A MULTI-DIMENSIONAL BLOW-UP PROBLEM
DUE TO A CONCENTRATED NONLINEAR SOURCE IN $\mathbb{R}^N$

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Abstract. Let $B$ be an $N$-dimensional ball $\{x \in \mathbb{R}^N : |x| < R\}$ centered at the origin with a radius $R$, and $\partial B$ be its boundary. Also, let $\nu(x)$ denote the unit inward normal at $x \in \partial B$, and let $\chi_B(x)$ be the characteristic function, which is 1 for $x \in B$, and 0 for $x \in \mathbb{R}^N \setminus B$. This article studies the following multi-dimensional semilinear parabolic problem with a concentrated nonlinear source on $\partial B$:

$$u_t - \Delta u = \alpha \frac{\partial \chi_B(x)}{\partial \nu} f(u) \quad \text{in} \quad \mathbb{R}^N \times (0, T],$$

$$u(x, 0) = \psi(x) \quad \text{for} \quad x \in \mathbb{R}^N, \ u(x, t) \to 0 \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad 0 < t \leq T,$$

where $\alpha$ and $T$ are positive numbers, $f$ and $\psi$ are given functions such that $f(0) \geq 0$, $f(u)$ and $f'(u)$ are positive for $u > 0$, $f''(u) \geq 0$ for $u > 0$, and $\psi$ is nontrivial on $\partial B$, nonnegative, and continuous such that $\psi \to 0$ as $|x| \to \infty$, $\int_{\mathbb{R}^N} \psi(x) \, dx < \infty$, and $\Delta \psi + \alpha \left( \frac{\partial \chi_B(x)}{\partial \nu} \right) f(\psi(x)) \geq 0$ in $\mathbb{R}^N$. It is shown that the problem has a unique nonnegative continuous solution before blowup occurs. We assume that $\psi(x) = M(0) > \psi(y)$ for $x \in \partial B$ and $y \notin \partial B$, where $M(t) = \sup_{x \in \mathbb{R}^N} u(x, t)$. It is proved that if $u$ blows up in a finite time, then it blows up everywhere on $\partial B$. If, in addition, $\psi$ is radially symmetric about the origin, then we show that if $u$ blows up, then it blows up on $\partial B$ only. Furthermore, if $f(u) \geq \kappa u^p$, where $\kappa$ and $p$ are positive constants such that $p > 1$, then it is proved that for any $\alpha$, $u$ always blows up in a finite time for $N \leq 2$; for $N \geq 3$, it is shown that there exists a unique number $\alpha^*$ such that $u$ exists globally for $\alpha \leq \alpha^*$ and blows up in a finite time for $\alpha > \alpha^*$. A formula for computing $\alpha^*$ is given.

1. Introduction. Let $H = \partial / \partial t - \Delta$, $T$ be a positive real number, $x = (x_1, x_2, \ldots, x_N)$ be a point in the $N$-dimensional Euclidean space $\mathbb{R}^N$, $\Omega = \mathbb{R}^N \times (0, T]$, $B$ be an
N-dimensional ball \{x \in \mathbb{R}^N : |x - \bar{b}| < R\} centered at a given point \(\bar{b}\) with a radius \(R\), \(\partial B\) be the boundary of \(B\), \(\nu(x)\) denote the unit inward normal at \(x \in \partial B\), and

\[
\chi_B(x) = \begin{cases} 
1 \text{ for } x \in B, \\
0 \text{ for } x \in \mathbb{R}^N \setminus B
\end{cases}
\]

be the characteristic function. Without loss of generality, let \(\bar{b}\) be the origin. We would like to study the following multi-dimensional semilinear parabolic problem with a source on the surface of the ball:

\[
\begin{aligned}
Hu &= \alpha \frac{\partial \chi_B(x)}{\partial \nu} f(u) \text{ in } \Omega, \\
u(x,0) &= \psi(x) \text{ for } x \in \mathbb{R}^N, \ u(x,t) \to 0 \text{ as } |x| \to \infty \text{ for } 0 < t \leq T,
\end{aligned}
\]

(1.1)

where \(\alpha\) is a positive constant, \(f\) and \(\psi\) are given functions such that \(f(0) \geq 0\), \(f(u)\) and \(f'(u)\) are positive for \(u > 0\), \(f''(u) \geq 0\) for \(u > 0\), and \(\psi\) is nontrivial on \(\partial B\), nonnegative, and continuous such that \(\psi \to 0\) as \(|x| \to \infty\), \(\int_{\mathbb{R}^N} \psi(x) dx < \infty\), and \(\Delta \psi + \alpha \partial \chi_B(x)/\partial \nu f(\psi(x)) \geq 0\) in \(\mathbb{R}^N\). We note that such a problem in a bounded domain, instead of \(\mathbb{R}^N\), was studied by Chan and Tian \([2, 3]\).

A solution \(u\) is said to blow up at the point \((x, t_b)\) if there exists a sequence \(\{(x_n, t_n)\}\) such that \(u(x_n, t_n) \to \infty\) as \((x_n, t_n) \to (x, t_b)\).

In Section 2, we show that the nonlinear integral equation corresponding to the problem (1.1) has a unique nonnegative continuous solution \(u\), which is a nondecreasing function of \(t\). We then prove that \(u\) is the unique solution of the problem (1.1). Let \(M(t)\) denote \(\sup_{x \in \mathbb{R}^N} u(x, t)\). We assume that

\[
\psi(x) = M(0) > \psi(y) \text{ for } x \in \partial B \text{ and } y \notin \partial B.
\]

(1.2)

If \(t_b\) is finite, we show that \(u\) blows up everywhere on \(\partial B\). If, in addition, \(\psi\) is radially symmetric about the origin, then we prove that if \(u\) blows up, then it blows up everywhere on \(\partial B\) only. Let \(\kappa\) and \(p\) be positive constants such that \(p > 1\). If \(f(u) \geq \kappa u^p\), then we prove, in Section 3, that for any \(\alpha\), \(u\) always blows up in a finite time if \(N \leq 2\). This behavior is completely different from that for \(N \geq 3\). In Section 4, we show that for \(N \geq 3\), there exists a unique number \(\alpha^*\) such that \(u\) exists globally for \(\alpha \leq \alpha^*\) and blows up in a finite time for \(\alpha > \alpha^*\). We also derive a formula for computing \(\alpha^*\). We note that whether a solution of the heat equation without a concentrated source in an unbounded domain blows up in a finite time was studied by Fujita [7], and Pinsky [8].

### 2. Existence, uniqueness, and blowup.

To derive the integral equation from the problem (1.1), we use the adjoint operator \((-\partial/\partial t - \Delta)\) of \(H\). Using Green’s second identity, we obtain

\[
\begin{aligned}
u(x,t) &= \int_{\mathbb{R}^N} g(x,t;\xi,0) \psi(\xi) d\xi + \alpha \int_0^t \int_{\mathbb{R}^N} g(x,t;\xi,\tau) \frac{\partial \chi_B(\xi)}{\partial \nu} f(u(\xi,\tau)) d\xi d\tau,
\end{aligned}
\]

where

\[
g(x,t;\xi,\tau) = \begin{cases} 
\frac{1}{[4\pi (t - \tau)]^{N/2}} \exp \left(-\frac{|x - \xi|^2}{4(t - \tau)}\right), & t > \tau, \\
0, & t < \tau
\end{cases}
\]

(2.1)
is Green’s function (cf. Stakgold [10, p. 198]) corresponding to the problem (1.1). Using integration by parts and the Divergence Theorem, we have

$$u(x,t) = \int_{\mathbb{R}^N} g(x,t;\xi,0) \psi(\xi) \, d\xi + \alpha \int_0^t \int_{\partial B} g(x,t;\xi,\tau) f(u(\xi,\tau)) \, dS_\xi \, d\tau$$  \hspace{1cm} (2.2)

(cf. Chan and Tragoonsirisak [4]). We note that \( \int_{\mathbb{R}^N} \psi(x) \, dx < \infty \) is used to show that the first term on the right-hand side is continuous, and Lemma 2.1 of Chan and Tragoonsirisak [4] is used to show that the second term is continuous for \( t > 0 \).

To establish the next two results, we modify the techniques in proving Theorems 3 and 4 of Chan and Tian [2] for a blow-up problem in a bounded domain, and Theorems 2.1 and 2.2 of Chan and Tragoonsirisak [4] for a quenching problem in \( \mathbb{R}^N \).

**Theorem 2.1.** There exists some \( t_0 \) such that for \( 0 \leq t < t_0, \) the integral equation (2.2) has a unique continuous nonnegative solution \( u \), and \( u \) is a nondecreasing function of \( t \). If \( t_0 \) is finite, then \( u \) is unbounded in \([0,t_0)\).

Our next result shows that the solution of the integral equation (2.2) is the solution of the problem (1.1).

**Theorem 2.2.** The problem (1.1) has a unique solution \( u \) for \( 0 \leq t < t_0 \).

Henceforth, we assume that (1.2) holds. Our next result shows that at \( t_0, \) \( u \) blows up everywhere on the surface of the ball.

**Theorem 2.3.** If \( t_0 \) is finite, then at \( t_0, \) \( u \) blows up everywhere on \( \partial B \).

**Proof.** By Theorems 2.1 and 2.2, there exists some \( t_0 \) such that for \( 0 \leq t < t_0, \) the problem (1.1) has a unique nonnegative continuous solution \( u \), which is a nondecreasing function of \( t \). Since \( u(x,t) \) on \( \partial B \times (0,t_0) \) is known, let us denote it by \( \tilde{g}(x,t) \), and rewrite the problem (1.1) as two initial-boundary value problems:

\[
\begin{array}{l}
HU = 0 \text{ in } B \times (0,t_0), \\
\{ u \text{ in } \partial B, u(x,t) = \tilde{g}(x,t) \text{ on } \partial B \times (0,t_0) \}; \\
\end{array}
\]  \hspace{1cm} (2.3)

\[
\begin{array}{l}
HU = 0 \text{ in } (\mathbb{R}^N \setminus \bar{B}) \times (0,t_0), \\
u(0,0) = \psi(x) \text{ on } B, u(x,t) = \tilde{g}(x,t) \text{ on } \partial B \times (0,t_0), \\
u(x,t) \rightarrow 0 \text{ as } x \rightarrow \infty \text{ for } 0 < t < t_0.
\end{array}
\]  \hspace{1cm} (2.4)

Let us consider the problem (2.3). From the strong maximum principle (cf. Friedman [5, p. 34]), \( u \) attains its maximum somewhere on \( \partial B \) for \( t > 0 \). For the problem (2.4), \( u(x,t) \rightarrow 0 \) as \( |x| \rightarrow \infty \). Since \( u \) is a nondecreasing function of \( t \), it follows from the Phragmén-Lindelöf Principle (cf. Protter and Weinberger [7, pp. 183-185]) that \( u \) attains its maximum somewhere on \( \partial B \) for \( t > 0 \). Thus for each given \( \rho \in (0,t_0) \), \( u \) attains its maximum for \( 0 \leq t \leq \rho \) somewhere on \( \partial B \times \{ \rho \} \).

Suppose that there exists a smallest positive value of \( t \), say \( t_1 \), and some \( \tilde{y} \notin \partial B \) such that \( u(\tilde{y},t_1) = \min_{x \in \partial B} u(x,t_1) \). We claim that for \( x \in \partial B, u(x,t_1) = u(\tilde{y},t_1) \). If this is not true, then there exists some \( \bar{x} \in \partial B \) such that \( u(\bar{x},t_1) > \min_{x \in \partial B} u(x,t_1) \). Since \( u \) is continuous, there exists some point \( (\tilde{y},t_1) \) in a neighborhood of \( (\bar{x},t_1) \) such that \( \tilde{y} \notin \partial B \) and \( u(\tilde{y},t_1) > \min_{x \in \partial B} u(x,t_1) \). This contradicts the definition of \( t_1 \). Thus, \( u \) attains
From (2.2), if $y \in B$, then it follows from the strong maximum principle and the continuity of $u$ that $u \equiv u (y, t_1)$ on $B \times [0, t_1]$. This contradicts (1.2). If $y \in (\mathbb{R}^N \setminus B)$, then let $\tilde{B}$ be an $N$-dimensional ball $\{ x \in \mathbb{R}^N : |x| < \tilde{R} \}$ such that $\bar{y} \in \tilde{B}$. By the strong maximum principle and the continuity of $u$, $u \equiv u (\bar{y}, t_1)$ on $\left( \tilde{B} \setminus B \right) \times [0, t_1]$. Again, this contradicts (1.2). Thus for any $t > 0$,

$$u (x, t) > u (y, t) \text{ for any } x \in \partial B \text{ and any } y \notin \partial B. \quad (2.5)$$

We claim that for each $t > 0$, $u$ attains the same value for $x \in \partial B$. If this is not true, then for some $t > 0$, there exists some $\tilde{x} \in \partial B$ such that $u (\tilde{x}, t) > \min_{x \in \partial B} u (x, t)$. By continuity, there exists some point $(\hat{y}, t)$ in a neighborhood of $(\tilde{x}, t)$ such that $\hat{y} \notin \partial B$ and $u (\hat{y}, t) > \min_{x \in \partial B} u (x, t)$. This contradicts (2.5). Hence for any $t > 0$,

$$u (x, t) = M (t) \text{ for } x \in \partial B, M (t) > u (y, t) \text{ for any } y \notin \partial B. \quad (2.6)$$

This implies that for each $t > 0$, $u$ attains its absolute maximum on $\partial B$. Thus, if $u$ blows up, then it blows up there. Since $t_b$ is finite, it follows from Theorem 2.1 that $u$ blows up everywhere on $\partial B$.

Our next result shows that for the symmetric case, $u$ blows up on $\partial B$ only.

**Theorem 2.4.** Under the additional assumption that $\psi$ is radially symmetric about the origin, if $t_b$ is finite, then at $t_b$, $u$ blows up on $\partial B$ only.

**Proof.** Let us construct a sequence $\{ u_n \}$ in $\Omega$ by $u_0 (x, t) = \psi (x)$, and for $n = 0, 1, 2, \ldots$,

$$H u_{n+1} = \alpha \frac{\partial \chi_B (x)}{\partial \nu} f (u_n) \text{ in } \Omega, \quad u_{n+1} (x, 0) = \psi (x) \text{ for } x \in \mathbb{R}^N, u_{n+1} (x, t) \to 0 \text{ as } |x| \to \infty \text{ for } 0 < t \leq T.$$ From (2.2),

$$u_{n+1} (x, t) = \int_{\mathbb{R}^N} g (x, t; \xi, 0) \psi (\xi) d\xi + \alpha \int_0^t \int_{\partial B} g (x, t; \xi, \tau) f (u_n (\xi, \tau)) dS_x d\tau. \quad (2.7)$$

We note that $\Delta \psi + \alpha (\partial \chi_B (x) / \partial \nu) f (\psi (x)) \geq 0$ in $\mathbb{R}^N$. Thus,

$$H (u_1 - u_0) \geq \alpha \frac{\partial \chi_B (x)}{\partial \nu} (f (u_0) - f (\psi (x))) = 0 \in \Omega,$$

$$(u_1 - u_0) (x, 0) = 0 \text{ for } x \in \mathbb{R}^N, (u_1 - u_0) (x, t) \to 0 \text{ as } |x| \to \infty \text{ for } 0 < t \leq T.$$ Since $g (x, t; \xi, \tau) > 0$ in $\{(x, t; \xi, \tau) : x \text{ and } \xi \text{ are in } \mathbb{R}^N, T \geq t > \tau \geq 0 \}$, it follows from (2.2) that $u_1 (x, t) \geq u_0 (x, t)$ in $\Omega$. Let us assume that for some positive integer $j$,

$$\psi \leq u_1 \leq u_2 \leq \cdots \leq u_{j-1} \leq u_j \text{ in } \Omega.$$ We have

$$H (u_{j+1} - u_j) = \alpha \frac{\partial \chi_B (x)}{\partial \nu} (f (u_j) - f (u_{j-1})) \text{ in } \Omega,$$

$$(u_{j+1} - u_j) (x, 0) = 0 \text{ for } x \in \mathbb{R}^N, (u_{j+1} - u_j) (x, t) \to 0 \text{ as } |x| \to \infty \text{ for } 0 < t \leq T.$$ Since $f$ is an increasing function and $u_j \geq u_{j-1}$, we have $f (u_j) - f (u_{j-1}) \geq 0$. It follows from (2.2) that $u_{j+1} \geq u_j$. By the principle of mathematical induction,

$$\psi \leq u_1 \leq u_2 \leq \cdots \leq u_{n-1} \leq u_n \text{ in } \Omega.$$
Since \( u_n \) is an increasing sequence as \( n \) increases, it follows from the Monotone Convergence Theorem that we have (2.2) with \( \lim_{n \to \infty} u_n (x, t) = u(x, t) \).

Since \( \psi(x) \) is radially symmetric about the origin, namely \( \psi(x) = \psi(|x|) \), it follows from (2.1) and the construction (2.7) that

\[
 u_1(x, t) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} \exp \left( -\frac{|x - \xi|^2}{4t} \right) \psi(|\xi|) d\xi \\
+ \alpha \int_0^t \frac{1}{4\pi (t - \tau)^{N/2}} \int_{\partial B} \exp \left( -\frac{|x - \xi|^2}{4(t - \tau)} \right) f(\psi(|\xi|)) d\xi d\tau
\]

\[
= \frac{1}{(4\pi t)^{N/2}} \lim_{r \to \infty} \int_{B(0, r)} \exp \left( -\frac{|x - \xi|^2}{4t} \right) \psi(|\xi|) d\xi \\
+ \alpha \int_0^t \frac{1}{4\pi (t - \tau)^{N/2}} \int_{\partial B} \exp \left( -\frac{|x - \xi|^2}{4(t - \tau)} \right) f(\psi(|\xi|)) d\xi d\tau,
\]

where \( B(0, r) \) is the \( N \)-dimensional ball centered at the origin with a radius \( r \). Thus, \( u_1(x, t) \) is radially symmetric about the origin. We assume that for some positive integer \( j \), \( u_j(x, t) \) is radially symmetric about the origin, namely \( u_j(x, t) = u_j(|x|, t) \). Then,

\[
u_{j+1}(x, t) = \frac{1}{(4\pi t)^{N/2}} \lim_{r \to \infty} \int_{B(0, r)} \exp \left( -\frac{|x - \xi|^2}{4t} \right) \psi(|\xi|) d\xi \\
+ \alpha \int_0^t \frac{1}{4\pi (t - \tau)^{N/2}} \int_{\partial B} \exp \left( -\frac{|x - \xi|^2}{4(t - \tau)} \right) f(u_j(|\xi|, \tau)) d\xi d\tau
\]
is also radially symmetric about the origin. By the principle of mathematical induction, \( u_n(x, t) \) is radially symmetric about the origin for \( n = 0, 1, 2, \ldots \). Hence, \( u(x, t) = \lim_{n \to \infty} u_n(x, t) \) is radially symmetric about the origin.

From the problem (2.3), we have

\[
u_t - \left( u_{rr} + \frac{N - 1}{r} u_r \right) = 0 \text{ in } (R, \infty) \times (0, t_b),
\]

\[
u(r, 0) = \psi(r) \text{ on } [R, \infty),
\]

\[
u(R, t) = M(t), \nu(r, t) \to 0 \text{ as } r \to \infty \text{ for } 0 < t < t_b.
\]

From Theorem 2.1, \( u_t(x, t) \geq 0 \) in \( (\mathbb{R}^N \setminus B) \times (0, t_b) \). Thus,

\[
u_{rr} + \frac{N - 1}{r} u_r = u_t \geq 0.
\]

We note from (2.6) and the parabolic version of Hopf’s lemma (cf. Friedman [6] p. 49) that \( u_r(R, t) \leq 0 \) for \( 0 < t < t_b \). Hence for \( 0 < t < t_b \), \( \lim_{r \to R^+} u_{rr}(r, t) \geq 0 \) for \( N \geq 1 \). Therefore, if \( u \) blows up, then it blows up on \( \partial B \) only.

For the problem (2.3), it follows from Theorem 2.1 that \( u_t(x, t) \geq 0 \) in \( B \times (0, t_b) \). By Corollary 2 of Friedman [6] p. 74, \( u \) is infinitely differentiable. Hence, \( Hu_t = 0 \) in \( B \times (0, t_b) \). If \( u_t = 0 \) somewhere in \( B \times (0, t_b) \), say at \( t = t_2 \), then it follows from the problem (2.3) and the strong maximum principle that \( u_t \equiv 0 \) in \( B \times (0, t_2] \), and hence \( u(x, t) = \psi(x) \) for \( (x, t) \in B \times (0, t_2] \). By continuity, we have for \( (x, t) \in \partial B \times [0, t_2] \), \( u(x, t) = M(0) \), which is bounded. Since the solution \( u \) is continuous on
∂B × [0, t_b), there exists some \( t_3 (\geq t_2) \) such that \( u_t > 0 \) in \( B \times [t_3, t_b) \). Because \( u \) is radially symmetric, we have

\[
\begin{align*}
  u_t - \left( u_{rr} + \frac{N-1}{r} u_r \right) &= 0 \quad \text{in} \quad (0, R) \times (0, t_b), \\
  u(r, 0) &= \psi(r) \quad \text{on} \quad [0, R], \\
  u_r(0, t) &= 0, \quad u(R, t) = M(t) \quad \text{for} \quad 0 < t < t_b.
\end{align*}
\]

Thus,

\[
 u_{rr} + \frac{N-1}{r} u_r = u_t > 0
\]

in \( B \times [t_3, t_b) \). Since \( \lim_{r \to 0} u_{rr} + (N-1) \lim_{r \to 0}(u_r/r) = Nu_{rr}(0, t) \), we have \( u_{rr}(0, t) > 0 \), implying that \( u \) is concave up near the origin \( r = 0 \). Because \( u_r(0, t) = 0 \) for \( t_3 \leq t < t_b \), we have \( u_r > 0 \) near the origin for \( t_3 \leq t < t_b \). We would like to show that \( u(0, t) \) is bounded as \( t \) tends to \( t_b \). Let us assume, on the contrary, that \( u(0, t) \) tends to infinity as \( t \) tends to \( t_b \). If \( u_t(0, t) \) is bounded, say by a constant \( k_1 \), then

\[
u(0, t) \leq u(0, 0) + k_1 t \quad \text{for} \quad 0 < t < t_b.
\]

Because \( u(0, 0) \) is bounded, we have a contradiction. Thus, \( u_t(0, t) \) tends to infinity as \( t \) tends to \( t_b \). Since \( u_t(0, t) = Nu_{rr}(0, t) \), we have \( u_{rr}(0, t) \) tending to infinity as \( t \) tends to \( t_b \). Thus for \( t_3 \leq t < t_b \), there are points in a neighborhood of the origin \( r = 0 \) with values larger than \( u(0, t) \), and hence, \( u \) should blow up before \( t_b \). This contradicts the definition of \( t_b \). Hence, \( u(0, t) \) is bounded as \( t \) tends to \( t_b \). Next, we would like to show that the graph of \( u \) is concave up near \( \partial B \). Since \( u(r, t) \) tends to infinity as \( r \) tends to \( R \) and \( t \) tends to \( t_b \), and \( u \) is a strictly increasing function of \( t \in [t_3, t_b) \), we have for any given number \( M_1 \) sufficiently large, that there exists \( \tilde{r} \) sufficiently close to \( R \) and some \( \tilde{t} \) such that \( u(r, t) > M_1 \) for \( r \in [\tilde{r}, R] \) and \( t \in [\tilde{t}, t_b) \). We claim that for any given large number \( M_2 \), we can choose \( \tilde{r} \) and \( \tilde{t} \) such that \( u_t(\tilde{r}, \tilde{t}) > M_2 \) for \( r \in [\tilde{r}, R] \) and \( t \in [\tilde{t}, t_b) \). To prove this, let us assume that \( u_t(\tilde{r}, \tilde{t}) \) is bounded, say by a constant \( M_2 \). Then, \( u(\tilde{r}, t) \leq u(\tilde{r}, 0) + M_2 t \). We note that for \( M_1 > u(R, 0) + M_2 t_b \), we have \( \tilde{r} \) sufficiently close to \( R \) and some \( \tilde{t} \) such that \( u(\tilde{r}, \tilde{t}) > M_1 \) for \( r \in [\tilde{r}, R] \) and \( t \in [\tilde{t}, t_b) \). Thus,

\[
 u(r, t) \leq u(r, 0) + M_2 t \leq u(R, 0) + M_2 t_b < M_1
\]

for \( r \in [\tilde{r}, R] \) and \( t \in [\tilde{t}, t_b) \). We have a contradiction. Hence, \( u_t(r, t) \) can be made as large as we please. By choosing \( r \) and \( t \) sufficiently close to \( R \) and \( t_b \) respectively, if \( u_{rr}(r, t) \leq 0 \), then it follows from (2.8) that \( u_r(r, t) \) can be made as large as we please. This gives a contradiction to \( u_{rr}(r, t) \leq 0 \) since \( u(r, t) \) can be made as large as we wish. Thus, \( u \) is concave up near \( \partial B \). Because \( t_b \) is finite, it follows from Theorem 2.3 that \( u \) blows up on \( \partial B \) only. \( \square \)

3. \( N \leq 2 \). In the sequel, we assume that \( f(u) \geq \kappa u^p \), where \( \kappa \) and \( p \) are positive constants such that \( p > 1 \). Let

\[
 I(x, t) = \int_{\partial B} g(x, t; \xi, 0) dS \xi.
\]
Lemma 3.1 of Chan and Tragoonsirisak [4] states that for \( t \geq 1 \) and any \( x \in \bar{B} \),

\[
(4\pi)^{-N/2} e^{-R^2} \omega_N R^{N-1} t^{-N/2} \leq I(x,t) \leq (4\pi)^{-N/2} \omega_N R^{N-1} t^{-N/2},
\]

where \( \omega_N \) denotes the surface area of an \( N \)-dimensional unit sphere.

**Theorem 3.1.** If \( N \leq 2 \), then for any \( \alpha \) and any \( \psi(x) \), the solution \( u \) of the problem (1.1) always blows up in a finite time.

**Proof.** Let

\[
h(x) = \frac{e^{-|x|^2}}{\pi^{N/2}}.
\]

We note that \( h(x) > 0, h(x) \rightarrow 0 \) as \( |x| \rightarrow \infty \),

\[
\int_{\mathbb{R}^N} h(x) \, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} e^{-x_1^2/\pi} \cdots e^{-x_N^2/\pi} \, dx_1 \cdots dx_N = 1,
\]

\[
\int_{\partial \Omega} h(x) \, dS_x = \int_{\partial \Omega} \frac{e^{-R^2}}{\pi^{N/2}} \, dS_x = \frac{e^{-R^2} \omega_N R^{N-1}}{\pi^{N/2}},
\]

\[
\int_{\Omega} h(x) \, dx < \int_{R^N} h(x) \, dx = 1,
\]

\[
\Delta h = \frac{4e^{-|x|^2} |x|^2}{\pi^{N/2}} - 2N h(x) \geq -2N h(x).
\]

Let

\[
F(t) = \int_{\Omega} u(x,t) h(x) \, dx.
\]

Since \( u \) is the solution of the problem (1.1), \( F(t) \) may be regarded as a distribution. Thus,

\[
F'(t) = \int_{\Omega} u_t(x,t) h(x) \, dx
\]

\[
= \int_{\Omega} \left( \Delta u(x,t) + \alpha \frac{\partial \chi_B(x)}{\partial \nu} f(u(x,t)) \right) h(x) \, dx
\]

\[
\geq \int_{\Omega} \Delta u(x,t) h(x) \, dx + \alpha \kappa \int_{\partial \Omega} \frac{\partial \chi_B(x)}{\partial \nu} w^p(x,t) h(x) \, dx
\]

\[
= \int_{\Omega} \Delta u(x,t) h(x) \, dx + \alpha \kappa \int_{\partial \Omega} w^p(x,t) h(x) \, dS_x.
\]
Using Green’s second identity and (3.2), we have
\[
\int_{\mathbb{R}^N} \nabla u(x,t) \cdot h(x) \, dx = \lim_{R \to \infty} \int_{|x| < R} \nabla u(x,t) \cdot h(x) \, dx = \lim_{R \to \infty} \int_{|x| < R} u(x,t) \Delta h(x) \, dx = \int_{\mathbb{R}^N} u(x,t) \Delta h(x) \, dx \geq -2N \int_{\mathbb{R}^N} u(x,t) h(x) \, dx = -2NF(t).
\]
From (2.6),
\[
F(t) \leq M(t) \int_{\mathbb{R}^N} h(x) \, dx = M(t).
\]
Thus,
\[
\int_{\partial B} u^p(x,t) h(x) \, dS_x = M^p(t) \int_{\partial B} h(x) \, dS_x \geq F^p(t) \int_{\partial B} h(x) \, dS_x = \frac{e^{-R^2 \omega_N R^{N-1}F^p(t)}}{\pi^{N/2}}.
\]
Hence,
\[
F'(t) + 2NF(t) \geq \frac{\alpha e^{-R^2 \omega_N R^{N-1}}}{\pi^{N/2}} F^p(t).
\]
Solving this Bernoulli inequality, we obtain
\[
F^{1-p}(t) \leq \frac{\alpha e^{-R^2 \omega_N R^{N-1}}}{2N\pi^{N/2}} + C e^{2N(p-1)t},
\]
where \(C\) is to be determined. We can choose for \(\tilde{t} \geq 0\),
\[
C = \left( F^{1-p}(\tilde{t}) - \frac{\alpha e^{-R^2 \omega_N R^{N-1}}}{2N\pi^{N/2}} \right) e^{2N(1-p)\tilde{t}}.
\]
Thus for \(t > \tilde{t} \geq 0\),
\[
F^{p-1}(t) \geq \left[ \frac{\alpha e^{-R^2 \omega_N R^{N-1}}}{2N\pi^{N/2}} + \left( F^{1-p}(\tilde{t}) - \frac{\alpha e^{-R^2 \omega_N R^{N-1}}}{2N\pi^{N/2}} \right) e^{2N(p-1)(t-\tilde{t})} \right]^{-1}.
\]
We would like to show that there exists \(\tilde{t}\) such that
\[
F^{1-p}(\tilde{t}) - \frac{\alpha e^{-R^2 \omega_N R^{N-1}}}{2N\pi^{N/2}} < 0.
\]
From (2.2) and (2.6),
\[ u(x,t) \geq \alpha \kappa \int_0^t \int_{\partial B} g(x,t;\xi,\tau) M^p(\tau) \ dS_\tau d\tau. \]

For \( t > 1 \),
\[ u(x,t) \geq \alpha \kappa \int_0^{t-1} \int_{\partial B} g(x,t;\xi,\tau) M^p(\tau) \ dS_\tau d\tau. \]

Since \( u \) is a nondecreasing function of \( t \), we have \( M^p(\tau) \geq M^p(0) > 0 \). Thus,
\[ u(x,t) \geq \alpha \kappa M^p(0) \int_0^{t-1} I(x,t-\tau) d\tau \]
\[ = \alpha \kappa M^p(0) \int_1^t I(x,\theta) d\theta. \]

Using (3.1), we have for any \( x \in \bar{B} \),
\[ u(x,t) \geq \alpha \kappa M^p(0) (4\pi)^{-N/2} e^{-R^2 \omega_N R^{N-1}} \int_1^t \theta^{-N/2} d\theta \]
\[ = \begin{cases} 
2\alpha \kappa M^p(0) (4\pi)^{-N/2} e^{-R^2 \omega_N R^{N-1}} (t^{1/2} - 1) & \text{if } N = 1, \\
\alpha \kappa M^p(0) (4\pi)^{-N/2} e^{-R^2 \omega_N R^{N-1}} \ln t & \text{if } N = 2.
\end{cases} \]

Thus, there exists \( \tilde{t} \) such that for \( t \geq \tilde{t} \),
\[ u(x,t) > \frac{(2N\pi^{N/2})^{1/(p-1)}}{\alpha^{1/(p-1)} (\kappa e^{-R^2 \omega_N R^{N-1}})^{1/(p-1)}} \left( \int_{\bar{B}} h(x) \ dx \right)^{p-1} \]
for any \( x \in \bar{B} \). Then,
\[ F^{p-1}(\tilde{t}) = \left( \int_{\mathbb{R}^N} u(x,\tilde{t}) h(x) \ dx \right)^{p-1} \]
\[ \geq \left( \int_{\bar{B}} u(x,\tilde{t}) h(x) \ dx \right)^{p-1} \]
\[ > \left[ \frac{(2N\pi^{N/2})^{1/(p-1)}}{\alpha^{1/(p-1)} (\kappa e^{-R^2 \omega_N R^{N-1}})^{1/(p-1)}} \left( \int_{\bar{B}} h(x) \ dx \right)^{p-1} \right]^{p-1} \left( \int_{\bar{B}} h(x) \ dx \right)^{p-1} \]
\[ = \frac{2N\pi^{N/2}}{\alpha \kappa e^{-R^2 \omega_N R^{N-1}}}, \]
which gives (3.4). From (3.3), there exists a finite time \( t_b (> \tilde{t}) \) such that \( \lim_{t \to t_b} F(t) = \infty \). Thus, \( u(x,t) \) blows up in a finite time. □
4. \( N \geq 3 \). In this section, we show that the blow-up behavior for \( N \geq 3 \) is completely different from that for \( N \leq 2 \).

**Theorem 4.1.** (i) For \( N \geq 3 \), if \( \alpha \) is sufficiently small, then the solution \( u \) of the problem (1.1) exists globally.

(ii) For \( N \geq 3 \), if \( \alpha \) is sufficiently large, then the solution \( u \) of the problem (1.1) blows up in a finite time.

**Proof.** (i) Since \( f'(u) > 0 \), we have \( f(u) \leq f(2M(0)) \) for \( u(x,t) \leq 2M(0) \). Thus for \( u(x,t) \leq 2M(0) \), it follows from (2.2) and \( \int_{\mathbb{R}^N} g(x,t;\xi,0) d\xi = 1 \) (cf. Evans [5, p. 46]) that

\[
\begin{align*}
    u(x,t) &\leq M(0) \int_{\mathbb{R}^N} g(x,t;\xi,0) d\xi + \alpha \int_0^t \int_{\partial B} g(x,t;\xi,\tau) f(2M(0)) dS_\xi d\tau \\
    &\leq M(0) + \alpha f(2M(0)) \int_0^t \int_{\partial B} g(x,t;\xi,\tau) dS_\xi d\tau.
\end{align*}
\]

Let \( \eta = (\xi_i - x_i)/(2\sqrt{t-\tau}) \). Using \( \int_{-\infty}^{\infty} e^{-\eta^2} d\eta = \sqrt{\pi} \), we have

\[\int_{\partial B} g(x,t;\xi,\tau) dS_\xi \leq \frac{1}{2\sqrt{\pi}} (t-\tau)^{1/2}.\]

For \( 0 < t \leq 1 \), we have

\[
u(x,t) \leq M(0) + \frac{\alpha f(2M(0))}{2\sqrt{\pi}} \int_0^t \frac{d\tau}{(t-\tau)^{1/2}}.
\]

For \( t > 1 \), and for any \( b \in \partial B \),

\[
v(x,t) \leq v(b,t) \leq M(0) + \alpha f(2M(0)) \left( \int_0^{t-1} \int_{\partial B} g(b,t;\xi,\tau) dS_\xi d\tau + \int_t^{t-1} \int_{\partial B} g(b,t;\xi,\tau) dS_\xi d\tau \right)
\]

\[
\leq M(0) + \alpha f(2M(0)) \left( \int_0^{t-1} I(b,t-\tau) d\tau + \int_t^{t-1} \frac{d\tau}{2\sqrt{\pi} (t-\tau)^{1/2}} \right)
\]

\[
= M(0) + \alpha f(2M(0)) \left( \int_0^t I(b,\theta) d\theta + \frac{1}{\sqrt{\pi}} \right)
\]

\[
\leq M(0) + \alpha f(2M(0)) \left( \int_1^{\infty} I(b,\theta) d\theta + \frac{1}{\sqrt{\pi}} \right). \tag{4.1}
\]

Using (3.1), we have for \( N \geq 3 \),

\[\int_1^{\infty} I(b,\theta) d\theta \leq \left(4\pi\right)^{-N/2} \omega_N R^{N-1} \int_1^{\infty} \theta^{-N/2} d\theta\]

\[= \left(4\pi\right)^{-N/2} \omega_N R^{N-1} \frac{N/2 - 1}{N/2 - 1} < \infty.\]

Thus, we can choose \( \alpha (>0) \) sufficiently small such that the right-hand side of (4.1) is less than or equal to \( 2M(0) \). Hence, the solution \( u \) of the problem (1.1) exists globally.
(ii) Let \( \hat{t} = 0 \) in (3.3). We have

\[
F^{p-1}(t) \geq \left[ \frac{\alpha R^{2} \omega_{N} R^{N-1}}{2 N \pi^{N/2}} + \left( F^{1-p}(0) - \frac{\alpha R^{2} \omega_{N} R^{N-1}}{2 N \pi^{N/2}} \right) e^{2N(p-1)t} \right]^{-1}.
\]

We note that \( F(0) = \int_{\mathbb{R}^N} \psi(x) h(x) \, dx \). We would like to choose \( \alpha \) sufficiently large such that \( F^{1-p}(0) - \frac{\alpha R^{2} \omega_{N} R^{N-1}}{2 N \pi^{N/2}} < 0 \). This can be accomplished by choosing

\[
\alpha > \frac{2 N \pi^{N/2} F^{1-p}(0)}{\kappa e^{-R^{2} \omega_{N} R^{N-1}}}. 
\]

Thus, there exists a finite time \( t_b \) such that \( \lim_{t \to t_b} F(t) = \infty \) and hence \( u(x,t) \) blows up in a finite time.

Let \( k \) denote the positive constant \( \int_{\mathbb{R}^N} \psi(\xi) \, d\xi \). Then,

\[
\int_{\mathbb{R}^N} \exp\left( -\frac{|x - \xi|^2}{4t} \right) \psi(\xi) \, d\xi \leq \int_{\mathbb{R}^N} \psi(\xi) \, d\xi = k.
\]

We have

\[
\int_{\mathbb{R}^N} g(x,t;\xi,0) \psi(\xi) \, d\xi = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} \exp\left( -\frac{|x - \xi|^2}{4t} \right) \psi(\xi) \, d\xi \leq \frac{k}{(4\pi t)^{N/2}},
\]

which tends to 0 as \( t \to \infty \). This shows that the initial data do not affect the solution as \( t \) tends to infinity. The fundamental solution (cf. Evans [5, pp. 22 and 615]) of the Laplace equation for \( N \geq 3 \) is given by

\[
G(x) = \frac{\Gamma \left( \frac{N}{2} + 1 \right)}{N (N-2) \pi^{N/2}} \frac{1}{|x|^{N-2}}.
\]

The proof of the following result is the same as that of Theorem 4.2 of Chan and Tragoonsirisak [4].

**Theorem 4.2.** If \( u(x,t) \leq C \) for some positive constant \( C \), then \( u(x,t) \) converges from below to a solution \( U(x) = \lim_{t \to \infty} u(x,t) \) of the nonlinear integral equation,

\[
U(x) = \alpha \int_{\partial B} G(x - \xi) f(U(\xi)) \, dS_\xi. \tag{4.2}
\]

The next result shows that there exists a critical value for \( \alpha \).

**Theorem 4.3.** For \( N \geq 3 \), there exists a unique \( \alpha^* \) such that \( u \) exists globally for \( \alpha < \alpha^* \), and \( u \) blows up in a finite time for \( \alpha > \alpha^* \).

**Proof.** To show that the larger the \( \alpha \), the larger the solution, let \( \alpha > \beta \), and consider the sequence \( \{ v_n \} \) given by \( v_0(x,t) = \psi(x) \), and for \( n = 0, 1, 2, \ldots \),

\[
v_{n+1}(x,t) = \int_{\mathbb{R}^N} g(x,t;\xi,0) \psi(\xi) \, d\xi + \beta \int_{0}^{t} \int_{\partial B} g(x,t;\xi,\tau) f(v_n(\xi,\tau)) \, dS_\xi \, d\tau.
\]
Similar to the construction of the sequence \( \{ u_n \} \) in \( \Omega \) in the proof of Theorem 2.4, we obtain

\[
v(x,t) = \lim_{n \to \infty} v_n(x,t) = \int_{\mathbb{R}^N} g(x,t;\xi,0) \psi(\xi) \, d\xi + \beta \int_{0}^{t} \int_{\partial B} g(x,t;\xi,\tau) f(v(\xi,\tau)) \, dS_{\xi} d\tau.
\]

Since \( u_n > v_n \) for \( n = 1, 2, 3, \ldots \), we have \( u \geq v \). Hence, the solution \( u \) is a nondecreasing function of \( \alpha \). It follows from Theorem 4.1 that there exists a unique \( \alpha^* \) such that \( u \) exists globally for \( \alpha < \alpha^* \) and \( u \) blows up in a finite time for \( \alpha > \alpha^* \). \( \square \)

We note that the critical value \( \alpha^* \) is determined as the supremum of all positive values \( \alpha \) for which a solution \( U \) of (4.2) exists. The proof of the next result (showing that the solution \( u \) exists globally when \( \alpha = \alpha^* \)) for the case \( f(0) > 0 \) is a modification of that for Theorem 7 of Chan and Jiang [1] for a degenerate one-dimensional problem in a bounded domain.

**Theorem 4.4.** For \( N \geq 3 \),

\[
\alpha^* = \frac{(N-2) \pi^{(N-3)/2}}{R \Gamma \left( \frac{N-1}{2} \right) \prod_{i=1}^{N-3} \int_{0}^{\pi} \sin^i \varphi d\varphi} \sup_{M(0) < s < \infty} \left( \frac{s}{f(s)} \right),
\]

where for \( N = 3 \),

\[
\prod_{i=1}^{N-3} \int_{0}^{\pi} \sin^i \varphi d\varphi = 1.
\]

Furthermore, \( u \) does not blow up in infinite time.

**Proof.** From (2.6), \( U(x) = \lim_{t \to \infty} u(x,t) \) attains its maximum at \( b \in \partial B \). From (4.2),

\[
U(b) = \alpha \int_{\partial B} G(b-\xi) f(U(b)) \, dS_{\xi}.
\]

Thus,

\[
\alpha = \left( \frac{1}{\int_{\partial B} G(b-\xi) \, dS_{\xi}} \right) \left( \frac{U(b)}{f(U(b))} \right),
\]

and hence,

\[
\alpha^* = \left( \frac{1}{\int_{\partial B} G(b-\xi) \, dS_{\xi}} \right) \sup_{M(0) < s < \infty} \left( \frac{s}{f(s)} \right).
\]

From the proof of Theorem 4.5 of Chan and Tragoonsirisak [4],

\[
\int_{\partial B} G(b-\xi) \, dS_{\xi} = \frac{R \Gamma \left( \frac{N-1}{2} \right) \prod_{i=1}^{N-3} \int_{0}^{\pi} \sin^i \varphi d\varphi}{(N-2) \pi^{(N-3)/2}}.
\]

Thus, we have (4.3).

Let us consider the function \( \varphi(s) = s/f(s) \).

**Case 1.** If \( f(0) = 0 \), then we claim that \( \varphi(s) \) is a decreasing function for \( s > 0 \). Since \( f \) is a convex function (cf. Stromberg [11, p. 199]) in \((0, \infty)\), we have for any \( 0 < s < s_2 \),

\[
f((1-t) s + ts_2) \leq (1-t) f(s) + tf(s_2), \quad t \in [0,1].
\]

Letting \( s \to 0 \), we have

\[
f(ts_2) \leq tf(s_2).
\]
Let \( t = s_1/s_2 \), where \( 0 < s_1 < s_2 \). Then,
\[
    f(s_1) \leq \frac{s_1}{s_2} f(s_2),
\]
which gives
\[
    \varphi(s_2) \leq \varphi(s_1),
\]
implying that \( \varphi(s) \) is a nonincreasing function of \( s (>0) \). It follows from (4.3) that
\[
    \alpha^* = \frac{(N - 2) \pi^{(N-3)/2}}{R \Gamma \left( \frac{N - 1}{2} \right) \prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi d\varphi} \left( \frac{M(0)}{f(M(0))} \right). \tag{4.4}
\]

**Case 2.** If \( f(0) > 0 \), then \( \varphi(s) > 0 \) for \( s > 0 \), and \( \varphi(0) = 0 = \lim_{s \to \infty} \varphi(s) \). We have \( \varphi'(s) = (f(s) - sf'(s))/f^2(s) \). Therefore, a relative maximum or minimum occurs at \( \tilde{s} \in (0, \infty) \), where \( f(\tilde{s}) = \tilde{s}f'(\tilde{s}) \). Since \( \varphi''(\tilde{s}) = -\tilde{s}f''(\tilde{s})/f^2(\tilde{s}) < 0 \), \( \varphi(s) \) attains its absolute maximum when \( \varphi(\tilde{s}) = 1/f'(\tilde{s}) \). Thus, \( \sup_{0<s<\infty} (s/f(s)) \) occurs at \( s = \tilde{s} \in (0, \infty) \). We note that the function \( \varphi(s) \) is a strictly increasing function for \( 0 \leq s < \tilde{s} \), and a strictly decreasing function for \( s > \tilde{s} \). Thus, if \( M(0) < \tilde{s} \), then
\[
    \alpha^* = \frac{(N - 2) \pi^{(N-3)/2}}{R \Gamma \left( \frac{N - 1}{2} \right) \prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi d\varphi} \left( \frac{\tilde{s}}{f(\tilde{s})} \right). \tag{4.5}
\]
If \( M(0) \geq \tilde{s} \), then it follows from \( \varphi(s) \) being a strictly decreasing function for \( s > \tilde{s} \) that \( \varphi(s) \) attains its supremum at \( M(0) \). Thus,
\[
    \alpha^* = \frac{(N - 2) \pi^{(N-3)/2}}{R \Gamma \left( \frac{N - 1}{2} \right) \prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi d\varphi} \left( \frac{M(0)}{f(M(0))} \right). \tag{4.6}
\]
From (4.4) to (4.6), \( \alpha^* \) occurs at some finite positive value. Hence for \( \alpha \leq \alpha^* \), \( u \) exists globally. Since \( u \) blows up in a finite time for \( \alpha > \alpha^* \), \( u \) does not blow up in infinite time. \( \square \)

For an illustration, we give below two examples on calculating \( \alpha^* \) for some given functions \( f \) and some given initial data on the surface of the ball \( M(0) \).

**Example 4.5.** Let \( f(u) = u^p \). Since \( f(0) = 0 \), it follows from (4.4) that
\[
    \alpha^* = \frac{(N - 2) \pi^{(N-3)/2}}{M^{p-1}(0) R \Gamma \left( \frac{N - 1}{2} \right) \prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi d\varphi}.
\]
Example 4.6. Let \( f(u) = (u + 1)^p \). Since \( f(0) > 0 \), we have \( \tilde{s} = 1/\( p-1 \) \). From (4.5) and (4.6),

\[
\alpha^* = \begin{cases} 
\frac{(p-1)^{p-1} (N-2) \pi^{(N-3)/2}}{p^p R \Gamma \left( \frac{N-1}{2} \right) \prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi d\varphi} & \text{if } M(0) < \frac{1}{p-1}, \\
\frac{M(0) (N-2) \pi^{(N-3)/2}}{(M(0) + 1)^p R \Gamma \left( \frac{N-1}{2} \right) \prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi d\varphi} & \text{if } M(0) \geq \frac{1}{p-1}.
\end{cases}
\]

References