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SHOCK WAVE FORMATION PROCESS FOR A MULTIDIMENSIONAL SCALAR CONSERVATION LAW

By

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Abstract. We construct a global smooth approximate solution to a multidimensional scalar conservation law describing the shock wave formation process for initial data with small variation. In order to solve the problem, we modify the method of characteristics by introducing "new characteristics", nonintersecting curves along which the (approximate) solution to the problem under study is constant. The procedure is based on the weak asymptotic method, a technique which appeared to be rather powerful for investigating nonlinear waves interactions.

1. Introduction. In the current paper, we construct an approximate (in a weak sense) solution $u_\varepsilon(t,x)$, $x \in \mathbf{R}^d$, $t \in \mathbf{R}^+$, where ε is a regularization parameter, corresponding to a multidimensional shock wave formation in the case of a Cauchy problem for a scalar conservation law.

In order to make the problem precise, assume that $\mathbf{R}^d$ is divided into three disjoint domains Ω_L, Ω_0 and Ω_R, i.e., $\mathbf{R}^d = \Omega_L \dot\cup \overline{\Omega}_0 \dot\cup \Omega_R$, where $\dot\cup$ denotes disjoint union. Let $\Gamma_L = \partial\Omega_L = \overline{\Omega}_L \cap \overline{\Omega}_0$ and $\Gamma_R = \partial\Omega_R = \overline{\Omega}_R \cap \overline{\Omega}_0$ (see Figure 1). Assume that Γ_L and Γ_R are $(d-1)$-dimensional manifolds admitting the following representation:

$$\Gamma_L = \{x = \chi_L(s) : \ s \in \mathbf{R}^{d-1}, \ \chi_L \in \mathrm{Lip}(\mathbf{R}^{d-1})\},$$
$$\Gamma_R = \{x = \chi_R(s) : \ s \in \mathbf{R}^{d-1}, \ \chi_R \in \mathrm{Lip}(\mathbf{R}^{d-1})\}.$$

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Now, take a Lipschitz piecewise smooth function $u_0 : \mathbf{R}^d \to \mathbf{R}$ such that

$$u_0(x) = \begin{cases} U_L, & x \in \Omega_L, \\ u_1(x), & x \in \bar{\Omega}_0 \\ U_R, & x \in \Omega_R, \end{cases} \in \mathrm{Lip}(\mathbf{R}^d), \tag{1.1}$$

where $u_1 \in C^2(\Omega_0)$, and U_L and U_R are constants such that $|U_L - U_R|$ is small enough. We are going to consider the following problem:

$$\frac{\partial u}{\partial t} + \langle \nabla, f(u) \rangle = 0, \tag{1.2}$$

$$u|_{t=0} = u_0(x) \in \mathrm{Lip}(\mathbf{R}^d), \tag{1.3}$$

where for $f \in C^3(\mathbf{R}; \mathbf{R}^d)$, $\langle \nabla, f(u) \rangle = \mathrm{div}_x\, f(u) = \sum_{i=1}^d \frac{\partial f(u)}{\partial x_i}$, $u = u(x,t)$. It is well known that problem (1.2), (1.3) in general does not admit a globally defined classical solution. This means that after some time the shock wave will occur, and we need to pass to the weak solution concept.

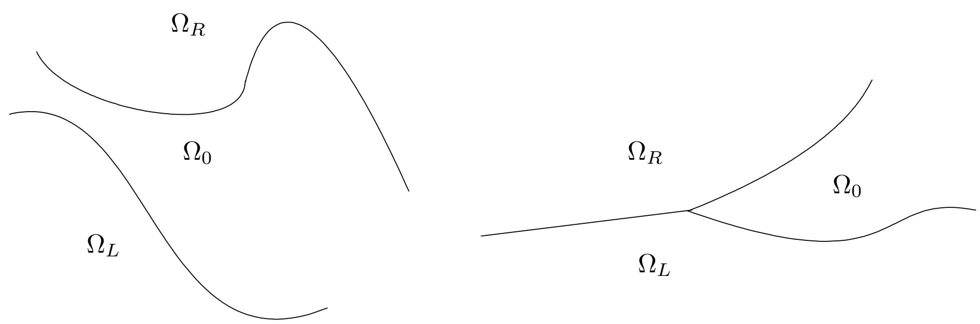

FIG. 1. The left-hand plot shows an admissible disposition of Ω_0, Ω_L and Ω_R. The right-hand profile is not admissible since the shock wave has already been formed.

Although it is well known that for any initial data $u_0 \in L^1(\mathbf{R}^d)$ there exists a unique entropy admissible weak solution to (1.2), (1.3) (see [20]), it is very important from geometrical [19, 26], analytical [17, 18], and practical (petroleum engineering) [1, 16, 28] points of view to explicitly construct the global (approximate) solution describing the formation and propagation of the shock wave. The mentioned construction is the main contribution of the paper.

To problem (1.2), (1.3), there corresponds the following system of characteristics:

$$\begin{aligned} \dot{x}_i &= f_i'(u), & x_i|_{t=0} &= x_0, \\ \dot{u} &= 0, & u|_{t=0} &= u_0(x). \end{aligned} \tag{1.4}$$

As is well known, problem (1.2), (1.3) will have the classical solution along the characteristics as long as the Jacobian $J = \det \frac{\partial x}{\partial x_0}$ is greater than zero. From, e.g., [26], we know that

$$J = \det \frac{\partial x}{\partial x_0} = 1 + t \sum_{i=1}^{d} f''(u_0) \frac{\partial u_0}{\partial x_{0i}}. \tag{1.5}$$

Therefore, if we want a shock wave to appear, we must assume that $J < 0$ for some $t > 0$ and $x_0 \in \mathbf{R}^d$. Actually, we shall assume a little bit more than the latter, that $u_1(x_0) = u_0(x_0)$, $x_0 \in \Omega_0$, satisfies the following problem:

$$\begin{gathered}\sum_{i=1}^{d} f_i''(u_1)\frac{\partial u_1}{\partial x_i} = -K,\\ u_1|_{\Gamma_L} = U_L, \quad u_1|_{\Gamma_R} = U_R,\end{gathered} \tag{1.6}$$

where $K = K(s)$, $s \in \mathbf{R}^{d-1}$, is a positive Lipschitz continuous function which is constant along the characteristics of problem (1.6). Furthermore, we shall assume that Ω_0 admits a complete fibration along the characteristics of problem (1.6) issuing from Γ_L (see Figure 2 and Remark 1.1).

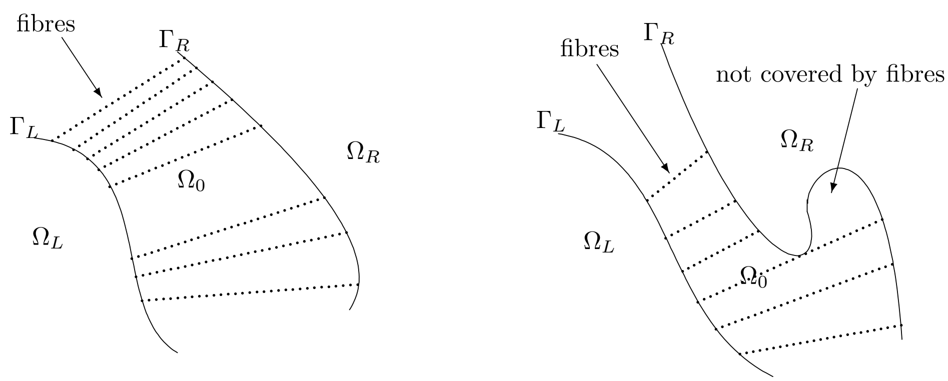

FIG. 2. On the left plot we have an admissible disposition of Ω_L, Ω_R and Ω_0. The situation on the right-hand side is not admissible since we have a part of Ω_0 that is not covered by the fibres. Observe that the fibres are characteristics corresponding to (1.6), and they are not necessarily straight lines.

This is a technical assumption which provides all the characteristics issuing from the same fibre of Ω_0 to intersect at the same point $(t^*(s), x(s)) \in \mathbf{R}^+ \times \mathbf{R}^d$, where $s \in \mathbf{R}^{d-1}$ is the parametrization argument of Γ_L (see Figure 3). It appears that this significantly simplifies the description of the shock wave formation process (see a further explanation below).

Notice that the problem of construction of a global smooth approximating solution to (1.2), (1.3) describing the formation of shock waves has already been studied by A. M. Il'in [22, 18, 23] and by the authors [7, 8, 9] in the case of one-dimensional scalar conservation laws with a convex flux $f \in C^2(\mathbf{R})$. In [22], the author considered a situation with initial data such that a classical solution blows up in a single point. His construction is based on a viscosity regularization of the considered conservation law. Using such a regularization, the author obtains a global approximating solution via a set of functional series which are defined in appropriate domains in $\mathbf{R}^+ \times \mathbf{R}$. Then, he shows that every two such series match in domains where they are both defined. Such a method is known as the matching method. The most difficult part in the matching method was the construction of the approximate solution in a neighborhood of the gradient catastrophe point, i.e. the point where the classical solution blows up (see Figure 4, a)). In [7], we noticed that it is much easier to construct a global smooth approximating solution if we have a line of the "gradient catastrophe" (see Figure 4, b)). In [7], we assumed that the function u_1 from (1.1) was defined by the following implicit equation:

$$f'(u_1(x_0)) = -Kx_0 + b, \quad a \le x_0 \le b, \tag{1.7}$$

where $K > 0$ and $b > 0$ were assumed to be constants. Calculating the derivative of both sides of (1.7) we get $f''(u_1(x_0))\frac{\partial u_1}{\partial x_0} = -K$. So, we see that (1.6) is a direct generalization of the one-dimensional situation. Actually, condition (1.7) provides all the characteristics issuing from the interval (a, b) to intersect at the same point. Thus, we have the line of the gradient catastrophe (Figure 4, b)), and not only the point of the gradient catastrophe (Figure 4, a)).

Also, notice that problem (1.2), (2.5) was only auxiliary. In [8, 24], we considered a one-dimensional system of generalized pressureless gas dynamics. We used the explicit form of the approximate solution to (1.2), (1.3) in order to describe the formation of the delta shock wave which appears as a natural part of a solution to the considered system. In [9], we used (1.2), (1.3) (with $|U_L - U_R|$ arbitrarily small) to describe the evolution of arbitrary initial data $u_0 \in C^1(\mathbf{R})$ corresponding to a one-dimensional scalar conservation law with a convex flux $f \in C^2(\mathbf{R})$.

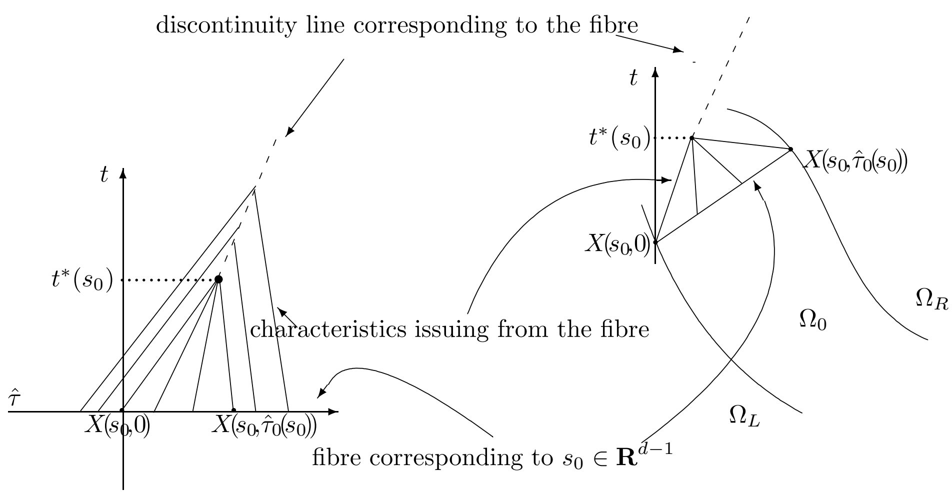

FIG. 3. Behavior of the standard characteristics issuing from the fibre corresponding to $s_0 \in \mathbf{R}^{d-1}$ plotted in $(t, \hat{\tau}) \in \mathbf{R}^+ \times \mathbf{R}$-plane (left plot) and in $(t, x) \in \mathbf{R}^+ \times \mathbf{R}^d$-space (right plot).

In order to find the desired approximate solution of (1.2), (1.3) in the one-dimensional case, we modified the method of characteristics. More precisely, we have defined the non-intersecting curves, so-called "new characteristics", along which the approximate solution is constant (see Figure 6). As the regularization parameter tends to zero, the "new characteristics" tend to the Dafermos generalized characteristics [2]. Therefore, the approximate solution tends to an admissible weak solution of the considered Cauchy problem (see [7, 8]). Here, we shall present a nontrivial generalization of the latter procedure to the multidimensional case.

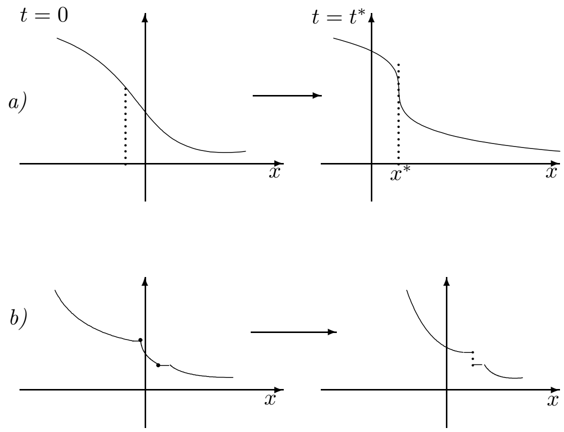

FIG. 4. Plot *a)* shows blowing up in a single point (the point (t^*, x^*)). Plot *b)* is a case when we have a line of the gradient catastrophe (the part between the dots straightens in the shock wave at the moment $t = t^*$).

Recall that in the case of initial data in the general position, at the moment of the gradient catastrophe, one point with vertical tangent arises on the graph of the solution. Moreover, small variations of arbitrary (smooth) initial data render the initial data in the general position. As one can see, we do not use the concept of construction of general position, which is standard in geometry. Actually, we replace the general position by a special one which allows us to construct an approximate solution.

REMARK 1.1. Notice that problem (1.6) is equivalent to the following system of characteristics:

$$\begin{aligned} \frac{dX_i}{d\hat{\tau}} &= f_i''(u_1), & X_i|_{\hat{\tau}=0} &= \chi_L^i(s), \\ \frac{du_1}{d\hat{\tau}} &= -K(s), & u_1|_{\hat{\tau}=0} &= U_L, \end{aligned} \tag{1.8}$$

and from here we have

$$\begin{aligned} &u_1 = U_L - K(s)\hat{\tau}, \\ &X = \int_0^{\hat{\tau}} f''(U_L - K(s)\tau')d\tau' + \chi_L(s) = -\frac{1}{K(s)}(f'(u_1) - f'(U_L)) + \chi_L(s). \end{aligned} \tag{1.9}$$

Furthermore, notice that from the boundary conditions given in (1.6), it follows that for every $s \in \mathbf{R}^{d-1}$ there exists $\hat{\tau} > 0$ such that $U_R = U_L - K(s)\hat{\tau}$. Thus, we see from (1.6) that it must be

$$U_L > U_R. \tag{1.10}$$

Notice that our assumption on the complete fibration of Ω_0 can be written as

$$\overline{\Omega}_0 = \dot{\bigcup}_{s\in\Gamma_L} \{X\,(s,\hat\tau):\ \hat\tau \in [0,\hat\tau_0(s)]\}, \tag{1.11}$$

where X is given by (1.8) and $\dot\cup$ denotes the disjoint union. By $\hat\tau_0(s)$ we denote a real number such that $X(\hat\tau_0(s),s)\in\Gamma_R$. It is clear that $X(0,s)\in\Gamma_L$. Also, since Ω_0 admits the complete fibration along the characteristics X, the change of coordinates $(\hat\tau,s)\mapsto x_0$, $(\hat\tau,s)\in\dot{\bigcup}_{s\in\mathbf{R}^{d-1}}[0,\hat\tau_0(s)]\times\{s\}$ is regular and the corresponding Jacobian $\det(\frac{\partial x_0}{\partial\hat\tau},\frac{\partial x_0}{\partial s})$ must be different from zero. Therefore, without losing generality, we can assume that

$$\det(\frac{\partial x_0}{\partial\hat\tau},\frac{\partial x_0}{\partial s}) \geq const > 0,\quad (\hat\tau,s)\in\dot{\bigcup}_{s\in\mathbf{R}^{d-1}}[0,\hat\tau_0(s)]\times\{s\}. \tag{1.12}$$

Example 1.2. We will give an example of problem (1.2), (1.3) satisfying the assumptions given above. A similar example is [26, Example 7.]. Consider the equation

$$\frac{\partial u}{\partial t} + p\frac{\partial u^2}{\partial x_1} + q\frac{\partial u^2}{\partial x_2} = 0, \tag{1.13}$$

where $p,q>0$. To assign initial data $u_0\in \mathrm{Lip}(\mathbf{R}^2)$, fix arbitrary constants $U_L>U_R>0$, and a positive function $K\in\mathrm{Lip}(\mathbf{R})$. Then, put

$$\Gamma_L = \{(x_1,x_2)\in\mathbf{R}^2:\ x_2=0\},$$
$$\Gamma_R = \{(x_1,x_2)\in\mathbf{R}^2:\ U_L - K(x_1-\frac{p}{q}x_2)\frac{x_2}{2q} = U_R,\ x_2\geq 0\}.$$

Finally, we take

$$u_0(x) = \begin{cases} U_L, & (x_1,x_2)\in\Omega_L,\\ U_L - K(x_1-\frac{p}{q}x_2)\frac{x_2}{2q}, & (x_1,x_2)\in\Omega_0,\\ U_R, & (x_1,x_2)\in\Omega_R,\end{cases} \tag{1.14}$$

where Ω_L, Ω_R and Ω_0 are plotted in Figure 5. Let us prove that the function $u_1(x_1,x_2) = U_L - K(x_1-\frac{p}{q}x_2)\frac{x_2}{2q}$ satisfies (1.6). In our case, (1.6) reduces to

$$2p\frac{\partial u_1}{\partial x_1} + 2q\frac{\partial u_1}{\partial x_2} = -K(x_1),$$
$$u_1|_{\Gamma_L} = U_L,\quad u_1|_{\Gamma_R} = U_R.$$

The corresponding system of characteristics is

$$\begin{cases} \dot x_1 = 2p,\quad \dot x_2 = 2q,\quad \dot u_1 = -K(x_{10}),\\ x_1(0) = x_{10},\quad x_2(0)=0,\quad u_1(0)=U_L,\end{cases}$$

whose solution is

$$x_1 = 2p\hat\tau + x_{10},\quad x_2 = 2q\hat\tau,$$
$$u_1 = U_L - K(x_{10})\hat\tau,\quad \hat\tau>0,$$

and from this system, we finally conclude $u_1(x_1,x_2) = U_L - K(x_1-\frac{p}{q}x_2)\frac{x_2}{2q}$, which is exactly the form of the function u_0 on Ω_0.

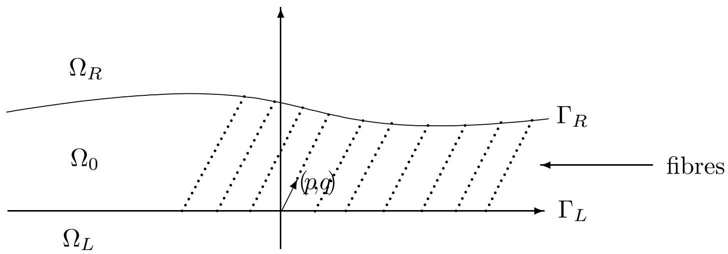

FIG. 5. If (1.13) models a traffic flow, then the vector (p, q) determines the direction of the traffic while the unknown function u represents the density of the vehicles.

For instance, if we take Γ_L as above and $\Gamma_R = \{(x, 2): \ x \in (-\infty, -1]\} \cup \{(x, x^2+1): x \in (-1, 0]\} \cup \{(x, 1): \ x \in (0, \infty)\}$, $p = 1$, $q = 1/2$, then we must have

$$U_R = U_L - K\left(x - \frac{1}{2}(x^2+1)\right)\frac{x^2+1}{4} \implies K(y) = \begin{cases} \frac{4(U_L-U_R)}{(1+\sqrt{2})^2+1}, & y \leq -2, \\ \frac{4(U_L-U_R)}{(1+\sqrt{-y})^2+1}, & -2 < y < -\frac{1}{2}, \\ \frac{4(U_L-U_R)}{(1+\sqrt{1/2})^2+1}, & y \geq -\frac{1}{2}. \end{cases}$$

The paper is organized as follows.

In Section 2, we solve an auxiliary Cauchy problem whose weak asymptotic solution actually represents the weak asymptotic solution to (1.2), (1.3). The latter fact is proved in Section 3.

2. Approximate equation. In this section, we shall introduce and solve a family of problems whose solutions will represent the wanted approximating solution to (1.2), (1.3). In the beginning, we introduce definitions and the fundamental theorem of the method that we are going to use: the weak asymptotic method. The family of approximating solutions constructed in such a way will be called the weak asymptotic solution. The method is intensively used in recent years for investigations of nonlinear wave phenomena. For instance, using this method, we are able to find explicit formulas describing the interaction of solitons in the case of generalized KdV equations [5, 10], the interaction of Sine-Gordon solitons [14, 21], the evolution of nonlinear waves in the case of scalar conservation laws [3, 7], the interaction [6, 11] and formation [8, 25] of δ-shock waves in the case of a triangular system of conservation laws, δ'-shock waves as a new type of singular solution of hyperbolic systems of conservation laws [27], the confluence of free boundaries in the Stefan problem with underheating [4], and different interactions of the shock waves appearing on the gas dynamics [12, 13, 15], etc.

In the sequel, we imply that $t \in \mathbf{R}^+$ and $x \in \mathbf{R}^d$. Also, for a differentiable function $f : \mathbf{R}^d \to \mathbf{R}$, we write

$$\nabla f = \nabla_x f = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_d}).$$

If $s = (s_1, \ldots, s_m) : \mathbf{R}^d \to \mathbf{R}^m$ is a differentiable function depending on $x \in \mathbf{R}^d$, then we denote

$$\frac{\partial s}{\partial x} = \begin{pmatrix} \frac{\partial s_1}{\partial x_1} \cdots & \frac{\partial s_1}{\partial x_d} \\ \vdots & \vdots \\ \frac{\partial s_m}{\partial x_1} \cdots & \frac{\partial s_m}{\partial x_d} \end{pmatrix} \in M^{m\times d},$$

where $M^{m\times d}$ is the space of the matrix with m rows and d columns.

By $\det(a,b)$ we denote the determinant of the matrix $(a,b) \in M^{d\times d}$, where $a \in M^{d\times k}$ and $b \in M^{d\times(d-k)}$.

For two vectors $x, y \in \mathbf{R}^d$, we write

$$\langle x, y\rangle = \sum_{i=1}^{d} x_i y_i \quad \text{and} \quad xy = [x_i y_j]_{i,j=1,\ldots,d} \in M^{d\times d}.$$

Definition 2.1. By $O_{\mathcal{D}'}(\varepsilon^\alpha) \subset \mathcal{D}'(\mathbf{R}^d)$, $\alpha \in \mathbf{R}$, we denote the family of distributions depending on $\varepsilon \in (0,1)$ and $t \in \mathbf{R}^+$ such that for any test function $\eta(x) \in C_0^1(\mathbf{R}^d)$, the estimate

$$\langle O_{\mathcal{D}'}(\varepsilon^\alpha), \eta(x)\rangle = O(\varepsilon^\alpha), \quad \varepsilon \to 0,$$

holds, where the term on the right-hand side is understood in the usual Landau sense and locally uniformly in t, i.e., $|O(\varepsilon^\alpha)| \le C_T \varepsilon^\alpha$ for $t \in [0,T]$, where C_T is a constant depending only on T.

By $o_{\mathcal{D}'}(1) \subset \mathcal{D}'(\mathbf{R}^d)$ we denote a family of distributions depending on $\varepsilon \in (0,1)$ and $t \in \mathbf{R}^+$ such that for any test function $\eta(x) \in C_0^1(\mathbf{R}^d)$, the estimate

$$\langle o_{\mathcal{D}'}(1), \eta(x)\rangle = o(1), \quad \varepsilon \to 0,$$

holds, where the estimate on the right-hand side is understood in the usual Landau sense and locally uniformly in t, i.e., $|o(1)| \le C_T g(\varepsilon)$ for $t \in [0,T]$, where g is a function tending to zero as $\varepsilon \to 0$, and C_T is a constant depending only on T.

Definition 2.2. The family of functions $(u_\varepsilon) = (u_\varepsilon(t,x)) \subset C^1(\mathbf{R}^+; \mathcal{D}'(\mathbf{R}^d))$ is called a weak asymptotic solution of problem (1.2), (1.3) if

$$\frac{\partial u_\varepsilon}{\partial t} + \langle \nabla, f(u_\varepsilon)\rangle = o_{\mathcal{D}'}(1),$$

$$u_\varepsilon\Big|_{t=0} - u_0 = o_{\mathcal{D}'}(1), \quad \varepsilon \to 0.$$

Observe that a sequence of solutions obtained by the standard vanishing viscosity approximation of (1.2) is a special case of the weak asymptotic solution. Indeed, in the framework of the vanishing viscosity approach, we consider the following sequence of equations:

$$\frac{\partial u_\varepsilon}{\partial t} + \langle \nabla, f(u_\varepsilon)\rangle = \varepsilon \partial_{xx} u_\varepsilon,$$

and, clearly, $\varepsilon \partial_{xx} u_\varepsilon = o_{\mathcal{D}'}(1)$. Using the vanishing viscosity approximation, we obtain a sequence of functions, say (u_ε), which converges toward an entropy admissible solution to (1.2), (1.3) (the initial data can be an arbitrary bounded function), but we cannot write down explicit formulas for (u_ε). Using the weak asymptotic method, we have more freedom in choosing the regularization terms, and we are able to obtain explicit formulas for the appropriate sequence of solutions to the regularized problem, but, first, we cannot be sure that the sequence will converge toward an entropy admissible solution of (1.2), (1.3), and second, we need additional assumptions on the flux and initial data. In order to overcome the first obstacle, we start from Lipschitz continuous initial data and construct a solution along characteristics. In the case considered in this contribution, we have the possibility of choosing an appropriate regularization which enables us to modify characteristics so that they become nonintersecting but still to approximate the process described by the conservation law accurately enough. Thus, we will necessarily obtain an entropy admissible solution in the limit (see Theorem 3.3).

The following theorem defines so called "switch" functions $B_i : \mathbf{R} \to (0,1)$, $i = 1, 2$, which are very important objects in the method.

THEOREM 2.3 ([7, 3]). Suppose that the functions $\omega_i \in C^\infty(\mathbf{R})$, $i = 1, 2$, satisfy $\lim\limits_{z\to+\infty} \omega_i(z) = 1$, $\lim\limits_{z\to-\infty} \omega_i(z) = 0$ and $\frac{d\omega_i(z)}{dz} \in \mathcal{S}(\mathbf{R})$, where $\mathcal{S}(\mathbf{R})$ is the Schwartz space of rapidly decreasing functions.

The functions $B_i \in C^\infty(\mathbf{R})$, $i = 1, 2$, defined for any $\rho \in \mathbf{R}$ by

$$B_1(\rho) = \int \dot{\omega}_1(z)\omega_2(z+\rho)dz \text{ and } B_2(\rho) = \int \dot{\omega}_2(z)\omega_1(z-\rho)dz, \tag{2.1}$$

satisfy

$$B_1(\rho) + B_2(\rho) = 1,$$
$$\lim_{\tau\to-\infty} \tau^N B_1(\tau) = 0, \quad \lim_{\tau\to\infty} \tau^N B_2(\tau) = 0 \Rightarrow \lim_{\tau\to\pm\infty} \tau^N B_1(\tau)B_2(\tau) = 0.$$

The functions ω_i, $i = 1, 2$, can be chosen so that

$$B_2(z) = 1 - B_1(z) \equiv 0, \quad z < 0. \tag{2.2}$$

In the sequel, by $t^*(s)$, $s \in \mathbf{R}^{d-1}$, we denote the moment of intersection of characteristics (1.4) emanating from the fibre connecting $x = X(0, s)$ and $x = X(\hat{\tau}_0(s), s)$, $s \in \mathbf{R}^{d-1}$, respectively, where X is given by (1.9). From (1.5) and (1.6), it follows that

$$t^*(s) = \frac{1}{K(s)}. \tag{2.3}$$

We shall also write

$$\tau = \frac{t - t^*(s)}{\varepsilon}, \quad B_i = B_i(\frac{t - t^*(s)}{\varepsilon}), \quad \dot{B}_i = (B_i(\tau))'_\tau, \quad i = 1, 2.$$

Now, we can motivate the approximate problem that we shall solve. First, recall that our aim is to find a globally defined smooth approximate solution to (1.2), (1.3). To do this we have to avoid the intersection of characteristics. A natural idea is to smear the discontinuity line corresponding to every fibre, i.e. to take an ε-neighborhood of the discontinuity line and to dispose of characteristics in that neighborhood in a way that they do not intersect, but, as $\varepsilon \to 0$, all of them lump together into the discontinuity line. Along such lines, the approximate solution to our problem will remain constant. Such lines we call the "new characteristics" (see Figure 6).

In order to accomplish this idea, we shall use the switch functions B_i, $i = 1, 2$, from Theorem 2.3. Namely, notice that $B_1(\frac{t-t^*(s)}{\varepsilon}) = 1 - B_2(\frac{t-t^*(s)}{\varepsilon})$ is close to zero for

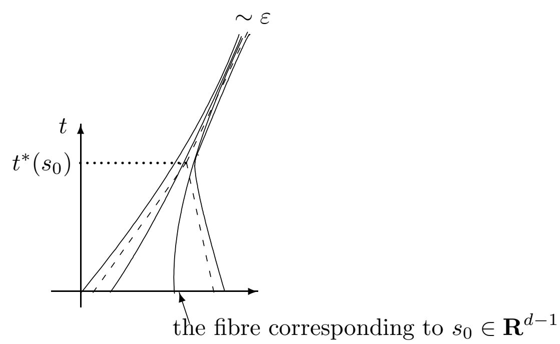

FIG. 6. The new characteristics issuing from the fibre corresponding to $s_0 \in \mathbf{R}^{d-1}$. The distance between the rear most new characteristics is of order ε for $t > t^*(s)$. The new characteristics are non-intersecting.

$t < t^*(s)$, and close to one for $t > t^*(s)$. Next, notice that for $t < t^*(s)$ we will have the classical solution to (1.2), (1.3), while for $t > t^*(s)$, a shock wave is formed. Therefore, it is natural to consider the following family of problems (compare with [8, Theorem 10]):

$$\partial_t u_\varepsilon + B_2(\frac{t-t^*(s)}{\varepsilon})\langle \nabla, f(u_\varepsilon)\rangle + B_1(\frac{t-t^*(s)}{\varepsilon})\sum_{i=1}^{d} c_i \frac{\partial u_\varepsilon}{\partial x_i} = 0, \tag{2.4}$$

$$u_\varepsilon|_{t=0} = u_0(x), \tag{2.5}$$

where $s = s(x) \in R^{d-1}$, $x \in \mathbf{R}^d$, are the last $d-1$ variables of the inverse of the function $(\hat{\tau}, s) \mapsto X(\hat{\tau}, s)$ given by (1.9), and $c = (c_1, \dots, c_d) \in \mathbf{R}^d$ is a constant to be determined in the next section. For simplicity, we shall also assume that B_i, $i = 1, 2$, satisfy (2.2).

Notice that for $t < t^*(s)$, we have $B_2 \sim 1$ and $B_1 \sim 0$, which means that (1.2) and (2.4) coincide in the direction of an appropriate fibre when $t < t^*(s)$. On the other hand, for $t > t^*(s)$, we have $B_2 \sim 0$ and $B_1 \sim 1$, which means that the nonlinear part of (2.4) disappears and the equation is governed by the linear part $\sum_{i=1}^{d} c_i \frac{\partial u_\varepsilon}{\partial x_i}$ in the direction of an appropriate fibre. It is well known that the linear part will not affect the profile of the corresponding initial data. In this case, the profile is a shock wave with the states U_L and U_R, i.e. the function u at the point of gradient catastrophe $(t^*(s), f'(U_L)t^*(s) + \chi_L(s)) \in \mathbf{R}^+ \times \mathbf{R}^d$. Since the variation $|U_L - U_R|$ is small, the form of a solution u_ε to (2.4), (2.5) will also not have an influence on the formed shock wave.

We will find the weak asymptotic solution to problem (2.4), (2.5) by using the usual method of characteristics. Since the characteristics issuing from Ω_L and Ω_R bear the same information, we can allow their intersection, Thus, we are interested only in the characteristics issuing from Ω_0.

The system of characteristics corresponding to (2.4), (2.5) has the following form:

$$\begin{aligned} \frac{dX_i}{dt} &= B_2 f_i'(u_\varepsilon) + B_1 c_i, \qquad && X_i(0) = x_{i0}, \\ \frac{du_\varepsilon}{dt} &= 0, && u_\varepsilon(0) = u_1(x_0), \quad x_0 \in \Omega_0, \end{aligned} \tag{2.6}$$

where $B_i = B_i(\frac{t-t^*(s)}{\varepsilon})$, $i = 1, 2$. It is easy to find the solution to the latter system:

$$X(t, x_0) = x_0 + f'(u_1)\int_0^t B_2 dt' + c\int_0^t B_1 dt', \tag{2.7}$$

where $B_i = B_i(\frac{t'-t^*(s)}{\varepsilon})$, $i = 1, 2$. In order to show that the characteristics from (2.7) determine the solution to (2.4), (2.5) we need to prove that the Jacobian $J = \det \frac{\partial X}{\partial x_0} > 0$ along the entire temporal axis. Indeed, if this is the case, then there exists the inverse function $x_0 = x_0(X, t, \varepsilon)$ of the function X. Then, the classical solution to (2.4), (2.5) is given by $u_1(x_0(x, t, \varepsilon))$ (again, we do not consider domains where the solution is constant).

According to assumptions (1.12), the transformation $x_0 = x_0(\hat{\tau}, s)$ is regular. Therefore, in order to prove global solvability of (2.4), (2.5), it is enough to prove that $\det\left(\frac{\partial X}{\partial \hat{\tau}}, \frac{\partial X}{\partial s}\right) > 0$ along the entire temporal axis. We shall prove the latter fact under conditions which are necessarily fulfilled if $|U_L - U_R|$ is small enough (see Remark 2.5).

Lemma 2.4. Assume that U_L and U_R are such that for any $(\hat{\tau}, s) \in \dot{\bigcup}_{s\in\mathbf{R}^{d-1}}[0, \hat{\tau}_0(s)]\times\{s\}$ and any $t \in \mathbf{R}^+$,

$$\det(\frac{\partial x_0}{\partial \hat{\tau}}, \frac{\partial x_0}{\partial s} + (f'(u_1) - c)\nabla_s t^*(s) B_1(\frac{t-t^*(s)}{\varepsilon})) \geq \alpha \det(\frac{\partial x_0}{\partial \hat{\tau}}, \frac{\partial x_0}{\partial s}), \tag{2.8}$$

for an $\alpha > 0$.

The Jacobian $\det(\frac{\partial X}{\partial \hat{\tau}}, \frac{\partial X}{\partial s})$ of the characteristics (2.7) computed with respect to the variables $(\hat{\tau}, s)$ satisfies:

$$\det\left(\frac{\partial X}{\partial \hat{\tau}}, \frac{\partial X}{\partial s}\right) > 0, \quad (\hat{\tau}, s) \in \dot{\bigcup}_{s\in\mathbf{R}^{d-1}}[0, \hat{\tau}_0(s)] \times \{s\}. \tag{2.9}$$

Proof. Consider the first column of the matrix $\left(\frac{\partial X}{\partial \hat{\tau}}, \frac{\partial X}{\partial s}\right)$, where X is given by (2.7),

$$\frac{\partial X}{\partial \hat{\tau}} = \frac{\partial x_0}{\partial \hat{\tau}} + f''(u_1)\frac{\partial u_1}{\partial \hat{\tau}}\int_0^t B_2 dt', \tag{2.10}$$

where we abbreviate $B_2 = B_2(\frac{t'-t^*(s)}{\varepsilon})$. According to (1.8), we have for $x_0 \in \Omega_0$:

$$\frac{\partial u_1(x_0)}{\partial \hat{\tau}} = -K(s), \quad \frac{\partial x_0}{\partial \hat{\tau}} = f''(u_1),$$

and thus, from (2.10):

$$\frac{\partial X}{\partial \hat{\tau}} = (1 - K(s)\int_0^t B_2 dt')\frac{\partial x_0}{\partial \hat{\tau}}. \tag{2.11}$$

Notice that since we assumed $B_2(z) = 0$, $z \leq 0$ (see (2.2)), it follows that

$$1 - K(s)\int_0^t B_2 dt' \geq 1 - K(s)\int_0^{t*} B_2 dt' > 0, \tag{2.12}$$

since $t^*(s) = 1/K(s)$ and $B_2(z) < 1$ on a nonzero subset of $(-\infty, 0)$.

Now, we pass to the columns $\frac{\partial X}{\partial s}$. From (2.7), we have

$$\frac{\partial X}{\partial s}=\frac{\partial x_0}{\partial s}+f''(u_1)\frac{\partial u_1}{\partial s}\int_0^t B_2dt' \\ +f'(u_1)\nabla_s t^*(s)\int_0^t\frac{1}{\varepsilon}\dot{B}_2dt'+c\nabla_s t^*(s)\int_0^t\frac{1}{\varepsilon}\dot{B}_1dt'.$$

From here, having in mind the change of variables $\frac{t'-t^*(s)}{\varepsilon}=z \implies dt'=\varepsilon dz$ and using $\dot{B}_2=-\dot{B}_1$, we obtain

$$\frac{\partial X}{\partial s}=\frac{\partial x_0}{\partial s}+f''(u_1)\frac{\partial u_1}{\partial s}\int_0^t B_2dt-f'(u_1)\nabla_s t^*(s)B_1+c\nabla_s t^*(s)B_1. \tag{2.13}$$

Now, we are able to estimate the determinant $J=\det\left(\frac{\partial x}{\partial\hat{\tau}},\frac{\partial x}{\partial s}\right)$. We have from (2.11), (2.12) and (2.13):

$$\begin{aligned}J&=\det\left(\frac{\partial X}{\partial\hat{\tau}},\frac{\partial X}{\partial s}\right) \qquad (2.14)\\ &=(1-\int_0^t B_2dt')\det\left(f''(u_1),\frac{\partial x_0}{\partial s}+f''(u_1)\frac{\partial u_1}{\partial s}\int_0^t B_2dt'-(f'(u_1)-c)\nabla_s t^*(s)B_1\right)\\ &=(1-\int_0^t B_2dt')\det\left(f''(u_1),\frac{\partial x_0}{\partial s}-(f'(u_1)-c)\nabla_s t^*(s)B_1\right)\\ &\overset{(2.8)}{\geq}\alpha(1-\int_0^t B_2dt')\det(\frac{\partial x_0}{\partial\hat{\tau}},\frac{\partial x_0}{\partial s})>0\end{aligned}$$

according to (1.12) and (2.12).

This concludes the proof. □

REMARK 2.5. Notice that, by relying on the continuity of the function det, the boundedness of the functions $\nabla_s t^*$ and B_1, the fact that $U_R<u_1<U_L$, and since we assumed $\det\left(\frac{\partial x_0}{\partial\hat{\tau}},\frac{\partial x_0}{\partial s}\right)\geq c>0$, if $|U_L-U_R|$ are sufficiently small, then (2.8) is satisfied. Indeed, notice that

$$\begin{aligned}f'(u_1)-c&=f'(u_1)-\frac{f(U_L)-f(U_R)}{U_L-U_R}=f'(u_1)-(f_1'(\tilde{u}_1),f_2'(\tilde{u}_2),\dots,f_d'(\tilde{u}_d))\\ &=((u_1-\tilde{u}_1)f_1''(\hat{u}_1),(u_1-\tilde{u}_2)f_2''(\hat{u}_2,\dots,(u_1-\tilde{u}_d)f_d''(\hat{u}_d)),\end{aligned}$$

for values $\tilde{u}_i\in[U_R,U_L]$ and $\hat{u}_i\in[\min\{u_i,\tilde{u}_i\},\max\{u_i,\tilde{u}_i\}]\subset[U_R,U_L]$, $i=1,\dots,d$, according to the Lagrange mean value theorem. From here, it further follows that

$$\|f'(u_1)-c\|\leq C|U_L-U_R|, \tag{2.15}$$

where $\|\cdot\|$ denotes the Euclidean norm and C a constant such that $C\geq\max_{U_R\leq u\leq U_R}|f''(u)|$. Now, using continuity of the function det, we conclude that

$$\det\left(\frac{\partial x_0}{\partial\hat{\tau}},\frac{\partial x_0}{\partial s}+(f'(u_1)-c)\nabla_s t^*(s)B_1(\frac{t-t^*(s)}{\varepsilon})\right)\\ =\det(\frac{\partial x_0}{\partial\hat{\tau}},\frac{\partial x_0}{\partial s})+\mathcal{O}(\|f'(u_1)-c\|),$$

since B_1 and $\nabla_s t^*$ are bounded functions. From here and (2.15), we conclude that

$$\begin{aligned}&\det\left(\frac{\partial x_0}{\partial\hat\tau}, \frac{\partial x_0}{\partial s} + (f'(u_1)-c)\nabla_s t^*(s)B_1(\frac{t-t^*(s)}{\varepsilon})\right)\\ &= \det\left(\frac{\partial x_0}{\partial\hat\tau}, \frac{\partial x_0}{\partial s}\right) + |U_L-U_R|\mathcal{O}(1).\end{aligned} \tag{2.16}$$

Since, according to (1.12), $\det(\frac{\partial x_0}{\partial\hat\tau}, \frac{\partial x_0}{\partial s}) \geq c > 0$, it follows that for any $0<\alpha<1$ we can choose $|U_L-U_R|$ small enough so that $|U_L-U_R|\mathcal{O}(1) < \alpha\det(\frac{\partial x_0}{\partial\hat\tau}, \frac{\partial x_0}{\partial s})$. For such chosen U_l and U_R, we have from (2.16):

$$\det\left(\frac{\partial x_0}{\partial\hat\tau}, \frac{\partial x_0}{\partial s} + (f'(u_1)-c)\nabla_s t^*(s)B_1(\frac{t-t^*(s)}{\varepsilon})\right) > (1-\alpha)\det\left(\frac{\partial x_0}{\partial\hat\tau}, \frac{\partial x_0}{\partial s}\right),$$

implying that (2.8) is satisfied.

In the sequel, we denote by Ω_0^t the set Ω_0 shifted along the new characteristics X from (2.7) for the time t (at the level t), i.e.

$$\Omega_0^t = \{y\in\mathbf{R}^d:\ y = X(x_0,t,\varepsilon),\quad x_0\in\Omega_0\}.$$

Furthermore, by Ω_L^t we denote the set Ω_L shifted along the standard characteristics $x = f'(U_L)t + x_0$, $x_0\in\Omega_L$, until the intersection with Ω_0^t. More precisely,

$$\Omega_L^t = \{y\in\mathbf{R}^d:\ y = f'(U_L)t + x_0,\ x_0\in\Omega_L,\ \text{ and }\ \nexists t'\in[0,t],\ f'(U_L)t'+x_0\in\Omega_0^{t'}\}.$$

Similarly, we let:

$$\Omega_R^t = \{y\in\mathbf{R}^d:\ y = f'(U_R)t + x_0,\ x_0\in\Omega_R,\ \text{ and }\ \nexists t'\in[0,t],\ f'(U_R)t'+x_0\in\Omega_0^{t'}\}.$$

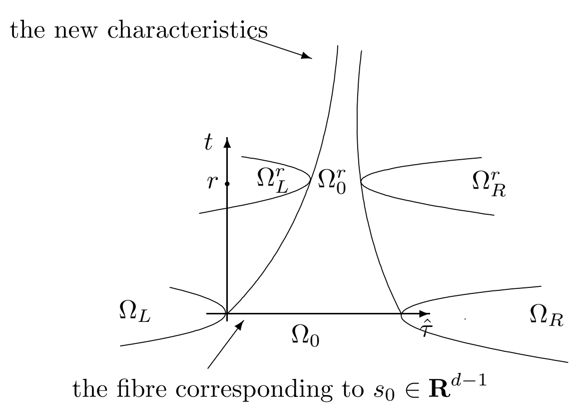

FIG. 7. Ω_0^r, Ω_L^r and Ω_R^r are the sets Ω_0, Ω_L and Ω_R moved along the new characteristics at the level $t=r$.

Notice that, since the characteristics are nonintersecting, we have

$$\mathbf{R}^d = \overline{\Omega_0^t}\dot\cup\Omega_L^t\dot\cup\Omega_R^t.$$

A simple corollary of Lemma 2.4 is the existence of the weak asymptotic solution to the Cauchy problem (2.4), (2.5):

Corollary 2.6. There exists a weak asymptotic solution to (2.4), (2.5).

Proof. It follows from Lemma 2.4 that there exists an inverse function $x_0(t,x,\varepsilon)$, $x \in \mathbf{R}^d$, of the function $X(x_0,t,\varepsilon)$, $x_0 \in \Omega_0$, defined by (2.7).

Thus, the weak asymptotic solution to (2.4), (2.5) is given by:

$$u_\varepsilon(x,t) = \begin{cases} U_L, & x \in \Omega_L^t, \\ u_1(x_0(t,x,\varepsilon)), & x \in \Omega_t^0, \\ U_R, & x \in \Omega_R^t. \end{cases} \tag{2.17}$$

□

3. The weak asymptotic equivalence. We will show that the weak asymptotic solution to (2.4), (2.5) is, at the same time, the weak asymptotic solution to (1.2), (1.3). More precisely, the following theorem holds.

Theorem 3.1. Let (u_ε) be the weak asymptotic solution to (2.4), (2.5). Then:

$$\partial_t u_\varepsilon + B_2\langle\nabla, f(u_\varepsilon)\rangle + B_1\sum_{i=1}^d c_i\frac{\partial u_\varepsilon}{\partial x_i} = \partial_t u_\varepsilon + \langle\nabla, f(u_\varepsilon)\rangle + o_{\mathcal{D}'}(1), \quad \varepsilon \to 0. \tag{3.1}$$

Proof. Since $\langle\nabla, f(u_\varepsilon)\rangle = (B_1+B_2)\langle\nabla, f(u_\varepsilon)\rangle$, (3.1) is equivalent to

$$B_1\langle\nabla, f(u_\varepsilon(t,\cdot)) - cu_\varepsilon(t,\cdot)\rangle = o_{\mathcal{D}'}(1).$$

We multiply this by $\varphi \in C_0^1(\mathbf{R}^d)$ and integrate over $\mathbf{R}^d$. We get after standard integration by parts (the Gauss-Ostrogradskii formula):

$$\begin{aligned} &\int_{\mathbf{R}^d} B_1\langle\nabla, f(u_\varepsilon(t,x)) - cu_\varepsilon(t,x)\rangle\varphi dx \\ &= -\int_{\mathbf{R}^d}\langle\nabla(B_1\varphi), f(u_\varepsilon(t,x)) - cu_\varepsilon(t,x)\rangle dx \\ &= -\int_{\Omega_L^t}\langle\nabla(B_1\varphi), f(U_L) - cU_L\rangle dx - \int_{\Omega_R^t}\langle\nabla(B_1\varphi), f(U_R) - cU_R\rangle dx \\ &- \int_{\Omega_0^t}\langle\nabla(B_1\varphi), f(u_\varepsilon(t,x)) - cu_\varepsilon(t,x)\rangle dx = o(1). \end{aligned} \tag{3.2}$$

Consider the integral in the latter formula corresponding to the domain Ω_R^t. Denote by $(\Omega_R^t)^C$ the complement of the set Ω_R^t. Notice that $(\Omega_R^t)^C = \Omega_L^t \cup \Omega_0^t$ and, using integration by parts again,

$$\begin{aligned} &\int_{\Omega_R^t}\langle\nabla(B_1\varphi), f(U_R) - cU_R\rangle dx = \int_{\partial\Omega_R^t}\langle B_1\varphi(f(U_R) - cU_R), \vec{n}\rangle ds \\ &= -\int_{(\Omega_R^t)^C}\langle\nabla(B_1\varphi), f(U_R) - cU_R\rangle dx \\ &= -\int_{\Omega_L^t}\langle\nabla(B_1\varphi), f(U_R) - cU_R\rangle dx - \int_{\Omega_0^t}\langle\nabla(B_1\varphi), f(U_R) - cU_R\rangle dx. \end{aligned} \tag{3.3}$$

We conclude from (3.2) and (3.3):

$$\begin{aligned}&\int_{\mathbf{R}^d} B_1\langle\nabla, f(u_\varepsilon(t,x)) - cu_\varepsilon(t,x)\rangle\varphi dx\\&= -\int_{\Omega_L^t}\langle\nabla(B_1\varphi), f(U_L) - f(U_R) - c(U_L - U_R)\rangle dx\\&\quad + \int_{\Omega_0^t}\langle\nabla(B_1\varphi), f(u_\varepsilon(t,x)) - f(U_R) - c(u_\varepsilon(t,x) - U_R)\rangle dx = o(1).\end{aligned}$$

Choosing here

$$c = \left(\frac{f_1(U_R) - f_1(U_L)}{U_R - U_L}, \dots, \frac{f_d(U_R) - f_d(U_L)}{U_R - U_L}\right), \tag{3.4}$$

we conclude that (3.2) is equivalent to

$$\begin{aligned}&\int_{\mathbf{R}^d} B_1\langle\nabla, f(u_\varepsilon(t,x)) - cu_\varepsilon(t,x)\rangle\varphi dx\\&= \int_{\Omega_0^t}\langle\nabla(B_1\varphi), f(u_\varepsilon(t,x)) - f(U_R) - c(u_\varepsilon(t,x) - U_R)\rangle dx = o(1).\end{aligned} \tag{3.5}$$

Let us prove the second equality in (3.5). Denote for simplicity

$$G(u_\varepsilon(t,x)) = f(u_\varepsilon(t,x)) - f(U_R) - c(u_\varepsilon(t,x) - U_R).$$

Then

$$\begin{aligned}&\int_{\Omega_0^t}\langle\nabla(B_1\varphi), G(u_\varepsilon(t,x))\rangle dx\\&= \int_{\Omega_0^t}\langle B_1\nabla\varphi, G(u_\varepsilon(t,x))\rangle dx - \int_{\Omega_0^t}\varphi\frac{\dot{B}_1}{\varepsilon}\langle\nabla t^*(s), G(u_\varepsilon(t,x))\rangle dx\\&= \int_{\Omega_0^t}\langle B_1\nabla\varphi, G(u_\varepsilon(t,x))\rangle dx - \int_{\Omega_0^t}\varphi\frac{\dot{B}_1}{\varepsilon}\langle\nabla_s t^*(s)\frac{\partial s}{\partial x}, G(u_\varepsilon(t,x))\rangle dx.\end{aligned} \tag{3.6}$$

For the function $X(t, x_0(\hat{\tau}, s), \varepsilon)$ from (2.7), introduce the following change of variables in the latter integrals:

$$X = X(t, x_0(\hat{\tau}, s), \varepsilon) \Longrightarrow \det(\frac{\partial X}{\partial\hat{\tau}}, \frac{\partial X}{\partial s}) \overset{(2.14)}{=} (1 - K(s)\int_0^t B_2 dt')\det(\frac{\partial x_0}{\partial\hat{\tau}}, \frac{\partial x_0}{\partial s}), \tag{3.7}$$

where $B_2 = B_2(\frac{t' - t^*(s)}{\varepsilon})$. To proceed, it is convenient to notice the following:

$$\begin{aligned}1 - K(s)\int_0^t B_2 dt' &= 1 - K(s)\left(\int_0^{t^*(s)} B_2 dt' + \int_{t^*(s)}^t B_2 dt'\right)\\&= 1 - K(s)\int_0^{t^*(s)}(1 - B_1)dt' - K(s)\int_{t^*(s)}^t B_2 dt'\\&= 1 - K(s)t^*(s) + K(s)(\int_0^{t*(s)} B_1 dt' + \int_{t^*(s)}^t B_2 dt')\\&= K(s)(\varepsilon\int_{-t^*(s)/\varepsilon}^0 B_1(z)dz + \varepsilon\frac{\tau}{t - t^*(s)}\int_{t^*(s)}^t B_2 dt'),\end{aligned}$$

where in the last step we introduced the change of variables $\frac{t'-t^*(s)}{\varepsilon} = z$, and, as usual, denoted $\tau = \frac{t-t^*(s)}{\varepsilon}$. Thus, from here and (3.7), we conclude that

$$\det(\frac{\partial X}{\partial\hat{\tau}}, \frac{\partial X}{\partial s}) = \varepsilon\tau\mathcal{O}(1)\det(\frac{\partial x_0}{\partial\hat{\tau}}, \frac{\partial x_0}{\partial s}), \quad \tau\to\infty. \tag{3.8}$$

We get from here and (3.6) (the $\cdot$ in the formula below stands for $x_0(\hat{\tau}, s)$ for x_0 from (1.12)):

$$\begin{aligned}
&\int_{\Omega_0^t}\langle\nabla(B_1\varphi), G(u_\varepsilon(t,x))\rangle dx \qquad (3.9)\\
&\overset{(3.8)}{=}\int_{\Gamma_L}\int_0^{\hat{\tau}_0(s)}\varepsilon\tau\mathcal{O}(1)B_1\langle\nabla\varphi, G(u_1(x_0(t,\cdot,\varepsilon)))\rangle\det(\frac{\partial x_0}{\partial\hat{\tau}}, \frac{\partial x_0}{\partial s})d\hat{\tau}ds\\
&\quad+\int_{\Gamma_L}\int_0^{\hat{\tau}_0(s)}\tau\mathcal{O}(1)\dot{B}_1\varphi\langle\nabla_s t^*(s)\frac{\partial s}{\partial x}, G(u_1(x_0(t,\cdot,\varepsilon)))\rangle\det(\frac{\partial x_0}{\partial\hat{\tau}}, \frac{\partial x_0}{\partial s})d\hat{\tau}ds\\
&=\mathcal{O}(\varepsilon)+\int_{\Gamma_L}\int_0^{\hat{\tau}_0(s)}\tau\mathcal{O}(1)\dot{B}_1\varphi\langle\nabla_s t^*(s)\frac{\partial s}{\partial x}, G(u_1(x_0(t,\cdot,\varepsilon)))\rangle\times\det(\frac{\partial x_0}{\partial\hat{\tau}}, \frac{\partial x_0}{\partial s})d\hat{\tau}ds,
\end{aligned}$$

where, as usual, $B_i = B_i(\frac{t-t^*(s)}{\varepsilon}) = B_i(\tau)$, $i=1,2$. So, we see that it remains to estimate

$$\begin{aligned}
&\int_{\Gamma_L}\int_0^{\hat{\tau}_0(s)}\tau\mathcal{O}(1)\dot{B}_1\varphi\langle\nabla_s t^*(s)\frac{\partial s}{\partial x}, G(u_1(x_0(t,\cdot,\varepsilon)))\rangle\det(\frac{\partial x_0}{\partial\hat{\tau}}, \frac{\partial x_0}{\partial s})d\hat{\tau}ds \qquad (3.10)\\
&=\int_{|\nabla_s t^*|\le\varepsilon}\int_0^{\hat{\tau}_0(s)}\tau\mathcal{O}(1)\dot{B}_1\varphi\langle\nabla_s t^*(s)\frac{\partial s}{\partial x}, G(u_1(x_0(t,\cdot,\varepsilon)))\rangle\det(\frac{\partial x_0}{\partial\hat{\tau}}, \frac{\partial x_0}{\partial s})d\hat{\tau}ds\\
&\quad+\int_{|\nabla_s t^*|\ge\varepsilon}\tau\mathcal{O}(1)\dot{B}_1\varphi\langle\nabla_s t^*(s)\frac{\partial s}{\partial x}, G(u_1(x_0(t,\cdot,\varepsilon)))\rangle\det(\frac{\partial x_0}{\partial\hat{\tau}}, \frac{\partial x_0}{\partial s})d\hat{\tau}ds.
\end{aligned}$$

Since

$$\int_{|\nabla_s t^*|\le\varepsilon}\int_0^{\hat{\tau}_0(s)}\tau\mathcal{O}(1)\dot{B}_1\varphi\langle\nabla_s t^*(s)\frac{\partial s}{\partial x}, G(u_1(x_0(t,\cdot,\varepsilon)))\rangle\det(\frac{\partial x_0}{\partial\hat{\tau}}, \frac{\partial x_0}{\partial s})d\hat{\tau}ds = \mathcal{O}(\varepsilon), \tag{3.11}$$

we only need to estimate the last term on the right-hand side of (3.10). First assume, without losing generality, that if $|\nabla_s t^*(s)|\ge\varepsilon$, then for a fixed $k\in\{1,\dots,d-1\}$,

$$\frac{\partial_{s_i}t^*(s)}{\partial_{s_k}t^*(s)}\le C<\infty, \quad i=1,\dots,d-1. \tag{3.12}$$

It is clear that the latter holds at least locally, which is enough for further consideration (we would involve the partition of unity argument if k would be different for different neighborhoods).

Then, take the following change of variables:

$$y_k = \frac{t-t^*(s)}{\varepsilon} = \tau, \quad y_i = s_i,\ i\neq k \implies \det(\frac{\partial s}{\partial y}) = \frac{\varepsilon}{\partial_{s_k}t^*(s(y))},$$

and denote $Y = y\left(\{s : \ |\nabla t^*(s)| \geq \varepsilon\}\right)$. Then

$$\begin{aligned}
&\int_{|\nabla t^*|\geq\varepsilon}\int_0^{\hat\tau_0(s)} \tau\mathcal{O}(1)\dot B_1\varphi\langle\nabla_s t^*(s)\frac{\partial s}{\partial x}, G(u_1(x_0(t,\cdot,\varepsilon)))\rangle \det(\frac{\partial x_0}{\partial\hat\tau},\frac{\partial x_0}{\partial s})d\hat\tau ds \qquad (3.13)\\
&= \int_Y\int_0^{\hat\tau_0(s(y))} \varepsilon\tau\mathcal{O}(1)y_k\dot B_1(y_k)\varphi\langle\frac{\nabla t^*(s)}{\partial_{s_k}t^*(s)}\frac{\partial s}{\partial x}, G(u_1(x_0(t,\cdot,\varepsilon)))\rangle \det(\frac{\partial x_0}{\partial\hat\tau},\frac{\partial x_0}{\partial s})d\hat\tau dy\\
&= \mathcal{O}(\varepsilon),
\end{aligned}$$

according to (3.12).

Noticing that $(3.13), (3.11) \implies (3.9) \implies (3.5) \implies (3.2)$, the proof of the theorem is completed. □

The last issue that we need to deal with is the entropy admissibility of the weak solution to (1.2), (1.3), which is obtained as a limit of the weak asymptotic solution. As expected, for a solution constructed from regular initial data along the characteristics, the solution obtained as a limit of the weak asymptotic solution is admissible (since characteristics must run into the shock surface; see the Lax-Oleinik admissibility conditions [2]). First, let us recall the notion of the entropy admissible weak solution introduced by S. N. Kruzhkov in his famous paper [20].

DEFINITION 3.2. We say that a weak solution to (1.2), (1.3) is entropy admissible if for every nonnegative $\varphi \in C_0^\infty(\mathbf{R}^+ \times \mathbf{R}^d)$ and every $\lambda \in \mathbf{R}$, it follows that

$$\iint_{\mathbf{R}\times\mathbf{R}^d} \operatorname{sgn}_\pm(u-\lambda)\left((u-\lambda)\frac{\partial\varphi}{\partial t} + \sum_{i=1}^d (f_i(u)-f_i(\lambda))\frac{\partial\varphi}{\partial x_i})\right)dxdt \geq 0, \qquad (3.14)$$

where

$$\operatorname{sgn}_+(u-\lambda) = (|u-\lambda|^+)'_u = \begin{cases} 1, & u>\lambda\\ 0, & u\leq\lambda\end{cases}$$

and

$$\operatorname{sgn}_-(u-\lambda) = (|u-\lambda|^-)'_u = \begin{cases} 0, & u>\lambda,\\ -1, & u\leq\lambda.\end{cases}$$

The following theorem holds.

THEOREM 3.3. Assume that $0 < U_L - U_R < \delta$ for a small enough constant δ. Then, the weak asymptotic solution (u_ε) to (1.2), (1.3) is given by (2.17). For every $s \in \mathbf{R}^{d-1}$ and $t \geq t^*(s)$, the a.e. pointwise limit of the weak asymptotic solution (u_ε) to problem (1.2), (1.3) contains an admissible shock wave of the strength $|U_L - U_R|$ which is formed at the point $X = f'(U_L)t^*(s) + \chi_L(s)$ at the moment $t = t^*(s)$. The corresponding shock surface moves by the law

$$x = f'(U_L)t^*(s) + \chi_L(s) + \frac{f(U_L)-f(U_R)}{U_L-U_R}(t-t^*(s)).$$

Proof. From (1.6), we see that the initial data (1.3) corresponding to equation (1.2) will evolve into the shock wave at the moment $t^*(s) = \frac{1}{K(s)}$ (see (1.5) and (1.6)). More precisely, any characteristic issuing from the point $x_0 = x_0(\hat\tau, s) \in \Omega_0$, where the function

x_0 is given by (1.11), will enter the shock surface at the moment $t^*(s)$. The shock surface then continues to move by the law

$$x = f'(U_L)t^*(s) + \chi_L(s) + \frac{f(U_L) - f(U_R)}{U_L - U_R}(t - t^*(s))$$

which is obtained by letting $\varepsilon \to 0$ in (2.7) for $t > t^*(s)$.

Next, notice that from the form of the weak asymptotic solution (2.17) and Theorem 3.1, it follows that it converges pointwisely a.e. toward a weak solution $u \in L^\infty(\mathbf{R}^+ \times \mathbf{R}^d)$ to (1.2), (1.3). Actually, in a neighborhood of any $(t, x) \in \mathbf{R}^+ \times \mathbf{R}^d$, the function u will be either the shock wave with values U_L and U_R (if $x_0(t, x, \varepsilon) = x_0(\hat{\tau}, s)$ such that $t \geq t^*(s)$, where x_0 is given by (1.11)), or it will be the classical solution to (1.2), (1.3) (if $x_0(t, x, \varepsilon) = x_0(\hat{\tau}, s)$ such that $t < t^*(s)$, where x_0 is given by (1.11)).

We shall prove that the weak solution u is entropy admissible in the sense of Definition 3.2 for the entropies $|\cdot - \lambda|^+$ (the situation with $|\cdot - \lambda|^-$ is analogical). Since u_ε are the Lipschitz continuous functions, from (2.4) we conclude that for an arbitrary $\varphi \in C_0^1(\mathbf{R}^+ \times \mathbf{R}^d)$, it follows that

$$\begin{aligned}
&\iint_{\mathbf{R}^+ \times \mathbf{R}^d} \operatorname{sgn}_+(u_\varepsilon - \lambda) \left((u_\varepsilon - \lambda) \frac{\partial \varphi}{\partial t} + \sum_{i=1}^d (f_i(u_\varepsilon) - f_i(\lambda)) \frac{\partial \varphi}{\partial x_i} \right) dxdt \qquad (3.15)\\
&= \iint_{\mathbf{R}^+ \times \mathbf{R}^d} \operatorname{sgn}_+(u_\varepsilon - \lambda) \Big((u_\varepsilon - \lambda) \frac{\partial \varphi}{\partial t} \\
&\qquad + \sum_{i=1}^d \left((f_i(u_\varepsilon) - f_i(\lambda)) \frac{\partial (B_2 \varphi)}{\partial x_i} + c_i (u_\varepsilon - \lambda) \frac{\partial (B_1 \varphi)}{\partial x_i} \right) \Big) dxdt \\
&+ \iint_{\mathbf{R}^+ \times \mathbf{R}^d} \operatorname{sgn}_+(u_\varepsilon - \lambda) \sum_{i=1}^d \left((f_i(u_\varepsilon) - f_i(\lambda)) - c_i (u_\varepsilon - \lambda) \right) \frac{\partial (B_1 \varphi)}{\partial x_i} dxdt \\
&= \iint_{\mathbf{R}^+ \times \mathbf{R}^d} \operatorname{sgn}_+(u - \lambda) \sum_{i=1}^d \left((f_i(u_\varepsilon) - f_i(\lambda)) - c_i (u_\varepsilon - \lambda) \right) \frac{\partial (B_1 \varphi)}{\partial x_i} dxdt,
\end{aligned}$$

where $c_i = \frac{[f_i]}{[u]} = \frac{f_i(U_L) - f_i(U_R)}{U_L - U_R}$, $i = 1, \ldots, d$. We are going to prove that

$$\iint_{\mathbf{R}^+ \times \mathbf{R}^d} \operatorname{sgn}_+(u_\varepsilon - \lambda) \sum_{i=1}^d \left((f_i(u_\varepsilon) - f_i(\lambda)) - c_i (u_\varepsilon - \lambda) \right) \frac{\partial (B_1 \varphi)}{\partial x_i} dxdt \geq o(1), \qquad (3.16)$$

for $U_L - U_R > 0$ small enough, which, combined with (3.15), will prove that the function u is an entropy solution to (1.2), (1.3).

Arguing in the completely same way as in the proof of Theorem 3.1, we reach to the following estimate:

$$\begin{aligned}
&\iint_{\mathbf{R}^+\times\mathbf{R}^d} \operatorname{sgn}_+(u_\varepsilon-\lambda)\sum_{i=1}^d \left((f_i(u_\varepsilon)-f_i(\lambda))-c_i(u_\varepsilon-\lambda)\right)\frac{\partial(B_1\varphi)}{\partial x_i}dxdt \qquad (3.17)\\
&=\int_{\mathbf{R}^+}\int_{\Omega_L^t}\Big[\operatorname{sgn}_+(U_L-\lambda)\sum_{i=1}^d\left((f_i(U_L)-f_i(\lambda))-c_i(U_L-\lambda)\right)\\
&\quad -\operatorname{sgn}_+(U_R-\lambda)\sum_{i=1}^d\left((f_i(U_R)-f_i(\lambda))-c_i(U_R-\lambda)\right)\Big]\frac{\partial(B_1\varphi)}{\partial x_i}dxdt+o(1).
\end{aligned}$$

Next, notice that if $\lambda>U_L>U_R$ or $U_L>U_R>\lambda$ (recall that $U_L>U_R$; see (1.10)), then the subintegral expression on the right-hand side of (3.17) is equal to zero, which means that (3.16) is fulfilled with the equality sign. So, assume that $U_R<\lambda<U_L$. Relation (3.17) reduces to

$$\begin{aligned}
&\iint_{\mathbf{R}^+\times\mathbf{R}^d} \operatorname{sgn}_+(u_\varepsilon-\lambda)\sum_{i=1}^d \left((f_i(u_\varepsilon)-f_i(\lambda))-c_i(u_\varepsilon-\lambda)\right)\frac{\partial(B_1\varphi)}{\partial x_i}dxdt \qquad (3.18)\\
&=\int_{\mathbf{R}^+}\int_{\Omega_L^t}\Big[\sum_{i=1}^d\left((f_i(U_L)-f_i(\lambda))-c_i(U_L-\lambda)\right)\Big]\frac{\partial(B_1\varphi)}{\partial x_i}dxdt+o(1).
\end{aligned}$$

Applying the Gauss-Ostrogradskii formula, we conclude that

$$\begin{aligned}
&\int_{\mathbf{R}^+}\int_{\Omega_L^t}\Big[\sum_{i=1}^d\left((f_i(U_L)-f_i(\lambda))-c_i(U_L-\lambda)\right)\Big]\frac{\partial(B_1\varphi)}{\partial x_i}dxdt \qquad (3.19)\\
&=(U_L-\lambda)\int_{\mathbf{R}^+}\int_{\Gamma_L^t}\langle\vec{n}_t,F(U_L,\lambda)\rangle B_1\varphi\, ds'dt,
\end{aligned}$$

where Γ_L^t is the boundary of the set Ω_L^t, $\vec{n}_t$ is the unit outer normal on Γ_L^t, and

$$\begin{aligned}
&F(U_L,\lambda)\\
&=\left(\frac{f_1(U_L)-f_1(\lambda)}{U_L-\lambda}-\frac{f_1(U_L)-f_1(U_R)}{U_L-U_R},\dots,\frac{f_d(U_L)-f_d(\lambda)}{U_L-\lambda}-\frac{f_d(U_L)-f_d(U_R)}{U_L-U_R}\right).
\end{aligned}$$

Next, notice that $\vec{n}_t$ is actually normal on $\Gamma_L^0=\Gamma_L$ translated along the new characteristics at the level t. Since the set Γ_L^0 is given by $\Gamma_L^0=\{x_0\in\mathbf{R}^d:\ u_1(x_0)=U_L\}$, we conclude that the normal on Γ_L^0 is given by $\vec{n}_0=\nabla u_1(x_0)$ (assume for simplicity that $\|\nabla u_1(x_0)\|=1$). Thus, $\vec{n}_t=\nabla_{x_0}u_1(x_0)\Big|_{x_0=x_0(t,x,\varepsilon)}$, $x\in\Gamma_L^t$, where $x_0(t,x,\varepsilon)$ is the inverse function of the new characteristics X given by (2.7) (the existence of the inverse function is proved in Lemma 2.4). Substituting such $\vec{n}_t$ into (3.19), we obtain

$$\begin{aligned}
&\int_{\mathbf{R}^+}\int_{\Omega_L^t}\Big[\sum_{i=1}^d((f_i(U_L)-f_i(\lambda))-c_i(U_L-\lambda))\Big]\frac{\partial(B_1\varphi)}{\partial x_i}dxdt \\
&=(U_L-\lambda)\int_{\mathbf{R}^+}\int_{\Gamma_L^t}\sum_{i=1}^d\left(\frac{f_i(U_L)-f_i(\lambda)}{U_L-\lambda}-\frac{f_i(U_L)-f_i(U_R)}{U_L-U_R}\right) \\
&\qquad\times\frac{\partial u_1}{\partial x_{0i}}\Big|_{x_0=x_0(t,x(s'),\varepsilon)}B_1\varphi ds'dt,
\end{aligned}\tag{3.20}$$

Using Taylor's formula with the integral remainder term, we have for any $i\in\{1,\dots,d\}$:

$$\begin{aligned}
\frac{f_i(U_L)-f_i(\lambda)}{U_L-\lambda}&=-f_i'(U_L)+\frac{1}{2}(U_L-\lambda)f_i''(U_L)+\frac{1}{2}\frac{1}{U_L-\lambda}\int_{U_L}^{\lambda}(\lambda-t)^2f_i'''(t)dt,\\
\frac{f_i(U_L)-f_i(U_R)}{U_L-U_R}&=-f_i'(U_L)+\frac{1}{2}(U_L-U_R)f_i''(U_L)+\frac{1}{2}\frac{1}{U_L-U_R}\int_{U_L}^{U_R}(U_R-t)^2f_i'''(t)dt.
\end{aligned}$$

Subtracting the latter terms and noticing that

$$\begin{aligned}
&\frac{1}{2}\frac{1}{U_L-\lambda}\int_{U_L}^{\lambda}(\lambda-t)^2f_i'''(t)dt-\frac{1}{2}\frac{1}{U_L-U_R}\int_{U_L}^{U_R}(U_R-t)^2f_i'''(t)dt\\
&=\mathcal{O}(|U_R-\lambda|^2+|(U_R-\lambda)(U_L-\lambda)|),
\end{aligned}$$

we get

$$\begin{aligned}
&\sum_{i=1}^d\left(\frac{f_i(U_L)-f_i(\lambda)}{U_L-\lambda}-\frac{f_i(U_L)-f_i(U_R)}{U_L-U_R}\right)\\
&=\left(\sum_{i=1}^d f_i''(U_L)\frac{U_R-\lambda}{2}+\mathcal{O}(|U_R-\lambda|^2+|(U_R-\lambda)(U_L-\lambda)|)\right).
\end{aligned}$$

From here and (3.20), we conclude that

$$\begin{aligned}
&\int_{\mathbf{R}^+}\int_{\Omega_L^t}\Big[\sum_{i=1}^d((f_i(U_L)-f_i(\lambda))-c_i(U_L-\lambda))\Big]\frac{\partial(B_1\varphi)}{\partial x_i}dxdt\\
&=-(U_L-\lambda)(\lambda-U_R)\int_{\mathbf{R}^+}\int_{\Gamma_L^t}\sum_{i=1}^d f''(U_L)\frac{\partial u_1}{\partial x_{0i}}\Big|_{x_0=x_0(t,x(s'),\varepsilon)}B_1\varphi ds'dt\\
&\quad+\int_{\mathbf{R}^+}\int_{\Gamma_L^t}\mathcal{O}(|(U_L-\lambda)(U_R-\lambda)^2|+|(U_R-\lambda)(U_L-\lambda)^2|)B_1\varphi ds'dt\\
&=-(U_L-\lambda)(\lambda-U_R)\int_{\mathbf{R}^+}\int_{\Gamma_L^t}\left(\sum_{i=1}^d f''(u_1)\frac{\partial u_1}{\partial x_{0i}}\right)\Big|_{x_0=x_0(t,x(s'),\varepsilon)}B_1\varphi ds'dt\\
&\quad+\int_{\mathbf{R}^+}\int_{\Gamma_L^t}\mathcal{O}(|(U_L-\lambda)(U_R-\lambda)^2|+|(U_R-\lambda)(U_L-\lambda)^2|)B_1\varphi ds'dt\\
&=\int_{\mathbf{R}^+}\int_{\Gamma_L^t}\Big(K(s)(U_L-\lambda)(\lambda-U_R)\\
&\qquad\qquad+\mathcal{O}(|(U_L-\lambda)(U_R-\lambda)^2|+|(U_R-\lambda)(U_L-\lambda)^2|)\Big)B_1\varphi ds'dt\geq 0,
\end{aligned}$$

for U_L-U_R small enough. Above, we used $u_1(x_0(t,x,\varepsilon))=U_L$ for $x\in\Gamma_L^t$ (the second equality) and (1.6) (the third equality).

Together with (3.18), this proves (3.16) and concludes the theorem. □

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