition 3.3], [4, Section III]; these are generalizations of materials considered in [17, Chapter I, Subsection 3.3, Item ii)] to a general stress range K and to nonisotropic elastic constants.

By (2.3), the total strain $\boldsymbol{E}$ is decomposed into the elastic and anelastic parts $\boldsymbol{E}^{\mathrm{e}}, \boldsymbol{E}^{\mathrm{a}}$ in such a way that the stress $\boldsymbol{T}$, depending linearly on the elastic strain, belongs to the stress range K and $\boldsymbol{E}^{\mathrm{a}}$ is in the *normal cone to* K at $\boldsymbol{T}$. See Proposition 4.1 (below) for additional properties of $\hat{\boldsymbol{T}}$ and $\hat{w}$.

DEFINITIONS 2.3. A normal material determined by C and K is said to be a

(i) *no–tension material* if $K = \mathrm{Sym}^-$;

(ii) *material with bounded compressive strength* if $K = \{\boldsymbol{T} \in \mathrm{Sym}^- : \boldsymbol{T} + \sigma^{\mathrm{c}}\mathbf{1} \in \mathrm{Sym}^+\}$ where σ^{c} is a nonnegative number;

(iii) *Hencky plastic material* if $K = \{\boldsymbol{S} - p\mathbf{1} : \boldsymbol{S} \in K_0,\, p \in \mathbb{R}\}$ where $K_0 \subset \mathrm{Sym}_0$ is a closed bounded convex set in Sym_0 such that $\mathbf{0}$ is in the relative interior of K_0.

We refer to [8], [2], [3], [5], [4], [10] for no–tension materials, to [11] and [12] for materials with bounded compressive strength and to [18], [17] for the Hencky plastic materials.

Let Ω be a reference configuration of a continuous body made of a normal material; it is assumed that Ω is a bounded connected open set with Lipschitz boundary $\partial\Omega$ of outer normal $\boldsymbol{n}$ in the sense of [1]. The body has a prescribed displacement $\boldsymbol{d}$ on an area measurable subset $\mathcal{D}$ of $\partial\Omega$ while on $\mathcal{S} := \partial\Omega \setminus \mathcal{D}$ the body is subjected to surface tractions depending on the loading multiplier as specified below. We assume that $\boldsymbol{d}$ is the restriction of the trace of some element $\boldsymbol{z} \in W$ of the Sobolev space

$$W := W^{1,2}(\Omega, \mathbb{R}^n)$$

of $\mathbb{R}^n$ valued functions on Ω [1]. Given $\boldsymbol{d}$, we choose and keep $\boldsymbol{z}$ fixed throughout the paper. We admit the cases $\mathcal{D} = \partial\Omega$ (the pure displacement problem) and $\mathcal{S} = \partial\Omega$ (the pure traction problem). We define the affine space U of admissible displacements and the linear space V of admissible variations of displacements by

$$U := \{\boldsymbol{u} \in W : \boldsymbol{u} = \boldsymbol{d} \text{ on } \mathcal{D}\},$$
$$V := \{\boldsymbol{v} \in W : \boldsymbol{v} = \boldsymbol{0} \text{ on } \mathcal{D}\}$$

where the equalities on $\mathcal{D}$ are understood in the sense of traces. We have

$$U = V + \boldsymbol{z}.$$

We assume that the body is subjected to loads consisting of the surface traction on $\mathcal{S}$ and a body force in Ω; both the surface traction and the body force depend affinely on a real parameter λ called the *loading multiplier.* Thus for a given $\lambda \in \mathbb{R}$, $\boldsymbol{s}(\lambda) : \mathcal{S} \to \mathbb{R}^n$ and $\boldsymbol{b}(\lambda) : \Omega \to \mathbb{R}^n$ are given by

$$\boldsymbol{s}(\lambda) = \boldsymbol{s}_0 + \lambda \bar{\boldsymbol{s}}, \quad \boldsymbol{b}(\lambda) = \boldsymbol{b}_0 + \lambda \bar{\boldsymbol{b}},$$

where

$$\boldsymbol{s}_0,\, \bar{\boldsymbol{s}} \in L^2(\mathcal{S}, \mathbb{R}^n), \quad \boldsymbol{b}_0,\, \bar{\boldsymbol{b}} \in L^2(\Omega, \mathbb{R}^n).$$

We call the pair $\mathfrak{L}(\lambda) = (\boldsymbol{s}(\lambda), \boldsymbol{b}(\lambda))$ the loads corresponding to λ. Denoting by $\langle \boldsymbol{f}, \boldsymbol{v} \rangle$ the value of an element $\boldsymbol{f} \in V^*$ of the dual V^* of V on $\boldsymbol{v} \in V$, we define the *work of the loads* $\mathfrak{L}(\lambda)$ as an element $\boldsymbol{l}(\lambda) \in V^*$ given by

$$\boldsymbol{l}(\lambda) = \boldsymbol{l}_0 + \lambda \bar{\boldsymbol{l}}$$

where

$$\langle \boldsymbol{l}_0, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{b}_0 \cdot \boldsymbol{v} \, d\mathcal{L}^n + \int_{\mathcal{S}} \boldsymbol{s}_0 \cdot \boldsymbol{v} \, d\mathcal{H}^{n-1},$$
$$\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = \int_\Omega \bar{\boldsymbol{b}} \cdot \boldsymbol{v} \, d\mathcal{L}^n + \int_{\mathcal{S}} \bar{\boldsymbol{s}} \cdot \boldsymbol{v} \, d\mathcal{H}^{n-1},$$

$\boldsymbol{v} \in V$, with $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$ the volume and area measures. The *potential energy* of the loads $\mathfrak{L}(\lambda)$ is a function $I(\cdot, \lambda) : U \to \mathbb{R}$ given by

$$I(\boldsymbol{u}, \lambda) = F(\boldsymbol{u}) - \langle \boldsymbol{l}(\lambda), \boldsymbol{u} - \boldsymbol{z} \rangle,$$

$\boldsymbol{u} \in U$, where $F : U \to \mathbb{R}$ is the *internal energy* given by

$$F(\boldsymbol{u}) = \int_\Omega \hat{w} \circ \hat{\boldsymbol{E}}(\boldsymbol{u}) \, d\mathcal{L}^n,$$

$\boldsymbol{u} \in U$ with $\hat{\boldsymbol{E}}(\boldsymbol{u}) := \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathsf{T}})$ the infinitesimal strain tensor of $\boldsymbol{u}$. The Lipschitz continuity of the stress function (see Proposition 4.1, below) implies that $|\hat{w}(\boldsymbol{E})| \le |\boldsymbol{E}|^2/2k$ for some $k > 0$ and all $\boldsymbol{E} \in \mathrm{Sym}$; hence $-\infty < I(\boldsymbol{u}, \lambda) < \infty$ and $-\infty < F(\boldsymbol{u}) < \infty$ for all $\lambda \in \mathbb{R}$ and all $\boldsymbol{u} \in U$.

Central to our considerations is the *infimum energy* of the loads $\mathfrak{L}(\lambda)$ defined as $I_0(\lambda) \in \mathbb{R} \cup \{-\infty\}$ by

$$I_0(\lambda) = \inf\{I(\boldsymbol{u}, \lambda) : \boldsymbol{u} \in U\}. \tag{2.5}$$

We say that $\boldsymbol{u} \in U$ is an *equilibrium state* for the loads $\mathfrak{L}(\lambda)$ if $I(\boldsymbol{u}, \lambda) = I_0(\lambda)$. Since $I(\cdot, \lambda)$ is not coercive on U, the infimum in (2.5) need not be attained and it can happen that $I_0(\lambda) = -\infty$. Our main concern is the relation $I_0(\lambda) > -\infty$, i.e., the boundedness from below of $I(\cdot, \lambda)$ on U. One can have $I_0(\lambda) > -\infty$ even if there is no equilibrium state. The set of all equilibrium states from the Sobolev space $W^{1,2}(\Omega, \mathbb{R}^n)$ may be empty and yet there may be equilibrium states in $BD(\Omega)$ [2], [8], [17].

We denote by

$$Y = L^2(\Omega, \mathrm{Sym})$$

the space of all Sym valued square integrable functions with respect to $\mathcal{L}^n$, endowed with the L^2 scalar product $(\boldsymbol{A}, \boldsymbol{B}) = \int_\Omega \boldsymbol{A} \cdot \boldsymbol{B}\, d\mathcal{L}^n$; we further denote by

$$Y_K$$

the set of all $\boldsymbol{T} \in Y$ such that for $\mathcal{L}^n$ a.e. point of Ω the corresponding value of $\boldsymbol{T}$ is in the stress range K. We say that $\boldsymbol{T} \in Y$ is an *admissible* stressfield if $\boldsymbol{T} \in Y_K$; we say that $\boldsymbol{T}$ is an *admissible equilibrating stressfield* for the loads $\mathfrak{L}(\lambda)$ if $\boldsymbol{T}$ is admissible and equilibrates the loads $\mathfrak{L}(\lambda)$ in the sense that

$$\big(\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})\big) = \langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \tag{2.6}$$

for each $\boldsymbol{v} \in V$ [3], [4], [2], [8]. We denote by $\mathcal{A}(\lambda)$ the set of all admissible stressfields equilibrating the loads $\mathfrak{L}(\lambda)$. The loads $\mathfrak{L}(\lambda)$ are said to be *compatible* [3] if $\mathcal{A}(\lambda) \neq \emptyset$.

PROPOSITION 2.4.

(i) The loads $\mathfrak{L}(\lambda)$ are compatible if and only if $I_0(\lambda) > -\infty$.

(ii) The function $I_0 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is concave and uppersemicontinuous, i.e.,

$$I_0(\alpha\lambda + (1-\alpha)\mu) \geq \alpha I_0(\lambda) + (1-\alpha) I_0(\mu)$$

for every $\lambda, \mu \in \mathbb{R}$ and $\alpha \in [0,1]$ and

$$I_0(\lambda) \geq \limsup_{k\to\infty} I_0(\lambda_k)$$

for every $\lambda \in \mathbb{R}$ and every sequence $\lambda_k \to \lambda$. Hence the set

$$\Lambda = \{\lambda \in \mathbb{R} : I_0(\lambda) > -\infty\} \equiv \{\lambda \in \mathbb{R} : \mathcal{A}(\lambda) \neq \emptyset\} \tag{2.7}$$

is an interval.

Since the notion of compatibility of loads is independent of the tensor of elastic constants C, the finiteness of $I_0(\lambda)$ is also independent of C [within the class specified by (2.1)], even though the concrete value of $I_0(\lambda)$ depends on C. We emphasize the role of the square integrability requirement of the stressfield in the definition of compatible loads; Example 3.4 (below) shows that for no–tension bodies there are loads $\mathfrak{L}(\lambda)$ with $I_0(\lambda) = -\infty$ and yet with $\mathfrak{L}(\lambda)$ being weakly equilibrated by a stressfield $\boldsymbol{T} \in L^1(\Omega, \mathrm{Sym}) \setminus L^2(\Omega, \mathrm{Sym})$ with values in $K = \mathrm{Sym}^-$.

Item (i) is an extension of [17, Theorem 4.1, Chapter I] to our more general case and Item (ii) is implicit in [17, Proof of Proposition 4.1, Chapter I].

For a normal material with the stress range K we define the function $\hat{m} : \mathrm{Sym} \to \mathbb{R} \cup \{\infty\}$ [17] by

$$\hat{m}(\boldsymbol{E}) = \sup\{\boldsymbol{T} \cdot \boldsymbol{E} : \boldsymbol{T} \in K\}, \tag{2.8}$$

$\boldsymbol{E} \in \mathrm{Sym}$. Clearly $\hat{m}$ is convex, positively 1 homogeneous, lowersemicontinuous, and bounded from below by an affine function (since K is nonempty); $\hat{m}$ is finite valued if and only if K is bounded. We furthermore define $G : V \to \mathbb{R} \cup \{\infty\}$ by

$$G(\boldsymbol{v}) = \int_\Omega \hat{m} \circ \hat{\boldsymbol{E}}(\boldsymbol{v})\, d\mathcal{L}^n,$$

$\boldsymbol{v} \in V$.

Definitions 2.5. Let Λ be given by (2.7). A loading multiplier $\lambda \in \mathbb{R}$ is said to

(i) be *statically admissible* if $\lambda \in \Lambda$; otherwise λ is said to be *statically inadmissible*;

(ii) be a *collapse multiplier* if it is a finite endpoint of Λ;

(iii) be *kinematically admissible* if there exists a $\boldsymbol{v} \in V$ such that $\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 1$ and

$$\lambda = G(\boldsymbol{v}) - \langle \boldsymbol{l}_0, \boldsymbol{v} \rangle; \tag{2.9}$$

(iv) *admit a collapse mechanism* if λ is kinematically admissible and $\lambda \le \sup \Lambda$.

Remarks 2.6.

(i) For materials with bounded compressive strength and for Hencky plastic materials, each collapse multiplier is statically admissible (see Theorem 2.9, below). For no–tension materials, the collapse multiplier can be statically admissible as well as statically inadmissible; see Remark 2.10 (below).

(ii) For a class of materials that includes no–tension materials, materials with bounded compressive strength and Hencky plastic materials, the definition of a kinematically admissible multiplier can be reformulated to a more standard form using normal cones to Y_K; see Remark 2.8, below.

(iii) If λ admits a collapse mechanism then there exists a $\boldsymbol{v} \in V$ with $\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 1$ and $\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle = G(\boldsymbol{v})$; each such $\boldsymbol{v}$ is said to be a *collapse mechanism for the loads* $\mathfrak{L}(\lambda)$.

(iv) If λ admits a collapse mechanism and if additionally λ is statically admissible then each admissible equilibrating stressfield for $\mathfrak{L}(\lambda)$ is called a *collapse stressfield.* A stronger version of the definition of collapse mechanism $\boldsymbol{v}$ in [4] requires that $\boldsymbol{v}$ be as in (iii) and that additionally λ be statically admissible. Example 3.4 (below) provides a statically inadmissible collapse multiplier with a collapse mechanism in our sense.

The number of collapse multipliers ranges from 0 to 2. In applications, one is interested in the larger of the possibly two collapse multipliers. Motivated by this, we introduce the multiplier

$$\lambda_{\mathrm{c}}^{+} := \sup\{\lambda \in \mathbb{R} : \lambda \text{ is statically admissible}\}$$

$-\infty \le \lambda_{\mathrm{c}}^{+} \le \infty$; thus if λ_{c}^{+} is finite, then λ_{c}^{+} is a collapse multiplier, and if there are two collapse multipliers, then λ_{c}^{+} is the larger of these two. Motivated by the kinematic theorem, we consider the multiplier

$$\bar{\lambda}_{\mathrm{c}}^{+} = \inf\{\lambda \in \mathbb{R} : \lambda \text{ is kinematically admissible}\};$$

equivalently ([17, Chapter I, Subsection 5.2], [18])

$$\bar{\lambda}_{\mathrm{c}}^{+} := \inf\{G(\boldsymbol{v}) - \langle \boldsymbol{l}_0, \boldsymbol{v} \rangle : \boldsymbol{v} \in V, \langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 1\}.$$

For the relationships among λ_{c}^{+} and $\bar{\lambda}_{\mathrm{c}}^{+}$, see Theorem 2.11, below.

Our first result shows that our definition of the collapse multiplier generalizes that based on the collapse mechanism:

THEOREM 2.7. If $\lambda \in \mathbb{R}$ admits a collapse mechanism, then $\lambda = \lambda_{\mathrm{c}}^{+}$.

We distinguish the following three special cases in the treatment below:

H_1: K is a cone;

H_2: K is bounded;

H_3:

$$K = \{\boldsymbol{S} - p\mathbf{1} : \boldsymbol{S} \in K_0, p \in \mathbb{R}\} \tag{2.10}$$

where $K_0 \subset \mathrm{Sym}_0$ is compact.

Thus $\mathrm{H}_1, \mathrm{H}_2$ and H_3 cover a no–tension material, materials with bounded compressive strength, and Hencky plasticity, respectively.

REMARK 2.8. If $\boldsymbol{T} \in Y_K$, we define the normal cone $\mathrm{N}(Y_K, \boldsymbol{T})$ to Y_K at $\boldsymbol{T}$ by

$$\begin{aligned}\mathrm{N}(Y_K, \boldsymbol{T}) &= \{\boldsymbol{D} \in Y : (\boldsymbol{D}, \boldsymbol{T} - \boldsymbol{S}) \geq 0 \text{ for each admissible stressfield } \boldsymbol{S}\} \\ &= \{\boldsymbol{D} \in Y : \boldsymbol{D} \cdot (\boldsymbol{T} - \boldsymbol{U}) \geq 0 \text{ for every } \boldsymbol{U} \in K \text{ and } \mathcal{L}^n \text{ a.e. point of } \Omega\}.\end{aligned}$$

Let $\lambda \in \mathbb{R}$.

(i) If one of H_1–H_3 holds, then λ is a kinematically admissible multiplier if and only if there exist $\boldsymbol{v} \in V$ and $\boldsymbol{T} \in Y_K$ with

$$\hat{\boldsymbol{E}}(\boldsymbol{v}) \in \mathrm{N}(Y_K, \boldsymbol{T}), \quad \langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 1 \tag{2.11}$$

and

$$\lambda = \big(\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})\big) - \langle \boldsymbol{l}_0, \boldsymbol{v} \rangle; \tag{2.12}$$

without the hypotheses H_1–H_3, Conditions (2.11) and (2.12) are only sufficient for λ to be kinematically admissible.

(ii) If $\boldsymbol{v} \in V$ satisfies (2.11) and (2.12) for some $\boldsymbol{T} \in \mathcal{A}(\lambda)$, then $\boldsymbol{v}$ is a collapse mechanism for the loads $\mathfrak{L}(\lambda)$.

THEOREM 2.9. If the material satisfies either H_2 or H_3, then any collapse multiplier is statically admissible; i.e., the interval Λ in (2.7) is closed.

Témam [17, Chapter I, Remark 5.1] proves the assertion for the Hencky plastic materials (covered by H_3) under slightly stronger additional hypotheses on $\mathcal{D}$ and $\mathcal{S}$; see Remark 7.2 (below).

REMARK 2.10. For no–tension materials, there are loads:

(i) with a statically admissible collapse multiplier without a collapse mechanism (see Example 3.3, below);

(ii) with a statically inadmissible collapse multiplier with a collapse mechanism (see Example 3.4, below).

THEOREM 2.11. We have

(i)

$$\lambda_{\mathrm{c}}^{+} \leq \bar{\lambda}_{\mathrm{c}}^{+}; \tag{2.13}$$

(ii) if $\lambda_{\rm c}^{+} > -\infty$ and if, in addition, either $\rm H_2$ holds or $\rm H_3$ holds and

$$\left.\begin{array}{c}\Omega \text{ is contained in a bounded Lipschitz set } \Sigma \subset \mathbb{R}^n \text{ with } \mathcal{D} \subset \partial\Sigma \\ \text{and with } R_0 := \{\boldsymbol{w}|\Omega : \boldsymbol{w} \in W_0^{1,2}(\Sigma, \mathbb{R}^n)\} \text{ dense in } V,\end{array}\right\} \tag{2.14}$$

then

$$\lambda_{\rm c}^{+} = \bar{\lambda}_{\rm c}^{+}. \tag{2.15}$$

Here $\boldsymbol{w}|\Omega$ denotes the restriction of $\boldsymbol{w}$ to Ω. Thus under the hypotheses of (ii) we have the kinematic theorem of limit analysis. For no–tension materials, Example 3.5 (below) shows that a strict inequality can hold in (2.13). Note, however, that if $\lambda_{\rm c}^{+}$ admits a collapse mechanism then $\lambda_{\rm c}^{+}$ is kinematically admissible and thus (2.15) holds. Témam [17, Theorem 5.1, Chapter I] proves (2.15) for Hencky plastic materials under slightly stronger additional hypotheses on $\mathcal{D}$ and $\mathcal{S}$; see Remark 7.2 (below). Recall that $\rm H_3$ covers Hencky plastic materials.

3. Loading no–tension bodies. In this section we consider a no–tension body Ω and loads $\mathfrak{L}(\lambda) = (\boldsymbol{s}(\lambda), \boldsymbol{b}(\lambda))$. In this case Y_K is the set Y^- of all $\boldsymbol{T} \in Y$ taking negative semidefinite values for $\mathcal{L}^n$ a.e. point of Ω; we furthermore denote by Y^+ the set of all $\boldsymbol{E} \in Y$ taking positive semidefinite values for $\mathcal{L}^n$ a.e. point of Ω.

If $\boldsymbol{E} \in \mathrm{Sym}$, we denote by $\boldsymbol{E}^{\pm}$ the positive and negative parts of $\boldsymbol{E}$, i.e., the unique pair of tensors in Sym^+ such that $\boldsymbol{E} = \boldsymbol{E}^+ - \boldsymbol{E}^-$ and $\boldsymbol{E}^+ \cdot \boldsymbol{E}^- = 0$. If $\boldsymbol{v} \in V$, we define $\hat{\boldsymbol{E}}^{\pm}(\boldsymbol{v})$ to be the positive and negative parts of $\hat{\boldsymbol{E}}(\boldsymbol{v})$ for $\mathcal{L}^n$ a.e. point of Ω; we furthermore denote by $\hat{\boldsymbol{E}}^{\rm e}(\boldsymbol{v}), \hat{\boldsymbol{E}}^{\rm a}(\boldsymbol{v})$ the elastic and anelastic parts of the strain $\hat{\boldsymbol{E}}(\boldsymbol{v})$ as determined in Proposition 2.1. Recall the outer normal $\boldsymbol{n}$ to Ω; we say that a point $\boldsymbol{x} \in \mathcal{S}$ is a class 2 point if $\mathcal{S}$ is locally of class 2 in a neighborhood of $\boldsymbol{x}$.

PROPOSITION 3.1. Consider loads $\mathfrak{L}(\cdot)$ for a no–tension body. Then:

(i) the following three conditions are equivalent for any $\lambda \in \mathbb{R}$:

(a) $\lambda \in \mathbb{R}$ is statically admissible;

(b) we have $\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \le \eta\, |\hat{\boldsymbol{E}}^{-}(\boldsymbol{v})|_Y$ for all $\boldsymbol{v} \in V$ and some η;

(c) we have $\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \le \eta\, |\hat{\boldsymbol{E}}^{\rm e}(\boldsymbol{v})|_Y$ for all $\boldsymbol{v} \in V$ and some η.

These conditions imply that

$$\boldsymbol{s}(\lambda) \cdot \boldsymbol{n} \le 0 \tag{3.1}$$

for $\mathcal{H}^{n-1}$ a.e. class 2 point of $\mathcal{S}$;

(ii) the multiplier $\lambda \in \mathbb{R}$ is kinematically admissible if and only if there exists a $\boldsymbol{v} \in V$ with

$$\hat{\boldsymbol{E}}(\boldsymbol{v}) \in Y^+ \quad \text{and} \quad \langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 1 \tag{3.2}$$

such that $\lambda = -\langle \boldsymbol{l}_0, \boldsymbol{v} \rangle$; hence

$$\bar{\lambda}_{\rm c}^{+} = \inf\{-\langle \boldsymbol{l}_0, \boldsymbol{v} \rangle : \boldsymbol{v} \in V, \hat{\boldsymbol{E}}(\boldsymbol{v}) \in Y^+, \langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 1\}; \tag{3.3}$$

(iii) if λ is statically admissible, then the following conditions are equivalent for each $\boldsymbol{v} \in V$:

(a) $\boldsymbol{v}$ is a collapse mechanism for λ;

(b) $\boldsymbol{v}$ satisfies (3.2) and

$$\boldsymbol{T} \cdot \hat{\boldsymbol{E}}(\boldsymbol{v}) = 0 \tag{3.4}$$

for $\mathcal{L}^n$ a.e. point of Ω and some admissible stressfield $\boldsymbol{T}$ equilibrating the loads $\mathfrak{L}(\lambda)$;

(c) $\boldsymbol{v}$ satisfies (3.2) and (3.4) for every admissible stressfield $\boldsymbol{T}$ equilibrating the loads $\mathfrak{L}(\lambda)$.

As mentioned in Section 2, the kinematic theorem generally does not hold. The reason is that the expression in (3.3) strictly excludes deformations $\boldsymbol{v} \in V$ with $\hat{\boldsymbol{E}}^{-}(\boldsymbol{v}) \neq \boldsymbol{0}$, no matter how small $\hat{\boldsymbol{E}}^{-}(\boldsymbol{v})$. A way to remedy the situation is to admit deformations with $\hat{\boldsymbol{E}}^{-}(\boldsymbol{v}) \neq \boldsymbol{0}$, but to penalize their occurrence. Motivated by this, we define

$$\hat{\lambda}_{\mathrm{c}}^{+} = \lim_{\eta \to \infty} \inf\{\eta |\hat{\boldsymbol{E}}^{-}(\boldsymbol{v})|_Y - \langle \boldsymbol{l}_0, \boldsymbol{v} \rangle : \boldsymbol{v} \in V, \langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 1\}. \tag{3.5}$$

The limit exists (possibly as ∞) since the infimum is a nondecreasing function of η; we note that by formally exchanging the order of the limit and infimum in (3.5), one obtains the problem (3.3). Also observe that the expression in (3.5) is not of the form of an integral of some density over Ω because of the occurrence of the L^2 norm $|\cdot|_Y$.

THEOREM 3.2. For a no–tension body we have

$$\lambda_{\mathrm{c}}^{+} = \hat{\lambda}_{\mathrm{c}}^{+}.$$

This can be interpreted as a weak version of the kinematic theorem of limit analysis for no–tension bodies.

We now consider four examples to clarify the relationships among the notions of limit analysis for masonry bodies. The first three of them deal with a rectangular panel in $\mathbb{R}^2$ of the form

$$\Omega = (0, b) \times (0, h)$$

where $b > 0, h > 0$, made of a no–tension material while the last deals with a circular ring made of a material with bounded compressive strength. In all examples we assume that body is free from body forces,

$$\boldsymbol{b}_0 = \bar{\boldsymbol{b}} = \boldsymbol{0} \quad \text{ in } \ \Omega, \tag{3.6}$$

and subject to different loads $\boldsymbol{s}_0, \bar{\boldsymbol{s}}$ on $\mathcal{S}$. We denote by $\boldsymbol{r} = (x, y)$ a general point of $\mathbb{R}^2$ and by $\boldsymbol{i}, \boldsymbol{j}$ the unit vectors in the x and y directions.

EXAMPLE 3.3 (Statically admissible collapse multiplier without a collapse mechanism). Assume that Ω has fixed base $\mathcal{D} = (0, b) \times \{0\}$, with $\boldsymbol{d} = \boldsymbol{0}$ on $\mathcal{D}$, its top $\mathcal{T} = (0, b) \times \{h\}$ is subjected to a constant permanent vertical load and to a tangential traction of a parabolic shape while the lateral sides $\mathcal{S} \setminus \mathcal{T}$ are free from forces; see Figure 1. We identify the loading multiplier with the intensity of the tangential part of the load. Specifically, we put

$$\boldsymbol{s}_0 = -p\boldsymbol{j} \quad \text{ on } \ \mathcal{T};$$
$$\bar{\boldsymbol{s}}(\boldsymbol{r}) = -\big(4px(b - x)/b^2\big)\boldsymbol{i} \quad \text{ if } \ \boldsymbol{r} = (x, y) \in \mathcal{T},$$

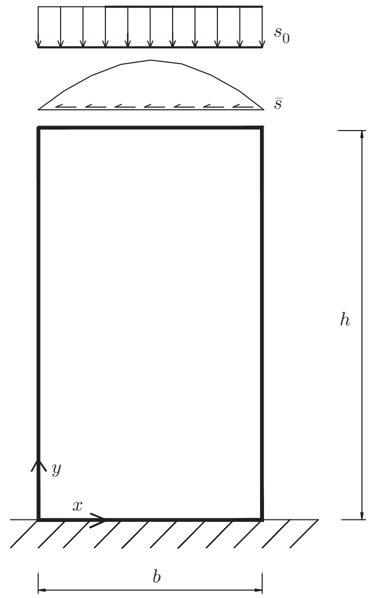

FIG. 1. Parabolic tangential traction

where $p > 0$ is a fixed number and

$$\boldsymbol{s}_0 = \bar{\boldsymbol{s}} = \mathbf{0} \quad \text{ on } \ \mathcal{S} \setminus \mathcal{T}.$$

If $\lambda \in \mathbb{R}$, then:

(i) λ *is statically admissible if and only if* $|\lambda| \leq \lambda_{\mathrm{c}} := b/4h$;

(ii) *if* $\lambda \in (\lambda_{\mathrm{c}}, 3\lambda_{\mathrm{c}})$*, then* λ *is kinematically admissible.* λ_{c} *is not kinematically admissible*; *hence* λ_{c} *is a statically admissible collapse multiplier which does not admit a collapse mechanism.*

EXAMPLE 3.4 (Statically inadmissible collapse multiplier with a collapse mechanism). Assume again that Ω has fixed base $\mathcal{D} = (0, b) \times \{0\}$, with $\boldsymbol{d} = \mathbf{0}$ on $\mathcal{D}$, its top $\mathcal{T} = (0, b) \times \{h\}$ is subjected to a permanent constant vertical load and to a tangential traction of a linear shape while the lateral sides $\mathcal{S} \setminus \mathcal{T}$ are free from forces; see Figure 2. We identify the loading multiplier with the slope of the linear tangential load. Accordingly,

$$\boldsymbol{s}_0(\boldsymbol{r}) = -p\boldsymbol{j}, \quad \bar{\boldsymbol{s}}(\boldsymbol{r}) = -x\boldsymbol{i} \quad \text{ if } \quad \boldsymbol{r} = (x, y) \in \mathcal{T},$$
$$\boldsymbol{s}_0 = \bar{\boldsymbol{s}} = \mathbf{0} \quad \text{ on } \ \mathcal{S} \setminus \mathcal{T},$$

where p is a given positive number. If $\lambda \in \mathbb{R}$, then:

(i) λ *is statically admissible if and only if* $|\lambda| < \lambda_{\mathrm{c}} := p/h$;

(ii) λ *is kinematically admissible if and only if* $|\lambda| \geq \lambda_{\mathrm{c}}$;

hence λ_{c} is statically inadmissible and kinematically admissible collapse multiplier [consequently, λ_{c} admits a collapse mechanism in the sense of Definition 2.5(iv) and Remark 2.6(iii)]. Moreover, despite of the fact that there is no admissible stressfield equilibrating the loads $\mathfrak{L}(\lambda_{\mathrm{c}})$, there exists a stressfield $\boldsymbol{T} \in L^1(\Omega, \mathrm{Sym}) \setminus L^2(\Omega, \mathrm{Sym})$ with values in

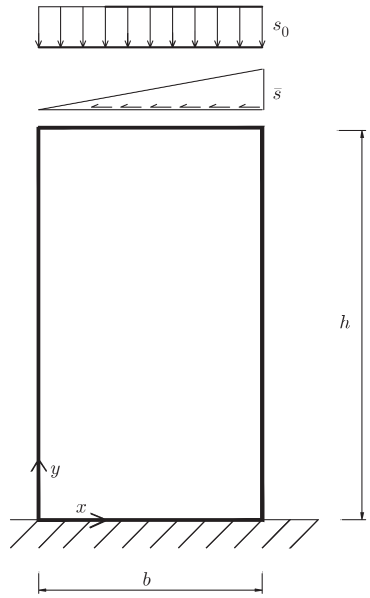

FIG. 2. Linear tangential traction

Sym^- such that

$$\langle \boldsymbol{l}(\lambda_{\mathrm{c}}), \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{T} \cdot \hat{\boldsymbol{E}}(\boldsymbol{v})\, d\mathcal{L}^2 \tag{3.7}$$

for every $\boldsymbol{v} \in V \cap W^{1,\infty}(\Omega, \mathbb{R}^2)$. A similar statement holds for $-\lambda_{\mathrm{c}}$.

EXAMPLE 3.5 (Violation of the kinematic theorem). Let $\mathcal{S} = (0, b) \times \{h\}$, $\mathcal{D} = \partial\Omega \setminus \mathcal{S}$, and $\boldsymbol{d} = \boldsymbol{0}$ on $\mathcal{D}$. Let the loads consist of a uniform pressure λ on $\mathcal{S}$, i.e.,

$$\boldsymbol{s}(\lambda) = \lambda \boldsymbol{n} \quad \text{ on } \ \mathcal{S};$$

see Figure 3. If $\lambda \in \mathbb{R}$, then:

(i) *λ is statically admissible if and only if $\lambda \le 0$;*

(ii) *there is no kinematically admissible multiplier;*

hence $0 = \lambda_{\mathrm{c}}^+ < \bar{\lambda}_{\mathrm{c}}^+ = \infty$.

EXAMPLE 3.6. Consider the ring

$$\Omega = \{\boldsymbol{r} \in \mathbb{R}^2 : a < |\boldsymbol{r}| < b\}$$

$(0 < a < b)$ made of a material with bounded compressive strength under pure traction conditions $\mathcal{D} = \emptyset$,

$$\mathcal{S} = \mathcal{S}_a \cup \mathcal{S}_b, \quad \mathcal{S}_a = \{\boldsymbol{r} \in \mathbb{R}^2 : |\boldsymbol{r}| = a\}, \quad \mathcal{S}_b = \{\boldsymbol{r} \in \mathbb{R}^2 : |\boldsymbol{r}| = b\},$$

free from body forces as in (3.6), subject to a uniform fixed pressure p on the inner part of its boundary and variable pressure λ on the outer part of its boundary, i.e.,

$$\boldsymbol{s}_0 = -p\boldsymbol{n}, \quad \bar{\boldsymbol{s}} = \boldsymbol{0} \quad \text{ on } \ \mathcal{S}_a,$$
$$\boldsymbol{s}_0 = \boldsymbol{0}, \quad \bar{\boldsymbol{s}} = -\boldsymbol{n} \quad \text{ on } \ \mathcal{S}_b,$$

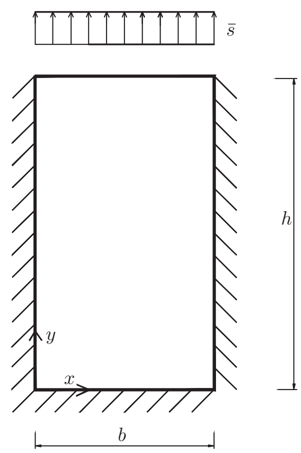

FIG. 3. Pulling load on the top

where p is a constant satisfying $0 \le p \le \sigma^{\mathrm{c}}$ and $\boldsymbol{n}$ is the outer normal to Ω. Putting

$$\lambda_{\mathrm{c}}^{-} = \eta p, \quad \lambda_{\mathrm{c}}^{+} = \eta p + \sigma^{\mathrm{c}}(1-\eta), \quad \eta := a/b, \tag{3.8}$$

we have the following assertions:

(i) *The multiplier $\lambda \in \mathbb{R}$ is statically admissible if and only if*

$$\lambda_{\mathrm{c}}^{-} \le \lambda \le \lambda_{\mathrm{c}}^{+};$$

(ii) *the multiplier λ_{c}^{+} is kinematically admissible and thus it admits a collapse mechanism.*

4. Proofs: normal materials. Here we prove Proposition 2.1 and the following additional properties of normal materials. If $F : W \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$ is a function on a topological linear space, then $F^* : W^* \to \bar{\mathbb{R}}$ is the convex conjugate function defined on the dual W^* of W by

$$F^*(\boldsymbol{\phi}) = \sup\{\langle \boldsymbol{\phi}, \boldsymbol{u} \rangle - F(\boldsymbol{u}) : \boldsymbol{u} \in W\},$$

$\boldsymbol{\phi} \in W^*$, where $\langle \cdot, \cdot \rangle$ is the duality pairing between W^* and W [6, Part One].

PROPOSITION 4.1. The response functions of a normal material satisfy

$$(\hat{\boldsymbol{T}}(\boldsymbol{F}) - \hat{\boldsymbol{T}}(\boldsymbol{E})) \cdot (\boldsymbol{F} - \boldsymbol{E}) \ge k|\hat{\boldsymbol{T}}(\boldsymbol{F}) - \hat{\boldsymbol{T}}(\boldsymbol{E})|^2, \tag{4.1}$$

$$|\hat{\boldsymbol{T}}(\boldsymbol{F}) - \hat{\boldsymbol{T}}(\boldsymbol{E})| \le k^{-1}|\boldsymbol{F} - \boldsymbol{E}|, \tag{4.2}$$

$$\hat{w}(\boldsymbol{F}) \ge \hat{w}(\boldsymbol{E}) + \hat{\boldsymbol{T}}(\boldsymbol{E}) \cdot (\boldsymbol{F} - \boldsymbol{E}) + \tfrac{1}{2}k|\hat{\boldsymbol{T}}(\boldsymbol{F}) - \hat{\boldsymbol{T}}(\boldsymbol{E})|^2 \tag{4.3}$$

for any $\boldsymbol{E}, \boldsymbol{F} \in \mathrm{Sym}$ where

$$k := \inf\{\boldsymbol{A} \cdot \mathsf{C}^{-1}\boldsymbol{A} : \boldsymbol{A} \in \mathrm{Sym}, |\boldsymbol{A}| = 1\} > 0. \tag{4.4}$$

We have

$$\hat{w}^*(\boldsymbol{T}) = \begin{cases} \frac{1}{2}\boldsymbol{T}\cdot\mathsf{C}^{-1}\boldsymbol{T} & \text{if } \boldsymbol{T}\in K, \\ \infty & \text{if } \boldsymbol{T}\in \mathrm{Sym}\setminus K. \end{cases} \tag{4.5}$$

Cf. [3, Proposition 4.4 and Lemma 5.1] for (4.1)–(4.3) in case $K = \mathrm{Sym}^-$ and [17, Chapter I, Eq. (3.41)] for (4.5) which defines $\hat{w}^*$ first and introduces $\hat{w}$ as the convex conjugate of $\hat{w}^*$ without giving the explicit form (2.4).

Proof of Propositions 2.1 *and* 4.1. We introduce the energetic scalar product on Sym by $(\boldsymbol{A},\boldsymbol{B})_E = \boldsymbol{A}\cdot\mathsf{C}^{-1}\boldsymbol{B}$, $\boldsymbol{A},\boldsymbol{B}\in\mathrm{Sym}$, denote by $\mathsf{P}:\mathrm{Sym}\to K$ the orthogonal projection onto K with respect to $(\cdot,\cdot)_E$, and put

$$\boldsymbol{T} = \mathsf{PC}\boldsymbol{E},\quad \boldsymbol{E}^{\mathrm{e}} = \mathsf{C}^{-1}\mathsf{PC}\boldsymbol{E},\quad \boldsymbol{E}^{\mathrm{a}} = \boldsymbol{E} - \mathsf{C}^{-1}\mathsf{PC}\boldsymbol{E},$$

which leads immediately to (2.3) and to the equality between the second and third terms in (2.4). To prove (4.1)–(4.3), let $\boldsymbol{E},\boldsymbol{F}\in\mathrm{Sym}$ and put $\boldsymbol{T} = \hat{\boldsymbol{T}}(\boldsymbol{E}), \boldsymbol{U} = \hat{\boldsymbol{T}}(\boldsymbol{F})$. From $(2.3)_4$ we obtain

$$(\boldsymbol{T}-\boldsymbol{U})\cdot(\boldsymbol{E}-\mathsf{C}^{-1}\boldsymbol{T}) \ge 0,\quad (\boldsymbol{U}-\boldsymbol{T})\cdot(\boldsymbol{F}-\mathsf{C}^{-1}\boldsymbol{U}) \ge 0; \tag{4.6}$$

summing these two inequalities and rearranging we obtain

$$(\boldsymbol{T}-\boldsymbol{U})\cdot\mathsf{C}^{-1}(\boldsymbol{T}-\boldsymbol{U}) \le (\boldsymbol{T}-\boldsymbol{U})\cdot(\boldsymbol{E}-\boldsymbol{F});$$

using (4.4) we obtain (4.1). Using the Schwarz inequality on the left-hand side of (4.1) we obtain (4.2). To prove (4.3), one finds that

$$\hat{w}(\boldsymbol{F}) - \hat{w}(\boldsymbol{E}) - \boldsymbol{T}\cdot(\boldsymbol{F}-\boldsymbol{E}) - \tfrac{1}{2}|\boldsymbol{T}-\boldsymbol{U}|_E^2 = (\boldsymbol{U}-\boldsymbol{T})\cdot(\boldsymbol{F}-\mathsf{C}^{-1}\boldsymbol{U}).$$

The last expression is nonnegative by $(4.6)_2$ and hence

$$\hat{w}(\boldsymbol{F}) - \hat{w}(\boldsymbol{E}) - \boldsymbol{T}\cdot(\boldsymbol{F}-\boldsymbol{E}) - \tfrac{1}{2}|\boldsymbol{T}-\boldsymbol{U}|_E^2 \ge 0;$$

a reference to (4.4) then yields (4.3) and hence also the convexity of $\hat{w}$. To prove that $\hat{w}$ is continuously differentiable and $\hat{\boldsymbol{T}}$ is its derivative, we note that using (4.3) twice we obtain

$$\hat{\boldsymbol{T}}(\boldsymbol{F})\cdot(\boldsymbol{F}-\boldsymbol{E}) \ge \hat{w}(\boldsymbol{F}) - \hat{w}(\boldsymbol{E}) \ge \hat{\boldsymbol{T}}(\boldsymbol{E})\cdot(\boldsymbol{F}-\boldsymbol{E})$$

for any $\boldsymbol{E},\boldsymbol{F}\in\mathrm{Sym}$; dividing by $|\boldsymbol{E}-\boldsymbol{F}|$, letting $\boldsymbol{F}\to\boldsymbol{E}$, using $\hat{\boldsymbol{T}}(\boldsymbol{F})\to\hat{\boldsymbol{T}}(\boldsymbol{E})$ and invoking the definition of the Fréchet derivative we obtain $D\hat{w}(\boldsymbol{E}) = \hat{\boldsymbol{T}}(\boldsymbol{E})$. To prove (4.5), let $\hat{h}:\mathrm{Sym}\to\mathbb{R}\cup\{\infty\}$ be the function defined by the right-hand side of (4.5). We calculate the convex conjugate $\hat{h}^*(\boldsymbol{E})$ of h at $\boldsymbol{E}\in\mathrm{Sym}$. We note that if $(\boldsymbol{T},\boldsymbol{E}^{\mathrm{e}},\boldsymbol{E}^{\mathrm{a}})$ is the triple associated with $\boldsymbol{E}$ as in Proposition 2.1, then algebraic manipulations show that $(2.3)_4$ can be rewritten as

$$\boldsymbol{T}\cdot\boldsymbol{E} - \hat{h}(\boldsymbol{T}) \ge \boldsymbol{S}\cdot\boldsymbol{E} - \hat{h}(\boldsymbol{S}) + \tfrac{1}{2}(\boldsymbol{T}-\boldsymbol{S})\cdot\mathsf{C}^{-1}(\boldsymbol{T}-\boldsymbol{S}) \tag{4.7}$$

for every $\boldsymbol{S}\in K$ with the equality if $\boldsymbol{S}=\boldsymbol{T}$. Since (4.7) also holds if $\boldsymbol{S}\notin K$ as the right-hand side is $-\infty$ in that case, we have

$$\boldsymbol{T}\cdot\boldsymbol{E} - \hat{h}(\boldsymbol{T}) \ge \boldsymbol{S}\cdot\boldsymbol{E} - \hat{h}(\boldsymbol{S})$$

for all $\boldsymbol{S}\in\mathrm{Sym}$ and thus the definition gives $\hat{h}^*(\boldsymbol{E}) = \boldsymbol{T}\cdot\boldsymbol{E} - \hat{h}(\boldsymbol{T}) \equiv \hat{w}(\boldsymbol{E})$ [by (2.4)]. Then $\hat{w}^* = \hat{h}^{**} = \hat{h}$ by [7, Theorem 4.92(iii)] since $\hat{h}$ is lowersemicontinuous, convex and bounded from below by an affine (continuous) function. The proof of (4.5) is complete. □

5. Proofs: compatibility of loads and collapse mechanisms. We base our considerations on the following version of the Hahn Banach theorem [7, Theorem A.35]:

THEOREM 5.1. Let $H : X \to \mathbb{R}$ be a convex function on a vectorspace X and let $L_0 : X_0 \to \mathbb{R}$ be a linear function on a linear subspace X_0 of X such that

$$L_0(x) \le H(x) \quad \text{for every} \quad x \in X_0.$$

Then L_0 has a linear extension $L : X \to \mathbb{R}$ such that

$$L(x) \le H(x) \quad \text{for every} \quad x \in X.$$

PROPOSITION 5.2. Let $H : Y \to \mathbb{R}$ be a continuous convex function, let $\lambda \in \mathbb{R}$ and $\boldsymbol{w} \in W$. Then we have

$$\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \le H(\hat{\boldsymbol{E}}(\boldsymbol{v}) + \hat{\boldsymbol{E}}(\boldsymbol{w})) + c \quad \text{for all} \quad \boldsymbol{v} \in V \quad \text{and some} \quad c \in \mathbb{R} \tag{5.1}$$

if and only if there exists a stressfield $\boldsymbol{T} \in Y$ equilibrating the loads $\mathfrak{L}(\lambda)$ such that

$$H^*(\boldsymbol{T}) < \infty. \tag{5.2}$$

If these conditions are satisfied, then also

$$(\boldsymbol{T}, \boldsymbol{A}) \le H(\boldsymbol{A}) + (\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{w})) + c \quad \text{for all} \quad \boldsymbol{A} \in Y. \tag{5.3}$$

The hypotheses on H are satisfied if

$$H(\boldsymbol{A}) = \int_\Omega h(\boldsymbol{A})\, d\mathcal{L}^n \tag{5.4}$$

for all $\boldsymbol{A} \in Y$ where $h : \mathrm{Sym} \to \mathbb{R}$ is a convex function such that $|h(\boldsymbol{E})|$ is bounded by some quadratic function of $|\boldsymbol{E}|$ for all $\boldsymbol{E} \in \mathrm{Sym}$; Condition (5.2) then reads

$$\int_\Omega h^*(\boldsymbol{T})\, d\mathcal{L}^n < \infty. \tag{5.5}$$

Proof. Let $X_0 := \{\hat{\boldsymbol{E}}(\boldsymbol{v}) : \boldsymbol{v} \in V\}$ so that $X_0 \subset Y$ and let $L_0 : X_0 \to \mathbb{R}$ be defined by

$$L_0(\hat{\boldsymbol{E}}(\boldsymbol{v})) = \langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \tag{5.6}$$

for each $\boldsymbol{v} \in V$. Then (5.1) reads

$$L_0(\boldsymbol{A}) \le H(\boldsymbol{A} + \hat{\boldsymbol{E}}(\boldsymbol{w})) + c \quad \text{for all} \quad \boldsymbol{A} \in X_0$$

and hence by Theorem 5.1 there exists a linear extension $L : Y \to \mathbb{R}$ of L_0 such that

$$L(\boldsymbol{A}) \le H(\boldsymbol{A} + \hat{\boldsymbol{E}}(\boldsymbol{w})) + c \quad \text{for all} \quad \boldsymbol{A} \in Y. \tag{5.7}$$

The continuity of H implies the continuity of L and hence L can be represented by an element $\boldsymbol{T} \in Y$ as a scalar product in Y; then (5.7) reads

$$(\boldsymbol{T}, \boldsymbol{A}) \le H(\boldsymbol{A} + \hat{\boldsymbol{E}}(\boldsymbol{w})) + c \quad \text{for all} \quad \boldsymbol{A} \in Y. \tag{5.8}$$

A replacement of $\boldsymbol{A} + \hat{\boldsymbol{E}}(\boldsymbol{w})$ by $\boldsymbol{A}$ then gives (5.3). Relation (5.6) then gives

$$(\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})) = \langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle$$

for each $\boldsymbol{v} \in V$ and thus $\boldsymbol{T}$ equilibrates the loads $\mathfrak{L}(\lambda)$. Relation (5.8) then gives $H^*(\boldsymbol{T}) \leq (\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{w})) + c$ and hence (5.2). To prove the converse part of the statement, we let $\boldsymbol{T}$ be a stressfield equilibrating the loads $\mathfrak{L}(\lambda)$ satisfying (5.2). Then

$$\infty > H^*(\boldsymbol{T}) := \sup\{(\boldsymbol{T}, \boldsymbol{A}) - H(\boldsymbol{A}) : \boldsymbol{A} \in Y\}$$

from which

$$H(\boldsymbol{A}) - (\boldsymbol{T}, \boldsymbol{A}) \geq -H^*(\boldsymbol{T}) \quad \text{for all} \quad \boldsymbol{A} \in Y.$$

Taking $\boldsymbol{A} = \hat{\boldsymbol{E}}(\boldsymbol{v}) + \hat{\boldsymbol{E}}(\boldsymbol{w})$ where $\boldsymbol{v} \in V$, this is rewritten as (5.1) [with $c = H^*(\boldsymbol{T}) - (\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{w}))$].

If H is of the form (5.4) with h satisfying the hypotheses of the proposition, then [7, Theorem 5.9] shows that H is a continuous, finite-valued function on Y, convex by the convexity of h. Furthermore, [6, Proposition IX.2.1] gives

$$H^*(\boldsymbol{T}) = \int_\Omega h^*(\boldsymbol{T})\, d\mathcal{L}^n$$

for all $\boldsymbol{T} \in Y$ and thus we have (5.5). □

Proof of Proposition 2.4. (i): Let λ be a loading multiplier such that $I_0(\lambda) > -\infty$. We apply Proposition 5.2 with H given by (5.4) where $h = \hat{w}$ is the stored energy of a normal material and $\boldsymbol{w} = \boldsymbol{z}$. The hypotheses on h are satisfied in view of Proposition 4.1. The condition

$$F(\boldsymbol{v} + \boldsymbol{z}) - I_0(\lambda) \geq \langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \quad \text{for all} \quad \boldsymbol{v} \in V$$

is equivalent to Condition (5.1) with $c = -I_0(\lambda)$; by Proposition 5.2 this is equivalent to the existence of a stressfield $\boldsymbol{T} \in Y$ equilibrating the loads $\mathfrak{L}(\lambda)$ such that

$$\int_\Omega \hat{w}^*(\boldsymbol{T})\, d\mathcal{L}^n < \infty.$$

By (4.5) the last condition is satisfied if and only if the (essential) range of $\boldsymbol{T}$ is contained in K, i.e., if and only if $\boldsymbol{T}$ is admissible. To summarize, the condition $I_0(\lambda) > -\infty$ is equivalent to $\mathcal{A}(\lambda) \neq \emptyset$. The proof of (i) is complete.

(ii): The affine dependence of $\boldsymbol{l}(\lambda)$ on λ implies that the function $\lambda \mapsto I(\boldsymbol{u}, \lambda)$ is affine for each $\boldsymbol{u} \in U$; thus the function $\lambda \mapsto I_0(\lambda)$, being the lower envelope of the family of affine continuous functions over the parameter set $\{\boldsymbol{u} \in U\}$, is concave and uppersemicontinuous [6, Chapter I, Section 2]. □

REMARK 5.3. Theorem 2.4(i) is proved in [18], [17] for Hencky plastic materials (Hypothesis H_3) and in [16] for no–tension materials (Hypothesis H_1) by evaluating the dual problem of the problem (2.5) in the sense of [6] and noting that the problem (2.5) is regular in the sense of [17]. The same can also be applied in our general case. However, we believe that the above proof is simpler and more direct.

Proof of Theorem 2.7. Since λ admits a collapse mechanism, λ is kinematically admissible and hence there exists a $\boldsymbol{v} \in V$ with

$$\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 1 \quad \text{and} \quad G(\boldsymbol{v}) = \langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle. \tag{5.9}$$

It follows from the definition of G that

$$\big(\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})\big) \leq \langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \quad \text{for every admissible stressfield } \boldsymbol{T}. \tag{5.10}$$

Prove that if $\boldsymbol{v} \in V$ satisfies (5.10) then

$$I(\boldsymbol{u}, \lambda) \geq I(\boldsymbol{u} + \boldsymbol{v}, \lambda) \tag{5.11}$$

for any $\boldsymbol{u} \in U$. Indeed (4.3) gives

$$\hat{w} \circ \hat{\boldsymbol{E}}(\boldsymbol{u}) \geq \hat{w} \circ \hat{\boldsymbol{E}}(\boldsymbol{u} + \boldsymbol{v}) - \boldsymbol{T} \cdot \hat{\boldsymbol{E}}(\boldsymbol{v})$$

for a.e. point of Ω where $\boldsymbol{T} := \hat{\boldsymbol{T}} \circ \hat{\boldsymbol{E}}(\boldsymbol{u} + \boldsymbol{v})$. Hence

$$F(\boldsymbol{u}) \geq F(\boldsymbol{u} + \boldsymbol{v}) - \int_\Omega \boldsymbol{T} \cdot \hat{\boldsymbol{E}}(\boldsymbol{v})\, d\mathcal{L}^n \geq F(\boldsymbol{u} + \boldsymbol{v}) - \langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle$$

by (5.10) and consequently we have (5.11). Next prove that $I_0(\mu) = -\infty$ for all $\mu > \lambda$. We have

$$I(\boldsymbol{u}, \mu) = I(\boldsymbol{u}, \lambda) - (\mu - \lambda)\langle \bar{\boldsymbol{l}}, \boldsymbol{u} - \boldsymbol{z} \rangle$$

for any $\boldsymbol{u} \in U$. In particular, letting $t > 0$ and putting $\boldsymbol{u} = t\boldsymbol{v} + \boldsymbol{z}$ where $\boldsymbol{v}$ is as in (5.9), we obtain

$$I(t\boldsymbol{v} + \boldsymbol{z}, \mu) = I(t\boldsymbol{v} + \boldsymbol{z}, \lambda) - t(\mu - \lambda)\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle$$

and applying (5.11) with $\boldsymbol{v}$ replaced by $t\boldsymbol{v}$,

$$I(t\boldsymbol{v} + \boldsymbol{z}, \lambda) \leq I(\boldsymbol{z}, \lambda)$$

and thus

$$I(t\boldsymbol{v} + \boldsymbol{z}, \mu) \leq I(\boldsymbol{z}, \lambda) - t(\mu - \lambda)\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle.$$

Letting $t \to \infty$ and using $\mu > \lambda$, $\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 1$ we thus obtain $I(t\boldsymbol{v} + \boldsymbol{z}, \mu) \to -\infty$ as $t \to \infty$ for all $\mu > \lambda$. Hence $\lambda_{\mathrm{c}}^+ \leq \lambda$; on the other hand, $\lambda \leq \lambda_{\mathrm{c}}^+$ as part of the definition of the multiplier admitting a collapse mechanism. $\square$

6. Proofs: static admissibility of collapse multipliers.

Proof of Theorem 2.9. Assume that λ_c is a collapse multiplier. Then there exists a sequence $\lambda_k \to \lambda_c$ such that all multipliers λ_k are statically admissible; i.e., there exists a stressfield $\boldsymbol{T}_k \in Y_K$ equilibrating the loads $\mathfrak{L}(\lambda_k)$. If we manage to prove that there is a sequence of admissible stressfields $\boldsymbol{T}_k$ such that the sequence $|\boldsymbol{T}_k|_Y$ is bounded independently of k, then (e.g., [7, Theorem A.52]) there is a $\boldsymbol{T} \in Y$ such that for some subsequence of $\boldsymbol{T}_k$, again denoted by $\boldsymbol{T}_k$, we have $\boldsymbol{T}_k \rightharpoonup \boldsymbol{T}$ in Y where $\rightharpoonup$ denotes the weak convergence. From $\langle \boldsymbol{l}(\lambda_k), \boldsymbol{v} \rangle = (\boldsymbol{T}_k, \hat{\boldsymbol{E}}(\boldsymbol{v}))$ for all $\boldsymbol{v} \in V$ we deduce that $\langle \boldsymbol{l}(\lambda_{\mathrm{c}}), \boldsymbol{v} \rangle = (\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v}))$ for all $\boldsymbol{v} \in V$; i.e., $\boldsymbol{T}$ equilibrates the loads $\mathfrak{L}(\lambda_{\mathrm{c}})$. The set Y_K is convex and closed under the norm convergence in Y; hence it is also closed under the weak convergence in Y (e.g., [7, Theorem A.47]). Consequently, $\boldsymbol{T} \in Y_K$, i.e., $\boldsymbol{T}$ is admissible. Hence under the assumption that there is a $|\cdot|_Y$ bounded sequence $\boldsymbol{T}_k$ of admissible stressfields equilibrating the loads $\mathfrak{L}(\lambda_k)$, we know that λ_{c} is statically admissible.

Let us verify the boundedness assumption if either H_2 or H_3 holds.

Assume that Hypothesis H_2 holds. Since K is bounded, it follows that the family $|\boldsymbol{T}_k|_Y$ of norms is uniformly bounded.

Assume that H_3 holds. Then

$$\boldsymbol{T}_k = -p_k \mathbf{1} + \boldsymbol{S}_k \tag{6.1}$$

where $p_k \in L^2(\Omega, \mathbb{R})$, $\boldsymbol{S}_k \in L^2(\Omega, \mathrm{Sym}_0)$, $\boldsymbol{S}_k$ ranges (essentially) in K_0, and

$$-(p_k, \operatorname{div} \boldsymbol{v}) = -\big(\boldsymbol{S}_k, \hat{\boldsymbol{E}}(\boldsymbol{v})\big) + \langle \boldsymbol{l}(\lambda_k), \boldsymbol{v} \rangle \tag{6.2}$$

for each $\boldsymbol{v} \in V$. If $|\boldsymbol{v}|_V := |\boldsymbol{v}|_{W^{1,2}(\Omega,\mathbb{R}^n)} \le 1$, then the first term on the right-hand side of (6.2) is bounded independently of k since the sequence $\boldsymbol{S}_k$ is pointwise bounded as K_0 is bounded; the second term is bounded as well in view of $\boldsymbol{l}(\lambda_k) = \boldsymbol{l}_0 + \lambda_k \bar{\boldsymbol{l}}$. Thus

$$-(p_k, \operatorname{div} \boldsymbol{v}) \le c \tag{6.3}$$

for some $c \in \mathbb{R}$, all k and all $\boldsymbol{v} \in V$ with $|\boldsymbol{v}|_V \le 1$. Testing (6.3) on $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^n) \subset V$, we deduce that the sequence of $W^{-1,2}(\Omega)$ norms of ∇p_k is bounded: $|\nabla p_k|_{W^{-1.2}(\Omega,\mathbb{R}^n)} \le c$ independently of k with the same constant c. By the theorem on negative norms ([14], [15]) for each k there exists a constant $c_k \in \mathbb{R}$ such that

$$|p_k + c_k|_{L^2(\Omega,\mathbb{R})} \le c \tag{6.4}$$

for all k and some $c \in \mathbb{R}$. From (6.2) and (6.4) one deduces that

$$c_k \int_\Omega \operatorname{div} \boldsymbol{v} \, d\mathcal{L}^n \le c \tag{6.5}$$

for some constant $c \in \mathbb{R}$, all k, and all $\boldsymbol{v} \in V$ with $|\boldsymbol{v}|_V \le 1$. If there is a $\boldsymbol{v} \in V$ with $\int_\Omega \operatorname{div} \boldsymbol{v} \, d\mathcal{L}^n \ne 0$, then (6.5) implies that the numerical sequence c_k is bounded and hence also the sequence p_k is bounded in $L^2(\Omega, \mathbb{R})$ by (6.4). If $\int_\Omega \operatorname{div} \boldsymbol{v} \, d\mathcal{L}^n = 0$ for all $\boldsymbol{v} \in V$, then calling by p_k what was previously denoted by $p_k + c_k$ we still have (6.1) giving an admissible equilibrating stressfield for $\mathfrak{L}(\lambda_k)$ and the new sequence p_k is bounded in $L^2(\Omega, \mathbb{R})$ by (6.4). We thus summarize that in each case there exists an admissible equilibrating stressfield $\boldsymbol{T}_k$ for $\mathfrak{L}(\lambda_k)$ such that $|\boldsymbol{T}_k|_Y$ is bounded. □

7. Proofs: kinematic theorem.

Proof of Remark 2.8. (i): If $\boldsymbol{v}, \boldsymbol{T}$ are as in (i), then the definition of $\mathrm{N}(Y_K, \boldsymbol{T})$ gives $\hat{m} \circ \hat{\boldsymbol{E}}(\boldsymbol{v}) = \hat{\boldsymbol{E}}(\boldsymbol{v}) \cdot \boldsymbol{T}$ for $\mathcal{L}^n$ a.e. point of Ω; hence $G(\boldsymbol{v}) = \big(\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})\big)$ and (2.12) reduces to (2.9). Thus λ is kinematically admissible. Next assume that one of H_1–H_3 holds, that λ is kinematically admissible, let $\boldsymbol{v}$ be as in the definition of a kinematically admissible multiplier, and prove that there exists a $\boldsymbol{T}$ such that the couple $\boldsymbol{v}, \boldsymbol{T}$ satisfies the requirements stated in Condition (i). If H_1 holds, then

$$\hat{m}(\boldsymbol{E}) = \begin{cases} 0 & \text{if } \boldsymbol{E} \in K^*, \\ \infty & \text{otherwise,} \end{cases}$$

$\boldsymbol{E} \in \mathrm{Sym}$, where

$$K^* = \{\boldsymbol{E} \in \mathrm{Sym} : \boldsymbol{E} \cdot \boldsymbol{T} \le 0 \ \text{ for each } \ \boldsymbol{T} \in K\}$$

is the cone dual to K. Equation (2.9) implies that $G(\boldsymbol{v}) < \infty$ and thus we have $\hat{m} \circ \hat{\boldsymbol{E}}(\boldsymbol{v}) \equiv \boldsymbol{0}$ identically. Then $\hat{\boldsymbol{E}}(\boldsymbol{v}) \in K^*$ for $\mathcal{L}^n$ a.e. point of Ω and the pair $\boldsymbol{v}, \boldsymbol{T} \equiv \boldsymbol{0}$ satisfies the requirements stated in Condition (i). If H_2 holds, then the definition of $\hat{m}$, the compactness of K and the theorem on a measurable selection ([6, Chapter VIII, Theorem 1.2]) implies the existence of a measurable $\boldsymbol{T} : \Omega \to K$ such that $\hat{m} \circ \hat{\boldsymbol{E}}(\boldsymbol{v}) = \hat{\boldsymbol{E}}(\boldsymbol{v}) \cdot \boldsymbol{T}$

for $\mathcal{L}^n$ a.e. point of Ω, and thus again the pair $\boldsymbol{v}, \boldsymbol{T}$ satisfies the requirements stated in Condition (i). If H_3 holds, then

$$\hat{m}(\boldsymbol{E}) = \begin{cases} \hat{m}_0(\boldsymbol{E}) & \text{if } \operatorname{tr} \boldsymbol{E} = 0, \\ \infty & \text{otherwise}, \end{cases} \tag{7.1}$$

$\boldsymbol{E} \in \mathrm{Sym}$, where

$$\hat{m}_0(\boldsymbol{E}) = \sup\{\boldsymbol{S} \cdot \boldsymbol{E} : \boldsymbol{S} \in K_0\}$$

for each $\boldsymbol{E} \in \mathrm{Sym}_0$. Hence $G(\boldsymbol{v}) < \infty$ means that $\hat{\boldsymbol{E}}(\boldsymbol{v}) \in L^2(\Omega, \mathrm{Sym}_0)$ and the theorem on a measurable selection then completes the proof as in Case H_2. (ii): If there exists a $\boldsymbol{v}, \boldsymbol{T}$ as in Condition (ii) then λ is statically admissible and from (i) we deduce that λ is also kinematically admissible; thus λ admits a collapse mechanism. □

LEMMA 7.1. Assume (2.14) and let $\boldsymbol{m} : V \to \mathbb{R}$ be a continuous linear function such that $\langle \boldsymbol{m}, \boldsymbol{v} \rangle = 0$ for every $\boldsymbol{v} \in V$ with $\operatorname{div} \boldsymbol{v} = 0$. Then there exists a $p \in L^2(\Omega, \mathbb{R})$ such that

$$\langle \boldsymbol{m}, \boldsymbol{v} \rangle = -\int_\Omega p \operatorname{div} \boldsymbol{v} \, d\mathcal{L}^n \tag{7.2}$$

for every $\boldsymbol{v} \in V$.

Proof. Let

$$\tilde{L} = \{\operatorname{div} \boldsymbol{w} : \boldsymbol{w} \in W_0^{1,2}(\Sigma, \mathbb{R}^n)\},$$

which is a subspace of $L^2(\Sigma, \mathbb{R}^n)$. Let $P : \tilde{L} \to \mathbb{R}$ be defined by

$$\langle P, \operatorname{div} \boldsymbol{w} \rangle = \langle \boldsymbol{m}, \boldsymbol{w} | \Omega \rangle$$

for each $\boldsymbol{w} \in W_0^{1,2}(\Sigma, \mathbb{R}^n)$. We note that P is well defined due to the hypothesis on $\boldsymbol{m}$. Since $\boldsymbol{m}$ is continuous, we have

$$\langle P, \operatorname{div} \boldsymbol{w} \rangle \le c |\boldsymbol{w}| \Omega|_{W^{1,2}(\Omega, \mathbb{R}^n)} \le c |\boldsymbol{w}|_{W_0^{1,2}(\Sigma, \mathbb{R}^n)} \tag{7.3}$$

for some c and all $\boldsymbol{w} \in W_0^{1,2}(\Sigma, \mathbb{R}^n)$. This means that the $W^{-1,2}(\Sigma, \mathbb{R})$ norm of the distributional derivative of P is bounded and hence by the theorem on negative norms ([14], [15]) there exists a $\tilde{p} \in L^2(\Sigma, \mathbb{R})$ such that

$$\langle P, \operatorname{div} \boldsymbol{w} \rangle = -\int_\Sigma \tilde{p} \operatorname{div} \boldsymbol{w} \, d\mathcal{L}^n$$

for every $\boldsymbol{w} \in W_0^{1,2}(\Sigma, \mathbb{R}^n)$. Note that the function $\tilde{p}$ is determined to within an additive constant. The definition of P gives $\langle P, \operatorname{div} \boldsymbol{w} \rangle = 0$ if $\operatorname{spt} \boldsymbol{w} \subset \Sigma \setminus \Omega$; thus the right-hand side of (7.3) vanishes for all such $\boldsymbol{w}$. This in turn implies that $\tilde{p}$ is (essentially) constant, of value d, on $\Sigma \setminus \Omega$. Thus replacing $\tilde{p}$ by $\tilde{p} - d$ we have $\tilde{p} = 0$ on $\Sigma \setminus \Omega$ and accordingly we replace the integration range Σ by Ω. This gives

$$\langle \boldsymbol{m}, \boldsymbol{w} | \Omega \rangle = -\int_\Omega \tilde{p} \operatorname{div}\big(\boldsymbol{w} | \Omega\big) \, d\mathcal{L}^n$$

for every $\boldsymbol{w} \in W^{1,2}(\Sigma, \mathbb{R}^n)$; in other words, we have (7.2) for every $\boldsymbol{v} \in R_0$ where $p = \tilde{p} | \Omega$. The assumed density of R_0 in V then extends (7.2) to all $\boldsymbol{v} \in V$. □

REMARK 7.2. Témam proves (2.15) for Hencky plastic materials (Definition 2.3(iii)) under the following hypotheses on $\partial\Omega$, $\mathcal{D}$ and $\mathcal{S}$. He assumes that

$$\left.\begin{array}{c}\text{the boundary of } \Omega \subset \mathbb{R}^3 \text{ is of class 2, and}\\ \mathcal{D} \text{ and } \mathcal{S} \text{ are relatively open subsets of } \partial\Omega,\end{array}\right\} \tag{7.4}$$

such that the common boundary $\Gamma_* := \partial\Omega \setminus (\mathcal{D} \cup \mathcal{S})$ of $\mathcal{D}$ and $\mathcal{S}$ has the following property (cf. [17, Equation (2.105), Chapter I]): for each $\boldsymbol{x} \in \Gamma_*$ there exists a class 2 diffeomorphism from some neighborhood $O_{\boldsymbol{x}}$ of $\boldsymbol{x}$ onto $\mathbb{R}^3$ which maps

$$\left.\begin{array}{lll} O_{\boldsymbol{x}} & \text{onto} & (-1,1)^3,\\ \mathcal{S} \cap O_{\boldsymbol{x}} & \text{onto} & (-1,1)^2 \times \{0\},\\ \Gamma_* \cap O_{\boldsymbol{x}} & \text{onto} & (-1,1) \times \{(0,0)\}.\end{array}\right\} \tag{7.5}$$

Let us show that these assumptions imply our hypothesis (2.14), which is therefore less restrictive. It is shown in [17, Section I.2] that under (7.4) and (7.5), the set

$$S_0 := \{\boldsymbol{v} \in V : \ \text{the trace of } \boldsymbol{v} \text{ on } \partial\Omega \text{ vanishes in some neighborhood of } \operatorname{cl}\mathcal{D}\}$$

is dense in V; here 'neighborhood' means a relative neighborhood in $\partial\Omega$. Equation (7.5) implies that Ω is contained in some Lipschitz region $\Sigma \subset \mathbb{R}^3$ with $\mathcal{D} \subset \partial\Sigma$ such that $\mathcal{S}$ is in the interior of Σ. Let us show that, for each such Σ, the set R_0 defined in $(2.14)_2$ is dense in V. For this it suffices to show that $S_0 \subset R_0$. Given $\boldsymbol{v} \in S_0$, standard extension theorems give an extension $\boldsymbol{t} \in W_0^{1,2}(\mathbb{R}^3, \mathbb{R}^3)$ of $\boldsymbol{v}$; since $\boldsymbol{v}$ vanishes in some neighborhood U of $\operatorname{cl}\mathcal{D}$, there exists $\varphi \in C_0^\infty(\Sigma, \mathbb{R})$ such that $\varphi = 1$ on $\partial\Omega \setminus U$. Then $\varphi\boldsymbol{t} \in W_0^{1,2}(\Sigma, \mathbb{R}^3)$ and $(1-\varphi)\boldsymbol{t}|\Omega \in W_0^{1,2}(\Omega, \mathbb{R}^3)$; hence $\boldsymbol{w} : \Sigma \to \mathbb{R}^3$ given by

$$\boldsymbol{w} = \begin{cases} \boldsymbol{v} = \varphi\boldsymbol{t} + (1-\varphi)\boldsymbol{t} & \text{on } \Omega,\\ \varphi\boldsymbol{t} & \text{on } \Sigma \setminus \Omega,\end{cases}$$

is the required extension.

Proof of Theorem 2.11. To prove (2.13), we note that the assumption $\bar\lambda_c^+ < \lambda_c^+$ implies that there exists a $\lambda \in (\bar\lambda_c^+, \lambda_c^+)$ which admits a collapse mechanism; Theorem 2.7 then gives a contradictory statement $\lambda = \lambda_c^+$.

We now prove (2.15) under Hypothesis H_2 and $\lambda_c^+ > -\infty$. From H_2 follows that $\hat m$ is a finite-valued function bounded from above by some multiple of the norm on Sym and bounded from below by some affine function on Sym. Then G is a finite-valued continuous convex function on V. If $\bar{\boldsymbol{l}} = \boldsymbol{0}$ then either all $\lambda \in \mathbb{R}$ are statically admissible or no $\lambda \in \mathbb{R}$ is statically admissible, leading, respectively, to $\lambda_c^+ = \infty$ and $\lambda_c^+ = -\infty$. The last possibility is excluded by the hypothesis; thus $\lambda_c^+ = \infty$ and then (2.13) gives (2.15). It remains to prove (2.15) under the assumption $\bar{\boldsymbol{l}} \neq 0$. This implies that there is a $\boldsymbol{v} \in V$ such that $\langle \bar{\boldsymbol{l}}, \boldsymbol{v}\rangle = 1$; hence $\bar\lambda_c^+ < \infty$. Since $-\infty < \lambda_c^+ \le \bar\lambda_c^+$ we have $\bar\lambda_c^+ \in \mathbb{R}$. The definition of $\bar\lambda_c^+$ gives

$$\langle \boldsymbol{l}(\bar\lambda_c^+), \boldsymbol{v}\rangle \le G(\boldsymbol{v}) \tag{7.6}$$

for each $\boldsymbol{v} \in V$ with $\langle \bar{\boldsymbol{l}}, \boldsymbol{v}\rangle = 1$. The positive 1 homogeneity and continuity of G then extends (7.6) to all $\boldsymbol{v} \in V$ with $\langle \bar{\boldsymbol{l}}, \boldsymbol{v}\rangle \ge 0$. Let us now prove that (7.6) holds also for all

$\boldsymbol{v} \in V$ with $\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle \le 0$. By the definition of $\bar{\lambda}_{\rm c}^{+}$ for each $\epsilon > 0$ there exists a $\boldsymbol{v} = \boldsymbol{v}_\epsilon \in V$ with $\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 1$ such that

$$\bar{\lambda}_{\rm c}^{+} + \epsilon + \langle \boldsymbol{l}_0, \boldsymbol{v} \rangle > G(\boldsymbol{v}). \tag{7.7}$$

Let $\bar{\boldsymbol{v}} := -\boldsymbol{v}$ and let $\boldsymbol{u} \in V$ be such that $\langle \bar{\boldsymbol{l}}, \boldsymbol{u} \rangle = 0$. Adding the term $G(\boldsymbol{u} + \bar{\boldsymbol{v}})$ to the two sides of (7.7) and using the convexity and positive 1 homogeneity of G to infer that $G(\boldsymbol{u} + \bar{\boldsymbol{v}}) + G(\boldsymbol{v}) \ge G(\boldsymbol{u})$, we obtain

$$\bar{\lambda}_{\rm c}^{+} + \epsilon + \langle \boldsymbol{l}_0, \boldsymbol{v} \rangle + G(\boldsymbol{u} + \bar{\boldsymbol{v}}) \ge G(\boldsymbol{u}). \tag{7.8}$$

Observing that for $\boldsymbol{u}$ the inequality (7.6) reduces to $G(\boldsymbol{u}) \ge \langle \boldsymbol{l}_0, \boldsymbol{u} \rangle$, we deduce from (7.8) that

$$G(\boldsymbol{u} + \bar{\boldsymbol{v}}) \ge \langle \boldsymbol{l}_0, \boldsymbol{u} + \bar{\boldsymbol{v}} \rangle - \bar{\lambda}_{\rm c}^{+} - \epsilon;$$

since the set $\{\boldsymbol{u} + \bar{\boldsymbol{v}} : \boldsymbol{u} \in V, \langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 0\}$ coincides with the set of all $\boldsymbol{w} \in V$ with $\langle \bar{\boldsymbol{l}}, \boldsymbol{w} \rangle = -1$, we have

$$G(\boldsymbol{w}) \ge \langle \boldsymbol{l}_0, \boldsymbol{w} \rangle - \bar{\lambda}_{\rm c}^{+} - \epsilon$$

for all such $\boldsymbol{w}$. The arbitrariness of $\epsilon > 0$ gives $G(\boldsymbol{w}) \ge \langle \boldsymbol{l}_0, \boldsymbol{w} \rangle - \bar{\lambda}_{\rm c}^{+}$ for all $\boldsymbol{w} \in V$ with $\langle \bar{\boldsymbol{l}}, \boldsymbol{w} \rangle = -1$; in other words, we have (7.6) for all $\boldsymbol{v} \in V$ with $\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = -1$. The positive 1 homogeneity and continuity extends (7.6) to all $\boldsymbol{v} \in V$ with $\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle \le 0$. Thus (7.6) holds for all $\boldsymbol{v} \in V$. We now apply Proposition 5.2 with H given by (5.4) with $h = \hat{m}$ and with $\boldsymbol{w} = \boldsymbol{0}$ and $\lambda = \bar{\lambda}_{\rm c}^{+}$. Inequality (7.6) then implies Condition (5.1) with $c = 0$, and Proposition 5.2 then says that there exists a stressfield $\boldsymbol{T} \in Y$ equilibrating the loads $\mathfrak{L}(\bar{\lambda}_{\rm c}^{+})$ such that

$$\int_\Omega \hat{m}^*(\boldsymbol{T})\, d\mathcal{L}^n < \infty.$$

From (2.8) one finds that

$$\hat{m}^* = \begin{cases} 0 & \text{on } K, \\ \infty & \text{on } \mathrm{Sym} \setminus K \end{cases}$$

and hence the (essential) range of $\boldsymbol{T}$ is contained in K; i.e., $\boldsymbol{T}$ is admissible. Thus the loads $\mathfrak{L}(\bar{\lambda}_{\rm c}^{+})$ are compatible and hence $\bar{\lambda}_{\rm c}^{+}$ is statically admissible. Hence $\bar{\lambda}_{\rm c}^{+} \le \lambda_{\rm c}^{+}$ and a combination with (2.13) gives (2.15).

If H_3 holds, then Proposition 5.2 cannot be applied directly since G takes infinite values in view of (7.1). If $\boldsymbol{v} \in V$, then $G(\boldsymbol{v}) < \infty$ if and only if $\boldsymbol{v} \in V_0$ where

$$V_0 = \{\boldsymbol{v} \in V : \mathrm{tr}\, \hat{\boldsymbol{E}}(\boldsymbol{v}) = 0\} \equiv \{\boldsymbol{v} \in V : \mathrm{div}\, \boldsymbol{v} = 0\}.$$

Then

$$\bar{\lambda}_{\rm c}^{+} = \inf\{G_0(\boldsymbol{v}) - \langle \boldsymbol{l}_0, \boldsymbol{v} \rangle : \boldsymbol{v} \in V_0, \langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 1\} \tag{7.9}$$

where G_0 is the restriction of G to V_0, with

$$G_0(\boldsymbol{v}) = \int_\Omega \hat{m}_0(\hat{\boldsymbol{E}}(\boldsymbol{v}))\, d\mathcal{L}^n$$

for every $\boldsymbol{v} \in V_0$. Assume now that H_3 and (2.14) hold and $\lambda_{\rm c}^{+} > -\infty$, and prove (2.15). If $\lambda_{\rm c}^{+} = \infty$ then (2.15) follows from (2.13). Since it is assumed that $\lambda_{\rm c}^{+} > -\infty$, only the case $\lambda_{\rm c}^{+} \in \mathbb{R}$ remains to be considered. Let us first show that $\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle \ne \boldsymbol{0}$ for some $\boldsymbol{v} \in V_0$. Indeed, otherwise we have $\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 0$ for all $\boldsymbol{v} \in V_0$, the functional $\boldsymbol{m} := \bar{\boldsymbol{l}}$ satisfies the hypothesis of Lemma 7.1 and hence there exists a $p \in L^2(\Omega, \mathbb{R})$ such that (7.2) holds for all $\boldsymbol{v} \in V$. Since $\lambda_c^+ \in \mathbb{R}$, there exists a statically admissible multiplier $\lambda_0 \in \mathbb{R}$ with the admissible equilibrating stressfield $\boldsymbol{T}_0$. From (2.10) one finds that for every $\lambda \in \mathbb{R}$ the stressfield $\boldsymbol{T}_\lambda := \boldsymbol{T}_0 - (\lambda - \lambda_0)p\mathbf{1}$ is admissible and clearly $\boldsymbol{T}_\lambda$ equilibrates the loads $\mathfrak{L}(\lambda)$. Thus all λ are statically admissible and hence $\lambda_c^+ = \infty$ in contradiction with the hypothesis. Thus $\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle \neq \mathbf{0}$ for some $\boldsymbol{v} \in V_0$, which in turn implies that $\bar{\lambda}_c^+ < \infty$ by (7.9). Repeating the steps of the proof in Case H_2 and using the obvious modification of Proposition 5.2 one obtains that there exists an $\boldsymbol{S} \in L^2(\Omega, \mathrm{Sym}_0)$ with the (essential) range in K_0 such that $\langle \boldsymbol{l}(\bar{\lambda}_c^+), \boldsymbol{v} \rangle = (\boldsymbol{S}, \hat{\boldsymbol{E}}(\boldsymbol{v}))$ for each $\boldsymbol{v} \in V_0$. Let $\boldsymbol{m} : V \to \mathbb{R}$ be a linear function defined by

$$\langle \boldsymbol{m}, \boldsymbol{v} \rangle = \langle \boldsymbol{l}(\bar{\lambda}_c^+), \boldsymbol{v} \rangle - (\boldsymbol{S}, \hat{\boldsymbol{E}}(\boldsymbol{v})) \tag{7.10}$$

for each $\boldsymbol{v} \in V_0$. Then $\langle \boldsymbol{m}, \boldsymbol{v} \rangle = 0$ if $\operatorname{div} \boldsymbol{v} = 0$ and hence there exists a $p \in L^2(\Omega, \mathbb{R})$ such that (7.2) holds, which together with (7.10) gives that $\boldsymbol{T} = -p\mathbf{1} + \boldsymbol{S}$ is an admissible stressfield equilibrating the loads $\boldsymbol{l}(\bar{\lambda}_c^+)$. Thus $\bar{\lambda}_c^+$ is statically admissible and (2.15) follows in the same way as in Case H_2. $\square$

8. Proofs: no–tension bodies.

Proof of Proposition 3.1. (i): We prove the equivalence of Conditions (a)–(c) as follows. Assume that (a) holds, i.e., that $\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle = \big(\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})\big)$ for all $\boldsymbol{v} \in V$ and some $\boldsymbol{T} \in Y^-$. Letting $\hat{\boldsymbol{E}}^\circ(\boldsymbol{v})$ stand for $\hat{\boldsymbol{E}}^-(\boldsymbol{v})$ or for $\hat{\boldsymbol{E}}^{\mathrm{e}}(\boldsymbol{v})$, we obtain

$$\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \le -\big(\boldsymbol{T}, \hat{\boldsymbol{E}}^\circ(\boldsymbol{v})\big) \le |\boldsymbol{T}|_Y |\hat{\boldsymbol{E}}^\circ(\boldsymbol{v})|_Y$$

for all $\boldsymbol{v} \in V$. Thus Conditions (b) and (c) hold. Conversely, assume that one of (b) or (c) holds and prove (a). We apply Proposition 5.2 with $H : V \to \mathbb{R}$ defined by

$$H(\boldsymbol{E}) = \eta\, |\boldsymbol{E}^\circ|_Y$$

for each $\boldsymbol{E} \in Y$, where $\boldsymbol{E}^\circ$ stands either for the negative part $\boldsymbol{E}^-$ of $\boldsymbol{E}$ or for $\boldsymbol{E}^{\mathrm{e}}$ with $\boldsymbol{E}^{\mathrm{e}}$ and $\boldsymbol{E}^{\mathrm{a}}$ the decomposition of $\boldsymbol{E}$ as in Proposition 2.1. One easily finds that H is a convex continuous function, and (b) or (c) implies Condition (5.1). Proposition 5.2 then gives the existence of a stressfield $\boldsymbol{T} \in Y$ equilibrating the loads such that Condition (5.2) holds, which reads

$$(\boldsymbol{T}, \boldsymbol{A}) \le \eta\, |\boldsymbol{A}^\circ|_Y \quad \text{for all} \quad \boldsymbol{A} \in Y.$$

If $\boldsymbol{A} \in Y^+$, the last condition reduces to $(\boldsymbol{T}, \boldsymbol{A}) \le 0$. Thus $\boldsymbol{T} \in Y^-$; i.e., $\boldsymbol{T}$ is an admissible stressfield equilibrating the loads $\mathfrak{L}(\lambda)$. Thus each of (b) and (c) is equivalent to (a).

To prove (3.1), we can assume that $\mathcal{S}$ is a class 2 surface; put $\hat{\boldsymbol{s}} = \boldsymbol{s}(\lambda)$. For each point of $\mathcal{S}$ there exists a neighborhood N of that point and a class 2 function $\delta : N \to \mathbb{R}$ with $\nabla\delta \neq \mathbf{0}$ on N such that

$$\Omega \cap N = \{\boldsymbol{x} \in N : \delta(\boldsymbol{x}) > 0\}, \quad \mathcal{S} \cap N = \{\boldsymbol{x} \in N : \delta(\boldsymbol{x}) = 0\}.$$

For each $\epsilon > 0$ let $L_\epsilon = \{\boldsymbol{x} \in N : 0 < \delta(\boldsymbol{x}) < \epsilon\}$ and let $\omega_\epsilon : \mathbb{R} \to \mathbb{R}$ be given by

$$\omega_\epsilon(t) = \begin{cases} 1 & \text{if } t \le 0, \\ 1 - t/\epsilon & \text{if } 0 < t < \epsilon, \\ 0 & \text{if } t \ge \epsilon, \end{cases}$$

$t \in \mathbb{R}$. Let $\varphi \in C_0^\infty(N, \mathbb{R})$ be any nonnegative function, put $\sigma_\epsilon := \omega_\epsilon \circ \delta$ and let $\boldsymbol{v}_\epsilon : \Omega \to \mathbb{R}^n$ be given by

$$\boldsymbol{v}_\epsilon = \begin{cases} \varphi\sigma_\epsilon \nabla\delta & \text{on } N, \\ \boldsymbol{0} & \text{on } \Omega \setminus N. \end{cases}$$

We have $\boldsymbol{v}_\epsilon \in V$,

$$\boldsymbol{v}_\epsilon|_{\partial\Omega} = \begin{cases} -\varphi|\nabla\delta|\boldsymbol{n} & \text{on } \mathcal{S}, \\ \boldsymbol{0} & \text{on } \mathcal{D}, \end{cases}$$

and

$$\nabla\boldsymbol{v}_\epsilon = \begin{cases} \sigma_\epsilon\nabla(\varphi\nabla\delta) - \epsilon^{-1}\varphi\nabla\delta\otimes\nabla\delta & \text{on } L_\epsilon, \\ \boldsymbol{0} & \text{on } \Omega \setminus L_\epsilon\ . \end{cases}$$

If $\boldsymbol{T}$ is an admissible equilibrating stressfield for the loads $\mathfrak{L}(\lambda)$, then (2.6) reads

$$A_\epsilon - \epsilon^{-1}\int_{L_\epsilon} \varphi\boldsymbol{T}\cdot(\nabla\delta\otimes\nabla\delta)\,d\mathcal{L}^n = -\int_{\mathcal{S}} |\nabla\delta|\varphi\hat{\boldsymbol{s}}\cdot\boldsymbol{n}\,d\mathcal{H}^{n-1}$$

where

$$A_\epsilon = \int_{L_\epsilon} \big(\sigma_\epsilon\boldsymbol{T}\cdot\nabla(\varphi\nabla\delta) - \boldsymbol{v}_\epsilon\cdot\boldsymbol{b}(\lambda)\big)\,d\mathcal{L}^n.$$

Since σ_ϵ and $\boldsymbol{v}_\epsilon$ are bounded independently of ϵ and $\mathcal{L}^n(L_\epsilon) \to 0$ as $\epsilon \to 0$, we have $A_\epsilon \to 0$ and hence

$$\lim_{\epsilon\to 0}\epsilon^{-1}\int_{L_\epsilon} \varphi\boldsymbol{T}\cdot(\nabla\delta\otimes\nabla\delta)\,d\mathcal{L}^n = \int_{\mathcal{S}} |\nabla\delta|\varphi\hat{\boldsymbol{s}}\cdot\boldsymbol{n}\,d\mathcal{H}^{n-1}.$$

Since $\boldsymbol{T}$ is negative semidefinite and φ nonnegative, the limit on the left-hand side is nonpositive and thus

$$\int_{\mathcal{S}} |\nabla\delta|\varphi\hat{\boldsymbol{s}}\cdot\boldsymbol{n}\,d\mathcal{H}^{n-1} \le 0.$$

The arbitrariness of φ then gives the assertion.

(ii): As in the proof of Remark 2.8, we have $G(\boldsymbol{v}) < \infty$ if and only if $\hat{\boldsymbol{E}}(\boldsymbol{v}) \in L^2(\Omega, \mathrm{Sym}^+)$ and then $G(\boldsymbol{v}) = 0$, which gives the assertion.

(iii): By Remark 2.6(iii) and (ii) above, we have $0 = G(\boldsymbol{v}) = \langle \boldsymbol{l}(\lambda), \boldsymbol{v}\rangle$ and $\hat{\boldsymbol{E}}(\boldsymbol{v}) \in Y^+$ for each collapse mechanism for the loads $\mathfrak{L}(\lambda)$. Thus if $\boldsymbol{T}$ is an admissible equilibrating stressfield for $\mathfrak{L}(\lambda)$, we have $\big(\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})\big) = 0$. Since $\hat{\boldsymbol{E}}(\boldsymbol{v}) \in Y^+$ and $\boldsymbol{T} \in Y_K$, we have $\hat{\boldsymbol{E}}(\boldsymbol{v})\cdot\boldsymbol{T} = 0$ for $\mathcal{L}^n$ a.e. point of Ω; thus we have (3.2) and (3.4) for every admissible stressfield $\boldsymbol{T}$ equilibrating the loads $\mathfrak{L}(\lambda)$. If we have (3.2) and (3.4) for some admissible stressfield $\boldsymbol{T}$ equilibrating the loads $\mathfrak{L}(\lambda)$, then $0 = \big(\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})\big) = \langle \boldsymbol{l}(\lambda), \boldsymbol{v}\rangle$ and thus $\boldsymbol{v}$ is a collapse mechanism for the loads $\mathfrak{L}(\lambda)$ by Remark 2.6(iii) and (ii) above. $\square$

Proof of Theorem 3.2. Let $\hat\lambda : (0,\infty) \to \mathbb{R}$ be defined by

$$\hat\lambda(\eta) = \inf\{\eta|\hat{\boldsymbol{E}}^-(\boldsymbol{v})|_Y - \langle \boldsymbol{l}_0, \boldsymbol{v}\rangle : \boldsymbol{v} \in V, \langle \bar{\boldsymbol{l}}, \boldsymbol{v}\rangle = 1\},$$

$\eta > 0$. If $\eta > 0$ then

$$\hat\lambda(\eta) \le \eta|\hat{\boldsymbol{E}}^-(\boldsymbol{v})|_Y - \langle \boldsymbol{l}_0, \boldsymbol{v}\rangle$$

for every $\boldsymbol{v} \in V$ satisfying $\langle \bar{\boldsymbol{l}}, \boldsymbol{v}\rangle = 1$, which can be rewritten as

$$\langle \boldsymbol{l}(\hat\lambda(\eta)), \boldsymbol{v}\rangle \le \eta|\hat{\boldsymbol{E}}^-(\boldsymbol{v})|_Y;$$

thus the loads $\boldsymbol{l}(\hat\lambda(\eta))$ are compatible by Proposition 3.1(i) and hence $\hat\lambda_c^+ \le \lambda_c^+$. Conversely, if λ is statically admissible, then

$$\lambda + \langle \boldsymbol{l}_0, \boldsymbol{v}\rangle \le \eta|\hat{\boldsymbol{E}}^-(\boldsymbol{v})|_Y$$

for all $\boldsymbol{v} \in V$ with $\langle \bar{\boldsymbol{l}}, \boldsymbol{v}\rangle = 1$ and some η by Proposition 3.1(i); thus $\lambda \le \hat\lambda(\eta)$, and hence $\lambda_c^+ \le \hat\lambda_c^+$. □

The admissible equilibrated stressfields for Examples 3.3 and 3.4 were determined in [13] by the method of characteristics of the equilibrium equations in $\mathbb{R}^2$. For bodies free from body forces the characteristics are straight lines that coincide with the active isostatic lines of $\boldsymbol{T}$, i.e., lines tangential to the eigenvectors of $\boldsymbol{T}$ corresponding to the negative eigenvalue of $\boldsymbol{T}$. We summarize and extend the essence of the method in case of vanishing body forces in the following definition of a regular family of straight lines and in Proposition 8.2, below.

Let $\mathfrak{l}$ be a system of straight lines in $\mathbb{R}^2$, let $\Omega \subset \mathbb{R}^2$ be a bounded open convex set and let $\mathcal{T} \subset \partial\Omega$. We say that $\mathfrak{l}$ is $(\Omega, \mathcal{T})$ *regular* if the following four conditions are satisfied:

(i) if $l_1, l_2 \in \mathfrak{l}$ and $l_1 \ne l_2$, then $l_1 \cap l_2 \cap (\Omega \cup \mathcal{T}) = \emptyset$. If $l \in \mathfrak{l}$, then $\Omega \cap l \ne \emptyset$;

(ii) $\Omega' := \Omega \cap \bigcup\{l : l \in \mathfrak{l}\}$ is an open set with Lipschitz boundary;

(iii) there exists a c such that for each $\boldsymbol{t} \in \mathbb{S}^1 := \{\boldsymbol{t} \in \mathbb{R}^2 : |\boldsymbol{t}| = 1\}$ the set of all lines $l \in \mathfrak{l}$ tangent to $\boldsymbol{t}$ has at most c elements;

(iv) there exists a class 2 function $\boldsymbol{t} : \Omega' \to \mathbb{S}^1$ such that for each $\boldsymbol{r} \in \Omega'$ the value $\boldsymbol{t}(\boldsymbol{r})$ is a tangent to the line $l \in \mathfrak{l}$ with $\boldsymbol{r} \in l$; moreover, $\boldsymbol{t}$ and $\nabla\boldsymbol{t}$ have continuous extensions to $\operatorname{cl}\Omega'$.

We put $(a,b)^\perp = (-b,a)$ for any $(a,b) \in \mathbb{R}^2$. If $\mathfrak{l}$ is an $(\Omega, \mathcal{T})$ regular system, we denote by $\boldsymbol{a} = -\boldsymbol{t}^\perp$ the normals to the lines from $\mathfrak{l}$ and by $\mathcal{M} = \{\boldsymbol{a}(\boldsymbol{r}) : \boldsymbol{r} \in \Omega'\}$ the set of all normals associated with the lines from $\mathfrak{l}$. For each $\boldsymbol{r} \in \Omega' \cup \mathcal{T}$ we denote by $\hat{l}(\boldsymbol{r})$ the unique line in $\mathfrak{l}$ that contains $\boldsymbol{r}$.

Lemma 8.1. If $\mathfrak{l}$ is an $(\Omega, \mathcal{T})$ regular system of lines, then there exists a $\varphi \in C^1(\Omega', \mathbb{R})$ such that

$$\nabla\boldsymbol{a} = \varphi\boldsymbol{t} \otimes \boldsymbol{a}; \tag{8.1}$$

we have

$$\nabla\varphi = \boldsymbol{a}(\nabla\varphi \cdot \boldsymbol{a}) + \boldsymbol{t}\varphi^2. \tag{8.2}$$

If $\varphi \ne 0$ everywhere on Ω' and if $\boldsymbol{w} := \boldsymbol{a}/\varphi$, then

$$\hat{\boldsymbol{E}}(\boldsymbol{w}) = -(\nabla\varphi \cdot \boldsymbol{a})\boldsymbol{a} \otimes \boldsymbol{a}/\varphi^2. \tag{8.3}$$

Proof. Since $\boldsymbol{a}$ is the same on each line $l \in \mathfrak{l}$, we have $\nabla \boldsymbol{a} = \boldsymbol{p} \otimes \boldsymbol{a}$ for some vector function $\boldsymbol{p}$ on Ω' and since $\boldsymbol{a}$ is unit, we have $\nabla \boldsymbol{a}^{\mathsf{T}} \boldsymbol{a} = 0$, which gives $\boldsymbol{p} \cdot \boldsymbol{a} = 0$ and hence $\boldsymbol{p} = \varphi \boldsymbol{t}$. We thus conclude that (8.1) holds. From (8.1) we deduce

$$\nabla^2 \boldsymbol{a} = \boldsymbol{t} \otimes \boldsymbol{a} \otimes \nabla \varphi - \varphi^2 \boldsymbol{a} \otimes \boldsymbol{a} \otimes \boldsymbol{a} + \varphi^2 \boldsymbol{t} \otimes \boldsymbol{t} \otimes \boldsymbol{a}$$

and hence the interchangeability of the second partial derivatives gives

$$\boldsymbol{a} \otimes \nabla \varphi + \varphi^2 \boldsymbol{t} \otimes \boldsymbol{a} = \nabla \varphi \otimes \boldsymbol{a} + \varphi^2 \boldsymbol{a} \otimes \boldsymbol{t};$$

the multiplication by $\boldsymbol{a}$ from the right gives (8.2). To prove (8.3), we differentiate using (8.1) to obtain

$$\nabla \boldsymbol{w} = (-\boldsymbol{a} \otimes \nabla \varphi + \varphi^2 \boldsymbol{t} \otimes \boldsymbol{a}) / \varphi^2$$

and hence

$$2 \hat{\boldsymbol{E}}(\boldsymbol{w}) = \big((\varphi^2 \boldsymbol{t} - \nabla \varphi) \otimes \boldsymbol{a} + \boldsymbol{a} \otimes (\varphi^2 \boldsymbol{t} - \nabla \varphi) \big) / \varphi^2.$$

A combination with (8.2) provides (8.3). □

In the next proposition we consider the loads $\mathfrak{L}(\cdot)$ with $\boldsymbol{s}(\lambda) : \mathcal{S} \to \mathbb{R}^2$ and $\boldsymbol{b}(\lambda) = \boldsymbol{0}$ on Ω. We fix λ and write $\hat{\boldsymbol{s}} = \boldsymbol{s}(\lambda)$.

Proposition 8.2. Assume that $\hat{\boldsymbol{s}} = \boldsymbol{0}$ on $\mathcal{S} \setminus \mathcal{T}$ for some subset $\mathcal{T}$ of $\mathcal{S}$ and that there exists an $(\Omega, \mathcal{T})$ regular family of lines $\mathfrak{l}$ and a function $\tau : \mathfrak{l} \to \mathbb{R}$ satisfying

$$\tau \circ \hat{l}\, \varphi \le 0 \quad \text{and} \quad \tau \circ \hat{l}\, \varphi\, \boldsymbol{t}(\boldsymbol{t} \cdot \boldsymbol{n}) = \hat{\boldsymbol{s}} \tag{8.4}$$

for $\mathcal{H}^1$ a.e. point of $\mathcal{T}$. Then the stressfield $\boldsymbol{T} : \Omega \to \mathrm{Sym}$ defined by

$$\boldsymbol{T} = \begin{cases} \tau \circ \hat{l}\, \varphi\, \boldsymbol{t} \otimes \boldsymbol{t} & \text{on } \Omega', \\ \boldsymbol{0} & \text{on } \Omega \setminus \Omega' \end{cases} \tag{8.5}$$

is in $L^1(\Omega, \mathrm{Sym}^-)$ and $\boldsymbol{T}$ equilibrates the loads $(\hat{\boldsymbol{s}}, \boldsymbol{0})$ in the sense that

$$\int_\Omega \boldsymbol{T} \cdot \hat{\boldsymbol{E}}(\boldsymbol{v})\, d\mathcal{L}^2 = \int_{\mathcal{T}} \boldsymbol{v} \cdot \hat{\boldsymbol{s}}\, d\mathcal{H}^1 \tag{8.6}$$

for every $\boldsymbol{v} \in V \cap C^1(\mathrm{cl}\, \Omega, \mathbb{R}^2)$.

The family $\mathfrak{l}$, if it exists, is uniquely determined by condition $(8.4)_2$ as the family of lines through points $\boldsymbol{r} \in \mathcal{T}$ of tangent $\boldsymbol{t} = \pm \hat{\boldsymbol{s}}(\boldsymbol{r}) / |\hat{\boldsymbol{s}}(\boldsymbol{r})|$.

Proof. For simplicity we assume that for each $\boldsymbol{a} \in \mathcal{M}$ there exists a unique $l \in \mathfrak{l}$ whose normal is $\boldsymbol{a}$. The general case with several lines in $\mathfrak{l}$ of the same normal is treated similarly; the only difference is the occurrence of the multiplicity functions in the change of variables formulas to be used below.

Let $\tilde{\tau} : \Omega' \cup \mathcal{T} \to \mathbb{R}$ be defined by $\tilde{\tau} = \tau \circ \hat{l}$. Our simplifying assumption yields that there exists a function $\eta : \mathcal{M} \to \mathbb{R}$ such that $\tilde{\tau}(\boldsymbol{r}) = \eta(\boldsymbol{a}(\boldsymbol{r}))$ for each $\boldsymbol{r} \in \Omega' \cup \mathcal{T}$. Prove that $\boldsymbol{T} \in L^1(\Omega, \mathrm{Sym})$. We pass from the variable $\boldsymbol{r} \in \Omega'$ to the variable $\theta(\boldsymbol{r}) := (\boldsymbol{a}(\boldsymbol{r}), d(\boldsymbol{r})) \in \mathbb{S}^1 \times \mathbb{R}$ where $d(\boldsymbol{r}) = |\boldsymbol{r} - \hat{l}(\boldsymbol{r}) \cap \mathcal{S}|$. We denote by $\tilde{\Omega}'$ the image of Ω' under θ, which is a relatively open subset of $\mathbb{S}^1 \times \mathbb{R}$. For a given point $\boldsymbol{r} \in \Omega$,

$$\nabla \theta = \big[\varphi \boldsymbol{t} \otimes \boldsymbol{a}, \nabla d \big]^{\mathsf{T}}, \quad \nabla \theta^{\mathsf{T}} = \big[\varphi \boldsymbol{a} \otimes \boldsymbol{t}, \nabla d \big];$$

consequently

$$\nabla \theta^{\mathsf{T}} \nabla \theta = \varphi^2 \boldsymbol{a} \otimes \boldsymbol{a} + \nabla d \otimes \nabla d.$$

Since $\boldsymbol{a}$ and ∇d are unit orthogonal tensors, the jacobian of the transformation is given by $J = |\det \nabla\theta| = |\varphi|$. Hence the change of variables formula gives

$$|\boldsymbol{T}|_{L^1(\Omega)} = \int_{\Omega'} |\tilde{\tau}||\varphi|\, d\mathcal{L}^2 = \int_{\tilde{\Omega}'} |\eta(\boldsymbol{a})|\, d\mu(\boldsymbol{a}, t), \tag{8.7}$$

where $\mu = \mathcal{H}^1 \llcorner \mathbb{S}^1 \otimes \mathcal{L}^1$ is the area measure on the cylinder $\mathbb{S}^1 \times \mathbb{R}$ and $(\boldsymbol{a}, t) \in \tilde{\Omega}'$ is the integration variable. Noting that η depends only on the $\boldsymbol{a}$ variable, we infer from the boundedness of $\tilde{\Omega}'$ that to prove the finiteness of the expressions in (8.7), it suffices to prove that

$$\int_{\mathcal{M}} |\eta(\boldsymbol{a})|\, d\mathcal{H}^1(\boldsymbol{a}) < \infty.$$

Consider a change $\boldsymbol{h}$ of variables from $\boldsymbol{r} \in \mathcal{T}$ to $\boldsymbol{a}(\boldsymbol{r}) \in \mathbb{S}^1$. The surface gradient $\nabla\boldsymbol{h}$ at a general point $\boldsymbol{r} \in \mathcal{S}$ satisfies

$$\nabla\boldsymbol{h} = \varphi\boldsymbol{t} \otimes \boldsymbol{o}, \quad \nabla\boldsymbol{h}^{\mathsf{T}} = \varphi\boldsymbol{o} \otimes \boldsymbol{t},$$

where $\boldsymbol{o}$ is the projection of $\boldsymbol{a}$ onto the tangent space of $\mathcal{T}$ at $\boldsymbol{r}$. Hence $\nabla\boldsymbol{h}^{\mathsf{T}}\nabla\boldsymbol{h} = \varphi^2\boldsymbol{o}\otimes\boldsymbol{o}$ and the jacobian J of $\boldsymbol{h}$ is $J = |\varphi||\boldsymbol{o}| = |\varphi||\boldsymbol{t}\cdot\boldsymbol{n}|$. Thus

$$\int_{\mathcal{M}} |\eta|\, d\mathcal{H}^1 = \int_{\mathcal{T}} |\tau||\varphi||\boldsymbol{t}\cdot\boldsymbol{n}|\, d\mathcal{H}^1 = \int_{\mathcal{T}} |\hat{\boldsymbol{s}}|\, d\mathcal{H}^1$$

by (8.4) and the last integral is finite since $\hat{\boldsymbol{s}} \in L^2(\mathcal{S}, \mathbb{R}^2)$. Thus $\boldsymbol{T} \in L^1(\Omega, \mathrm{Sym})$.

Prove that $\boldsymbol{T}$ is negative semidefinite. In view of (8.5) this amounts to proving that

$$\eta(\boldsymbol{a}(\boldsymbol{r}))\varphi(\boldsymbol{r}) \le 0 \tag{8.8}$$

for every $\boldsymbol{r} \in \Omega'$. By the nonpositivity of τ this is true if $\boldsymbol{r} \in \mathcal{T}$. Thus the proof of (8.8) will be complete if we show that φ does not change its sign on each line segment of the form $l \cap \Omega$ where $l \in \mathfrak{l}$. Letting γ be the restriction of φ to l and denoting by prime the differentiation with respect to the length parameter starting from the point on $l \cap \mathcal{T}$, we obtain by multiplying (8.2) by $\boldsymbol{t}$ the equation $\gamma' = \gamma^2$ from which $\gamma(s) = \gamma(0)/(1 - s\gamma(0))$. Thus γ can change its sign only at the point of singularity of γ; however, γ is regular on $l \cap \Omega$ and thus of constant sign which completes the proof of (8.8).

We finally prove (8.6). Assume first that the function $\eta : \mathcal{N} \to \mathbb{R}$ is of class C^1 so that also $\boldsymbol{T}$ is of class 2 on Ω'. If $\boldsymbol{v} \in V \cap C^1(\mathrm{cl}\,\Omega, \mathbb{R}^2)$ then the use of the divergence theorem gives

$$\int_{\Omega} \boldsymbol{T} \cdot \hat{\boldsymbol{E}}(\boldsymbol{v})\, d\mathcal{L}^2 = \int_{\Omega'} \boldsymbol{T} \cdot \hat{\boldsymbol{E}}(\boldsymbol{v})\, d\mathcal{L}^2 = -\int_{\Omega'} \boldsymbol{v} \cdot \mathrm{div}\,\boldsymbol{T}\, d\mathcal{L}^2 + \int_{\partial\Omega'} \boldsymbol{v} \cdot \boldsymbol{T}\boldsymbol{\nu}\, d\mathcal{H}^1, \tag{8.9}$$

where $\boldsymbol{\nu}$ is the outer normal to $\partial\Omega'$. From $\boldsymbol{T} = \tilde{\tau}\varphi\boldsymbol{t} \otimes \boldsymbol{t} = \eta \circ \boldsymbol{a}\varphi\boldsymbol{t} \otimes \boldsymbol{t}$ we obtain

$$\mathrm{div}\,\boldsymbol{T} = \tilde{\tau}\,\mathrm{div}(\varphi\boldsymbol{t} \otimes \boldsymbol{t}) + \varphi\boldsymbol{t}(\boldsymbol{t} \cdot \nabla\tilde{\tau}) = 0$$

since $\mathrm{div}(\varphi\boldsymbol{t} \otimes \boldsymbol{t}) = 0$ as a consequence of $\nabla\boldsymbol{t} = -\varphi\boldsymbol{a} \otimes \boldsymbol{a}$ (see (8.1)) and $\boldsymbol{t} \cdot \nabla\tilde{\tau} = 0$ as a consequence of $\tilde{\tau} = \eta \circ \boldsymbol{a}$. Thus (8.9) reduces to

$$\int_{\Omega} \boldsymbol{T} \cdot \hat{\boldsymbol{E}}(\boldsymbol{v})\, d\mathcal{L}^2 = \int_{\partial\Omega'} \boldsymbol{v} \cdot \boldsymbol{T}\boldsymbol{\nu}\, d\mathcal{H}^1.$$

To see that the right-hand side reduces to

$$\int_{\mathcal{T}} \boldsymbol{v} \cdot \hat{\boldsymbol{s}}\, d\mathcal{H}^1,$$

we note that $\boldsymbol{T\nu} = \boldsymbol{0}$ on $\partial\Omega' \cap \Omega$ since $\boldsymbol{\nu} \cdot \boldsymbol{t} = 0$; on the other hand, $\boldsymbol{\nu} = \boldsymbol{n}$ on $\partial\Omega' \cap \partial\Omega$ and thus it suffices to note that $\boldsymbol{Tn} = \hat{\boldsymbol{s}}$ on $\mathcal{S}$ as a consequence of $(8.4)_2$. The general case of a nonsmooth η is obtained by applying the above smooth case to smooth approximations of η. □

REMARK 8.3. Let $n = 2$ and let $\boldsymbol{A} \in \mathrm{Sym}^-$ be orthogonal to the tensor $\boldsymbol{e} \otimes \boldsymbol{e}$, where $\boldsymbol{e}$ is a unit vector. Then $\boldsymbol{A} = \sigma \boldsymbol{e}^\perp \otimes \boldsymbol{e}^\perp$ where $\sigma \le 0$. Indeed, if $t \in \mathbb{R}$ and $\boldsymbol{a} \in \mathbb{R}^2$, then $0 \ge \boldsymbol{A}(\boldsymbol{a} + t\boldsymbol{e}) \cdot (\boldsymbol{a} + t\boldsymbol{e}) = \boldsymbol{Aa} \cdot \boldsymbol{a} + 2t\boldsymbol{Ae} \cdot \boldsymbol{a}$; as t and $\boldsymbol{a}$ are arbitrary, we obtain $\boldsymbol{Ae} = \boldsymbol{0}$. □

PROPOSITION 8.4. Let $\mathfrak{l}$ be an $(\Omega, \mathcal{T})$ regular system, let $\mathcal{D} \subset \partial\Omega$ be such that

$$\mathcal{D} = \{l \cap \mathcal{D} : l \in \mathfrak{l}\} \tag{8.10}$$

and let $\varphi \ne 0$ for $\mathcal{L}^2$ a.e. point of Ω and for $\mathcal{H}^1$ a.e. point of $\mathcal{D}$. If $\boldsymbol{v} \in V$ satisfies

$$\hat{\boldsymbol{E}}(\boldsymbol{v}) = \alpha \boldsymbol{a} \otimes \boldsymbol{a}, \quad \boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad \mathcal{D} \tag{8.11}$$

where $\alpha : \Omega \to \mathbb{R}$ is some function, then $\boldsymbol{v} = \boldsymbol{0}$ on Ω.

Proof. Differentiating the function $\boldsymbol{v} \cdot \boldsymbol{t}$ at $\boldsymbol{r} \in \Omega$ in the direction $\boldsymbol{t}(\boldsymbol{r})$ and using (8.11), we obtain

$$\nabla(\boldsymbol{v} \cdot \boldsymbol{t})\boldsymbol{t} = 0.$$

Integrating along a line $l \in \mathfrak{l}$ containing $\boldsymbol{r}$ from $\boldsymbol{q} := \mathcal{D} \cap l$ to $\boldsymbol{r}$ and using that $\boldsymbol{v} \cdot \boldsymbol{t} = 0$ at $\boldsymbol{q}$, we obtain $\boldsymbol{v} \cdot \boldsymbol{t} = 0$ at $\boldsymbol{r}$ and thus there exists a function $\beta : \Omega \to \mathbb{R}$ such that $\boldsymbol{v} = \beta \boldsymbol{w}$ for every $\boldsymbol{r} \in \Omega$ where $\boldsymbol{w} = \boldsymbol{a}/\varphi$. Hence

$$\hat{\boldsymbol{E}}(\boldsymbol{v}) = \tfrac{1}{2}(\boldsymbol{w} \otimes \nabla\beta + \nabla\beta \otimes \boldsymbol{w}) + \beta \hat{\boldsymbol{E}}(\boldsymbol{w}).$$

A multiplication by $\boldsymbol{t}$ from the right and the use of (8.11) and (8.3) gives $\nabla\beta \cdot \boldsymbol{t} = 0$ on Ω. The integration along lines $l \in \mathfrak{l}$ gives that there exists a function $\tau : \mathfrak{l} \to \mathbb{R}$ such that

$$\beta(\boldsymbol{r}) = \tau(\hat{l}(\boldsymbol{r}))$$

for $\mathcal{L}^2$ a.e. point of Ω and hence

$$\boldsymbol{v}(\boldsymbol{r}) = \tau(\hat{l}(\boldsymbol{r}))\boldsymbol{a}(\boldsymbol{r})/\varphi(\boldsymbol{r}).$$

The boundary condition $\boldsymbol{v} = \boldsymbol{0}$ on $\mathcal{D}$ and (8.10) then yields $\tau = 0$ on $\mathfrak{l}$ identically. □

Proof of Example 3.3. It suffices to consider only nonnegative values of the loading multiplier; the considerations about negative values can be converted to those for positive values by changing the orientation of the x axis.

(ii): If $\omega : \mathbb{R} \to \mathbb{R}$ is any nonincreasing C^1 function vanishing on $(b/h, \infty)$ that does not vanish identically on $(0, b/h)$ then $\boldsymbol{v} : \Omega \to \mathbb{R}^2$, given by

$$\boldsymbol{v}(\boldsymbol{r}) = \omega(x/y)\boldsymbol{r}^\perp, \tag{8.12}$$

$\boldsymbol{r} = (x, y) \in \Omega, \boldsymbol{r}^\perp := (-y, x)$, satisfies $\boldsymbol{v} \in V$ and $\hat{\boldsymbol{E}}(\boldsymbol{v}) \in Y^+$. Indeed, one finds that $\boldsymbol{v} \in W^{1,2}(\Omega, \mathbb{R}^2)$ and since ω vanishes on $(b/h, \infty)$, $\boldsymbol{v}$ vanishes on

$$\Omega^- := \{\boldsymbol{r} \in \Omega : x/y > b/h\}$$

and thus in particular on $\mathcal{D}$ (in the sense of trace). Hence $\boldsymbol{v} \in V$. Furthermore,

$$\hat{\boldsymbol{E}}(\boldsymbol{v})(\boldsymbol{r}) = -y^{-2}\omega'(x/y)\boldsymbol{r}^{\perp} \otimes \boldsymbol{r}^{\perp} \tag{8.13}$$

$\boldsymbol{r} \in \Omega$, and as $\omega' \le 0$ we have $\hat{\boldsymbol{E}}(\boldsymbol{v}) \in Y^{+}$. One has

$$\langle \boldsymbol{l}_0, \boldsymbol{v} \rangle = -p \int_{\mathcal{T}} \omega(x/y)x \, d\mathcal{H}^1(\boldsymbol{r}) = -p \int_0^b \omega(x/h)x \, dx$$

$$\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 4pb^{-2} \int_{\mathcal{T}} \omega(x/y)yx(b-x) \, d\mathcal{H}^1(\boldsymbol{r}) = 4pb^{-2}h \int_0^b \omega(x/h)x(b-x) \, dx;$$

noting that the last expression and the hypotheses on ω imply that $\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle > 0$, we thus deduce that the value

$$\lambda = -\langle \boldsymbol{l}_0, \boldsymbol{v} \rangle / \langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 4^{-1}b^2 \int_0^b \omega(x/h)x \, dx \Big/ \int_0^b \omega(x/h)hx(b-x) \, dx \tag{8.14}$$

is a kinematically admissible multiplier. Fixing $\epsilon \in (0, b/h)$ and taking a sequence of the functions of the type of ω that converges to the function ω_ϵ, given by

$$\omega_\epsilon(t) = \begin{cases} 1 & \text{if } t \le \epsilon, \\ 0 & \text{otherwise,} \end{cases}$$

$t \in \mathbb{R}$, we deduce from (8.14) by evaluating the integrals that the value

$$\lambda = b^2/4h(b - 2h\epsilon/3)$$

is kinematically admissible. Varying $\epsilon \in (0, b/h)$ we obtain the interval $(\lambda_{\mathrm{c}}, 3\lambda_{\mathrm{c}})$. This completes the proof of the first part of (ii); the proof of the second part will be given later.

(i): For the rest of the proof we put the origin of the coordinate system to the middle of the top of the panel with the positive x axis pointing to the left and the positive y axis pointing down. Thus

$$\begin{gathered} \Omega = I \times (0, h), \quad I := (-b/2, b/2), \\ \mathcal{D} = I \times \{h\}, \quad \mathcal{T} = I \times \{0\}, \\ \boldsymbol{s}_0(\boldsymbol{r}) = p\boldsymbol{j}, \quad \bar{\boldsymbol{s}}(\boldsymbol{r}) = \big(p(b^2 - 4x^2)/b^2\big)\boldsymbol{i} \quad \text{ if } \ \boldsymbol{r} = (x, y) \in \mathcal{T}, \\ \boldsymbol{s}_0 = \bar{\boldsymbol{s}} = \boldsymbol{0} \quad \text{ on } \ \mathcal{S} \setminus \mathcal{T}. \end{gathered}$$

Let $\lambda > 0$ be fixed and consider the family

$$\mathfrak{l} = \{l_\rho : \rho \in I\}$$

of lines

$$l_\rho := \{(x, y) \in \mathbb{R}^2 : y = b^2(x - \rho)/\lambda(b^2 - 4\rho^2)\}$$

where $\rho \in I$. The line l_ρ passes through the point $\boldsymbol{q} := (\rho, 0) \in \mathcal{T}$ and the tangent to l_ρ is $\boldsymbol{s}(\lambda)(\boldsymbol{q})/|\boldsymbol{s}(\lambda)(\boldsymbol{q})|$; thus it satisfies $(8.4)_2$. If $\rho_1, \rho_2 \in I$, $\rho_1 \ne \rho_2$, then the lines l_{ρ_1} and l_{ρ_2} share a common point (x, y) where

$$x = (4\rho_1\rho_2 + b^2)/4(\rho_1 + \rho_2), \quad y = b^2/4(\rho_1 + \rho_2)\lambda.$$

One finds that if $\lambda \le \lambda_c$, then the intersection point (x, y) is outside Ω for all values $\rho_1, \rho_2 \in I, \rho \ne \rho_2$. Under the same condition on λ, one finds that the family $\mathfrak{l}$ is $(\Omega, \mathcal{T})$ regular with $\Omega' = \Omega$. One finds that the tangent to l_ρ at $\boldsymbol{q} = (\rho, 0)$ is

$$\boldsymbol{t} = \big(\lambda(b^2 - 4\rho^2), b^2\big)/\omega,$$

where

$$\omega = \sqrt{\lambda^2(b^2 - 4\rho^2)^2 + b^4}.$$

A differentiation with respect to ρ and a comparison with the formula

$$t_{y,x} = -\varphi(\boldsymbol{a} \otimes \boldsymbol{a})\boldsymbol{i} \cdot \boldsymbol{j} = -\varphi a_x a_y$$

(see (8.1)) gives

$$\varphi = 8\lambda\rho/\omega;$$

hence $\varphi \ne 0$ everywhere on $\mathcal{T}$ except at the point $\boldsymbol{q} = \boldsymbol{0}$. Thus there exists a function τ as in (8.4), viz.,

$$\tau(l_\rho) = p\omega^3/8\rho\lambda b^4.$$

The stressfield (8.5) is given by [13]

$$\boldsymbol{T}(\boldsymbol{r}) = \sigma\big(\kappa^2 \boldsymbol{i} \otimes \boldsymbol{i} + \kappa(\boldsymbol{i} \otimes \boldsymbol{j} + \boldsymbol{j} \otimes \boldsymbol{i}) + \boldsymbol{j} \otimes \boldsymbol{j}\big)$$

for any $\boldsymbol{r} = (x, y) \in \Omega$, where

$$\sigma = -bp/\zeta, \quad \kappa = \frac{b\zeta + 8\lambda xy - b^2}{8\lambda y^2}, \quad \zeta := \sqrt{-16\lambda xy + 16\lambda^2 y^2 + b^2}.$$

One finds that if $\lambda < \lambda_c$, then $\boldsymbol{T}$ is bounded on Ω while if $\lambda = \lambda_c$, then $\boldsymbol{T}$ is unbounded but $\boldsymbol{T} \in L^2(\Omega, \mathrm{Sym})$. This proves that if $0 < \lambda \le \lambda_c$, then λ is statically admissible. On the other hand, if $\lambda > \lambda_c$, then λ is not statically admissible by the first part of (ii). This completes the proof of (i).

We finally complete the proof of (ii) by showing that λ_c is not kinematically admissible. Assume that $\boldsymbol{v} \in V$ is a collapse mechanism for λ_c. By Proposition 3.1(iii) we have $\boldsymbol{T} \cdot \hat{\boldsymbol{E}}(\boldsymbol{v}) = 0$ for $\mathcal{L}^2$ a.e. point of Ω and for any admissible stressfield equilibrating the loads corresponding to λ_c. Taking for $\boldsymbol{T}$ the stressfield from (8.5) and combining with Remark 8.3 we deduce that $\hat{\boldsymbol{E}}$ is of the form (8.11) and as the hypotheses of Proposition 8.4 are satisfied, we obtain that $\boldsymbol{v} = \boldsymbol{0}$; hence, $\boldsymbol{v}$ is not a collapse mechanism. $\square$

Proof of Example 3.4. It suffices to consider only nonnegative values of the loading multiplier; the considerations about negative values can be converted to those for positive values by changing the orientation of the x axis.

(i): We shall prove that λ is statically admissible if $0 \le \lambda < \lambda_c$ by showing that $\mathcal{A}(\lambda) \ne \emptyset$ for these values of λ; then we shall show that λ_c is kinematically admissible by exhibiting the $\boldsymbol{v}$ required by the definition of a kinematically admissible multiplier and use $\boldsymbol{v}$ to show that λ_c is statically inadmissible. Then the inadmissibility of all $\lambda > \lambda_c$ is a consequence of Proposition 2.4. For $\lambda = 0$ there is an admissible equilibrated stressfield $\boldsymbol{T} = -p\boldsymbol{j} \otimes \boldsymbol{j}$. Assume that $0 < \lambda < \lambda_c$. Employing Proposition 8.2 in the same way as in the proof of Example 3.3, we find that an admissible equilibrating stressfield $\boldsymbol{T} \in \mathcal{A}(\lambda)$ is given by [13]:

$$\boldsymbol{T}(\boldsymbol{r}) = \begin{cases} c(\boldsymbol{r} - \alpha\boldsymbol{j}) \otimes (\boldsymbol{r} - \alpha\boldsymbol{j})/(y-\alpha)^3 & \text{if } (y-\alpha)/x \le (h-\alpha)/b, \\ \boldsymbol{0} & \text{otherwise,} \end{cases} \tag{8.15}$$

$\boldsymbol{r} \in \Omega$, where

$$c = -p^2/\lambda, \quad \alpha = h - p/\lambda.$$

Clearly, $\boldsymbol{T}$ is bounded on Ω and hence in Y.

Next prove that $\lambda = \lambda_{\mathrm{c}}$ is kinematically admissible by showing that if $\omega : \mathbb{R} \to \mathbb{R}$ is any nonincreasing C^1 function vanishing on $(b/h, \infty)$ such that

$$h \int_0^b \omega(x/h) x \, dx = 1, \tag{8.16}$$

then $\boldsymbol{v} : \Omega \to \mathbb{R}^2$ defined in (8.12) satisfies $\boldsymbol{v} \in V$, $\hat{\boldsymbol{E}}(\boldsymbol{v}) \in Y^+$ (see the proof of Example 3.3) and

$$\lambda_{\mathrm{c}} = -\langle \boldsymbol{l}_0, \boldsymbol{v} \rangle, \quad \langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 1 \tag{8.17}$$

since

$$-\langle \boldsymbol{l}_0, \boldsymbol{v} \rangle = p \int_{\mathcal{T}} \omega(x/y) x \, d\mathcal{H}^1(\boldsymbol{r}) = p/h = \lambda_{\mathrm{c}},$$
$$\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = \int_{\mathcal{T}} \omega(x/y) xy \, d\mathcal{H}^1(\boldsymbol{r}) = 1$$

by $y = h$ on $\mathcal{T}$ and by (8.16). Thus λ_{c} is kinematically admissible and hence no $\lambda > \lambda_{\mathrm{c}}$ can be statically admissible.

We now complete the proof of (i) by showing that $\mathcal{A}(\lambda_{\mathrm{c}}) = \emptyset$. We reason by contradiction, assuming that there is a $\boldsymbol{T} \in \mathcal{A}(\lambda_{\mathrm{c}})$ and deducing that $\boldsymbol{T}$ cannot be in $\mathcal{A}(\lambda_{\mathrm{c}})$. Thus let $\boldsymbol{T} \in \mathcal{A}(\lambda_{\mathrm{c}})$. Let $\boldsymbol{v} \in V$ be defined through ω as above, but assume additionally, as we can, that $\omega'(t) < 0$ for every $t \in (0, b/h)$. Then $\big(\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})\big) = \langle \boldsymbol{l}(\lambda_{\mathrm{c}}), \boldsymbol{v} \rangle = 0$ with the first equality since $\boldsymbol{T}$ equilibrates the loads and the second by (8.17). By Proposition 3.1(iii) then, $\boldsymbol{T} \cdot \hat{\boldsymbol{E}}(\boldsymbol{v}) = 0$ for $\mathcal{L}^2$ a.e. point of Ω. We have $\omega'(x/y) \ne 0$ for every point of

$$\Omega^+ := \{\boldsymbol{r} \in \Omega : x/y < b/h\}.$$

By (8.13) then, $\boldsymbol{T}(\boldsymbol{r}) \cdot (\boldsymbol{r}^\perp \otimes \boldsymbol{r}^\perp) = 0$ for a.e. point of Ω^+ and by Remark 8.3, $\boldsymbol{T}(\boldsymbol{r})$ must be proportional to $\boldsymbol{r} \otimes \boldsymbol{r}$; hence, we write

$$\boldsymbol{T}(\boldsymbol{r}) = \varphi(\boldsymbol{r}) \boldsymbol{r} \otimes \boldsymbol{r}/y^3$$

for $\mathcal{L}^2$ a.e. $\boldsymbol{r} = (x, y) \in \Omega^+$, where $\varphi : \Omega^+ \to \mathbb{R}$ is a $\mathcal{L}^2$ measurable function. Let $0 \le \alpha < \beta \le b$ and $0 < s < t < h$ and put

$$S_1 = \{\boldsymbol{r} = (x, h) \in \mathcal{S} : \alpha < x < \beta\},$$
$$U = \{\boldsymbol{r} \in \Omega^+ : \boldsymbol{r} = \tau\boldsymbol{q}, \, \boldsymbol{q} \in S_1, \, s < \tau < t\}.$$

Since $\boldsymbol{T}$ balances the loads and the body force vanishes, the weak divergence of $\boldsymbol{T}$ vanishes; letting $p > 0$, denoting the p mollification of $\boldsymbol{T}$ by $\boldsymbol{T}_p$ and using that $\operatorname{cl} U \subset \Omega$, we find that if p is sufficiently small, then $\operatorname{div} \boldsymbol{T}_p = \boldsymbol{0}$ on U. The divergence theorem therefore yields $\int_{\partial U} \boldsymbol{T}_p \boldsymbol{o} \, d\mathcal{H}^1 = 0$, where $\boldsymbol{o}$ denotes the normal to U. Since $\boldsymbol{T}_p$ converges to $\boldsymbol{T}$ for $\mathcal{L}^2$ a.e. point of Ω, we obtain that for $\mathcal{L}^1$ a.e. α, β, s, t, the product $\boldsymbol{T}_p \boldsymbol{o}$ converges to $\boldsymbol{T}\boldsymbol{o}$, these α, β, s, t being determined by the requirement that $\mathcal{H}^1$ a.e. point of ∂U is a Lebesgue $\mathcal{L}^2$ point of $\boldsymbol{T}\boldsymbol{o}$. Thus we have

$$\int_{\partial U} \boldsymbol{T}\boldsymbol{o}\, d\mathcal{H}^1 = 0 \tag{8.18}$$

for all such α, β, s, t. The form of $\boldsymbol{T}$ shows that $\boldsymbol{T}\boldsymbol{o}$ is nonzero only on the subsets sS_1 and tS_1 of ∂U, where $\boldsymbol{o} = \pm \boldsymbol{j}$, respectively. Then (8.18) reads

$$s^{-2} \int_{sS_1} \varphi(\boldsymbol{r})\boldsymbol{r}\, d\mathcal{H}^1(\boldsymbol{r}) = t^{-2} \int_{tS_1} \varphi(\boldsymbol{r})\boldsymbol{r}\, d\mathcal{H}^1(\boldsymbol{r})$$

which by a change of variables reduces to

$$\int_{S_1} \varphi(s\boldsymbol{r})\boldsymbol{r}\, d\mathcal{H}^1(\boldsymbol{r}) = \int_{S_1} \varphi(t\boldsymbol{r})\boldsymbol{r}\, d\mathcal{H}^1(\boldsymbol{r}).$$

As this must hold for $\mathcal{L}^1$ a.e. α, β, s, t, we deduce that $\varphi(s\boldsymbol{r}) = \varphi(t\boldsymbol{r})$ for $\mathcal{L}^1$ a.e. s, t and $\mathcal{H}^1$ a.e. $\boldsymbol{r} \in \mathcal{T}$ where, it will be recalled, $\mathcal{T} = (0, b) \times \{h\}$. Thus by changing φ on a set of $\mathcal{L}^2$ measure 0, we have

$$\varphi(t\boldsymbol{r}) = \varphi(\boldsymbol{r})$$

for all $t \in (0,1)$ and $\mathcal{L}^2$ a.e. $\boldsymbol{r} \in \Omega$. Then $\mathcal{H}^1$ a.e. point of $\mathcal{T}$ is a $\mathcal{L}^2$ Lebesgue point of $\boldsymbol{T}$, and employing the mollification argument as before, we obtain

$$\int_{\Omega^+} \boldsymbol{T} \cdot \hat{\boldsymbol{E}}(\boldsymbol{v}) = h^{-2} \int_{\mathcal{T}} \varphi(\boldsymbol{r})\boldsymbol{r} \cdot \boldsymbol{v}(\boldsymbol{r})\ d\mathcal{H}^1(\boldsymbol{r})$$

for all $\boldsymbol{v} \in W^{1,2}(\Omega, \mathbb{R}^2)$. Letting $\boldsymbol{v} \in V$ with $\boldsymbol{v} = \boldsymbol{0}$ on $\Omega \setminus \Omega^+$, we find that the relation $\langle \boldsymbol{l}(\lambda_{\rm c}), \boldsymbol{v} \rangle = (\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v}))$ reads

$$h^{-2} \int_{\mathcal{T}} \varphi(\boldsymbol{r})\boldsymbol{r} \cdot \boldsymbol{v}(\boldsymbol{r})\, d\mathcal{H}^1(\boldsymbol{r}) = -p/h \int_{\mathcal{T}} \boldsymbol{r} \cdot \boldsymbol{v}(\boldsymbol{r})\, d\mathcal{H}^1(\boldsymbol{r}).$$

From the arbitrariness of $\boldsymbol{v}$ subject to conditions listed above, we deduce that

$$\varphi(\boldsymbol{r}) = -ph$$

for $\mathcal{H}^1$ almost all $\boldsymbol{r} \in \mathcal{T}$ and hence for $\mathcal{L}^2$ almost all $\boldsymbol{r} \in \Omega^+$. Thus

$$\boldsymbol{T}(\boldsymbol{r}) = -ph\boldsymbol{r} \otimes \boldsymbol{r}/y^3,$$

$\boldsymbol{r} \in \Omega^+$. In particular, $T_{yy}(\boldsymbol{r}) = -ph/y$ for $\boldsymbol{r} \in \Omega^+$ and one easily finds that $T_{yy} \notin L^2(\Omega^+, \mathbb{R})$. Thus $\boldsymbol{T} \notin Y$ and hence $\mathcal{A}(\lambda_{\rm c}) = \emptyset$.

(ii): Prove that each $\lambda \geq \lambda_{\rm c}$ is kinematically admissible. Let $\boldsymbol{v}_1 : \Omega \to \mathbb{R}^2$ be given by

$$\boldsymbol{v}_1(\boldsymbol{r}) = (0, y)$$

for each $\boldsymbol{r} = (x, y) \in \Omega$. One has $\boldsymbol{v}_1 \in V$, $\hat{\boldsymbol{E}}(\boldsymbol{v}_1) \in Y^+$ and

$$-\langle \boldsymbol{l}_0, \boldsymbol{v}_1 \rangle = ph, \quad \langle \bar{\boldsymbol{l}}, \boldsymbol{v}_1 \rangle = 0. \tag{8.19}$$

Let $\boldsymbol{v}_\lambda = \boldsymbol{v} + (\lambda - \lambda_{\rm c})\boldsymbol{v}_1/ph$, where $\boldsymbol{v}$ is as in the proof of (i). One has $\boldsymbol{v}_\lambda \in V$ and $\hat{\boldsymbol{E}}(\boldsymbol{v}_\lambda) \in Y^+$ for $\lambda \geq \lambda_{\rm c}$. From (8.17) and (8.19) one finds that

$$-\langle \boldsymbol{l}_0, \boldsymbol{v}_\lambda \rangle = \lambda, \quad \langle \bar{\boldsymbol{l}}, \boldsymbol{v}_\lambda \rangle = 1.$$

To prove the last statement in the example, viz., there exists a stressfield $\boldsymbol{T} \in L^1(\Omega, \mathrm{Sym}) \setminus L^2(\Omega, \mathrm{Sym})$ with values in Sym^- such that (3.7) holds for every $\boldsymbol{v} \in V \cap W^{1,\infty}(\Omega, \mathbb{R}^2)$, we let $\boldsymbol{T} : \Omega \to \mathrm{Sym}$ be defined by

$$\boldsymbol{T}(\boldsymbol{r}) = \begin{cases} -ph\boldsymbol{r} \otimes \boldsymbol{r}/y^3 & \text{if } \boldsymbol{r} = (x, y) \in \Omega^+, \\ \boldsymbol{0} & \text{otherwise,} \end{cases}$$

$\boldsymbol{r} \in \Omega$, and $\boldsymbol{T}_\lambda$ be defined by the right-hand side of (8.15) for $\lambda < \lambda_{\mathrm{c}}$. We find that $\boldsymbol{T}_\lambda \rightharpoonup \boldsymbol{T}$ in $L^1(\Omega, \mathrm{Sym})$ as $\lambda \to \lambda_{\mathrm{c}}$; thus if $\boldsymbol{v} \in V \cap W^{1,\infty}(\Omega, \mathbb{R}^2)$ then the limit in

$$\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{T}_\lambda \cdot \hat{\boldsymbol{E}}(\boldsymbol{v})\, d\mathcal{L}^2$$

as $\lambda \to \lambda_{\mathrm{c}}$ gives (3.7). One easily finds that $\boldsymbol{T} \in L^1(\Omega, \mathrm{Sym})$ and we already know that $\boldsymbol{T} \notin L^2(\Omega, \mathrm{Sym})$. □

Proof of Example 3.5. (i): If $\lambda \le 0$, then $\boldsymbol{T} = \lambda \boldsymbol{1}$ on Ω is an admissible equilibrating stressfield for the loads $\mathfrak{L}(\lambda)$; if $\lambda > 0$, then $\boldsymbol{s}(\lambda)$ does not satisfy (3.1). (ii): Prove that $\{\boldsymbol{v} \in V : \hat{\boldsymbol{E}}(\boldsymbol{v}) \in Y^+\} = \{\boldsymbol{0}\}$. Indeed, if $\boldsymbol{v} = (v_x, v_y) \in V$ satisfies $\hat{\boldsymbol{E}}(\boldsymbol{v}) \in Y^+$, then

$$v_{x,x} \ge 0, \quad v_{y,y} \ge 0, \quad (v_{x,y} + v_{y,x})^2 \le 4 v_{x,x} v_{y,y}. \tag{8.20}$$

Condition $(8.20)_1$ and the boundary condition $v_x(0, y) = v_x(b, y) = 0$ give $0 = v_x(0, y) \le v_x(b, y) = 0$ and thus $v_{x,x} = 0$; i.e., v_x is a function of y only, which, by invoking the boundary condition again, gives $v_x = 0$ identically. Then the condition $(8.20)_3$ gives $v_{y,x} = 0$ and thus v_y is a function of y only and the boundary condition $v_y(0, y) = v_y(b, y) = 0$ leads to $v_y = 0$ identically. Thus $\boldsymbol{v} \equiv \boldsymbol{0}$. Proposition 3.1(ii) then shows that there is no kinematically admissible multiplier. □

Proof of Example 3.6. (i): Assume that λ is statically admissible. Because of the rotational symmetry of the loads one can prove that if the loads are equilibrated by an admissible stressfield, then they are also equilibrated by a rotationally symmetric stressfield, i.e., a stressfield of the form

$$\boldsymbol{T}(\boldsymbol{r}) = \alpha(\rho)\boldsymbol{1} + (\beta(\rho) - \alpha(\rho))\boldsymbol{r} \otimes \boldsymbol{r}/\rho^2 \tag{8.21}$$

$\boldsymbol{r} \in \Omega$, where α, β are functions on (a, b) and $\rho = |\boldsymbol{r}|$. The eigenvalues of $\boldsymbol{T}$ are α and β, and we have $\boldsymbol{T} \in L^2(\Omega, \mathrm{Sym})$ if and only if $\alpha, \beta \in L^2((a, b), \mathbb{R})$. The stressfield equilibrates the loads $\mathfrak{L}(\lambda)$ if and only if β is absolutely continuous on $[a, b]$ and

$$(\rho\beta)' = \alpha \quad \text{a.e.} \quad \text{on } (a, b) \quad \text{and} \quad \beta(a) = -p, \quad \beta(b) = -\lambda; \tag{8.22}$$

$\boldsymbol{T}$ is admissible if and only if

$$-\sigma^{\mathrm{c}} \le \alpha \le 0, \quad -\sigma^{\mathrm{c}} \le \beta \le 0 \tag{8.23}$$

a.e. on (a, b). From (8.22) and (8.23) then, $-\sigma^{\mathrm{c}} \le (\rho\beta)' \le 0$; the integration from a to b using the boundary conditions gives

$$\sigma^{\mathrm{c}}(a - b) \le ap - b\lambda \le 0$$

and these inequalities imply (3.8). Conversely, assume that (3.8) hold and show that the stressfield (8.21) with

$$\alpha = (\lambda_{\mathrm{c}}^- - \lambda)/(1 - \eta), \quad \beta = \big((\lambda - p)a/\rho + \lambda_{\mathrm{c}}^- - \lambda\big)/(1 - \eta) \tag{8.24}$$

is admissible and equilibrates the loads. These functions satisfy (8.22) and thus $\boldsymbol{T}$ equilibrates the loads. To show that $\boldsymbol{T}$ is admissible, we have to verify the inequalities (8.23). One finds that the first pair of inequalities in (8.23) is equivalent to the inequalities in (3.8). To verify the second pair of inequalities in (8.23), we note that since β is a monotone function of ρ, it suffices to verify this pair of inequalities at the endpoints $\rho = a, b$. There the function β is equal to $-p, -\lambda$, respectively, and thus we have to verify $0 \le p \le \sigma^{\mathrm{c}}$, which we assume, and $0 \le \lambda \le \sigma^{\mathrm{c}}$, which follows from (3.8). (ii): The admissible equilibrating stressfield $\boldsymbol{T}^+$ as in (8.21), (8.24) corresponding to the multiplier λ_{c}^+ is given by

$$\boldsymbol{T}^+ = -\sigma^{\mathrm{c}}\mathbf{1} + (\sigma^{\mathrm{c}} - p)a\boldsymbol{r} \otimes \boldsymbol{r}/\rho^3;$$

putting

$$\boldsymbol{v}^+(\boldsymbol{r}) = -\boldsymbol{r}/2\pi b\rho,$$

$\boldsymbol{r} \in \Omega$, we find that

$$\hat{\boldsymbol{E}}(\boldsymbol{v}^+) = -\boldsymbol{r}^\perp \otimes \boldsymbol{r}^\perp/2\pi b\rho^3.$$

We have $\hat{\boldsymbol{E}}(\boldsymbol{v}^+) \cdot \boldsymbol{T}^+ = \sigma^{\mathrm{c}}/2\pi b\rho$ and thus if $\boldsymbol{S} \in K$, then the inclusion $\boldsymbol{S} - \sigma^{\mathrm{c}}\mathbf{1} \in \mathrm{Sym}^+$ provides

$$(\boldsymbol{S} - \boldsymbol{T}^+) \cdot \hat{\boldsymbol{E}}(\boldsymbol{v}^+) \le 0$$

for every $\boldsymbol{S} \in K$ and almost every point of Ω. Hence $\boldsymbol{v}^+$ satisfies $(2.11)_2$. One easily finds that $\boldsymbol{v}^+$ also satisfies $(2.11)_1$; thus $\boldsymbol{v}^+$ is a collapse mechanism for the loads $\mathfrak{L}(\lambda_{\mathrm{c}}^+)$ by (2.12). □

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