QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXX, NUMBER 1
MARCH 2012, PAGES 159–169
S 0033-569X(2011)01255-0
Article electronically published on August 30, 2011

BLOW-UP CRITERIA FOR A PARABOLIC PROBLEM DUE TO A CONCENTRATED NONLINEAR SOURCE ON A SEMI-INFINITE INTERVAL

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Abstract. Let α, b and T be positive numbers, $D = (0, \infty)$, $\bar{D} = [0, \infty)$, and $\Omega = D \times (0, T]$. This article studies the first initial-boundary value problem,

$$u_t - u_{xx} = \alpha\delta(x - b) f(u(x, t)) \text{ in } \Omega,$$
$$u(x, 0) = \psi(x) \text{ on } \bar{D},$$
$$u(0, t) = 0 = \lim_{x\to\infty} u(x, t) \text{ for } 0 < t \leq T,$$

where $\delta(x)$ is the Dirac delta function, and f and ψ are given functions. We assume that $f(0) \geq 0$, $f(u)$ and its derivatives $f'(u)$ and $f''(u)$ are positive for $u > 0$, and $\psi(x)$ is nontrivial, nonnegative and continuous such that $\psi(0) = 0 = \lim_{x\to\infty} \psi(x)$, and

$$\psi'' + \alpha\delta(x - b) f(\psi) \geq 0 \text{ in } D.$$

It is shown that if u blows up, then it blows up in a finite time at the single point b only. A criterion for u to blow up in a finite time and a criterion for u to exist globally are given. It is also shown that there exists a critical position b^* for the nonlinear source to be placed such that no blowup occurs for $b \leq b^*$, and u blows up in a finite time for $b > b^*$. This also implies that u does not blow up in infinite time. The formula for computing b^* is also derived. For illustrations, two examples are given.

1. Introduction. Let $\alpha, b,$ and T be positive numbers, $D = (0, \infty)$, $\bar{D} = [0, \infty)$, $\Omega = D \times (0, T]$, and $Lu = u_t - u_{xx}$. We consider the following semilinear parabolic first

Received August 26, 2010.
2010 *Mathematics Subject Classification.* Primary 35K60, 35K57.
Key words and phrases. Semilinear parabolic first initial-boundary value problem, semi-infinite interval, concentrated nonlinear source, single blow-up point, critical position, global existence.
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initial-boundary value problem on a semi-infinite interval with a concentrated nonlinear source:

$$\left.\begin{array}{c} Lu = \alpha\delta(x-b)f\left(u(x,t)\right) \text{ in } \Omega, \\ u(x,0) = \psi(x) \text{ on } \bar{D}, \\ u(0,t) = 0 = \lim_{x\to\infty} u(x,t) \text{ for } 0 < t \le T, \end{array}\right\} \tag{1.1}$$

where $\delta(x)$ is the Dirac delta function, and f and ψ are given functions. We assume that $f(0) \ge 0$, $f(u)$ and its derivatives $f'(u)$ and $f''(u)$ are positive for $u > 0$, and $\psi(x)$ is nontrivial, nonnegative and continuous such that $\psi(0) = 0 = \lim_{x\to\infty}\psi(x)$, and

$$\psi'' + \alpha\delta(x-b)f(\psi) \ge 0 \text{ in } D. \tag{1.2}$$

The condition (1.2) is used to show that before u blows up, u is a nondecreasing function of t.

A solution of the problem (1.1) is a continuous function satisfying (1.1). A solution u of the problem (1.1) is said to blow up at the point $(\hat{x}, t_b)$ if there exists a sequence $\{(x_n, t_n)\}$ such that $u(x_n, t_n) \to \infty$ as $(x_n, t_n) \to (\hat{x}, t_b)$.

The position b^* is called the critical position of the nonlinear source if a unique global solution u exists for $b < b^*$ and if the solution u blows up in a finite time for $b > b^*$.

Olmstead and Roberts [6] studied a similar problem on a bounded domain. By analyzing its corresponding nonlinear Volterra equation at the site of the concentrated source, they showed that blowup can be prevented by locating the nonlinear source sufficiently close to the boundary of the domain. Chan and Boonklurb [1] studied the following degenerate semilinear parabolic first initial-boundary value problem with a concentrated source at $b \in (0,1)$ on a bounded domain:

$$\left.\begin{array}{c} x^q w_t - w_{xx} = a\delta(x-b)s\left(w(x,t)\right) \text{ for } 0 < x < 1,\ 0 < t \le T, \\ w(x,0) = w_0(x) \text{ for } 0 \le x \le 1,\ w(0,t) = w(1,t) = 0 \text{ for } 0 < t \le T, \end{array}\right\} \tag{1.3}$$

where a is a positive number which represents the length of the domain, q is a nonnegative number, and $w_0(x)$ and $s(w)$ are given functions. They showed that for a sufficiently large, there exists a unique $b^* \in (0, 1/2)$ such that u never blows up for $b \in (0, b^*] \cup [1-b^*, 1)$, and u always blows up in a finite time for $b \in (b^*, 1-b^*)$. To obtain the formula for calculating b^*, they made use of existence of a solution for the steady-state problem corresponding to the problem (1.3).

Our main purpose here is to find the exact position b^* for the problem (1.1) such that for $b \le b^*$, u exists for $0 \le t < \infty$, and for $b > b^*$, u blows up in a finite time. From this, we can conclude that u does not blow up in infinite time. We also note that the proof does not depend on the existence of a solution for the steady-state problem corresponding to the problem (1.1). A formula for calculating b^* is derived. For illustrations, two examples are given.

Green's function $G(x,t;\xi,\tau)$ corresponding to the problem (1.1) is given by

$$G(x,t;\xi,\tau) = \frac{e^{-\frac{(x-\xi)^2}{4(t-\tau)}} - e^{-\frac{(x+\xi)^2}{4(t-\tau)}}}{\sqrt{4\pi(t-\tau)}} \text{ for } t > \tau$$

(cf. Duffy [4, p. 183]). To derive the integral equation from the problem (1.1), we consider the adjoint operator L^*, which is given by $L^*u = -u_t - u_{xx}$. Using Green's second identity, we obtain

$$u(x,t) = \int_0^\infty G(x,t;\xi,0)\psi(\xi)d\xi + \alpha \int_0^t G(x,t;b,\tau) f\left(u(b,\tau)\right) d\tau. \tag{1.4}$$

For ease of reference, we state Theorems 2.2 and 2.3 of Chan and Treeyaprasert [3] below as Theorem 1.1.

Theorem 1.1. There exists some t_b such that the integral equation (1.4) has a unique nonnegative continuous solution u for $0 \leq t < t_b$, and u is a nondecreasing function of t. If t_b is finite, then u blows up at t_b. Furthermore, u is the solution of the problem (1.1).

2. Single-point blowup and the critical position b^*. We modify the technique used in proving Theorems 2.6 and 3.2 of Chan and Tian [2] for a bounded domain to obtain the following result.

Theorem 2.1. If ψ attains its maximum value at b, then the solution $u(x,t)$ of the problem (1.1) attains its maximum value at (b,θ) for $0 < t \leq \theta < t_b$. If in addition, u blows up, then b is the single blow-up point.

Proof. By Theorem 1.1, $u(b,t)$ is known. Let it be denoted by $g(t)$. Then, the problem (1.1) can be written as follows:

$$\left.\begin{array}{c} Lu = 0 \text{ in } (0,b)\times(0,t_b),\ u(x,0) = \psi(x) \text{ for } 0 \leq x \leq b, \\ u(0,t) = 0 \text{ and } u(b,t) = g(t) \text{ for } 0 < t < t_b, \end{array}\right\} \tag{2.1}$$

$$\left.\begin{array}{c} Lu = 0 \text{ in } (b,\infty)\times(0,t_b),\ u(x,0) = \psi(x) \text{ for } b \leq x < \infty, \\ u(b,t) = g(t) \text{ and } \lim_{x\to\infty} u(x,t) = 0 \text{ for } 0 < t < t_b. \end{array}\right\} \tag{2.2}$$

Since ψ attains its maximum value at b, it follows from the strong maximum principle (cf. Friedman [5, p. 34]) and Theorem 1.1 that the solution $u(x,t)$ of the problem (2.1) attains its maximum value at (b,θ) for $0 < t \leq \theta < t_b$. For the problem (2.2), it follows from the Phragmèn-Lindelöf Principle and Remark (ii) of Protter and Weinberger [7, pp. 183-185] that u attains its maximum value at $x = b$ since $\lim_{x\to\infty} u(x,t) = 0$. We claim that the solution $u(x,t)$ of the problem (2.2) attains its maximum value at (b,θ) for $0 < t \leq \theta < t_b$. To show this, let us assume that $u(x,t)$ attains its maximum value at (r,t) for some positive number $r > b$ and $t \in (0,\theta]$. Let l be a positive number such that $l > r$ and $\psi(x)$ is not identically a constant for $b \leq x \leq l$. By Theorem 1.1, $u(l,t)$ is known. Let us denote it by $q(t)$. We then consider the following problem:

$$\begin{array}{c} Lu = 0 \text{ in } (b,l)\times(0,t_b),\ u(x,0) = \psi(x) \text{ for } b \leq x \leq l, \\ u(b,t) = g(t) \text{ and } u(l,t) = q(t) \text{ for } 0 < t < t_b. \end{array}$$

Since $u(x,t)$ attains its maximum value at (r,t), we have by the strong maximum principle and the continuity of u that $u(x,t) \equiv u(r,t)$ for all $(x,t) \in [b,l]\times[0,t]$, for which we have a contradiction to $\psi(x)$ being not identically a constant for $b \leq x \leq l$. Since r is an arbitrary point in (b,∞), we conclude that $u(x,t)$ of the problem (2.2) attains its maximum value only at (b,θ) for $0 < t \leq \theta < t_b$. Therefore, the solution $u(x,t)$ attains its maximum value at (b,θ) for $0 < t \leq \theta < t_b$ for each of the problems (2.1) and (2.2). Hence, if u blows up, then it blows up at $x = b$.

To show that b is the single blow-up point, let us consider the problem (2.1). By the parabolic version of Hopf's lemma (cf. Friedman [5, p. 49]), $u_x(0,t) > 0$ for any arbitrarily fixed $t \in (0, t_b)$. For any $x \in (0,b)$, $u_{xx} = u_t$, which is nonnegative by Theorem 1.1. Hence, u is concave up. For the problem (2.2), it follows from the parabolic version of Hopf's lemma that for any arbitrarily fixed $t \in (0,t_b)$, $u_x(b,t) < 0$. For any $x \in (b,\infty)$, $u_{xx} = u_t \geq 0$, and hence, u is concave up. Thus if u blows up, then b is the single blow-up point. □

What is going to happen if the initial data ψ does not attain its maximum value at $x = b$? We claim that for any $t > 0$, u attains its maximum value at $x = b$. We prove this by contradiction. Let us assume that there exists some $t_0 > 0$ such that u attains its maximum value at some point $x_0 \neq b$. By Theorem 1.1, u is a nondecreasing function of t. If $x_0 \in (0,b)$, then by using the strong maximum principle and the continuity of u, it follows from the problem (2.1) that $u(x,t) \equiv u(x_0,t_0)$ for all $(x,t) \in [0,b] \times [0,t_0]$, for which we have a contradiction to $\psi(x)$ being not identically a constant for $0 \leq x \leq b$. If $x_0 \in (b,\infty)$, then in the problem (2.2), let ℓ be a positive number such that $\ell > x_0$, and $\psi(x)$ is not identically a constant for $b \leq x \leq \ell$. By Theorem 1.1, $u(\ell,t)$ is known. It follows from the strong maximum principle and the continuity of u that $u(x,t) \equiv u(x_0,t_0)$ for all $(x,t) \in [b,\ell] \times [0,t_0]$, for which we have a contradiction to $\psi(x)$ being not identically a constant for $b \leq x \leq \ell$. This proves our claim. Thus, even if the initial data ψ does not attain its maximum value at $x = b$, b is still the single blow-up point.

Let

$$h(x) = 2kxe^{-kx^2},\ x \geq 0,$$

where k is a positive number. Then, $\int_0^\infty h(x)dx = 1$, $h'(x) = 2ke^{-kx^2}\left(1 - 2kx^2\right)$, and

$$h''(x) \geq -6kh(x). \tag{2.3}$$

Let

$$\mu(t) = \int_0^\infty h(x)u(x,t)\,dx.$$

Modifying the technique in proving Theorem 3.3 of Chan and Tian [2] for a bounded domain, we obtain the following sufficient conditions for u to blow up in a finite time. Chan and Tian made use of the fundamental eigenfunction, which is not available for our unbounded domain. □

THEOREM 2.2. If ψ attains its maximum value at b,

$$\mu(0) > \left(\frac{6k}{\alpha}\right)^{1/(p-1)}, \tag{2.4}$$

$$h(b)\,f(s) \geq s^p, \tag{2.5}$$

where p is a number greater than 1 and $s \in [0,\infty)$, then the solution u of the problem (1.1) blows up in a finite time.

Proof. From Theorem 2.1,

$$\mu(t) < u(b,t)\int_0^\infty h(x)\,dx = u(b,t). \tag{2.6}$$

Multiplying the differential equation in the problem (1.1) by h, and integrating over x from 0 to ∞, we obtain

$$\mu'(t) - \int_0^\infty h(x)\, u_{xx} dx = \alpha h(b)\, f(u(b,t)).$$

Since $h(0) = 0$ and $h(x) \to 0$ as $x \to \infty$, it follows from integration by parts that

$$\mu'(t) + \int_0^\infty h'(x)\, u_x(x,t)\, dx = \alpha h(b)\, f(u(b,t)).$$

Using integration by parts, $u(0,t) = 0$ and $h'(x) \to 0$ as $x \to \infty$, we have

$$\mu'(t) = \alpha h(b)\, f(u(b,t)) + \int_0^\infty h''(x)\, u(x,t)\, dx.$$

From (2.6) and f being a nondecreasing function for $u > 0$,

$$\mu'(t) \geq \alpha h(b)\, f(\mu(t)) + \int_0^\infty h''(x)\, u(x,t)\, dx.$$

By (2.3),

$$\mu'(t) \geq \alpha h(b)\, f(\mu(t)) - 6k\mu(t).$$

By (2.5),

$$\mu'(t) + 6k\mu(t) \geq \alpha \mu^p(t).$$

Solving this Bernoulli inequality, we obtain

$$\mu^{1-p}(t) \leq \frac{\alpha}{6k} + \left(\mu^{1-p}(0) - \frac{\alpha}{6k}\right) e^{6k(p-1)t}.$$

From (2.4), $\mu^{1-p}(0) < \alpha/(6k)$. Thus, μ tends to infinity for some finite t_b. This implies that u blows up at t_b. □

We remark that for any given functions ψ sufficiently large and f sufficiently superlinear, we can choose appropriate values for k and b so that (2.4) and (2.5) hold. For illustration, let $\alpha = 1$, $f(s) = (1+s)^2$, and

$$\psi(x) = \begin{cases} rx \text{ for } 0 \leq x \leq b, \\ rbe^{-(x-b)} \text{ for } b < x < \infty, \end{cases}$$

where r is a positive number. Then

$$\psi'(x) = \begin{cases} r \text{ for } 0 \leq x \leq b, \\ -rbe^{-(x-b)} \text{ for } b < x < \infty, \end{cases}$$

$$\psi''(x) = \begin{cases} 0 \text{ for } 0 \leq x < b, \\ -r(1+b)\,\delta(x-b) \text{ for } x = b, \\ rbe^{-(x-b)} \text{ for } b < x < \infty. \end{cases}$$

The condition (1.2) is satisfied if

$$[-r(1+b) + f(\psi(b))]\, \delta(x-b) \geq 0.$$

A sufficient condition for this to hold is

$$-r(1+b) + f(\psi(b)) \geq 0.$$

The sufficient conditions for u to blow up in a finite time in Theorem 2.2 become

$$\mu(0) > 6k, \tag{2.7}$$

$$2kbe^{-kb^2} \geq \frac{s^2}{(1+s)^2}, \tag{2.8}$$

where we choose $p = 2$. We note that

$$\begin{aligned}
\mu(0) &= \int_0^\infty h(x)u(x,0)\,dx \\
&= \int_0^\infty h(x)\psi(x)\,dx \\
&= \int_0^b h(x)rxdx + \int_b^\infty h(x)rbe^{-(x-b)}dx \\
&= \frac{r\sqrt{\pi}}{2\sqrt{k}}\left[\mathrm{Erf}\left(b\sqrt{k}\right) + be^{b+\frac{1}{4k}}\left(\mathrm{Erf}\left(\frac{1+2bk}{2\sqrt{k}}\right) - 1\right)\right].
\end{aligned}$$

For examples, for $r = 50$, (2.7) and (2.8) are satisfied by choosing $k = 2$ and $b = 0.5$ while for $r = 30$, we can choose $k = 1.5$ and $b = 0.5$.

To find a sufficient condition for u to exist globally, let $l = (x-\xi)/\left(2\sqrt{t}\right)$, and $r = (x+\xi)/\left(2\sqrt{t}\right)$. Then,

$$\begin{aligned}
\int_0^\infty G(x,t;\xi,0)d\xi &= \frac{1}{\sqrt{\pi}}\left(-\int_{\frac{x}{2\sqrt{t}}}^{-\infty} e^{-l^2}dl - \int_{\frac{x}{2\sqrt{t}}}^\infty e^{-r^2}dr\right) \\
&= \frac{1}{\sqrt{\pi}}\left(\int_{-\frac{x}{2\sqrt{t}}}^{\infty} e^{-l^2}dl - \int_{\frac{x}{2\sqrt{t}}}^\infty e^{-r^2}dr\right) \\
&= \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{t}}} e^{-r^2}dr \\
&= \mathrm{Erf}\left(\frac{x}{2\sqrt{t}}\right).
\end{aligned} \tag{2.9}$$

Let k_0 denote $\sup_{x\in\bar{D}}\psi(x)$. We have

$$\int_0^\infty G(x,t;\xi,0)\psi(\xi)d\xi \leq k_0\int_0^\infty G(x,t;\xi,0)\,d\xi = k_0\mathrm{Erf}\left(\frac{x}{2\sqrt{t}}\right) \leq k_0. \tag{2.10}$$

From (1.4),

$$u(b,t) = \int_0^\infty G(b,t;\xi,0)\psi(\xi)d\xi + \alpha\int_0^t G(b,t;b,\tau)f(u(b,\tau))\,d\tau.$$

By Mathematica version 7.01.0,

$$\int_0^t G(b,t;b,\tau)d\tau = b + \left(1 - e^{-b^2/t}\right)\sqrt{\frac{t}{\pi}} - b\mathrm{Erf}\left(\frac{b}{\sqrt{t}}\right),$$

$$\frac{d}{dt}\int_0^t G(b,t;b,\tau)d\tau = \frac{1-e^{-\frac{b^2}{t}}}{2\sqrt{\pi t}} > 0. \tag{2.11}$$

By the L'Hôpital rule,

$$\lim_{t\to\infty}\left(1-e^{-b^2/t}\right)\sqrt{\frac{t}{\pi}}=\lim_{t\to\infty}\frac{2b^2}{\sqrt{\pi t}}e^{-b^2/t}=0.$$

Since $\lim_{t\to\infty}\mathrm{Erf}\big(b/\sqrt{t}\big)=0$, we have

$$\lim_{t\to\infty}\int_0^t G(b,t;b,\tau)d\tau=b. \tag{2.12}$$

Given any positive number $M>k_0$, we would like to choose b such that $u(b,t)\le M$ for all $t>0$. From (1.4), (2.10), (2.11), and (2.12), we have $u(b,t)\le k_0+\alpha f\,(M)\,b$. Thus, if

$$k_0+\alpha f\,(M)\,b\le M, \tag{2.13}$$

then u exists globally. The above discussion gives us the following sufficient condition for global existence of u.

THEOREM 2.3. Given any number $M>k_0$, if (2.13) holds, then u exists for all $t>0$.

The next result shows that the initial condition has no effect on u as $t\to\infty$.

LEMMA 2.4. $\lim_{t\to\infty}\int_0^\infty G(x,t;\xi,0)\psi\,(\xi)\,d\xi=0$.

Proof. From (2.9),

$$\int_0^\infty G(x,t;\xi,0)\psi\,(\xi)\,d\xi\le k_0\int_0^\infty G(x,t;\xi,0)d\xi=k_0\mathrm{Erf}\left(x/\left(2\sqrt{t}\right)\right),$$

which tends to 0 as $t\to\infty$. □

THEOREM 2.5. If $\lim_{t\to\infty}u\,(b,t)<\infty$, then

$$U\,(b)=\alpha bf\,(U\,(b))\,, \tag{2.14}$$

where $U\,(b)$ denotes $\lim_{t\to\infty}u\,(b,t)$. Furthermore, $u\,(b,t)\le U\,(b)$ for $t\in(0,\infty)$.

Proof. From (1.4) and Lemma 2.4,

$$U\,(b)=\lim_{t\to\infty}u\,(b,t)=\lim_{t\to\infty}\alpha\int_0^t G(b,t;b,\tau)f\,(u\,(b,\tau))\,d\tau.$$

We want to show that

$$\lim_{t\to\infty}\alpha\int_0^t G(b,t;b,\tau)f\,(u\,(b,\tau))\,d\tau=\alpha bf\,(U\,(b))\,. \tag{2.15}$$

By using Mathematica version 7.01.0,

$$\int_{\frac{t}{2}}^t G(b,t;b,\tau)d\tau=b+\sqrt{\frac{t}{2\pi}}\left(1-e^{-\frac{2b^2}{t}}\right)-b\mathrm{Erf}\left(\frac{b\sqrt{2}}{\sqrt{t}}\right),$$

which is an increasing function of t since

$$\frac{d}{dt}\left[b+\frac{\sqrt{t}}{\sqrt{2\pi}}\left(1-e^{-\frac{2b^2}{t}}\right)-b\mathrm{Erf}\left(\frac{b\sqrt{2}}{\sqrt{t}}\right)\right]=\frac{1-e^{-\frac{2b^2}{t}}}{2\sqrt{2\pi t}}>0.$$

By the L'Hôpital rule,

$$\lim_{t\to\infty}\sqrt{\frac{t}{2\pi}}\left(1-e^{-\frac{2b^2}{t}}\right)=\lim_{t\to\infty}\frac{2\sqrt{2}b^2e^{-b^2/t}}{\sqrt{\pi t}}=0.$$

Since $\lim_{t\to\infty}\mathrm{Erf}\left(b\sqrt{2/t}\right)=0$, we have

$$\lim_{t\to\infty}\int_{\frac{t}{2}}^{t}G(b,t;b,\tau)d\tau=b.$$

It follows from f being continuous that

$$\lim_{t\to\infty}f\left(u\left(b,t\right)\right)=f\left(\lim_{t\to\infty}u\left(b,t\right)\right)=f\left(U\left(b\right)\right).$$

Thus given any positive number ε, there exists some positive number $\tilde{t}$ such that for $t>\tilde{t}$,

$$0<b-\int_{\frac{t}{2}}^{t}G(b,t;b,\tau)d\tau<\frac{\varepsilon}{2f\left(U\left(b\right)\right)},\tag{2.16}$$

$$0<f\left(U\left(b\right)\right)-f\left(u\left(b,t\right)\right)<\frac{\varepsilon}{2b}.\tag{2.17}$$

Hence,

$$\begin{aligned}
&bf\left(U\left(b\right)\right)-\int_{0}^{t}G(b,t;b,\tau)f\left(u\left(b,\tau\right)\right)d\tau\\
&=bf\left(U\left(b\right)\right)-f\left(U\left(b\right)\right)\int_{0}^{t}G(b,t;b,\tau)d\tau+\int_{0}^{t}G(b,t;b,\tau)\left(f\left(U\left(b\right)\right)-f\left(u\left(b,\tau\right)\right)\right)d\tau\\
&=bf\left(U\left(b\right)\right)-f\left(U\left(b\right)\right)\int_{0}^{\frac{t}{2}}G(b,t;b,\tau)d\tau-f\left(U\left(b\right)\right)\int_{\frac{t}{2}}^{t}G(b,t;b,\tau)d\tau\\
&+\int_{0}^{\frac{t}{2}}G(b,t;b,\tau)\left(f\left(U\left(b\right)\right)-f\left(u\left(b,\tau\right)\right)\right)d\tau+\int_{\frac{t}{2}}^{t}G(b,t;b,\tau)\left(f\left(U\left(b\right)\right)-f\left(u\left(b,\tau\right)\right)\right)d\tau\\
&=bf\left(U\left(b\right)\right)-f\left(U\left(b\right)\right)\int_{\frac{t}{2}}^{t}G(b,t;b,\tau)d\tau-\int_{0}^{\frac{t}{2}}G(b,t;b,\tau)f\left(u\left(b,\tau\right)\right)d\tau\\
&+\int_{\frac{t}{2}}^{t}G(b,t;b,\tau)\left[f\left(U\left(b\right)\right)-f\left(u\left(b,\tau\right)\right)\right]d\tau\\
&\leq f\left(U\left(b\right)\right)\left(b-\int_{\frac{t}{2}}^{t}G(b,t;b,\tau)d\tau\right)+\int_{\frac{t}{2}}^{t}G(b,t;b,\tau)\left(f\left(U\left(b\right)\right)-f\left(u\left(b,\tau\right)\right)\right)d\tau.
\end{aligned}$$

It follows from (2.16), f being an increasing function of τ, and (2.17) that for $t > 2\tilde{t}$,

$$\begin{aligned}
& bf\left(U\left(b\right)\right) - \int_0^t G(b,t;b,\tau) f\left(u\left(b,\tau\right)\right) d\tau \\
& < f\left(U\left(b\right)\right) \frac{\varepsilon}{2f\left(U\left(b\right)\right)} + \left(f\left(U\left(b\right)\right) - f\left(u\left(b,\frac{t}{2}\right)\right)\right) \int_{\frac{t}{2}}^t G(b,t;b,\tau) d\tau \\
& < \frac{\varepsilon}{2} + \frac{\varepsilon}{2b} \int_{\frac{t}{2}}^t G(b,t;b,\tau) d\tau \\
& \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2b} b = \varepsilon.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& bf\left(U\left(b\right)\right) - \int_0^t G(b,t;b,\tau) f\left(u\left(b,\tau\right)\right) d\tau \\
& = f\left(U\left(b\right)\right)\left(b - \int_0^t G(b,t;b,\tau) d\tau\right) + \int_0^t G(b,t;b,\tau)\left(f\left(U\left(b\right)\right) - f\left(u\left(b,\tau\right)\right)\right) d\tau,
\end{aligned}$$

which, for $t > 2\tilde{t}$, is positive by (2.11), (2.12), and (2.17). Hence for $t > 2\tilde{t}$,

$$0 < bf\left(U\left(b\right)\right) - \int_0^t G(b,t;b,\tau) f\left(u\left(b,\tau\right)\right) d\tau < \varepsilon.$$

Since ε is arbitrary, we have (2.15), and the theorem is proved. Since $u\left(b,t\right)$ is a nondecreasing function of t, $u\left(b,t\right) \leq U\left(b\right)$ for $t \geq 0$. □

Let $\phi\left(s\right) = s/f\left(s\right)$ for $0 \leq s < \infty$. If $f\left(s\right) \neq 0$ for $0 \leq s < \infty$, and $\lim_{s\to\infty} \phi\left(s\right) = 0$, then it follows from $\phi'\left(s\right) = \left(f\left(s\right) - sf'\left(s\right)\right)/f^2\left(s\right)$ that the critical value $\hat{s}$ of $\phi\left(s\right)$ is given by $\hat{s} = f\left(\hat{s}\right)/f'\left(\hat{s}\right)$. Since

$$\phi''\left(\hat{s}\right) = -\frac{f\left(\hat{s}\right) f''\left(\hat{s}\right)}{f'\left(\hat{s}\right)\left(f\left(\hat{s}\right)\right)^2} < 0,$$

$\phi\left(s\right)$ attains its relative (namely in this case, absolute) maximum at this critical value $\hat{s}$. Thus, $\phi\left(s\right)$ is strictly increasing for $0 \leq s < \hat{s}$, and strictly decreasing for $s > \hat{s}$. From (2.14), $b = U\left(b\right)/\left(\alpha f\left(U\left(b\right)\right)\right)$, where $\psi\left(b\right) \leq U\left(b\right) < \infty$. Let

$$b^* = \frac{1}{\alpha} \sup_{\psi(b) \leq U(b) < \infty} \frac{U\left(b\right)}{f\left(U\left(b\right)\right)}. \tag{2.18}$$

If $\psi\left(b\right) \leq \hat{s}$, which is finite, then from (2.18),

$$b^* = \frac{1}{\alpha} \frac{\hat{s}}{f\left(\hat{s}\right)}. \tag{2.19}$$

If $\hat{s} < \psi\left(b\right)$, which is finite, then from (2.18),

$$b^* = \frac{1}{\alpha} \frac{\psi\left(b\right)}{f\left(\psi\left(b\right)\right)} \tag{2.20}$$

since $\phi\left(s\right)$ is strictly decreasing for $s > \hat{s}$. For $b \leq b^*$, $U\left(b\right)$ exists, and hence u exists for $0 \leq t < \infty$. For $b > b^*$, it follows from Theorem 2.5 that $U\left(b\right)$ does not exist, and hence, u blows up in a finite time by Theorem 1.1. Thus, we obtain the following result.

THEOREM 2.6. If $f(s) \neq 0$ for $0 \leq s < \infty$, and $\lim_{s\to\infty} \phi(s) = 0$, then for $b \leq b^*$, u exists for all $t > 0$, and for $b > b^*$, u blows up in a finite time.

The next result follows from Theorem 2.6.

COROLLARY 2.7. Under the assumptions of Theorem 2.6, the solution u of the problem (1.1) does not blow up in infinite time.

We remark that if $f(u) = u^p$, where p is a real number greater than 1, then for $s > 0$, $d\phi(s)/ds = (1-p)/s^p < 0$. Since $\psi(b) > 0$, it follows from (2.18) that

$$b^* = \frac{\psi(b)}{\alpha f(\psi(b))},$$

and u does not blow up in infinite time.

For illustrations, we give below two examples on finding b^* for some given functions f and $\psi(b)$.

EXAMPLE 2.1. Let us consider the problem (1.1) with $f(u) = (1+u)^p$, where p is a positive number greater than 1. Since

$$\frac{d}{ds}\left(\frac{s}{(1+s)^p}\right) = \frac{(1+s)^{p-1}(1+s-ps)}{(1+s)^{p^2}},$$

the critical value is given by

$$\hat{s} = \frac{1}{p-1}.$$

If $\psi(b) \leq \hat{s}$, then from (2.19),

$$b^* = \frac{(p-1)^{p-1}}{\alpha p^p}.$$

If $\psi(b) > \hat{s}$, then b^* is given by (2.20).

EXAMPLE 2.2. Let us consider the problem (1.1) with $f(u) = \rho e^u$, where ρ is a positive number. Since

$$\frac{d}{ds}\left(\frac{s}{\rho e^s}\right) = \frac{e^s(1-s)}{\rho e^{2s}},$$

the critical value is given by $\hat{s} = 1$. If $\psi(b) \leq 1$, then from (2.19),

$$b^* = \frac{1}{\alpha\rho e}.$$

If $\psi(b) > 1$, then b^* is given by (2.20).

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