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THE DIRECT SCATTERING PROBLEM: UNIQUENESS AND EXISTENCE FOR ANISOTROPIC MEDIA

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Abstract. In this paper, we investigate the problem of the transmission of plane acoustic waves through a penetrable inhomogeneous body with an impenetrable core. Firstly, we discretize the inhomogeneous body via means of a multi-layer, multi-subdivision approach. Then, we prove the uniqueness and existence of solutions to this problem. Finally, we conclude with a discussion of potential applications and what remains to be done.

1. Introduction. Scattering theory has long established itself as one of the most prominent and interesting areas of applied mathematics. It has applications in areas as diverse as military aerial and ground vehicle design, medical imaging and brain tumor detection, oil extraction and space exploration, etc. It is interesting, though, that there are many applications of scattering theory yet to be fulfilled. For instance, oil detection has been a pretty challenging problem for many decades, if not centuries. With the introduction of scattering theory, we were better able to pinpoint the approximate locations of potential oil cavities, and we were thus closer to meeting the world's oil and energy needs. Despite these profound advances, we are still not able to precisely detect oil cavities around the world. This is partially due to the fact that, up to this point, scientists in the field had generally restricted themselves to the treatment of either impenetrable or penetrable homogeneous bodies and thus applications involving inhomogeneous media (e.g., the surface of the earth) had remained at an infant stage. While there has been a treatment of anisotropic media in [3] as well as in [1], it is not sufficiently generalized to enjoy widespread applications. It is, thus, natural, to say, that a simpler, more elegant approach would be needed to treat the problem of the scattering by inhomogeneous bodies.

In the literature, there has been an extensive study of the multi-layered or stratified scatterer. Specifically, the problem of the scattering of acoustic and electromagnetic waves by an infinitely stratified scatterer has been investigated by C. Athanasiadis in [4]

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and C. Athanasiadis and I. G. Stratis in [5], while the problem of the acoustic scattering by a multi-layered ellipsoid is studied in [4]. Some general transmission problems have been studied by R. Kress and G. F. Roach in [8]. Results for an infinitely stratified scatterer in low-frequency electromagnetic scattering theory are given in [6]. Finally, an excellent monograph in scattering theory and general transmission problems is the one in [2].

It is the purpose of this work to introduce some uniqueness and existence results for the aforementioned scattering problem. This paper extends the multi-layer approach to a further dimension, by subdividing the multi-layered scatterer along the polar angle. This results in the formation of a number of four-sided regions in $\mathbb{R}^2$ which, for a sufficiently large number of layers and subdivisions, give a good approximation of an inhomogeneous medium. It was this property that motivated our study of this problem.

This paper is organized as follows. Section 2 states the problem in two dimensions and gives a thorough description of the scatterer. It firmly sets the foundations for this work. Next, in Section 3, we prove the existence of a unique solution to the 2-dimensional problem. To show that this problem has at most one solution we use a heavily modified version of the method employed in [5], which in turn is based on an approach by R. Kress and G. F. Roach in [8]. Then, we show that this problem has a unique solution via means of a novel integral equation based approach. Finally, in the conclusion, we discuss some potential applications of our method as well as what remains to be done.

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2. Statement of the problem. Let $\widetilde{\Omega}$ be a bounded surface in $\mathbb{R}^2$ with boundary S_0. We assume that a core Ω_c, containing the origin of the coordinates, with boundary S_c, lies in the interior of $\widetilde{\Omega}$. The set $\Omega = \widetilde{\Omega} - \Omega_c$ is divided into annuli-like regions Ω_i by smooth surfaces S_i, $i = 1, 2, \ldots, N+1$, where S_i surrounds S_{i+1}. We also suppose that $S_{N+1} = S_c$. We further assume that Ω is evenly subdivided along the polar angle, by lines l_j, $j = 0, 1, \ldots, M$, containing the origin of the coordinates, into subregions D_{ij}, $i = 1, 2, \ldots, N$, $j = 0, 1, \ldots, M : \bigcup_{j=0}^{M} D_{ij} = \Omega_i \ \forall \ i = 1, 2, \ldots, N$. The set $\widetilde{\Omega}$, as described above, is called a multi-layered, polarly subdivided or web-like scatterer, which we will denote by W. Furthermore, let the boundary of D_{ij}, ∂D_{ij}, be decomposed into four C^2-surfaces, namely, $\Gamma_{ij}^{front}, \Gamma_{ij}^{right}, \Gamma_{ij}^{back}, \Gamma_{ij}^{left}$. In addition, let the unit normal to each layer surface point to the exterior of the corresponding region and the unit normal to l_j be directed clockwise. For convenience, we will denote both unit normals by ν. The exterior Ω_0 of $\widetilde{\Omega}$ as well as each D_{ij}, $i = 1, 2, \ldots, N$, $j = 0, 1, \ldots, M$, are assumed to be homogeneous isotropic media with densities ρ_0 and ρ_{ij}, $i = 1, 2, \ldots, N$, $j = 0, 1, \ldots, M$. Furthermore, let the wavenumbers k_{ij}, k_0, corresponding to D_{ij} and Ω_0 respectively, satisfy $\mathrm{Im}(k_{ij}) \geq 0$ and $\mathrm{Im}(k_0) \geq 0$.

Moreover, let the core of the scatterer, S_c, be sound soft. The total acoustic field in D_{ij} is defined as u_{ij}, $i = 1, 2, \ldots, N$, $j = 0, 1, \ldots, M$, while the field in $\Omega_0 := D_{0j}$ is

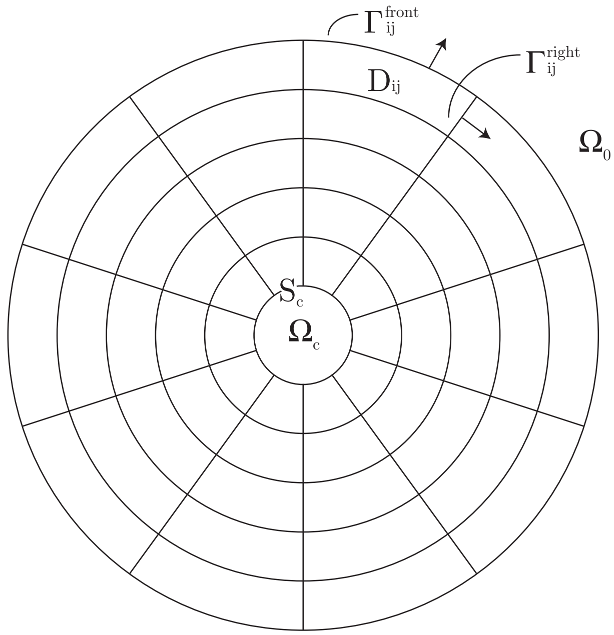

FIG. 2.1. The web-like scatterer.

defined as $u_0^{total} = u_{0j}^{total} = u^{inc} + u_{0j}$, where u_{0j} denotes the scattered acoustic field. Then we have the following problem:

$$\triangle u_{ij} + k_{ij}^2 u_{ij} = 0 \quad \text{in} \quad D_{ij},\ i = 0, 1, \ldots, N, \tag{2.1}$$

$$\left.\begin{aligned} & u_{ij} - u_{(i+1)j} = f_{ij} \\ & \rho_{(i+1)j}\frac{\partial u_{ij}}{\partial \nu} - \rho_{ij}\frac{\partial u_{(i+1)j}}{\partial \nu} = g_{ij} \end{aligned}\right\} \quad \text{on} \quad \Gamma_{(i+1)j}^{front},\ i = 0, 1, \ldots, N-1, \tag{2.2}$$

$$u_{Nj} = f_{Nj} \quad \text{on} \quad \Gamma_{(N+1)j}^{front} \tag{2.3}$$

$$\left.\begin{aligned} & u_{ij} - u_{i(j+1)} = f_{ij}^{'} \\ & \rho_{i(j+1)}\frac{\partial u_{ij}}{\partial \nu} - \rho_{ij}\frac{\partial u_{i(j+1)}}{\partial \nu} = g_{ij}^{'} \end{aligned}\right\} \quad \text{on} \quad \Gamma_{ij}^{right},\ i = 1, 2, \ldots, N, \tag{2.4}$$

$$\frac{\partial u_{0j}}{\partial \nu} - ik_0 u_{0j} = o\left(\frac{1}{\sqrt{r}}\right), \quad r \to \infty \tag{2.5}$$

for $j = 0, 1, \ldots, M$.

We will denote this transmission problem by (A).

3. Solvability of the problem. In this section we prove the existence and uniqueness of solutions for this acoustic transmission problem. In what follows, we shall make the following assumptions on the wavenumbers k_0, k_{ij}, and the corresponding densities ρ_0, ρ_{ij} :

Let $k \in \mathbb{C} - \{0\}$ with $0 \leq \arg(k) \leq \pi$, $i = 1, 2, \ldots, N$, $j = 0, 1, \ldots, M$, be such that $\frac{k_{ij}^2}{k_0^2} \in \mathbb{R}$, with $\sup(\frac{k_{ij}^2}{k_0^2}) < +\infty$. Moreover, let $\sup(\frac{\rho_0}{\rho_{ij}}) < +\infty$.

Let us denote by (H) the homogeneous form of problem (A) with $f_{ij}=g_{ij}=f_{ij}^{'}=g_{ij}^{'}=0$. Then we have the following important theorem:

THEOREM 3.1. (H) has only the trivial solution.

Proof. Let $\Omega_{0,R}=\{r\in\Omega_0 : r<R\}$, $R>0$. By Green's first theorem on $\Omega_{0,R}$ we obtain

$$\begin{aligned}\int_{r=R} u_{0j}\frac{\partial\overline{u_{0j}}}{\partial\eta}ds &= \int_{\Omega_{0,R}} u_{0j}\triangle\overline{u_{0j}}dv+\int_{S_0} u_{0j}\frac{\partial\overline{u_{0j}}}{\partial\eta}ds+\int_{\Omega_{0,R}} \mid\nabla u_{0j}\mid^2 dv\\ &= \int_{\Omega_{0,R}} u_{0j}\triangle\overline{u_{0j}}dv+\sum_{j=0}^{M}\int_{\Gamma_{1j}^{front}} u_{0j}\frac{\partial\overline{u_{0j}}}{\partial\eta}ds+\int_{\Omega_{0,R}} \mid\nabla u_{0j}\mid^2 dv,\end{aligned}$$

which again by the transmission conditions (2.2) and Green's first theorem becomes

$$\begin{aligned}\int_{r=R} u_{0j}\frac{\partial\overline{u_{0j}}}{\partial\eta}ds &= \int_{\Omega_{0,R}} u_{0j}\triangle\overline{u_{0j}}dv+\int_{\Omega_{0,R}} \mid\nabla u_{0j}\mid^2 dv+\sum_{j=0}^{M}\frac{\rho_0}{\rho_{1j}}\int_{\Gamma_{1j}^{front}} u_{1j}\frac{\partial\overline{u_{1j}}}{\partial\eta}ds\\ &=\sum_{j=0}^{M}\frac{\rho_0}{\rho_{1j}}\int_{\Gamma_{2j}^{front}} u_{1j}\frac{\partial\overline{u_{1j}}}{\partial\eta}ds+\sum_{j=0}^{M}\frac{\rho_0}{\rho_{1j}}\int_{\Gamma_{1(j-1)}^{right}} u_{1j}\frac{\partial\overline{u_{1j}}}{\partial\eta}ds\\ &\quad-\sum_{j=0}^{M}\frac{\rho_0}{\rho_{1j}}\int_{\Gamma_{1j}^{right}} u_{1j}\frac{\partial\overline{u_{1j}}}{\partial\eta}ds+\sum_{j=0}^{M}\frac{\rho_0}{\rho_{1j}}\int_{D_{1j}} u_{1j}\triangle\overline{u_{1j}}dv\\ &\quad+\sum_{j=0}^{M}\frac{\rho_0}{\rho_{1j}}\int_{D_{1j}} \mid\nabla u_{1j}\mid^2 dv+\int_{\Omega_{0,R}} u_{0j}\triangle\overline{u_{0j}}dv+\int_{\Omega_{0,R}} \mid\nabla u_{0j}\mid^2 dv. \qquad (3.1)\end{aligned}$$

By using the transmission conditions (2.4) and applying Green's first theorem over $D_{1(j+1)}$ we obtain

$$\begin{aligned}\int_{r=R} u_{0j}\frac{\partial\overline{u_{0j}}}{\partial\eta}ds &= \int_{\Omega_{0,R}} u_{0j}\triangle\overline{u_{0j}}dv+\int_{\Omega_{0,R}} \mid\nabla u_{0j}\mid^2 dv+\sum_{j=0}^{M}\frac{\rho_0}{\rho_{1j}}\int_{\Gamma_{2j}^{front}} u_{1j}\frac{\partial\overline{u_{1j}}}{\partial\eta}ds\\ &\quad+\sum_{j=0}^{M}\frac{\rho_0}{\rho_{1M}}\int_{\Gamma_{1(j+M)}^{right}} u_{1(j+M)}\frac{\partial\overline{u_{1(j+M)}}}{\partial\eta}ds-\sum_{j=0}^{M}\frac{\rho_0}{\rho_{1(j+1)}}\int_{\Gamma_{1(j+1)}^{front}} u_{1(j+1)}\frac{\partial\overline{u_{1(j+1)}}}{\partial\eta}ds\\ &\quad-\sum_{j=0}^{M}\frac{\rho_0}{\rho_{1(j+1)}}\int_{\Gamma_{1(j+1)}^{right}} u_{1(j+1)}\frac{\partial\overline{u_{1(j+1)}}}{\partial\eta}ds+\sum_{j=0}^{M}\frac{\rho_0}{\rho_{1(j+1)}}\int_{\Gamma_{2(j+1)}^{front}} u_{1(j+1)}\frac{\partial\overline{u_{1(j+1)}}}{\partial\eta}ds\\ &\quad+\sum_{j=0}^{M}\frac{\rho_0}{\rho_{1(j+1)}}\int_{D_{1(j+1)}} u_{1(j+1)}\triangle\overline{u_{1(j+1)}}dv+\sum_{j=0}^{M}\frac{\rho_0}{\rho_{1(j+1)}}\int_{D_{1(j+1)}} \mid\nabla u_{1(j+1)}\mid^2 dv\\ &\quad+\sum_{j=0}^{M}\frac{\rho_0}{\rho_{1j}}\int_{D_{1j}} u_{1j}\triangle\overline{u_{1j}}dv+\sum_{j=0}^{M}\frac{\rho_0}{\rho_{1j}}\int_{D_{1j}} \mid\nabla u_{1j}\mid^2 dv. \qquad (3.2)\end{aligned}$$

By repeated application of Green's first theorem and the transmission conditions (2.4) we get from (3.2) that

$$\begin{aligned}&\int_{r=R} u_{0j}\frac{\partial\overline{u_{0j}}}{\partial\eta}ds=\\ &\int_{\Omega_{0,R}} u_{0j}\triangle\overline{u_{0j}}dv+\int_{\Omega_{0,R}} \mid\nabla u_{0j}\mid^2 dv+\sum_{j=0}^{M}\sum_{k=0}^{M}\frac{\rho_0}{\rho_{1(j+k)}}\int_{\Gamma_{2(j+k)}^{front}} u_{1(j+k)}\frac{\partial\overline{u_{1(j+k)}}}{\partial\eta}ds\end{aligned}$$

$$-\sum_{j=0}^{M}\sum_{k=0}^{M}\frac{\rho_0}{\rho_{1(j+k)}}\int_{\Gamma_{1(j+k)}^{front}} u_{1(j+k)}\frac{\partial\overline{u_{1(j+k)}}}{\partial\eta}ds+\sum_{j=0}^{M}\frac{\rho_0}{\rho_{1j}}\int_{\Gamma_{1j}^{front}} u_{1j}\frac{\partial\overline{u_{1j}}}{\partial\eta}ds$$
$$+\sum_{j=0}^{M}\sum_{k=0}^{M}\frac{\rho_0}{\rho_{1(j+k)}}\int_{D_{1(j+k)}} u_{1(j+k)}\triangle\overline{u_{1(j+k)}}dv+\sum_{j=0}^{M}\sum_{k=0}^{M}\frac{\rho_0}{\rho_{1(j+k)}}\int_{D_{1(j+k)}} \mid\nabla u_{1(j+k)}\mid^2 dv, \tag{3.3}$$

which by virtue of (3.1) and the general fact that $\sum_{a_1=0}^{M}\cdots\sum_{a_n=0}^{M}G_{a_1+a_2+\ldots+a_n} = (M+1)^{n-1}\sum_{i=0}^{M}G_i$, for every G_i satisfying $G_{i+M+1}=G_i$, gives

$$\int_{r=R}u_{0j}\frac{\partial\overline{u_{0j}}}{\partial\eta}ds = \int_{\Omega_{0,R}}u_{0j}\triangle\overline{u_{0j}}dv+\int_{\Omega_{0,R}}\mid\nabla u_{0j}\mid^2 dv+\sum_{j=0}^{M}\frac{\rho_0}{\rho_{1j}}\int_{\Gamma_{2j}^{front}}u_{1j}\frac{\partial\overline{u_{1j}}}{\partial\eta}ds$$
$$+\sum_{j=0}^{M}\frac{\rho_0}{\rho_{1j}}\int_{D_{1j}}u_{1j}\triangle\overline{u_{1j}}dv+\sum_{j=0}^{M}\frac{\rho_0}{\rho_{1j}}\int_{D_{1j}}\mid\nabla u_{1j}\mid^2 dv. \tag{3.4}$$

By repeated use of Green's first theorem, the transmission conditions (2.2) and (2.4) and the boundary behavior (2.3) on $\Gamma_{(N+1)j}^{front}$, we get from (3.4),

$$\int_{r=R}u_{0j}\frac{\partial\overline{u_{0j}}}{\partial\eta}ds = \int_{\Omega_{0,R}}u_{0j}\triangle\overline{u_{0j}}dv+\int_{\Omega_{0,R}}\mid\nabla u_{0j}\mid^2 dv+\sum_{i=1}^{N}\sum_{j=0}^{M}\frac{\rho_0}{\rho_{ij}}\int_{D_{ij}}u_{ij}\triangle\overline{u_{ij}}dv$$
$$+\sum_{i=1}^{N}\sum_{j=0}^{M}\frac{\rho_0}{\rho_{ij}}\int_{D_{ij}}\mid\nabla u_{ij}\mid^2 dv$$

or

$$\frac{1}{\overline{k_0^2}}\int_{r=R}u_{0j}\frac{\partial\overline{u_{0j}}}{\partial\eta}ds=-\int_{\Omega_{0,R}}\mid u_{0j}\mid^2 dv+\frac{1}{\overline{k_0^2}}\int_{\Omega_{0,R}}\mid\nabla u_{0j}\mid^2 dv-\sum_{i=1}^{N}\sum_{j=0}^{M}\frac{\rho_0}{\rho_{ij}}\frac{\overline{k_{ij}^2}}{\overline{k_0^2}}\int_{D_{ij}}\mid u_{ij}\mid^2 dv$$
$$+\frac{1}{\overline{k_0^2}}\sum_{i=1}^{N}\sum_{j=0}^{M}\frac{\rho_0}{\rho_{ij}}\int_{D_{ij}}\mid\nabla u_{ij}\mid^2 dv. \tag{3.5}$$

Taking imaginary parts in (3.5) and taking into account the assumptions on k_0, k_{ij} and ρ_0, ρ_{ij}, we obtain

$$\operatorname{Im}\left(\frac{1}{\overline{k_0^2}}\int_{r=R}u_{0j}\frac{\partial\overline{u_{0j}}}{\partial\eta}ds\right)$$
$$=\operatorname{Im}\left(\frac{1}{\overline{k_0^2}}\right)\int_{\Omega_{0,R}}\mid\nabla u_{0j}\mid^2 dv+\operatorname{Im}\left(\frac{1}{\overline{k_0^2}}\right)\sum_{i=1}^{N}\sum_{j=0}^{M}\frac{\rho_0}{\rho_{ij}}\int_{D_{ij}}\mid\nabla u_{ij}\mid^2 dv. \tag{3.6}$$

As $R\to\infty$, $\Omega_{0,R}$ tends to Ω_0. Furthermore, since u_{0j} satisfies the radiation condition (2.5), it follows, [8], that the left-hand side of (3.6) tends to zero as $R\to\infty$. Since $\operatorname{Im}\left(\frac{1}{\overline{k_0^2}}\right)=\frac{\operatorname{Im}\left(k_0^2\right)}{|k_0|^4}$, and $\operatorname{Im}\left(k_0^2\right)=2\operatorname{Re}(k_0)\operatorname{Im}(k_0)$, we obtain from (3.6) that

$$\operatorname{Im}(k_0)\int_{\Omega_{0,R}}\mid\nabla u_{0j}\mid^2 dv = -\operatorname{Im}(k_0)\sum_{i=1}^{N}\sum_{j=0}^{M}\frac{\rho_0}{\rho_{ij}}\int_{D_{ij}}\mid\nabla u_{ij}\mid^2 dv. \tag{3.7}$$

If $\operatorname{Im}(k_0)>0$, then it follows that, since both integrals are positive, they must vanish, and thus both u_{0j} and u_{ij} must remain constant throughout Ω_0 and D_{ij}, $i=1,2,\ldots,N$,

$j = 0, 1, \ldots, M$, respectively. But by the radiation condition (2.5), u_{0j} turns out to be zero in Ω_0 and then, by conditions (2.2) and (2.4), every other u_{ij}, $i = 1, 2, \ldots, N$, $j = 0, 1, \ldots, M$ must also be equal to zero in D_{ij}. Hence, in this case, (H) has only the trivial solution.

On the other hand, if $\operatorname{Im}(k_0) = 0$, then we have from (3.6) that

$$\operatorname{Im}\left(\int_{r=R} u_{0j} \frac{\partial \overline{u_{0j}}}{\partial \eta} ds\right) = 0. \tag{3.8}$$

From (2.5) we have that

$$k_0^2 \int_{r=R} | u_{0j} |^2 \, ds + \operatorname{Im}\left(\int_{r=R} u_{0j} \frac{\partial \overline{u_{0j}}}{\partial \eta} ds\right) = o\,(1)\,, \quad R \to \infty \tag{3.9}$$

and hence by (3.8),

$$\int_{r=R} | u_{0j} |^2 \, ds = o\,(1)\,, \quad R \to \infty, \tag{3.10}$$

which, in turn, by Rellich's theorem [9], gives $u_{0j} = 0$ in Ω_0. If we prove that $u_{10} = 0$ in D_{10}, then by the same argument, u_{20} will turn out to be zero in D_{20}, etc.

By Holmgren's uniqueness theorem [10], we have that the solution of the Cauchy problem

$$\triangle u_{10} + k_{10}^2 u_{10} = 0 \quad \text{in} \quad D_{10}, \tag{3.11}$$

$$u_{10} = \frac{\partial u_{10}}{\partial \eta} = 0 \quad \text{on} \quad \Gamma_{10}^{front} \tag{3.12}$$

is equal to zero, in $D_{10} \cap G$, where G is a neighborhood of any point of Γ_{10}^{front}. Since u_{10} is analytic [9], it follows by the unique continuation principle that $u_{10} = 0$ in D_{10}. Similarly by posing the conditions (3.12) on the surface Γ_{10}^{right} we also derive that $u_{11} = 0$ in D_{11} and generally, $u_{ij} = 0$ in D_{ij}, $i = 1, 2, \ldots, N$, $j = 0, 1, \ldots, M$. The proof is, therefore, complete. □

To prove the existence of a unique solution to problem (A), we will employ a combined single-layer and double-layer approach, which makes use of two independent densities ψ_{ij}, ϕ_{ij} for each boundary arc $\Gamma_{(i+1)j}^{front}$, and two for each boundary arc Γ_{ij}^{right}. Therefore, we define

$$u_{ij}(x) = \int_{\partial D_{ij}} \left\{ \frac{\partial \Phi_{ij}(x,y)}{\partial v(y)} \psi_{ij}(y) + \rho_{ij} \Phi_{ij}(x,y) \phi_{ij}(y) \right\} ds(y), \quad x \in D_{ij} \tag{3.13}$$

for $i = 0, 1, \ldots, N$, $j = 0, 1, \ldots, M$, where $\Phi_{ij}(x, y)$ denotes the fundamental solution to the Helmholtz equation, with the wave number k replaced by k_{ij}, and ψ_{ij}, ϕ_{ij} are defined on ∂D_{ij}, $i = 0, 1, \ldots, N$, $j = 0, 1, \ldots, M$. Then, from the jump relations for single-layer and double-layer potentials [9], we have that u_{ij} defines a solution to problem (A) if the densities ψ_{ij}, ϕ_{ij} satisfy the following system, which we will denote by (B):

$$(\rho_{ij} + \rho_{(i+1)j}) \begin{pmatrix} \psi_{ij}^a \\ \phi_{ij}^a \end{pmatrix} - A_{ij}^1 \begin{pmatrix} \psi_{ij}^a \\ \phi_{ij}^a \end{pmatrix} = 2 \begin{pmatrix} f_{ij} \\ -g_{ij} \end{pmatrix} \tag{3.14}$$

for $i = 0, 1, \ldots, N-1$, $j = 0, 1, \ldots, M$,

$$(\rho_{ij} + \rho_{i(j+1)}) \begin{pmatrix} \psi^b_{ij} \\ \phi^b_{ij} \end{pmatrix} - A^2_{ij} \begin{pmatrix} \psi^b_{ij} \\ \phi^b_{ij} \end{pmatrix} = 2 \begin{pmatrix} f^{'}_{ij} \\ -g^{'}_{ij} \end{pmatrix} \tag{3.15}$$

for $i = 1, 2, \ldots, N$, $j = 0, 1, \ldots, M$, and

$$\psi^a_{Nj} - A^3_{ij}\psi^a_{Nj} = 2f_{Nj} \tag{3.16}$$

for $j = 0, 1, \ldots, M$, where

$$A^1_{ij} := \begin{pmatrix} -(\rho_{ij}K^a_{ij} - \rho_{(i+1)j}K^a_{(i+1)j}) & -(\rho^2_{ij}S^a_{ij} - \rho^2_{(i+1)j}S^a_{(i+1)j}) \\ T^a_{ij} - T^a_{(i+1)j} & \rho_{ij}K^{a'}_{ij} - \rho_{(i+1)j}K^{a'}_{(i+1)j} \end{pmatrix}, \tag{3.17}$$

$$A^2_{ij} := \begin{pmatrix} -(\rho_{ij}K^b_{ij} - \rho_{i(j+1)}K^b_{i(j+1)}) & -(\rho^2_{ij}S^b_{ij} - \rho^2_{i(j+1)}S^b_{i(j+1)}) \\ T^b_{ij} - T^b_{i(j+1)} & \rho_{ij}K^{b'}_{ij} - \rho_{i(j+1)}K^{b'}_{i(j+1)} \end{pmatrix}, \tag{3.18}$$

$$A^3_{ij} := -K^a_{Nj} + i\eta_j S^a_{Nj} \tag{3.19}$$

and where ψ^a_{ij}, ϕ^a_{ij} and ψ^b_{ij}, ϕ^b_{ij}, denote the restriction of the corresponding densities ψ_{ij}, ϕ_{ij}, to the boundary arcs $\Gamma^{front}_{(i+1)j}$ and Γ^{right}_{ij}, respectively, where we have set $\phi^a_{Nj} := -i\frac{\eta_j}{\rho_{Nj}}\psi^a_{Nj}$, and where the operators K^a_{ij}, $K^{a'}_{ij}$, S^a_{ij}, T^a_{ij} and K^b_{ij}, $K^{b'}_{ij}$, S^b_{ij}, T^b_{ij} denote the restriction of the corresponding operators, K_{ij}, $K^{'}_{ij}$, S_{ij} and T_{ij}, defined by

$$\left.\begin{aligned} (K_{ij}\psi)(x) :=&\quad 2\int_{\partial D_{ij}} \frac{\partial \Phi_{ij}(x,y)}{\partial \nu(y)}\psi(y)ds(y) \\ (K^{'}_{ij}\psi)(x) :=&\quad 2\int_{\partial D_{ij}} \frac{\partial \Phi_{ij}(x,y)}{\partial \nu(x)}\psi(y)ds(y) \\ (S_{ij}\psi)(x) :=&\quad 2\int_{\partial D_{ij}} \Phi_{ij}(x,y)\psi(y)ds(y) \\ (T_{ij}\psi)(x) :=&\quad 2\frac{\partial}{\partial \nu(x)}\int_{\partial D_{ij}} \frac{\partial \Phi_{ij}(x,y)}{\partial \nu(y)}\psi(y)ds(y) \end{aligned}\right\} x \in \partial D_{ij} \tag{3.20}$$

for $i = 0, 1, \ldots, N$, $j = 0, 1, \ldots, M$, to the boundary parts $\Gamma^{front}_{(i+1)j}$ and Γ^{right}_{ij}, respectively.

We then define the operator

$$A := \begin{pmatrix} A^1 & 0 & 0 \\ 0 & A^2 & 0 \\ 0 & 0 & A^3 \end{pmatrix}, \tag{3.21}$$

where

$$A^1 := \begin{pmatrix} A^1_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A^1_{N-1} \end{pmatrix}, \quad A^1_i := \begin{pmatrix} A^1_{i0} \\ \vdots \\ A^1_{iM} \end{pmatrix}, \tag{3.22}$$

$$A^2 := \begin{pmatrix} A^2_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A^2_N \end{pmatrix}, \quad A^2_i := \begin{pmatrix} A^2_{i0} \\ \vdots \\ A^2_{iM} \end{pmatrix}, \tag{3.23}$$

$$A^3 := \begin{pmatrix} A^3_{N0} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A^3_{NM} \end{pmatrix}. \tag{3.24}$$

Let us further define

$$\chi := \begin{pmatrix} \chi^1 \\ \chi^2 \\ \chi^3 \end{pmatrix}, \tag{3.25}$$

with

$$\chi^1 := \begin{pmatrix} \chi^1_0 \\ \vdots \\ \chi^1_{(N-1)} \end{pmatrix}, \quad \chi^1_i := \begin{pmatrix} \chi^1_{i0} \\ \vdots \\ \chi^1_{iM} \end{pmatrix}, \tag{3.26}$$

$$\chi^2 := \begin{pmatrix} \chi^2_1 \\ \vdots \\ \chi^2_N \end{pmatrix}, \quad \chi^2_i := \begin{pmatrix} \chi^2_{i0} \\ \vdots \\ \chi^2_{iM} \end{pmatrix}, \tag{3.27}$$

$$\chi^3 := \begin{pmatrix} \chi^3_{N0} \\ \vdots \\ \chi^3_{NM} \end{pmatrix}, \tag{3.28}$$

where

$$\chi^1_{ij} := \begin{pmatrix} \psi^a_{ij} \\ \phi^a_{ij} \end{pmatrix}, \tag{3.29}$$

$$\chi^2_{ij} := \begin{pmatrix} \psi^b_{ij} \\ \phi^b_{ij} \end{pmatrix}, \tag{3.30}$$

and

$$\chi^3_{ij} := \psi^a_{ij}. \tag{3.31}$$

In addition, we define the matrix M by (3.21) with $A \leftrightarrow M$, and the matrices M^k and M^k_i by (3.22)–(3.24) with $A \leftrightarrow M$, $k = 1, 2, 3$, where we have set

$$M^1_{ij} := \rho_{ij} + \rho_{(i+1)j}, \tag{3.32}$$

$$M^2_{ij} := \rho_{ij} + \rho_{i(j+1)}, \tag{3.33}$$

$$M^3_{Nj} := 1. \tag{3.34}$$

Finally, we define

$$H := \begin{pmatrix} H^1 \\ H^2 \\ H^3 \end{pmatrix}, \tag{3.35}$$

with

$$H^1 := \begin{pmatrix} H^1_0 \\ \vdots \\ H^1_{(N-1)} \end{pmatrix}, \quad H^1_i := \begin{pmatrix} H^1_{i0} \\ \vdots \\ H^1_{iM} \end{pmatrix}, \tag{3.36}$$

$$H^2 := \begin{pmatrix} H_1^2 \\ \vdots \\ H_N^2 \end{pmatrix}, \quad H_i^2 := \begin{pmatrix} H_{i0}^2 \\ \vdots \\ H_{iM}^2 \end{pmatrix}, \tag{3.37}$$

$$H^3 := \begin{pmatrix} H_{N0}^3 \\ \vdots \\ H_{NM}^3 \end{pmatrix}, \tag{3.38}$$

where

$$H_{ij}^1 := \begin{pmatrix} f_{ij} \\ -g_{ij} \end{pmatrix}, \tag{3.39}$$

$$H_{ij}^2 := \begin{pmatrix} f'_{ij} \\ -g'_{ij} \end{pmatrix}, \tag{3.40}$$

and

$$H_{Nj}^3 := f_{Nj}. \tag{3.41}$$

We can now write (B) in the abbreviated form

$$M\chi - A\chi = 2H. \tag{3.42}$$

From the above definition, it is easily seen that A is a compact operator, since each of its components is compact [9]. Then, we have the following main theorem:

THEOREM 3.2. The transmission problem (A) has a unique solution.

Proof. To prove the theorem, it suffices to show that (3.42) is uniquely solvable. Let χ be a solution to the homogeneous equation $M\chi - A\chi = 0$. Then each u_{ij}, $i = 0, 1, \ldots, N$, $j = 0, 1, \ldots, M$, solves the homogeneous form of problem (A). Hence by Theorem 3.1, we have that $u_{ij} = 0$ in D_{ij}, $i = 0, 1, \ldots, N$, $j = 0, 1, \ldots, M$. We now compute the jumps across $\Gamma_{(i+1)j}^{front}$ and Γ_{ij}^{right}:

$$\left.\begin{aligned} &u_{ij}^+ - u_{ij}^- = -u_{ij}^- = \psi_{ij}^a, \\ &\frac{\partial u_{ij}^+}{\partial \nu} - \frac{\partial u_{ij}^-}{\partial \nu} = -\frac{\partial u_{ij}^-}{\partial \nu} = -\rho_{ij}\phi_{ij}^a \end{aligned}\right\} \text{ on } \Gamma_{(i+1)j}^{front} \tag{3.43}$$

and

$$\left.\begin{aligned} &u_{(i+1)j}^+ - u_{(i+1)j}^- = u_{(i+1)j}^+ = \psi_{ij}^a, \\ &\frac{\partial u_{(i+1)j}^+}{\partial \nu} - \frac{\partial u_{(i+1)j}^-}{\partial \nu} = \frac{\partial u_{(i+1)j}^+}{\partial \nu} = -\rho_{(i+1)j}\phi_{ij}^a \end{aligned}\right\} \text{ on } \Gamma_{(i+1)j}^{front} \tag{3.44}$$

for $i = 0, 1, \ldots, N-1$, $j = 0, 1, \ldots, M$,

$$\left.\begin{aligned} &u_{ij}^r - u_{ij}^l = u_{ij}^r = \psi_{ij}^b, \\ &\frac{\partial u_{ij}^r}{\partial \nu} - \frac{\partial u_{ij}^l}{\partial \nu} = \frac{\partial u_{ij}^r}{\partial \nu} = -\rho_{ij}\phi_{ij}^b \end{aligned}\right\} \text{ on } \Gamma_{ij}^{right} \tag{3.45}$$

and

$$\left.\begin{aligned} & u^r_{i(j+1)} - u^l_{i(j+1)} = -u^l_{i(j+1)} = \psi^b_{ij}, \\ & \frac{\partial u^r_{i(j+1)}}{\partial \nu} - \frac{\partial u^l_{i(j+1)}}{\partial \nu} = -\frac{\partial u^l_{i(j+1)}}{\partial \nu} = -\rho_{i(j+1)}\phi^b_{ij} \end{aligned}\right\} \text{ on } \Gamma^{right}_{ij} \tag{3.46}$$

for $i = 1, 2, \dots, N$, $j = 0, 1, \dots, M$, and

$$\left.\begin{aligned} & u^+_{Nj} - u^-_{Nj} = -u^-_{Nj} = \psi^a_{Nj}, \\ & \frac{\partial u^+_{Nj}}{\partial \nu} - \frac{\partial u^-_{Nj}}{\partial \nu} = -\frac{\partial u^-_{Nj}}{\partial \nu} = i\eta_j \psi^a_{Nj} \end{aligned}\right\} \text{ on } \Gamma^{front}_{(N+1)j} \tag{3.47}$$

for $j = 0, 1, \dots, M$. We note that $u^\pm(x) := \lim_{h\to 0} u(x \pm h\nu(x))$ on Γ^{front} and $u^{r,l}(x) := \lim_{h\to 0} u(x \pm h\nu(x))$ on Γ^{right}, where we have dropped the indices i, j for convenience. By now eliminating ψ^a_{ij}, ϕ^a_{ij}, and ψ^b_{ij}, ϕ^b_{ij} in (3.43)–(3.47), we obtain

$$\left.\begin{aligned} & u^-_{ij} + u^+_{(i+1)j} = 0 \\ & \rho_{(i+1)j}\frac{\partial u^-_{ij}}{\partial \nu} + \rho_{ij}\frac{\partial u^+_{(i+1)j}}{\partial \nu} = 0 \end{aligned}\right\} \text{ on } \Gamma^{front}_{(i+1)j}, \tag{3.48}$$

for $i = 0, 1, \dots, N-1$, $j = 0, 1, \dots, M$, and

$$\left.\begin{aligned} & u^r_{ij} + u^l_{i(j+1)} = 0 \\ & \rho_{i(j+1)}\frac{\partial u^r_{ij}}{\partial \nu} + \rho_{ij}\frac{\partial u^l_{i(j+1)}}{\partial \nu} = 0 \end{aligned}\right\} \text{ on } \Gamma^{right}_{ij}, \tag{3.49}$$

for $i = 1, 2, \dots, N$, $j = 0, 1, \dots, M$.

Let w_{ij} be a solution to the Helmholtz equation $\triangle w_{ij} + k^2_{ij} w_{ij} = 0$, in D_{ij}, for $i = 0, 1, \dots, N$, $j = 0, 1, \dots, M$, satisfying

$$w_{0j} = -u^+_{1j} \text{ on } \Gamma^{front}_{1j} \tag{3.50}$$

for $0, 1, \dots, M$ and

$$w_{ij} = \begin{cases} u^-_{(i-1)j} & \text{on } \Gamma^{front}_{ij} \\ -u^l_{i(j+1)} & \text{on } \Gamma^{right}_{ij} \\ -u^+_{(i+1)j} & \text{on } \Gamma^{front}_{(i+1)j} \\ u^r_{i(j-1)} & \text{on } \Gamma^{right}_{i(j-1)} \end{cases} \tag{3.51}$$

for $i = 1, 2, \dots, N$, $j = 0, 1, \dots, M$, where we have set $u^+_{(N+1)j} := 0$. Then, by conditions (3.48), (3.49), we have that the w_{ij} also satisfy

$$\left.\begin{aligned} & w_{ij} - w_{(i+1)j} = 0 \\ & \rho_{(i+1)j}\frac{\partial w_{ij}}{\partial \nu} - \rho_{ij}\frac{\partial w_{(i+1)j}}{\partial \nu} = 0 \end{aligned}\right\} \text{ on } \Gamma^{front}_{(i+1)j}, \tag{3.52}$$

for $i = 0, 1, \dots, N-1$, $j = 0, 1, \dots, M$,

$$w_{Nj} = 0 \text{ on } \Gamma^{front}_{(N+1)j}, \tag{3.53}$$

for $j = 0, 1, \dots, M$, and

$$\left.\begin{aligned} & w_{ij} - w_{i(j+1)} = 0 \\ & \rho_{i(j+1)}\frac{\partial w_{ij}}{\partial \nu} - \rho_{ij}\frac{\partial w_{i(j+1)}}{\partial \nu} = 0 \end{aligned}\right\} \text{ on } \Gamma^{right}_{ij}, \tag{3.54}$$

for $i = 1, 2, \dots, N$, $j = 0, 1, \dots, M$. Therefore, by Theorem 3.1, we have that each w_{ij} vanishes in D_{ij}, $i = 0, 1, \dots, N$, $j = 0, 1, \dots, M$, and hence, by (3.51), we also have that $u_{ij}^- = u_{(i+1)j}^+ = 0$ on $\Gamma_{(i+1)j}^{front}$, for $i = 0, 1, \dots, N-1$, $j = 0, 1, \dots, M$ and $u_{ij}^r = u_{i(j+1)}^l = 0$ on Γ_{ij}^{right}, $i = 1, 2, \dots, N$, $j = 0, 1, \dots, M$. We thus obtain that each ψ_{ij}^a, ϕ_{ij}^a, vanishes for $i = 0, 1, \dots, N-1$, $j = 0, 1, \dots, M$, and that each ψ_{ij}^b, ϕ_{ij}^b vanishes for $i = 1, 2, \dots, N$, $j = 0, 1, \dots, M$. Therefore, it remains to show that $\psi_{Nj}^a = 0$ for every $j = 0, 1, \dots, M$. To this end, by using conditions (3.47) and applying Green's first theorem, we obtain

$$\begin{aligned}\sum_{j=0}^{M} i\eta_j \int_{\Gamma_{(N+1)j}^{front}} \mid \psi_{Nj}^a(y) \mid^2 ds(y) &= \sum_{j=0}^{M} \int_{\Gamma_{(N+1)j}^{front}} \overline{u_{Nj}^-(y)} \frac{\partial u_{Nj}^-(y)}{\partial \nu(y)} ds \\ &= -k_{Nj}^2 \int_{\Omega_c} \mid u_{Nj} \mid^2 dv + \int_{\Omega_c} \mid \nabla u_{Nj} \mid^2 dv. \qquad (3.55)\end{aligned}$$

By taking imaginary parts in (3.55) and taking into account that η_j is real, we obtain that $\psi_{Nj}^a = 0$ for $j = 0, 1, \dots, M$. We thus have $\chi = 0$, and by the Riesz-Fredholm theory, (3.42) has a unique solution.

□

4. Conclusions. In the preceding sections, we proved the uniqueness and existence theorems for the finite case of the multi-layered, polarly subdivided scatterer. It would, thus, be the subject matter of a future work to establish the aforementioned results for the case of infinite layers and subdivisions, which is of further mathematical and physical interest. The extension of this method to the 3-dimensional case is relatively straightforward; while the calculations are more complex, the main principle remains the same. For example, in the 3-dimensional case, the multi-layered scatterer would be subdivided along the polar and azimuthal angles, forming some six-sided regions, D_{ijl}.

It is interesting to note that this scattering problem is a very general one. That is, it can potentially reduce to a number of direct acoustic scattering problems. For instance, with zero layers, this problem can be reduced to the case of the scattering by an impenetrable obstacle, while, with just one very thin layer, we can approximate an impedance problem by solving a transmission one. Finally, it is straightforward to see that this problem can also reduce to the simple multi-layered scatterer with equal densities on each layer.

From a physical perspective, this method and potential extensions of it have a plethora of applications ranging from space exploration to electroencephalography and brain tumor detection, to oil and metal extraction, to nuke detection to passenger screening, etc.

Of course, this method just treats the direct acoustic problem. The extension of this method to the electromagnetic and elastic problems as well as the corresponding inverse problems would be of further interest in terms of real-world applications.

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