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A NOTE ON YOUNG MEASURES AND CORRECTORS IN Γ-CONVERGENCE AND HOMOGENIZATION

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Abstract. We explore a general strategy to determine the Young measure associated with sequences of pairs of conductivity coefficients and corresponding gradient fields. We also relate this issue to corrector results and investigate various examples.

1. Introduction. Our goal in this note is to determine, under suitable conditions and assumptions, the joint (full) Young measure ([2], [14]) associated with pairs of sequences $\{(a_j, \nabla u_j)\}$ where there is an intimate relationship between those two sequences of functions to the point that, in typical situations, u_j is determined in a unique way from a_j. The simplest situation where we can explain our objective comes directly from homogenization. Suppose that given $\{a_j\}$, a sequence of scalar functions bounded uniformly away from zero in $\Omega \subset \mathbf{R}^N$, we solve the conductivity equation

$$\operatorname{div}[a_j(x)\nabla u_j(x)] = 0 \text{ in } \Omega, \quad u_j = u_0 \text{ on } \partial\Omega,$$

where $u_j \in H^1(\Omega)$, $u_j - u_0 \in H_0^1(\Omega)$. We know that $u_j \rightharpoonup u$ in $H^1(\Omega)$ where u is the solution of the homogenized problem

$$\operatorname{div}[A(x)\nabla u(x)] = 0 \text{ in } \Omega, \quad u = u_0 \text{ on } \partial\Omega,$$

and the matrix A is appropriately determined from the sequence $\{a_j\}$ (A is the H-limit of $\{a_j\}$ [8]). We might be interested in knowing the limit behavior of certain physical quantities of interest which can be easily computed for each problem j. One such example is the dissipated heat ([7]), if we think in terms of electric conductivity, given by the integral

$$H_j = \int_\Omega \frac{1}{a_j(x)} |\nabla u_j(x)|^2 \, dx.$$

The issue we would like to address is how to find the limit of such quantities as $j \to \infty$, so that we can say what the "true" dissipated heat is for the limit homogenized problem.

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In fact, our objective is much more ambitious. We pretend to give a rule to calculate the limit of integrals of the form

$$\int_\Omega F(x, a_j(x), \nabla u_j(x))\, dx$$

for arbitrary Carathéodory integrands $F(x, \lambda, \rho)$ (continuous in (λ, ρ) and measurable in x), provided that such compositions $\{F(x, a_j(x), \nabla u_j(x))\}$ make up an equiintegrable family of functions in $L^1(\Omega)$. This issue is nothing but finding the Young measure corresponding to the sequence (or an appropriate subsequence) of pairs $\{(a_j, \nabla u_j)\}$.

In an equivalent format in the context of Γ-convergence of functionals ([5]), we may consider the sequence of quadratic functionals

$$I_j(v) = \int_\Omega \frac{a_j(x)}{2} |\nabla v(x)|^2\, dx.$$

We know that the Γ-limit is given by the functional

$$I(v) = \int_\Omega \frac{1}{2} \nabla v(x)^T A(x) \nabla v(x)\, dx$$

for an appropriate matrix field $A(x)$. Suppose that for any given $u \in H^1(\Omega)$, not just the solution of the homogenized problem, we have a sequence $\{u_j\}$ such that $u_j \rightharpoonup u$ in $H^1(\Omega)$ and

$$\lim_{j\to\infty} I_j(u_j) = I(u).$$

Can we say something about the limits

$$\lim_{j\to\infty} \int_\Omega F(x, a_j(x), \nabla u_j(x))\, dx$$

for an arbitrary Carathéodory integrand F as before? That is, can we determine the full Young measure corresponding to the sequence of pairs $\{(a_j, \nabla u_j)\}$? Notice that this question is a bit more general than the one considered above in the language of homogenization, as the sequence $\{u_j\}$ does not have to be the sequence of minimizers for I_j.

As a matter of fact, we would like to place ourselves in a more general context in which non-linear problems may eventually be examined. We will consider, dwelling in the context of Γ-convergence, a sequence of functionals of the type

$$I_j(v) = \int_\Omega W(a_j(x), \nabla v(x))\, dx$$

under suitable assumptions for the integrands W. The Γ-limit of such functionals has already been addressed in some previous papers ([11], [12]), and given in the form of an integral functional

$$I(v) = \int_\Omega \psi(x, \nabla v(x))\, dx,$$

where the integrand ψ is given in a precise way from W and the sequence a_j. Here we would like to deepen that analysis to determine the full Young measure corresponding to pairs $\{(a_j, \nabla u_j)\}$ under the main hypotheses that $u_j \rightharpoonup u$ and $I_j(u_j) \to I(u)$.

The assumptions on $W(\lambda, \rho) : \mathbf{R}^m \times \mathbf{R}^N \to \mathbf{R}$ are:

- growth of order $p > 1$ on ρ:
$$c\,(|\rho|^p - 1) \le W(\lambda, \rho) \le C\,(|\rho|^p - 1)\,, \quad 0 < c < C;$$
- uniform continuity with respect to λ:
$$|W(\lambda_1, \rho) - W(\lambda_2, \rho)| \le w(|\lambda_1 - \lambda_2|)|\rho|^p,$$
where w is continuous and $w(0) = 0$;
- strict convexity of W on ρ for every λ;
- $a_j : \Omega \to \mathbf{R}^m$ is a sequence of vector- (matrix-) valued functions which we assume, for simplicity, is uniformly bounded in $L^\infty(\Omega; \mathbf{R}^m)$.

Further to this last requirement on $\{a_j\}$, we mainly assume the AGP (average gradient property) condition. This is a condition specifically designed so that Γ-limits can be computed or precisely determined. The main example corresponds to the periodic setting. See [11] and [12] for more information. We can say that such a sequence $\{a_j\}$ verifies the AGP condition if, loosely speaking, "averages of gradients over sets where the sequence is essentially constant are gradients themselves". The precise definition is a bit involved and can be found in [12]. To avoid unnecessary repetition, we do not include it here as it is not a main issue in this work. All we need to know for our purpose here is that this assumption guarantees that Γ-limits in the gradient case can be computed through the identity

$$\psi(x,\rho) = \inf_{\phi} \left\{ \int_{\mathbf{R}^m} W(\lambda, \phi(\lambda))\, d\sigma_x(\lambda) : \rho = \int_{\mathbf{R}^m} \phi(\lambda)\, d\sigma_x(\lambda), \text{ and} \right.$$
$$\text{whenever } r_j \searrow 0 \text{ is such that } \{a_j(x + r_j y)\} \text{ generates } \sigma_x,$$
$$\left. \text{then } \|\text{curl } (\phi(a_j(x + r_j y)))\,\|_{W^{-1,q}(B)} \to 0 \right\},$$

where B is the unit ball in $\mathbf{R}^N$, the family of probability measures $\sigma = \{\sigma_x\}_{x\in\Omega}$ is the Young measure associated with $\{a_j\}$, and q is the dual exponent of p (see Lemma 4.2 of [11]).

Our main results follow.

Theorem 1. Under the preceding assumptions on the sequence $\{a_j\}$ and the integrand $W(\lambda, \rho)$, for a.e. $x \in \Omega$, every $u \in W^{1,p}(\Omega)$ such that
$$\int_\Omega \psi(y, \nabla u(y))\, dy < +\infty,$$
and every sequence $\{u_j\}$ such that $u_j \rightharpoonup u$ and $I_j(u_j) \to I(u)$, we have:

(1) $\{|\nabla u_j|^p\}$ is equiintegrable;

(2) there exists a field $\varphi(\lambda, x)$ such that for a.e. $x \in \Omega$, $\varphi(\cdot, x) \in L^1(\sigma_x)$ and
$$\nabla u(x) = \int_{\mathbf{R}^m} \varphi(\lambda, x)\, d\sigma_x(\lambda), \quad \psi(x, \nabla u(x)) = \int_{\mathbf{R}^m} W(\lambda, \varphi(\lambda, x))\, d\sigma_x(\lambda);$$

(3) for every Carathéodory integrand F
$$\lim_{j\to\infty} \int_\Omega F(x, a_j(x), \nabla u_j(x))\, dx = \int_\Omega \int_{\mathbf{R}^m} F(x, \lambda, \varphi(\lambda, x))\, d\sigma_x(\lambda)\, dx,$$
provided that $\{F(x, a_j(x), \nabla u_j(x))\}$ is equiintegrable.

Notice that this result can be equivalently stated by saying that the Young measure associated with pairs $\{(a_j, \nabla u_j)\}$ is given by

$$\delta_{\varphi(\lambda,x)} \otimes \sigma_x. \tag{1}$$

When the sequence $\{u_j\}$ is precisely the sequence of minimizers for I_j, then the determination of the Young measure can be understood through corrector results in homogenization ([4]). Indeed, a corrector result is, at first sight, more than determining the Young measure, though we will see that, at least in our framework, it is equivalent. In our scenario in Γ-convergence, we will prove a more general result.

THEOREM 2. Let $U_j : \Omega \subset \mathbf{R}^N \to \mathbf{R}^N$ be a sequence of fields (not necessarily gradients) such that $\{|U_j|^p\}$ is equiintegrable, and the Young measure associated with pairs $\{(a_j, U_j)\}$ is given by (1) for a certain Carathéodory field $\varphi : \mathbf{R}^m \times \Omega \to \mathbf{R}^N$ (as in the preceding theorem). Then

$$U_j - \varphi(a_j(x), x) \to 0 \text{ strong in } L^p(\Omega).$$

As an immediate corollary we get the following.

COROLLARY 3. Suppose that $u \in W^{1,p}(\Omega)$ is such that

$$\int_\Omega \psi(x, \nabla u(x))\, dx < +\infty,$$

and $\{u_j\}$ is as in Theorem 1 with the field φ, a Carathéodory integrand. Then

$$\nabla u_j(x) - \varphi(a_j(x), x) \to 0 \text{ strong in } L^p(\Omega),$$

so that $\{\varphi(a_j(x), x)\}$ are the correctors.

Whenever the field $\varphi(\lambda, x)$ can be explicitly computed, we will have the corresponding sequence of correctors. As far as we can tell, there are not many such situations. See [6], [9].

Our objective in this note is to provide an alternative point of view on some issues in homogenization that may be helpful in some situations. We explore the most typical examples in Section 3. The proofs of our main results are described in Section 2.

2. Proofs. Suppose that the sequence of pairs $\{(a_j, \nabla u_j)\}$ generate the joint Young measure

$$\Lambda = \{\Lambda_x\}_{x\in\Omega}\,, \quad \Lambda_x = \mu_{\lambda,x}(\rho) \otimes \sigma_x(\lambda),$$

where we are using the slicing decomposition onto the Young measure corresponding to $\{a_j\}$. $\{u_j\}$ converges weakly in $W^{1,p}(\Omega)$ to a certain u. The representation in terms of Young measures always yields something smaller (Theorem 6.11 in [10]), so that

$$\lim_{j\to\infty} \int_\Omega W(a_j(x), \nabla u_j(x))\, dx \geq \int_\Omega \int_{\mathbf{R}^m} \int_{\mathbf{R}^N} W(\lambda, \rho)\, d\mu_{\lambda,x}(\rho)\, d\sigma_x(\lambda)\, dx.$$

By using Jensen's inequality, we can write

$$\int_\Omega \int_{\mathbf{R}^m} \int_{\mathbf{R}^N} W(\lambda,\rho)\, d\mu_{\lambda,x}(\rho)\, d\sigma_x(\lambda)\, dx$$
$$\geq \int_\Omega \int_{\mathbf{R}^m} W\left(\lambda, \int_{\mathbf{R}^N} \rho\, d\mu_{\lambda,x}(\rho)\right) d\sigma_x(\lambda)\, dx.$$

We know from [11] that the Γ-limit is an infimum over fields of the form

$$\int_{\mathbf{R}^N} \rho\, d\mu_{\lambda,x}(\rho)$$

coming from all feasible sequences $\{\nabla v_j\}$ having the weak limit ∇u. In particular, we definitely have

$$\int_\Omega \int_{\mathbf{R}^m} W\left(\lambda, \int_{\mathbf{R}^N} \rho\, d\mu_{\lambda,x}(\rho)\right) d\sigma_x(\lambda)\, dx \geq \int_\Omega \psi(x,\nabla u(x))\, dx,$$

since the right-hand side is the minimum over expressions of the form occurring on the left-hand side. By hypothesis, the starting and final points of all these inequalities,

$$\lim_{j\to\infty} \int_\Omega W(a_j(x),\nabla u_j(x))\, dx = \int_\Omega \psi(x,\nabla u(x))\, dx,$$

are in fact equal. Hence all the intermediate inequalities are, indeed, equalities. In particular, the equality

$$\lim_{j\to\infty} \int_\Omega W(a_j(x),\nabla u_j(x))\, dx = \int_\Omega \int_{\mathbf{R}^m} \int_{\mathbf{R}^N} W(\lambda,\rho)\, d\mu_{\lambda,x}(\rho)\, d\sigma_x(\lambda)\, dx$$

implies that $\{(a_j,\nabla u_j)\}$ is equiintegrable because the family of probability measures on the right-hand side is the Young measure associated with the sequence on the left, and due to the growth assumed on W with respect to ρ, the sequence $\{|\nabla u_j|^p\}$ is equiintegrable ([10]).

By the strict convexity of W with respect to the same gradient variable ρ, we conclude that

$$\mu_{\lambda,x} = \delta_{\varphi(\lambda,x)}$$

and, thus, the Young measure corresponding to $\{(a_j,\nabla u_j)\}$ is

$$\Lambda_x = \delta_{\varphi(\lambda,x)} \otimes \sigma_x$$

for a certain field $\varphi(\lambda,x)$ such that $\varphi(\cdot,x)\in L^1(\sigma_x)$ for a.e. $x\in\Omega$. Finally, this optimal field should be such that

$$\nabla u(x) = \int_{\mathbf{R}^m} \varphi(\lambda,x)\, d\sigma_x(\lambda), \quad \psi(x,\nabla u(x)) = \int_{\mathbf{R}^m} W(\lambda,\varphi(\lambda,x))\, d\sigma_x(\lambda).$$

This proves the theorem.

For the proof of Theorem 2, under our assumptions, the sequence of fields

$$U_j(x) - \varphi(a_j(x),x)$$

is such that its p-th power is equiintegrable. Recall that the sequence $\{a_j\}$ is assumed to be uniformly bounded in $L^\infty(\Omega)$. Let us examine its associated Young measure. To this end, let $G(\lambda)$ be any smooth and bounded integrand. By assumption, because $\delta_{\varphi(\lambda,x)}\otimes\sigma_x$ is the Young measure associated with $\{(a_j,U_j)\}$ and φ is a Carathéodory integrand,

$$\lim_{j\to\infty}\int_E G(U_j(x)-\varphi(a_j(x),x))\,dx=\int_E\int_{\mathbf{R}^m}G(\varphi(\lambda,x)-\varphi(\lambda,x))\,d\sigma_x(\lambda)\,dx=G(0)|E|$$

for arbitrary measurable $E\subset\Omega$. The arbitrariness of G and E yields that the Young measure for $\{U_j(x)-\varphi(a_j(x),x)\}$ is δ_0. It is well known (see [10] for instance) that this implies the strong convergence claimed in the statement of the Corollary.

3. Examples. We focus on the typical periodic case

$$I_j(v)=\int_Q\frac{a_j(x)}{2}|\nabla v(x)|^2\,dx,\quad a_j(x)=a(jx),\ x\in Q=(0,1)^N,$$

where a is bounded, bounded away from zero, and Q-periodic. Define the density

$$\psi:\mathbf{R}^N\to\mathbf{R},\quad \psi(\rho)=\int_Q\frac{a(\lambda)}{2}|\rho+\nabla_\lambda\xi(\lambda,\rho)|^2\,d\lambda,$$

where

$$\xi(\lambda,\rho):\mathbf{R}^N\times\mathbf{R}^N\to\mathbf{R},$$

for each fixed ρ, is the unique Q-periodic, weak solution in $H^1(Q)$ of the cell problem

$$\text{div }[a(\lambda)(\rho+\nabla_\lambda\xi(\lambda,\rho))]=0,\int_Q\xi(\lambda,\rho)\,d\lambda=0.$$

The Γ-limit of $\{I_j\}$ is

$$I(u)=\int_\Omega\psi(\nabla u(x))\,dx.$$

Observe that $\sigma_x=d\lambda$, the Lebesgue measure, for a.e. $x\in Q$. [1] It is easy to check ([12]) that the field $\varphi(\lambda,x)$ in Theorem 1 is

$$\varphi(\lambda,x)=\nabla u(x)+\nabla_\lambda\xi(\lambda,\nabla u(x)).$$

A direct application of Theorem 1 yields that if $\{u_j\}$ is such that

$$u_j\rightharpoonup u\text{ in }H^1(Q),\quad\int_Q\frac{a_j(x)}{2}|\nabla u_j(x)|^2\,dx\to\int_Q\psi(\nabla u(x))\,dx,$$

then the joint Young measure associated with pairs $\{(a(jx),\nabla u_j(x))\}$ is given by

$$\delta_{\nabla u(x)+\nabla_\lambda\xi(\lambda,\nabla u(x))}\otimes\ d\lambda;$$

that is to say

$$\int_Q F(x,a(jx),\nabla u_j(x))\,dx\to\int_Q\int_Q F(x,a(\lambda),\nabla u(x)+\nabla_\lambda\xi(\lambda,\nabla u(x)))\,d\lambda\,dx \tag{2}$$

for any Carathéodory integrand F such that the sequence $\{F(x,a(jx),\nabla u_j(x))\}$ is equi-integrable in $L^1(Q)$.

As remarked above, in the particular case in which

$$\text{div }[a(jx)\nabla u_j(x)]=0\text{ in }Q,\quad u=u_0\text{ on }\partial\Omega,$$

[1] There is a typical abuse of language here. This σ_x is NOT the Young measure associated with $\{a(jx)\}$ but rather with the Q-periodic extension of jx. We hope this will not cause any confusion.

and

$$\operatorname{div}\left[\frac{\partial\psi}{\partial\rho}(\nabla u(x))\right]=0 \text{ in } Q,\quad u=u_0 \text{ on } \partial\Omega,$$

the determination of the associated Young measure is a direct consequence of classical corrector results. Notice that if $\nabla u_j - U_j$ converges strongly to 0, then the Young measures corresponding to the two sequences $\{(a(jx),\nabla u_j(x))\}$ and $\{(a(jx),U_j(x))\}$ are the same.

We would like to apply Corollary 3 in this situation. This amounts to finding a constructive way of generating the optimal field in the definition of the integrand for the Γ-limit, or equivalently, the solution of the cell problem

$$\operatorname{div}\left[a(\lambda)(\rho+\nabla_\lambda\xi(\lambda,\rho))\right]=0.$$

It is elementary to realize that the dependence of $\rho+\nabla_\lambda\xi(\lambda,\rho)$ on ρ is linear so that

$$\rho+\nabla_\lambda\xi(\lambda,\rho)=H(\lambda)\rho.$$

Therefore, the sequence of fields

$$U_j(x)=H(jx)\nabla u(x)$$

generates, together with $a_j(x)=a(jx)$, the optimal underlying measure for the Γ-limit. In fact, notice that, by the classical Riemann-Lebesgue lemma,

$$\lim_{j\to\infty}\int_Q F(x,a(jx),H(jx)\nabla u(x))\,dx=\int_Q\int_Q F(x,\lambda,H(\lambda)\nabla u(x))\,d\lambda\,dx.$$

By comparing this to (2), we deduce that the joint Young measure for $\{(a_j,U_j)\}$ is the same as the one for $\{(a_j,\nabla u_j)\}$, and as a consequence of Theorem 2, we have

$$\nabla u_j-U_j\to 0 \text{ strong in } L^2(\Omega).$$

As a new example where these ideas can be applied, consider the case of the p-laplacian ($p>1$) still in a periodic setting. Put

$$I_j(v)=\int_Q\frac{a(jx)}{p}|\nabla v(x)|^p\,dx,\quad v\in W^{1,p}(Q),\ Q=(0,1)^N.$$

The integrand for the Γ-limit, $\psi(\rho)$ ([11]), is given by

$$\psi(\rho)=\int_Q\frac{a(\lambda)}{p}|\rho+\nabla_\lambda\xi(\lambda,\rho)|^p\,d\lambda,$$
$$\operatorname{div}\left[a(\lambda)|\rho+\nabla_\lambda\xi(\lambda,\rho)|^{p-2}(\rho+\nabla_\lambda\xi(\lambda,\rho))\right]=0,$$

where $\xi(\lambda,\rho)$ is the Q-periodic solution for fixed ρ. Therefore, by keeping track of the proof of Theorem 1 in this particular situation (see [11]), we conclude that the field whose existence is guaranteed in Theorem 1 is again of the form

$$\varphi(\lambda,x)=\nabla u(x)+\nabla_\lambda\xi(\lambda,\nabla u(x)).$$

The Young measure associated with pairs $\{(jx,\nabla u_j(x))\}$ is

$$\delta_{\nabla u(x)+\nabla_\lambda\xi(\lambda,\nabla u(x))}\otimes d\lambda.$$

In other words,

$$\lim_{j\to\infty}\int_Q F(x,jx,\nabla u_j(x))\,dx=\int_Q\int_Q F(x,\lambda,\nabla u(x)+\nabla_\lambda\xi(\lambda,\nabla u(x)))\,d\lambda\,dx,$$

provided that F is a Carathéodory integrand, Q-periodic in its second variable, for which $\{F(x,jx,\nabla u_j(x))\}$ is equiintegrable, and

$$\lim_{j\to\infty}\int_Q \frac{a(jx)}{p}|\nabla u_j(x)|^p\,dx=\int_Q\psi(\nabla u(x))\,dx.$$

Concerning correctors, we can write

$$H(\lambda,\nabla u(x))\equiv\varphi(\lambda,x)=\nabla u(x)+\nabla_\lambda\xi(\lambda,\nabla u(x)),$$

where $H(\lambda,\rho)=\rho+\nabla_\lambda\xi(\lambda,\rho)$. By directly applying Theorem 2, we conclude that if

$$U_j(x)=H(jx,\nabla u(x)),$$

then

$$U_j-\nabla u_j\to 0\text{ strong in }L^p(\Omega).$$

More explicit formulae can be given for a first-order laminate

$$a(\lambda)=\alpha\chi(\lambda\cdot n)+\beta(1-\chi(\lambda\cdot n))$$

for specific, positive values α,β, unit vector n, and $\chi=\chi_t$, the characteristic function of the interval $(0,t)$ in $(0,1)$ extended by periodicity. In this case, all of the above computations simplify to ([13])

$$\psi(\rho)=\frac{1}{p}\min_{c\in\mathbf{R}}\left\{t\alpha|\rho+(1-t)cn|^p+(1-t)\beta|\rho-tcn|^p\right\}.$$

Let $c(\rho)$ be the optimal solution of this one-dimensional minimization problem so that

$$\psi(\rho)=\frac{1}{p}\left(t\alpha|\rho+(1-t)c(\rho)n|^p+(1-t)\beta|\rho-tc(\rho)n|^p\right).$$

In this case, the field φ is given by

$$\varphi(\lambda,x)=\nabla u(x)+(\chi_t(\lambda\cdot n)-t)\,c(\nabla u(x))n.$$

If $\{u_j\}$ is such that $u_j\rightharpoonup u$ in $W^{1,p}(Q)$ and

$$\int_Q\frac{\alpha\chi_t(jx\cdot n)+\beta(1-\chi_t(jx\cdot n))}{p}|\nabla u_j(x)|^p\,dx\to\int_Q\psi(\nabla u(x))<+\infty,$$

then the Young measure corresponding to pairs

$$\{(\alpha\chi_t(jx\cdot n)+\beta(1-\chi_t(jx\cdot n)),\nabla u_j)\}$$

is

$$\delta_{\nabla u(x)+(\chi_t(\lambda\cdot n)-t)c(\nabla u(x))n}\otimes\left(\chi_t(\lambda\cdot n)\delta_\alpha+(1-\chi_t(\lambda\cdot n))\delta_\beta\right)\,d\lambda,$$

so that

$$\int_Q F(x,\alpha\chi_t(jx\cdot n)+\beta(1-\chi_t(jx\cdot n)),\nabla u_j(x))\,dx\to$$

$$\int_Q\left[tF(x,\alpha,\nabla u(x)+(1-t)c(\nabla u(x))n)+(1-t)F(x,\beta,\nabla u(x)-tc(\nabla u(x))n)\right]\,dx,$$

provided that $\{F(x, \alpha\chi_t(jx \cdot n) + \beta(1 - \chi_t(jx \cdot n)), \nabla u_j(x))\}$ is equiintegrable.

As before, if we put

$$H(\lambda, \rho) = \rho + (\chi_t(\lambda \cdot n) - t)\, c(\rho)n, \quad \varphi(\lambda, x) = H(\nabla u(x), \lambda)$$

and

$$U_j(x) = H(jx, \nabla u(x)) = \nabla u(x) + (\chi_t(jx \cdot n) - t)\, c(\nabla u(x))n,$$

then

$$U_j - \nabla u_j \to 0 \text{ strong in } L^p(\Omega).$$

For the particular case $p = 2$, explicit expressions for the mapping $c(\rho)$ can be obtained which lead to corrector results similar to the ones in [3].

References

[1] Allaire, G. *Shape optimization by the homogenization method*, 2002, Springer. MR1859696 (2002h:49001)

[2] Balder, E. J. (2000), Lectures on Young measure theory and its applications in economics, in: Proceedings Grado School on Measure Theory and Real Analysis, Rend. Sem. Matem. Trieste 31, suppl. 1, 1–69. MR1798830 (2001m:49069)

[3] Briane, M., Correctors for the homogenization of a laminate, Adv. Math. Sci. Appl., Gakk. Tokyo, 4, n. 2 (1994), 357–379. MR1294225 (95h:35023)

[4] Cioranescu, D., Donato, P., *An Introduction to Homogenization*, 1999, Oxford Univ. Press. MR1765047 (2001j:35019)

[5] Dal Maso, G. (1993), *Introduction to Γ-convergence*, Boston MA, Birkhäuser. MR1201152 (94a:49001)

[6] Dal Maso, G., Defranceschi, A., Correctors for the homogenization of monotone operators, Diff. Int. Equations 3 (1990), no. 6, 1151-1166. MR1073064 (91k:35032)

[7] Kohn, R. V., Strang, G. (1986), Optimal design and relaxation of variational problems, I, II and III, CPAM, 39, 113-137, 139-182 and 353-377. MR820342 (87d:49019a)

[8] Murat, F., Tartar, L., H-convergence, in Topics in the mathematical modelling of composite materials, Cherkaev, A., Kohn, R. V., eds., Boston, Birkhäuser, 1997, 21-44. MR1493039

[9] Pankov, A., *G-convergence and homogenization of nonlinear partial differential operators*, Mathematics and its Applications, 422. Kluwer Academic Publishers, Dordrecht, 1997. MR1482803 (99a:35017)

[10] Pedregal, P. (1997), *Parametrized Measures and Variational Principles*, Birkhäuser, Basel. MR1452107 (98e:49001)

[11] Pedregal, P., Γ-convergence through Young measures, SIAM J. Math. Anal., 36, n. 2 (2004), 423–440. MR2111784 (2005i:49016)

[12] Pedregal, P., Young measures associated with homogenization, SIAM J. Math. Anal., 37, n. 5 (2006), 1454-1464. MR2215272 (2006m:49023)

[13] Serrano, H., Ph.D. Thesis, U. Complutense, 2007.

[14] Valadier M. (1990), Young measures, Methods of Nonconvex Analysis, Lect. Notes in Math., 1446, Springer, 152-188. MR1079763 (91j:28006)