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SUITABLE WEAK SOLUTIONS AND LOW STRATIFICATION SINGULAR LIMIT FOR A FLUID PARTICLE INTERACTION MODEL

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Dedicated to Constantine M. Dafermos on the occasion of his 70th birthday

Abstract. This article deals with a fluid-particle interaction model for the evolution of particles dispersed in a viscous compressible fluid within the physical space $\Omega \subset \mathbb{R}^3$. The system is expressed by the continuity equation, the balance of momentum and the so-called Smoluchowski equation for the evolution of particles. The coupling between the dispersed and dense phases is obtained through the drag forces that the fluid and the particles exert mutually by the action-reaction principle. Using the relative entropy method of Dafermos and DiPerna, the global-in-time existence of *suitable weak solutions* is presented under reasonable physical assumptions on the initial data, the physical domain, and the external potential. In addition, the low Mach number and low stratification limits of the system are established rigorously for both bounded and unbounded domains.

1. Introduction.

Fluid-particle interactions arise in many practical applications in science and engineering [1, 2, 5, 6, 7, 31, 33, 34, 35]. The state of such flows is characterized by the macroscopic variables: the total mass density $\varrho(t,x)$, the velocity field $\mathbf{u}(t,x)$, and the density of particles dispersed in the mixture $\eta(t,x)$, which depend on the

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Eulerian spatial coordinate $x \in \Omega \subset \mathbb{R}^3$ and on time $t \in (0,T)$. The governing equations express the conservation of mass, the balance of momentum, and the balance of particle densities often referred to as the *Smoluchowski equation*:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(p(\varrho) + \eta) - \mu \Delta \mathbf{u} - \lambda \nabla_x \mathrm{div}_x \mathbf{u} = -(\eta + \beta \varrho)\nabla_x \Phi, \tag{1.2}$$

$$\partial_t \eta + \mathrm{div}_x(\eta(\mathbf{u} - \nabla_x \Phi)) - \Delta \eta = 0. \tag{1.3}$$

Here, p denotes the pressure $p(\varrho) = a\varrho^\gamma$, $a > 0$, $\gamma > 1$, $\beta \neq 0$, and Φ denotes the external potential (typically incorporating gravity and buoyancy).

In this paper, we consider potentials that satisfy suitable confinement conditions (**HC**) (see Section 2). The total pressure $P = P(\varrho, \eta)$ in the mixture depends on the density of the particles and the density of the fluid and is given by

$$P(\varrho, \eta) = p(\varrho) + \eta.$$

The viscosity parameters $\mu > 0$ and $\lambda + \frac{2}{3}\mu \geq 0$ are nonnegative constants, while $\beta > 0$ if Ω is unbounded.

We impose the no-slip boundary condition for the velocity vector leading to no flux for the fluid density through the boundaries and the no-flux condition for the particle density

$$\mathbf{u}|_{\partial\Omega} = \nabla_x \eta \cdot \nu + \eta \nabla_x \Phi \cdot \nu = 0 \ \text{ on } \ (0,T) \times \partial\Omega, \tag{1.4}$$

with ν denoting the outer normal vector to the boundary $\partial\Omega$. Our problem is supplemented with the initial data $\{\varrho_0, \mathbf{m}_0, \eta_0\}$ such that

$$\begin{aligned} \varrho(0,x) &= \varrho_0 \in L^\gamma(\Omega) \cap L^1_+(\Omega), \\ (\varrho \mathbf{u})(0,x) &= \mathbf{m}_0 \in L^{\frac{6}{5}}(\Omega) \cap L^1(\Omega), \\ \eta(0,x) &= \eta_0 \in L^2(\Omega) \cap L^1_+(\Omega). \end{aligned} \tag{1.5}$$

Motivated by the stability arguments in [8], the numerical investigation presented in [9], the existence and asymptotic results presented in [10] as well as a number of studies on numerical experiments and scale analysis on the proposed model (see [4]), we investigate the issues of regularity and singular limits for fluid-particle interaction flows providing a rigorous mathematical theory based on the principle of balance laws. The total energy of the mixture is given by

$$E(\eta, \varrho, \mathbf{u})(t) := \int_\Omega \left[\frac{1}{2} \varrho(t) |\mathbf{u}(t)|^2 + \frac{a}{\gamma - 1} \varrho^\gamma(t) + (\eta \log \eta)(t) + (\beta \varrho + \eta)(t)\Phi \right] \mathrm{d}x. \tag{1.6}$$

At the formal level, the total energy can be viewed as a Lyapunov function satisfying the *energy inequality*

$$\frac{dE}{dt} + \int_\Omega \left[\mu |\nabla_x \mathbf{u}|^2 + \lambda |\mathrm{div}_x \mathbf{u}|^2 + |2\nabla_x \sqrt{\eta} + \sqrt{\eta} \nabla_x \Phi|^2 \right] \mathrm{d}x \leq 0. \tag{1.7}$$

The results and ingredients of our approach can be formulated as follows:

- First a *variational framework* of the underlying physical principles is presented followed by a description of results on the global existence of *free energy solutions* and their long time asymptotics. For the details we refer the reader to the manuscript by Carrillo, Karper and Trivisa [10].

- An inherent definition of *suitable weak solutions* to the Navier-Stokes Smoluchowski system (1.1)–(1.3) is introduced satisfying a relative entropy inequality with respect to any hypothetical strong solution to the problem. Our analysis is motivated by the pioneering work of Dafermos [11] and DiPerna [13], the results of Germain [19], the analysis of Mellet and Vasseur [27] as well as the approach of Feireisl et al. [18]. This analysis shows that the relative entropy method is essential in establishing conditional regularity of the solutions to models of compressible fluids. For the details we refer the reader to the manuscript by Ballew and Trivisa [3].
- Next, results on the low Mach number and low stratification limits are established. Our strategy can be summarized as follows. First, using the *free energy inequality*, uniform bounds in ε are obtained. Using these bounds convergence results on some relevant quantities are established. In order to establish rigorously the limits of the sequence of solution $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}_{\varepsilon>0}$ we need to show that the divergence of the term $\overline{\varrho}\overline{\mathbf{u}} \otimes \overline{\mathbf{u}} - \overline{\varrho\mathbf{u} \otimes \mathbf{u}}$ converges weakly to a gradient. This can be established by employing the standard Helmholtz decomposition in order to decompose this quantity into a divergence-free and a gradient part.
- Physically grounded hypotheses are imposed on the domain Ω and the external potential Φ (confinement hypotheses **(HC)**). The analysis in the present article treats both the case of a *bounded* physical domain Ω as well as the case of an *unbounded* domain. The confinement hypothesis **(HC)** on (Ω, Φ) plays a crucial role in providing control of the negative contribution of the physical entropy $\eta \log \eta$ in the free-energy bounds for unbounded domains.

We remark that system (1.1)–(1.3) was derived by formal asymptotics from a mesoscopic description [8]. This is based on a kinetic equation for the particle distribution of Fokker-Planck type coupled to fluid equations. In this scaling limit, particles are supposed to have a negligible density with respect to the fluid, and thus, due to buoyancy effects, they typically move upwards in a system under gravity. For that reason, this scaling regime is known as the *bubbling* regime.

The coupling between the kinetic and the fluid equations is obtained through the friction forces that the fluid and the particles exert mutually. The friction force is assumed to follow the Stokes law and thus is proportional to the relative velocity vector, namely

$$F_\varepsilon = \int_{\mathbb{R}^3} \left(\frac{\xi}{\sqrt{\varepsilon}} - u_\varepsilon(t,x) \right) f(t,x,\xi) \,\mathrm{d}\xi.$$

This forcing term affects the momentum equation in the Navier-Stokes system which is now enhanced by an additional forcing term taking into account the action of the cloud of particles on the fluid. The cloud of particles is described by its distribution function $f_\varepsilon(t,x,\xi)$ on phase space, which is the solution to the dimensionless Vlasov-Fokker-Planck equation

$$\partial_t f_\varepsilon + \frac{1}{\sqrt{\varepsilon}} \Big(\xi \cdot \nabla_x f_\varepsilon - \nabla_x \Phi \cdot \nabla_\xi f_\varepsilon \Big) = \frac{1}{\varepsilon} \operatorname{div}_\xi \Big(\big(\xi - \sqrt{\varepsilon} u_\varepsilon \big) f + \nabla_\xi f_\varepsilon \Big). \tag{1.8}$$

Here, $\varepsilon > 0$ is a dimensionless parameter and we have a drag force independent of the fluid density ϱ_ε, but proportional to the relative velocity of the fluid and the particles.

The paper is organized as follows. In Section 2 we collect all the necessary hypotheses imposed on the external potential (confinement hypotheses **(HC)**) and we discuss their consequence. In Section 3 we present the notion of *free energy solutions* and results on the global existence and asymptotic analysis of these solutions. In Section 4 the notion of *suitable weak solutions* is introduced, which is based on a *relative entropy inequality*. In Section 5 the scaled Navier-Stokes-Smoluchowski system is derived and a formal derivation of the low stratification limit is presented. In Section 6 the main results are stated for both bounded and unbounded domains $\Omega \in \mathbb{R}^3$. In Section 7 the rigorous derivation of the low Mach number and low stratification limits for the scaled Navier-Stokes-Smoluchowski system are established. In the heart of the analysis lies the introduction of a *relative entropy-type functional* and the use of the standard Helmholtz decomposition.

2. The confinement hypothesis. In this work, we analyze the existence of suitable weak solutions to the two-phase flow problem (1.1)–(1.3), as well as the rigorous low Mach number and low stratification limit of this system in two different geometrical constraints of interest in the applications: for bounded domains and for unbounded domains under confinement conditions due to the external potential. We will collect all assumptions concerning the geometry Ω and the external potential Φ under the generic name of *confinement conditions*. Let us remark that the external potential Φ is always defined up to a constant. Therefore, for bounded from below external potentials Φ, we can always assume without loss of generality, by adding a suitable constant, that

$$\inf_{x\in\Omega} \Phi(x) = 0. \tag{2.9}$$

Definition 2.1. Given a domain $\Omega \in C^{2,\nu}$, $\nu > 0$, $\Omega \subset \mathbb{R}^3$, and given a bounded from below external potential $\Phi : \Omega \longrightarrow \mathbb{R}_0^+$ satisfying (2.9), we will say that (Ω, Φ) verifies the confinement hypotheses **(HC)** for the two-phase flow system (1.1)–(1.3) coupled with no-flux boundary conditions (1.4) whenever:

(HC-Bounded) If Ω is bounded, Φ is bounded and Lipschitz continuous in $\bar{\Omega}$ and the sub-level sets $[\Phi < k]$ are connected in Ω for any $k > 0$.

(HC-Unbounded) If Ω is unbounded, we assume that $\Phi \in W^{1,\infty}_{loc}(\Omega)$, $\beta > 0$, the sub-level sets $[\Phi < k]$ are connected in Ω for any $k > 0$,

$$e^{-\Phi/2} \in L^1(\Omega),$$

and

$$|\Delta\Phi(x)| \leq c_1 |\nabla_x \Phi(x)| \leq c_2 \Phi(x), \; |x| > R, \tag{2.10}$$

for some large $R > 0$.

Remark 2.1. The confinement assumption **(HC)** has physical relevance in our setting as it is verified for several domains Ω with Φ being the gravitational potential. For instance,

(1) when $\Omega = \{x \in \mathbb{R}^3 \,|\, (x_1, x_2) \in [a, b]^2, \, x_3 \in [0, H]\}$ and $\Phi(x) = g x_3$, where $\beta = 1 - \frac{\varrho_F}{\varrho_P}$;

(2) when $\Omega = \{x \in \mathbb{R}^3 \,|\, (x_1, x_2) \in [a, b]^2, \, x_3 > 0\}$ and $\Phi(x) = g x_3$, where $\beta = 1 - \frac{\varrho_F}{\varrho_P}$ and $\varrho_F < \varrho_P$;

(3) when $\Omega = \mathbb{R}^3 \setminus \overline{B(0,R)}$ and $\Phi(x) = g|x|$, where $B(0,R)$ is the ball centered at the origin with radius R and $\beta > 0$.

Here, ϱ_F and ϱ_P are the typical mass density of fluid and particles, respectively. Note that (1) corresponds to the standard bubbling case (see [8]) in which particles move upwards due to buoyancy.

2.1. *Consequences of confinement: Ω unbounded.* One of the *key* issues in this context is providing a control for the negative contribution of the physical entropy $\eta \log \eta$ in the free-energy bounds for unbounded domains Ω. Here, the confinement conditions **(HC)** on (Ω, Φ) are crucial. Most of these lemmas can be seen in [10], [14] but we include them here for the sake of completeness. We first start with a classical lemma in kinetic theory.

LEMMA 2.1. Assume that (Ω, Φ) satisfy the hypotheses **(HC)**. For any density $\eta \in L^1_+(\Omega)$,

$$\int_\Omega \eta(x) \log_- \eta(x) \,\mathrm{d}x \le \frac{1}{2} \int_\Omega \Phi(x)\eta(x)\,\mathrm{d}x + \frac{1}{e}\int_\Omega e^{-\Phi(x)/2}\,\mathrm{d}x\,.$$

Proof. Let $\bar\eta := \eta\,\chi_{\{\eta\le 1\}}$ and $\bar M = \int_\Omega \bar\eta(x)\,\mathrm{d}x \le \int_\Omega \eta(x)\,\mathrm{d}x = M$. Then

$$\int_\Omega \bar\eta(x)\left(\log\bar\eta(x) + \frac{1}{2}\Phi(x)\right)\mathrm{d}x = \int_\Omega [U(x)\log U(x)]\mu\,\mathrm{d}x - \bar M \log Y,$$

where $U := \bar\eta/\mu$, $\mu(x) = e^{-\Phi(x)/2}/Y$ with $Y = \int_\Omega e^{-\Phi(x)/2}\,\mathrm{d}x$. The Jensen inequality yields

$$\int_\Omega [U(x)\,\log U(x)]\mu\,\mathrm{d}x \ge \left(\int_\Omega U(x)\mu\,\mathrm{d}x\right)\,\log\left(\int_\Omega U(x)\mu\,\mathrm{d}x\right) = \bar M\,\log\bar M$$

and

$$\begin{aligned} -\int_\Omega \eta(x)\,\log_-\eta(x)\,\mathrm{d}x = \int_\Omega \bar\eta(x)\,\log\bar\eta(x)\,\mathrm{d}x &\ge \bar M\log\bar M - \bar M\log Y - \frac{1}{2}\int_\Omega \Phi(x)\,\bar\eta(x)\,\mathrm{d}x \\ &\ge -\frac{Y}{e} - \frac{1}{2}\int_\Omega \Phi(x)\,\eta(x)\,\mathrm{d}x\,, \end{aligned}$$

from which the desired claim follows. □

We can immediately use this previous lemma to conclude the following consequence.

COROLLARY 2.1. Assume that (Ω, Φ) satisfy the hypotheses **(HC)**. For any density $\eta \in L^1_+(\Omega)$, if

$$\int_\Omega \eta(x)\,\log\eta(x)\,\mathrm{d}x + \int_\Omega \Phi(x)\eta(x)\,\mathrm{d}x \le C\,,$$

then $\eta\log\eta \in L^1(\Omega)$ and there exists $D > 0$ depending on C and Φ such that

$$\int_\Omega \eta(x)\,\log_+\eta(x)\,\mathrm{d}x \le D \qquad \text{and} \qquad \int_\Omega \Phi(x)\eta(x)\,\mathrm{d}x \le D\,.$$

Finally, the above estimates can be used to control the mass of the densities η outside a large ball to avoid loss of mass at infinity.

Lemma 2.2. Given any domain Ω such that $e^{-\Phi} \in L^1_+(\Omega)$ and any density $\eta \in L^1_+(\Omega)$, then

$$\int_\Omega \eta(x) \log \eta(x) \,\mathrm{d}x + \int_\Omega \Phi(x)\eta(x) \,\mathrm{d}x \geq \int_\Omega \eta(x) \,\mathrm{d}x \log \left(\frac{\int_\Omega \eta(x) \,\mathrm{d}x}{\int_\Omega e^{-\Phi(x)} \,\mathrm{d}x} \right).$$

As a consequence, if $e^{-\Phi} \in L^1_+(\Omega)$ and

$$\int_\Omega \eta(x) \log \eta(x) \,\mathrm{d}x + \int_\Omega \Phi(x)\eta(x) \,\mathrm{d}x \leq C,$$

then, for any $\epsilon > 0$ there exists $R > 0$ depending on C and Φ only such that

$$\int_{\Omega \cap (\mathbb{R}^3/B(0,R))} \eta(x) \,\mathrm{d}x < \epsilon.$$

Proof. A direct use of Jensen's inequality shows the first inequality by using the convexity of $x \mapsto x \log x$. To show the second claim we start by applying the first inequality to the domain $\Omega_R^c := \Omega \cap (\mathbb{R}^3/B(0,R))$ from which we obtain

$$\int_{\Omega_R^c} \eta(x) \,\mathrm{d}x \log \left(\frac{\int_{\Omega_R^c} \eta(x) \,\mathrm{d}x}{\int_{\Omega_R^c} e^{-\Phi(x)} \,\mathrm{d}x} \right) \leq D \tag{2.11}$$

for some $D > 0$, where Lemma 2.1 and Corollary 2.1 were used. Now, we argue by contradiction; if the second claim were not true, we would have

$$\exists \epsilon_0 > 0 \; \forall R_0 > 0 \; \exists R > R_0 \text{ such that } \int_{\Omega_R^c} \eta(x) \,\mathrm{d}x \geq \epsilon_0.$$

Since $e^{-\Phi} \in L^1_+(\Omega)$, we can always assume that R_0 is large such that

$$\int_{\Omega_R^c} e^{-\Phi(x)} \,\mathrm{d}x \leq \int_{\Omega_{R_0}^c} e^{-\Phi(x)} \,\mathrm{d}x < \epsilon_0 \leq \int_{\Omega_R^c} \eta(x) \,\mathrm{d}x$$

and thus due to (2.11),

$$\int_{\Omega_R^c} \eta(x) \,\mathrm{d}x \leq \int_{\Omega_R^c} e^{-\Phi(x)} \,\mathrm{d}x \, e^{D/\epsilon_0} \leq \int_{\Omega_{R_0}^c} e^{-\Phi(x)} \,\mathrm{d}x \, e^{D/\epsilon_0}.$$

This leads to a contradiction since the right-hand side can be made arbitrarily small by taking R_0 large enough. □

3. Free-energy solutions: Global-in-time existence.

We start with the notion of a free-energy solution to the two-phase flow system (1.1)–(1.3) that we will be dealing with.

Definition 3.1. Let us assume that (Ω, Φ) satisfy the confinement hypotheses **(HC)**. We say that $\{\varrho, \mathbf{u}, \eta\}$ is a free-energy solution of problem (1.1)–(1.3) supplemented with boundary data for which (1.4) holds and initial data $\{\varrho_0, \mathbf{m}_0, \eta_0\}$ satisfying (1.5) provided that the following hold:

- $\varrho \geq 0$ represents a renormalized solution of equation (1.1) on $(0,\infty) \times \Omega$: For any test function $\varphi \in \mathcal{D}([0,T) \times \overline{\Omega})$, any $T > 0$, and any b such that

$$b \in L^\infty \cap C[0,\infty), \; B(\varrho) = B(1) + \int_1^\varrho \frac{b(z)}{z^2} \,\mathrm{d}z,$$

the following integral identity holds:

$$\int_0^\infty \int_\Omega \Big(B(\varrho)\partial_t \varphi + B(\varrho)\mathbf{u}\cdot\nabla_x\varphi - b(\varrho)\mathrm{div}_x\mathbf{u}\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t = -\int_\Omega B(\varrho_0)\varphi(0,\cdot)\,\mathrm{d}x. \quad (3.12)$$

- The balance of momentum holds in the distributional sense, namely

$$\int_0^\infty \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\mathbf{v} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\mathbf{v} + (p(\varrho)+\eta)\ \mathrm{div}_x\mathbf{v}\Big)\,\mathrm{d}x\ \mathrm{d}t \quad (3.13)$$

$$= \int_0^\infty \int_\Omega (\mu\nabla_x\mathbf{u}\nabla_x\mathbf{v} + \lambda\mathrm{div}_x\mathbf{u}\mathrm{div}_x\mathbf{v} - (\eta+\beta\varrho)\nabla_x\Phi\cdot\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \mathbf{m}_0\cdot\mathbf{v}(0,\cdot)\,\mathrm{d}x,$$

for any test function $\mathbf{v}\in\mathcal{D}([0,T);\mathcal{D}(\overline{\Omega};\mathbb{R}^3))$ and any $T>0$ satisfying $\vec{\varphi}|_{\partial\Omega}=0$.

All quantities appearing in (3.13) are supposed to be at least integrable. In particular, the velocity field $\mathbf{u}$ belongs to the space $L^2(0,T;W^{1,2}(\Omega;R^3))$; therefore it is legitimate to require $\mathbf{u}$ to satisfy the boundary conditions (1.4) in the sense of traces.

- $\eta\geq 0$ is a weak solution of (1.3). That is, the integral identity

$$\int_0^\infty\int_\Omega \eta\partial_t\varphi + \eta\mathbf{u}\cdot\nabla_x\varphi - \eta\nabla_x\Phi\cdot\nabla_x\varphi - \nabla_x\eta\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t = -\int_\Omega \eta_0\varphi(0,\cdot)\ \mathrm{d}x \quad (3.14)$$

is satisfied for test functions $\varphi\in\mathcal{D}([0,T)\times\bar{\Omega})$ and any $T>0$.

All quantities appearing in (3.14) must be at least integrable on $(0,T)\times\Omega$. In particular, η belongs to $L^2([0,T];L^3(\Omega))\cap L^1(0,T;W^{1,\frac{3}{2}}(\Omega))$.

- Given the total free-energy of the system by

$$E(\varrho,\mathbf{u},\eta)(t) := \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma + \eta\log\eta + (\beta\varrho+\eta)\Phi\right)\ \mathrm{d}x,$$

then $E(\varrho,\mathbf{u},\eta)(t)$ is finite and bounded by the initial energy of the system, i.e., $E(\varrho,\mathbf{u},\eta)(t)\leq E(\varrho_0,\mathbf{u}_0,\eta_0)$ a.e. $t>0$. Moreover, the following free energy-dissipation inequality holds:

$$\int_0^\infty\int_\Omega \left(\mu|\nabla_x\mathbf{u}|^2 + \lambda|\mathrm{div}_x\mathbf{u}|^2 + |2\nabla_x\sqrt{\eta}+\sqrt{\eta}\nabla_x\Phi|^2\right)\ \mathrm{d}x\,\mathrm{d}t \leq E(\varrho_0,\mathbf{u}_0,\eta_0). \quad (3.15)$$

The global existence and asymptotic analysis of the solutions are presented below.

Theorem 3.1 (Global existence). Let us assume that (Ω,Φ) satisfy the confinement hypotheses **(HC)**. Then, the problem (1.1)–(1.3) supplemented with boundary conditions (1.4) and initial data satisfying (1.5) admits a weak solution $\{\varrho,\mathbf{u},\eta\}$ on $(0,\infty)\times\Omega$ in the sense of Definition 3.1. In addition,

i) the total fluid mass and particle mass given by

$$M_\varrho(t) = \int_\Omega \varrho(t,\cdot)\ dx \qquad \text{and} \qquad M_\eta(t) = \int_\Omega \eta(t,\cdot)\ dx,$$

respectively, are constants of motion;

ii) the density satisfies the higher integrability result

$$\varrho\in L^{\gamma+\Theta}((0,T)\times\Omega),\ \text{for any } T>0,$$

where $\Theta=\min\{\frac{2}{3}\gamma-1,\frac{1}{4}\}$.

Proof. For the proof we refer the reader to the article [10]. □

We can now completely characterize the large-time behavior of free-energy solutions to (1.1)–(1.6).

THEOREM 3.2 (Large-time asymptotics). Let us assume that (Ω, Φ) satisfy the confinement hypotheses **(HC)**. Then, for any free-energy solution $(\varrho, \mathbf{u}, \eta)$ of the problem (1.1)–(1.3), in the sense of Definition 3.1, there exist universal stationary states $\varrho_s(x)$, $\eta_s(x)$, such that

$$\begin{cases} \varrho(t) \to \varrho_s \text{ strongly in } L^\gamma(\Omega), \\ \\ \operatorname*{ess\,sup}_{\tau>t} \displaystyle\int_\Omega \varrho(\tau)|\mathbf{u}(\tau)|^2 \,\mathrm{d}x \to 0, \\ \\ \eta(t) \to \eta_s \text{ strongly in } L^{p_2}(\Omega) \text{ for } p_2 > 1, \end{cases}$$

as $t \to \infty$, where (η_s, ϱ_s) are characterized as the unique free-energy solution of the stationary state problem:

$$\begin{cases} \nabla_x p(\varrho_s) = -\beta\varrho_s \nabla_x \Phi, \\ \nabla_x \eta_s = -\eta_s \nabla_x \Phi, \end{cases} \qquad \int_\Omega \varrho_s \,\mathrm{d}x = \int_\Omega \varrho_0 \,\mathrm{d}x, \quad \int_\Omega \eta_s \,\mathrm{d}x = \int_\Omega \eta_0 \,\mathrm{d}x, \tag{3.16}$$

given by the formulas

$$\varrho_s = \left(\frac{\gamma - 1}{a\gamma} [-\beta\Phi + C_\varrho]^+ \right)^{\frac{1}{\gamma-1}}, \qquad \eta_s = C_\eta \exp(-\Phi),$$

where C_η and C_ϱ are uniquely given by the initial masses due to (3.16).

Proof. For the proof we refer the reader to the article [10]. ☐

4. Suitable weak solutions: Global-in-time existence. In the spirit of Dafermos [11], given an entropy $\mathcal{E}(U)$ we can define the relative entropy by

$$\mathcal{H}(U|\overline{U}) := \mathcal{E}(U) - \mathcal{E}(\overline{U}) - D\mathcal{E}(\overline{U}) \cdot (U - \overline{U}), \tag{4.17}$$

where D stands for the total differentiation operator with respect to $\varrho, \mathbf{m},$ and η. In the present context,

$$U = \begin{bmatrix} \varrho \\ \mathbf{m} := \varrho\mathbf{u} \\ \eta \end{bmatrix},$$

$$\overline{U} = \begin{bmatrix} r \\ \overline{\mathbf{m}} := r\mathbf{U} \\ s \end{bmatrix}$$

and

$$\mathcal{E}(U) := \frac{|\mathbf{m}|^2}{2\varrho} + \frac{a}{\gamma - 1}\varrho^\gamma + \eta \ln \eta + (\beta\varrho + \eta)\Phi. \tag{4.18}$$

Thus, from the definition, the relative entropy is

$$\begin{aligned}
\mathcal{H}(U|\overline{U}) &= \frac{|\mathbf{m}|^2}{2\varrho} + \frac{a}{\gamma-1}\varrho^\gamma + \eta\ln\eta + (\beta\varrho+\eta)\Phi \\
&- \frac{|\overline{\mathbf{m}}|^2}{2r} - \frac{a}{\gamma-1}r^\gamma - s\ln s - (\beta r + s)\Phi \\
&- \begin{bmatrix} -\frac{|\mathbf{U}|^2}{2} + \frac{a\gamma}{\gamma-1}r^{\gamma-1} + \beta\Phi \\ \mathbf{U} \\ \ln s + 1 + \Phi \end{bmatrix} \cdot \begin{bmatrix} \varrho - r \\ \varrho\mathbf{u} - r\mathbf{U} \\ \eta - s \end{bmatrix} \\
&= \frac{\varrho|\mathbf{u}|^2}{2} + \frac{a}{\gamma-1}\varrho^\gamma + \eta\ln\eta + \beta\varrho\Phi + \eta\Phi \\
&- \frac{r|\mathbf{U}|^2}{2} - \frac{a}{\gamma-1}r^\gamma - s\ln s - \beta r\Phi - s\Phi \\
&+ \frac{\varrho|\mathbf{U}|^2}{2} - \frac{r|\mathbf{U}|^2}{2} - \frac{a\gamma}{\gamma-1}r^{\gamma-1}\varrho + \frac{a\gamma}{\gamma-1}r^\gamma - \beta\varrho\Phi + \beta r\Phi \\
&- \varrho\mathbf{u}\cdot\mathbf{U} + r|\mathbf{U}|^2 - \eta\ln s + s\ln s - \eta + s - \eta\Phi + s\Phi.
\end{aligned} \tag{4.19}$$

After some basic calculations, the relative entropy is calculated to be

$$\mathcal{H}(U|\overline{U}) = \frac{\varrho}{2}|\mathbf{u}-\mathbf{U}|^2 + \frac{a}{\gamma-1}(\varrho^\gamma - r^\gamma) - \frac{a\gamma}{\gamma-1}r^{\gamma-1}(\varrho-r) + \eta\ln\eta - s\ln s - (\ln s+1)(\eta-s), \tag{4.20}$$

or equivalently,

$$\mathcal{H}(U|\overline{U}) = \frac{\varrho}{2}|\mathbf{u}-\mathbf{U}|^2 + E_F(\varrho, r) + E_P(\eta, s),$$

where

$$\begin{aligned}
&E_F(\varrho, r) := H_F(\varrho) - H_F'(r)(\varrho - r) - H_F(r), \\
&E_P(\eta, s) := H_P(\eta) - H_P'(s)(\eta - s) - H_P(s), \\
&H_F(\varrho) := \frac{a}{\gamma-1}\varrho^\gamma, \\
&H_P(\eta) := \eta\log\eta, \\
&P_F = H_F', P_P = H_P'.
\end{aligned}$$

Note that the relative entropy does not contain any information regarding the external potential Φ. This is expected, as the relative entropy reflects information about quadratic terms, but not linear terms.

Letting

$$r = r(t,x),\ \mathbf{U} = \mathbf{U}(t,x),\ s = s(t,x)$$

be smooth functions on $[0,T]\times\overline{\Omega}$ with $r, s > 0$ on $[0,T]\times\overline{\Omega}$ and

$$\mathbf{U}|_{\partial\Omega} = 0,$$

it is shown in [3] that for smooth $\{\varrho, \mathbf{u}, \eta\}$, the following *relative entropy inequality* holds:

$$\int_\Omega \frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + E_F(\varrho, r) + E_P(\eta, s)\,\mathrm{d}x(\tau) + \int_0^\tau \int_\Omega [\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})] : \nabla_x(\mathbf{u}-\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq \int_\Omega \frac{1}{2}\varrho_0|\mathbf{u}_0-\mathbf{U}_0|^2 + E_F(\varrho_0, r_0) + E_P(\eta_0, s_0)\,\mathrm{d}x + \int_0^\tau \mathcal{R}(\varrho, \mathbf{u}, \eta, r, \mathbf{U}, s)\,\mathrm{d}t, \tag{4.21}$$

where

$$\begin{aligned}\mathcal{R}(\varrho, \mathbf{u}, \eta, r, \mathbf{U}, s) := &\int_\Omega \frac{1}{r}\mathrm{div}_x(\mathbb{S}(\nabla_x \mathbf{U})\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x - \int_\Omega \varrho(\partial_t \mathbf{U} + \mathbf{u}\cdot\mathbf{U})\cdot(\mathbf{u}-\mathbf{U})\,\mathrm{d}x\\
&- \int_\Omega \partial_t P_F(r)(\varrho - r) + \nabla_x P_F(r)\cdot(\varrho\mathbf{u} - r\mathbf{U})\,\mathrm{d}x\\
&- \int_\Omega [\varrho(P_F(\varrho) - P_F(r)) - E_F(\varrho, r)]\,\mathrm{div}_x \mathbf{U}\,\mathrm{d}x\\
&- \int_\Omega \partial_t P_P(s)(\eta - s) + \nabla_x P_P(s)\cdot(\eta\mathbf{u} - s\mathbf{U})\,\mathrm{d}x\\
&- \int_\Omega [\eta(P_P(\eta) - P_P(s)) - E_P(\eta, s)]\,\mathrm{div}_x \mathbf{U}\,\mathrm{d}x\\
&- \int_\Omega \nabla_x(P_P(\eta) - P_P(s))\cdot(\nabla_x \eta + \eta\nabla_x \Phi)\,\mathrm{d}x\\
&- \int_\Omega (\beta\varrho + \eta)\nabla_x \Phi\cdot(\mathbf{u}-\mathbf{U})\,\mathrm{d}x - \int_\Omega \frac{\eta\nabla_x s}{s}(\mathbf{u}-\mathbf{U})dx.\end{aligned} \tag{4.22}$$

The definition of *suitable weak solutions* follows:

Definition 4.1. $\{\varrho, \mathbf{u}, \eta\}$ is a *suitable weak solution* of (1.1)–(1.4) with initial data $\{\varrho_0, \mathbf{u}_0, \eta_0\}$ if and only if

- $\{\varrho, \mathbf{u}, \eta\}$ is a weak solution in the sense of Definition 3.1.
- $\{\varrho, \mathbf{u}, \eta\}$ obeys inequality (4.21) for any smooth functions $\{r, \mathbf{U}, s\}$.

Next we present the theorem establishing the global existence of *suitable weak solutions.*

Theorem 4.1 (Suitable weak solutions). Let us assume that (Ω, Φ) satisfy the confinement hypotheses **(HC)** with $\Omega \subset \mathbb{R}^3$ a domain of class $C^{2+\nu}, \nu > 0$. Suppose the initial data $\{\varrho_0, \mathbf{u}_0, \eta_0\}$ satisfy

$$0 < \varrho_0 \in L^\gamma(\Omega), \varrho_0|\mathbf{u}_0|^2 \in L^1(\Omega), \eta_0 \log\eta_0 \in L^1(\Omega)$$

in addition to the conditions on the initial data specified in Section 1. Then the Navier-Stokes-Smoluchowski system in (1.1)–(1.7) has a suitable weak solution in the sense of Definition 4.1.

Proof. For the proof and further implications of Theorem 4.1 we refer the reader to the article by Ballew and Trivisa [3]. It is worth mentioning that the proof relies on the construction of a suitable approximating scheme, which relies on an approximate *relative entropy inequality*, the addition of the *artificial pressure* in the momentum equation as well as other regularizations in the spirit of [15]. The fundamental issue in that context is the establishment of the appropriate compactness of the sequence of approximate solutions $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$. The main issues can be summarized as follows:

- The strong convergence of the density is established by showing the *weak continuity* of the effective viscous pressure.
- High integrability properties for the density need to be established for the limit passage in the family of approximate solutions and in particular in taking the vanishing artificial pressure limit. We remark that in the present context, the potential Φ is not integrable on unbounded domains. To deal with this new difficulty we employ the Fourier multipliers in the spirit of [15], while taking into consideration the new features of our problem.
- The confinement hypothesis **(HC)** is crucial in providing a control for the negative contribution of the physical entropy $\eta \log \eta$ in the free-energy bounds for unbounded domains Ω.

Considering an unbounded domain Ω and an external potential Φ satisfying the assumptions **(HC)**, we can always find an increasing sequence of domains Ω_r, with $r > 0$ such that Ω_r are bounded and (Ω_r, Φ) satisfies **(HC)** approximating Ω in the sense that $\bigcup_{r>0} \Omega_r = \Omega$. Having obtained the result for a bounded Ω_r, for any $r > 0$ and using the results in section 2.1 we can show that we can send $r \to \infty$ to obtaining the earlier results in Ω. We refer the reader to [3].

□

REMARK 4.1. Basic properties of the suitable weak solutions to the Navier-Stokes Smoluchowski system (1.1)–(1.3), including the problem of weak-strong uniqueness and conditional regularity issues, are presented in [3].

5. Low stratification singular limits. Before scaling the system (1.1)–(1.3), the values D and ζ must be added to ensure consistency of the physical units in the equations. Specifically, the pressure term in the momentum equation becomes

$$\nabla_x \left(a\varrho^\gamma + \frac{D}{\zeta}\eta \right),$$

the Smoluchowski equation becomes

$$\partial_t \eta + \operatorname{div}_x(\eta \mathbf{u}) - \operatorname{div}_x(\zeta \eta \Phi) - D\Delta_x \eta = 0$$

and the energy inequality becomes

$$\begin{aligned} &\int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma + \frac{D}{\zeta}\eta\log\eta + (\beta\varrho + \eta)\Phi \, \mathrm{d}x(\tau) \\ &+ \int_0^\infty \int_\Omega \mu|\nabla_x \mathbf{u}|^2 + \lambda|\operatorname{div}_x \mathbf{u}|^2 + \left| 2\frac{D}{\sqrt{\zeta}}\nabla_x\sqrt{\eta} + \sqrt{\zeta\eta}\nabla_x\Phi \right|^2 \mathrm{d}x \, \mathrm{d}t \\ &\le \int_\Omega \frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{a}{\gamma-1}\varrho_0^\gamma + \frac{D}{\zeta}\eta_0\log\eta_0 + (\beta\varrho_0 + \eta_0)\Phi \, \mathrm{d}x. \end{aligned}$$

5.1. *Scaling.* To begin the scaling of the Navier-Stokes-Smoluchowski model, the quantities

$$\varrho, \mathbf{u}, \eta, \zeta, D, p_F, p_P, \Phi, \mu, \text{ and } \lambda,$$

as well as the time and length scales, must be made nondimensional. This is done by defining for each quantity A a reference value A_{ref} which also reflects the physical unit of measurement for that quantity, such as meter, second, meter per second, et cetera. Then, the dimensionless value A' is defined as

$$A' := \frac{A}{A_{ref}}.$$

After some application of the chain rule and some straightforward algebra, the formal dimensionless Navier-Stokes-Smoluchowski system becomes (omitting the primes for the sake of notational simplicity)

$$\mathrm{Sr}\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{5.23}$$

$$\mathrm{Sr}\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\mathrm{Ma}^2}\nabla_x\left(a\varrho^\gamma + \mathrm{Pc}\frac{D}{\zeta}\eta\right) = \frac{1}{\mathrm{Re}}(\mu\Delta_x\mathbf{u} + \lambda\nabla_x \mathrm{div}_x\mathbf{u}) - \frac{1}{\mathrm{Fr}^2}(\beta\varrho + \mathrm{Dc}\eta)\nabla_x\Phi, \tag{5.24}$$

$$\mathrm{Sr}\partial_t\eta + \mathrm{div}_x(\eta\mathbf{u}) - \mathrm{Za}\,\mathrm{div}_x(\zeta\eta\nabla_x\Phi) - \mathrm{Da}D\Delta_x\eta = 0. \tag{5.25}$$

The total energy inequality for the scaled system takes the form

$$\mathrm{Sr}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\mathrm{Ma}^2}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma + \mathrm{Pc}\frac{D\eta}{\zeta}\ln\eta + \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}(\beta\varrho + \mathrm{Dc}\eta)\Phi\,\mathrm{d}x + \int_\Omega \frac{\mathrm{Ma}^2}{\mathrm{Re}}\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\,\mathrm{d}x + \int_\Omega \mathrm{PcDa}D^2\frac{|\nabla_x\eta|^2}{\zeta\eta} + 2\mathrm{Za}D\nabla_x(\eta)\cdot\nabla_x\Phi + \frac{\mathrm{Za}^2}{\mathrm{Da}}\zeta\eta|\nabla_x\Phi|^2\,\mathrm{d}x \le 0. \tag{5.26}$$

Here, the nondimensional parameters used in (5.23)–(5.26) are defined as follows:

TABLE 5.1. Definitions of the Dimensionless Parameters

$\mathrm{Sr} := \frac{L_{ref}}{\mathbf{u}_{ref}t_{ref}}$	$\mathrm{Ma} := \frac{\mathbf{u}_{ref}}{\sqrt{p_{F_{ref}}/\varrho_{ref}}}$	$\mathrm{Re} := \frac{\varrho_{ref}\mathbf{u}_{ref}L_{ref}}{\mu_{ref}}$
$\mathrm{Fr} := \frac{\mathbf{u}ref}{\sqrt{L_{ref}f_{ref}}}$	$\mathrm{Za} := \frac{\zeta_{ref}f_{ref}}{\mathbf{u}_{ref}}$	$\mathrm{Da} := \frac{D_{ref}}{L_{ref}\mathbf{u}_{ref}}$
$\mathrm{Pc} := \frac{p_{P_{ref}}}{p_{F_{ref}}}$	$\mathrm{Dc} := \frac{\eta_{ref}}{\varrho_{ref}}.$	

Here, the quantities

$$L_{ref}, \mathbf{u}_{ref}, t_{ref}, p_{F_{ref}}, \varrho_{ref}, \mu_{ref}, f_{ref}, \zeta_{ref}, D_{ref}, p_{P_{ref}}, \text{ and } \eta_{ref}$$

represent the reference values for the length, velocity, time, fluid pressure, fluid density, viscosity coefficient, force (equal to $\nabla_x\Phi$), drag coefficient, diffusivity coefficient, particle pressure (equal to $D\zeta/\eta$), and particle density, respectively. The compatibility conditions $\mu_{ref} = \lambda_{ref}$ and $p_{F_{ref}} = \varrho_{ref}e_{F_{ref}}$, $p_{P_{ref}} = \eta_{ref}e_{P_{ref}}$ are also imposed to obtained the scaling, the second and third of which follow naturally from Maxwell's relation. Note also that Ma represents the Mach number, Sr the Strouhal number, Re the Reynolds number, and Fr the Froude number used in other works on singular limits (see [16]). Since existence of solutions to the scaled system follows from [3] and [10] for any choices of positive values of the dimensionless parameters, various singular limits can be explored. The current paper considers a low-Mach-number limit, with Ma taken to be a small parameter ε, Za scaled as Ma, and $\mathrm{Fr} = \sqrt{\varepsilon}$.

5.2. *Formal derivation of the low stratification limit.* The low stratification case is represented by $\mathrm{Ma} = \mathrm{Za} = \varepsilon$ and $\mathrm{Fr} = \sqrt{\varepsilon}$. As such, the formal system is as follows:

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0, \tag{5.27}$$

$$\varepsilon^2[\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)] + \nabla_x \left(a\varrho_\varepsilon^\gamma + \frac{D}{\zeta}\eta_\varepsilon \right)$$
$$= \varepsilon^2(\mu\Delta_x \mathbf{u}_\varepsilon + \lambda\nabla_x \mathrm{div}_x \mathbf{u}_\varepsilon) - \varepsilon(\beta\varrho_\varepsilon + \eta_\varepsilon)\nabla_x\Phi, \tag{5.28}$$

$$\partial_t \eta_\varepsilon + \mathrm{div}_x(\eta_\varepsilon \mathbf{u}_\varepsilon) - \varepsilon \mathrm{div}_x(\zeta\eta_\varepsilon\nabla_x\Phi) - D\Delta_x\eta_\varepsilon = 0. \tag{5.29}$$

The total energy inequality for the scaled system takes the form

$$\int_\Omega \frac{\varepsilon^2}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{a}{\gamma-1}\varrho_\varepsilon^\gamma + \frac{D\eta_\varepsilon}{\zeta}\ln\eta_\varepsilon + \varepsilon(\beta\varrho_\varepsilon + \eta_\varepsilon)\Phi \,\mathrm{d}x(T)$$
$$+ \int_0^T \int_\Omega \varepsilon^2(\mu|\nabla_x \mathbf{u}_\varepsilon|^2 + \lambda|\mathrm{div}_x \mathbf{u}_\varepsilon|^2)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^T \int_\Omega \left| 2\frac{D}{\sqrt{\zeta}}\nabla_x\sqrt{\eta_\varepsilon} + \varepsilon\sqrt{\zeta\eta_\varepsilon}\nabla_x\Phi \right|^2 \mathrm{d}x\,\mathrm{d}t$$
$$\leq \int_\Omega \frac{\varepsilon^2}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{a}{\gamma-1}\varrho_0^\gamma + \frac{D\eta_0}{\zeta}\ln\eta_0 + \varepsilon(\beta\varrho_0 + \eta_0)\Phi\,\mathrm{d}x. \tag{5.30}$$

The free energy solutions to the scaled Navier-Stokes Smoluchowski system $(\varrho_\varepsilon, \mathbf{u}_\varepsilon . \eta_\varepsilon)$ satisfy the boundary conditions:

$$\mathbf{u}_\varepsilon|_{\partial\Omega} = 0, \quad \left(D\nabla_x\eta_\varepsilon + \varepsilon\,\zeta\eta_\varepsilon\nabla_x\Phi \right)\cdot\nu = 0. \tag{5.31}$$

The formal technique is to expand $\varrho_\varepsilon, \mathbf{u}_\varepsilon,$ and η_ε as

$$\varrho_\varepsilon = \overline{\varrho} + \sum_{i=1}^\infty \varepsilon^i \varrho^{(i)},$$
$$\mathbf{u}_\varepsilon = \overline{\mathbf{u}} + \sum_{i=1}^\infty \varepsilon^i \mathbf{u}^{(i)},$$
$$\eta_\varepsilon = \overline{\eta} + \sum_{i=1}^\infty \varepsilon^i \eta^{(i)},$$

plug these expansions into (5.27)–(5.30), and equate terms of equal orders of ε. In doing so, it becomes clear that since the right side of (5.30) is bounded uniformly in ε, as it is just the initial energy, it must be true that

$$\nabla_x\sqrt{\overline{\eta}} = 0.$$

Thus, $\overline{\eta}$ is constant on Ω for each time t and

$$\overline{\eta} = \frac{1}{|\Omega|}\int_\Omega \eta_0\,\mathrm{d}x$$

in the formal limit. Moving to the momentum equation (5.28) and equating terms of order one, the formal equation becomes

$$\nabla_x \left(a\overline{\varrho}^\gamma + \frac{D}{\zeta}\overline{\eta} \right) = 0.$$

Since $\overline{\eta}$ is constant, it follows formally that

$$\overline{\varrho} = \frac{1}{|\Omega|} \int_\Omega \varrho_0 \,\mathrm{d}x.$$

Using this fact in the continuity equation (5.27) and equating terms of order one yields the incompressibility condition for the limit velocity

$$\mathrm{div}_x \overline{\mathbf{u}} = 0.$$

Returning to (15) and equating terms of order ε^2, it is easy to show formally that

$$\overline{\varrho}[\partial_t \overline{\mathbf{u}} + \mathrm{div}_x(\overline{\mathbf{u}} \otimes \overline{\mathbf{u}})] + \nabla_x \Pi = \mu \Delta_x \overline{\mathbf{u}} - (\beta r + \theta)\nabla_x \Phi,$$

where r, θ are related to the limit quantities by

$$\nabla_x \left(a r^\gamma + \frac{D}{\zeta}\theta \right) = -(\beta\overline{\varrho} + \overline{\eta})\nabla_x \Phi,$$

which is found by equating terms of order ε in (5.28) and relabeling $\varrho^{(1)}$ and $\eta^{(1)}$. Thus, the formal low stratification and low Mach number limit for the Navier-Stokes-Smoluchowski system become

$$\overline{\eta} = \frac{1}{|\Omega|} \int_\Omega \eta_0(x) \,\mathrm{d}x, \tag{5.32}$$

$$\overline{\varrho} = \frac{1}{|\Omega|} \int_\Omega \varrho_0(x) \,\mathrm{d}x, \tag{5.33}$$

$$\mathrm{div}_x \overline{\mathbf{u}} = 0, \tag{5.34}$$

$$\overline{\varrho}[\partial_t \overline{\mathbf{u}} + \mathrm{div}_x(\overline{\mathbf{u}} \otimes \overline{\mathbf{u}})] + \nabla_x \Pi = \mu \Delta_x \overline{\mathbf{u}} - (\beta r + \theta)\nabla_x \Phi, \tag{5.35}$$

where r, θ satisfy

$$\nabla_x \left(a r^\gamma + \frac{D}{\zeta}\theta \right) = -(\beta\overline{\varrho} + \overline{\eta})\nabla_x \Phi \tag{5.36}$$

and Π is a function incorporating the terms for which a gradient is taken.

5.3. *Rigorous derivation of the low stratification limit.* In this section, the formal limit derived in Subsection 5.2 is rigorously proven.

First, we need to introduce the notion of solution for the scaled system (5.27)–(5.30).

5.3.1. *Free energy solutions.*

Definition 5.1. Let us assume that (Ω, Φ) satisfy the confinement hypotheses **(HC)** with $\Omega \subset \mathbb{R}^3$ a domain of class $C^{2+\nu}, \nu > 0$. Also, assume that μ, λ, ζ, and D are positive constants. Then $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$ represent a weak solution of the low stratification system with Mach number ε if and only if

- $\varrho_\varepsilon \geq 0$ represents a renormalized solution of the continuity equation on $(0,\infty)\times\Omega$; i.e., for any test function $\phi \in \mathcal{D}([0,T) \times \overline{\Omega}), T > 0$ and any b, B such that

$$b \in L^\infty \cap C[0,\infty), B(\varrho) := B(1) + \int_1^\varrho \frac{b(z)}{z^2} \,\mathrm{d}z,$$

the renormalized continuity equation

$$\int_0^T \int_\Omega B(\varrho_\varepsilon)\partial_t\phi + B(\varrho_\varepsilon)\mathbf{u}_\varepsilon\cdot\nabla_x\phi - b(\varrho_\varepsilon)\phi\mathrm{div}_x\mathbf{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t = -\int_\Omega B(\varrho_0)\phi(0,\cdot)\,\mathrm{d}x \quad (5.37)$$

holds.

- The balance of momentum holds in the sense of distributions; i.e., for any $\mathbf{v}\in\mathcal{D}([0,T);\mathcal{D}(\overline{\Omega};\mathbb{R}^3))$,

$$\int_0^T\int_\Omega \varepsilon^2(\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t\mathbf{v} + \varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon:\nabla_x\mathbf{v}) + \left(p_F(\varrho_\varepsilon)+\frac{D}{\zeta}\eta_\varepsilon\right)\mathrm{div}_x\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_0^T\int_\Omega \varepsilon^2(\mu\nabla_x\mathbf{u}_\varepsilon\nabla_x\mathbf{v} + \lambda\mathrm{div}_x\mathbf{u}_\varepsilon\mathrm{div}_x\mathbf{v}) - \varepsilon(\beta\varrho_\varepsilon+\eta_\varepsilon)\nabla_x\Phi\cdot\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t$$
$$-\varepsilon^2\int_\Omega \mathbf{m}_0\cdot\mathbf{v}(0,\cdot)\,\mathrm{d}x. \quad (5.38)$$

- $\eta_\varepsilon\geq 0$ is a weak solution of (3); i.e.,

$$\int_0^T\int_\Omega \eta_\varepsilon\partial_t\phi + \eta_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x\phi - \varepsilon\zeta\eta_\varepsilon\nabla_x\Phi\cdot\nabla_x\phi - D\nabla_x\eta_\varepsilon\cdot\nabla_x\phi\,\mathrm{d}x\,\mathrm{d}t = -\int_\Omega \eta_0\phi(0,\cdot)\,\mathrm{d}x \quad (5.39)$$

for any test function $\phi\in\mathcal{D}([0,T)\times\overline{\Omega})$.

- The energy inequality below holds:

$$\int_\Omega \frac{\varepsilon^2}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{a}{\gamma-1}\varrho_\varepsilon^\gamma + \frac{D\eta_\varepsilon}{\zeta}\ln\eta_\varepsilon + \varepsilon(\beta\varrho_\varepsilon+\eta_\varepsilon)\Phi\,\mathrm{d}x(T)$$
$$+\int_0^T\int_\Omega \varepsilon^2(\mu|\nabla_x\mathbf{u}_\varepsilon|^2 + \lambda|\mathrm{div}_x\mathbf{u}_\varepsilon|^2)\mathrm{d}x\mathrm{d}t$$
$$+\int_0^T\int_\Omega \left|2\frac{D}{\sqrt{\zeta}}\nabla_x\sqrt{\eta_\varepsilon} + \varepsilon\sqrt{\zeta\eta_\varepsilon}\nabla_x\Phi\right|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq \int_\Omega \frac{\varepsilon^2}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{a}{\gamma-1}\varrho_0^\gamma + \frac{D\eta_0}{\zeta}\ln\eta_0 + \varepsilon(\beta\varrho_0+\eta_0)\Phi\,\mathrm{d}x. \quad (5.40)$$

By the existence results in [3] and [10], it is clear that such $\{\varrho_\varepsilon,\mathbf{u}_\varepsilon,\eta_\varepsilon\}$ exist for each $\varepsilon>0$. We now introduce the notion of weak solutions of the target system (5.32)–(5.36) called the Oberbeck-Boussinesq approximation.

Definition 5.2. We say that $\{\bar{\mathbf{u}},r,s\}$ is a variational solution of the target system (5.32)–(5.35) supplemented with the boundary conditions

$$\bar{\mathbf{u}} = 0 \text{ on } \partial\Omega \quad (5.41)$$

and the initial conditions

$$\bar{\mathbf{u}}(0,\cdot) = \mathbf{u}_0 \quad (5.42)$$

if the following conditions hold:

- $\bar{\mathbf{u}}\in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))\cap L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$,

[Incompressibility condition]:

$$\mathrm{div}_x\bar{\mathbf{u}} = 0 \text{ weakly on } (0,T)\times\Omega,$$

- and the integral identity

$$\int_0^T \int_\Omega \Big(\bar{\varrho}\bar{\mathbf{u}} \cdot \partial_t \varphi + \bar{\varrho}(\bar{\mathbf{u}} \otimes \bar{\mathbf{u}}) : \nabla_x \varphi\Big) \mathrm{d}x\mathrm{d}t \\ = \int_0^T \int_\Omega \Big(\bar{\mu}\nabla_x \bar{\mathbf{u}} - (\beta r + s)\nabla_x \Phi\Big) \cdot \varphi \mathrm{d}x\mathrm{d}t - \int_\Omega \bar{\varrho}\bar{\mathbf{u}} \cdot \varphi(0,\cdot)\mathrm{d}x \tag{5.43}$$

  holds for any test function

$$\varphi \in \mathcal{D}((0,T)\times\Omega;\mathbb{R}^3),\ \mathrm{div}_x \varphi = 0 \text{ in } \Omega,\ \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

- The quantities r and s are interrelated via the so-called **[Boussinesq relation]:**

$$r = -\frac{1}{a\gamma\overline{\varrho}^{\gamma-1}}\left[(\beta\overline{\varrho} + \overline{\eta})\Phi + \frac{D}{\zeta}s\right].$$

6. Main results. We now introduce a geometric condition on Ω which plays a crucial role in the study of propagation of the acoustic waves. Let us consider the following problem:

$$-\Delta\phi = \lambda\phi \text{ in } \Omega,\ \frac{\partial\phi}{\partial\mathbf{n}} = 0 \text{ on } \partial\Omega, \tag{6.44}$$

where ϕ is constant on $\partial\Omega$. We call a solution of the problem (6.44) trivial if $\lambda = 0$ and ϕ is constant. We also define that Ω verifies assumption (H) if all solutions of the problem (6.44) are trivial. Notice that Schiffer's conjecture shows that every Ω satisfies (H) except the ball, and Feireisl, Novotný, Petzeltová [17] gives an example of a domain Ω which is trivial. In two-dimensional space, it is proven that every bounded, simply connected open domain $\Omega \subset \mathbb{R}^2$ whose boundary is Lipschitz but not real analytic satisfies (H).

6.1. *Result on bounded domains.* We first mention a result of incompressible limit problems on bounded domains with Dirichlet boundary conditions.

THEOREM 6.1 (Low stratification limit). Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a boundary of class $C^{2+\nu}$, $\nu > 0$ and verify the suitable assumption (H) in the problem (6.44). Also let (Ω, Φ) satisfy the confinement hypothesis **(HC)** and assume $\mathrm{Ma} = \varepsilon, \mathrm{Fr} = \sqrt{\varepsilon}$ and $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}_{\varepsilon>0}$ is a family of free energy solutions to the scaled Navier-Stokes-Smoluchowski system in the sense of Definition 5.1 with the boundary conditions (5.31).

Assume the initial condition as follows:

$$\varrho_\varepsilon(0,\cdot) = \varrho_{\varepsilon,0} = \bar{\varrho} + \varepsilon\varrho^{(1)}_{\varepsilon,0},\ \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{\varepsilon,0},\ \eta(0,\cdot) = \bar{\eta} + \varepsilon\eta^{(1)}_{\varepsilon,0}, \tag{6.45}$$

where

$$\bar{\varrho} = \frac{1}{|\Omega|}\int_\Omega \varrho_{\varepsilon,0}\,\mathrm{d}x,\ \bar{\eta} = \frac{1}{|\Omega|}\int_\Omega \eta_{\varepsilon,0}\,\mathrm{d}x, \tag{6.46}$$

and

$$\varrho^{(1)}_{\varepsilon,0} \rightharpoonup \varrho^{(1)}_0,\ \mathbf{u}_{\varepsilon,0} \rightharpoonup \mathbf{u}_0,\ \eta^{(1)}_{\varepsilon,0} \rightharpoonup \eta^{(1)}_0, \tag{6.47}$$

as ε tends to 0, where we have used weak$-*$ convergence in $L^\infty(\Omega)$. Then, up to a subsequence,

$$\begin{cases} \varrho_n \to \bar{\varrho} \text{ in } C([0,T];L^1(\Omega)) \cap L^\infty(0,T;L^q(\Omega)), \\ \\ \eta_n \to \bar{\eta} \text{ in } L^2(0,T;L^2(\Omega)), \\ \\ \mathbf{u}_n \to \bar{\mathbf{u}} \text{ strongly in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)), \\ \end{cases} \tag{6.48}$$

and

$$\begin{cases} \varrho_\varepsilon^{(1)} = \dfrac{\varrho_\varepsilon - \bar{\varrho}}{\varepsilon} \to \varrho^{(1)} \text{ weakly}-* \text{ in } L^\infty(0,T;L^q(\Omega)),\ q = \min\{2,\gamma\}, \\ \\ \eta_\varepsilon^{(1)} = \dfrac{\eta_\varepsilon - \bar{\eta}}{\varepsilon} \to \eta^{(1)} \text{ weakly in } L^2(0,T;L^2(\Omega)), \end{cases} \tag{6.49}$$

where $\{\bar{\mathbf{u}},\ \varrho^{(1)},\eta^{(1)}\}$ solves the target system in the sense of Definition 5.2 with the boundary condition $\mathbf{u}|_{\partial\Omega} = 0$ and the initial data

$$\mathbf{U}(0) = \mathbf{H}[\mathbf{U}_0], \tag{6.50}$$

where the Helmholtz projection $\mathbf{H}$ is defined by

$$\mathbf{H} = \mathbf{I} - \mathbf{H}^\perp,\ \mathbf{H}^\perp = \nabla\Delta^{-1}\text{div}. \tag{6.51}$$

6.2. *Result on unbounded domains.* We next study incompressible limit problems for the fluid-particle interaction model on unbounded domains. Consider an unbounded domain $\Omega \subset \mathbb{R}^3$ with a compact regular boundary $\partial\Omega$ and a family of bounded domains $\{\Omega_\epsilon\}_{\epsilon>0}$ satisfying:

$$\Omega_\epsilon \subset \Omega,\ \partial\Omega \subset \partial\Omega_\epsilon,\ \epsilon\text{dist}[x,\partial\Omega_\epsilon] \to \infty \text{ as } \epsilon \to 0, \tag{6.52}$$

for any $x \in \Omega$. We consider a variational solution to the system in the sense of Definition 5.1 on Ω_ϵ. Then the main result is the following:

Theorem 6.2 (Low stratification limit for unbounded domains). Let $\Omega \subset \mathbb{R}^3$ be an unbounded domain with a compact boundary $\partial\Omega$ of class $C^{2+\nu}$, $\nu > 0$ and a family of bounded domains $\{\Omega_\epsilon\}_{\epsilon>0}$ satisfies (6.52). Also let (Ω,Φ) satisfy the confinement hypothesis **(HC)** and assume $\text{Ma} = \varepsilon, \text{Fr} = \sqrt{\varepsilon}$ and $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$, is a family of free energy solutions on $(0,T) \times \Omega_\epsilon$ to the Navier-Stokes-Smoluchowski system in the sense of Definition 5.1 on Ω_ϵ with the same initial conditions given in Theorem 6.1 and the boundary conditions (5.31), on Ω_ϵ. Let us assume that all of the hypotheses in Theorem 6.1 hold. Then we have the same convergence of $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$ on any compact $K \subset \Omega$ as given in Theorem 6.1 such that the limit $\bar{\mathbf{u}}$ of $\{\mathbf{u}_\varepsilon\}$ solves the target system in the sense of Definition 5.2 and the initial data (6.50).

7. Rigorous derivation of the low stratification limit of the system.

7.1. Ω *bounded domain in* $\mathbb{R}^3$.

7.1.1. *Free energy inequality and uniform bounds.* The first step in rigorously deriving the convergence stated in Theorem 6.1 is to obtain bounds uniform in ε which will yield the weak limits. To do this, analogs of the Helmholtz free-energy function defined below are utilized:

$$E_F(\varrho) := \frac{a}{\gamma-1}\varrho^\gamma - (\varrho-\overline{\varrho})\frac{a\gamma}{\gamma-1}\overline{\varrho}^{\gamma-1} - \frac{a}{\gamma-1}\overline{\varrho}^\gamma$$

and

$$E_P(\eta) := \frac{D}{\zeta}\eta\ln\eta - \frac{D}{\zeta}(\eta-\overline{\eta})(\ln\overline{\eta}+1) - \frac{D}{\zeta}\overline{\eta}\ln\overline{\eta}.$$

Basic calculations show that E_F and E_P have global minima at $\overline{\varrho}$ and $\overline{\eta}$ respectively, and are both convex, facts that will be used later in the proof. Thus after some analysis, the energy inequality given by (5.40) can be rewritten as

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}(E_F(\varrho_\varepsilon)+E_P(\eta_\varepsilon)) + \frac{1}{\varepsilon}(\beta\varrho_\varepsilon+\eta_\varepsilon)\Phi \mathrm{d}x(T)\\
&+\int_0^T\int_\Omega \mu|\nabla_x\mathbf{u}_\varepsilon|^2 + \lambda|\mathrm{div}_x\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\left|\frac{2D\nabla_x\sqrt{\eta_\varepsilon}}{\sqrt{\zeta}} + \varepsilon\sqrt{\zeta\eta_\varepsilon}\nabla_x\Phi\right|^2 \mathrm{d}x\,\mathrm{d}t\\
&\leq \int_\Omega \frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{1}{\varepsilon^2}(E_F(\varrho_0)+E_P(\eta_0)) + \frac{1}{\varepsilon}(\beta\varrho_0+\eta_0)\Phi\,\mathrm{d}x. \qquad (7.53)
\end{aligned}$$

By the hypotheses on the initial data, the right side of this equation is bounded by a constant (cf. Chapter 5.1 in [16]). Thus, the following uniform in ε bounds are obtained:

$$\begin{aligned}
&\{\mathbf{u}_\varepsilon\}_{\varepsilon>0} \in_b L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),\\
&\{\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\}_{\varepsilon>0} \in_b L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)),\\
&\left\{\frac{1}{\varepsilon}\left(\frac{2D\nabla_x\sqrt{\eta_\varepsilon}}{\sqrt{\zeta}} + \varepsilon\sqrt{\zeta\eta_\varepsilon}\nabla_x\Phi\right)\right\}_{\varepsilon>0} \in_b L^2(0,T;L^2(\Omega;\mathbb{R}^3)).
\end{aligned}$$

Next, the following sets are defined:

$$\begin{aligned}
&\mathcal{O}_{ess} := \{(\varrho,\eta)\in\mathbb{R}^2|\overline{\varrho}/2 \leq \varrho,\eta \leq 2\overline{\eta}\},\\
&\mathcal{M}^\varepsilon_{\mathrm{ess}} := \{(t,x)\in(0,T)\times\Omega|(\varrho_\varepsilon(t,x),\eta_\varepsilon(t,x))\in\mathcal{O}_{ess}\},\\
&\mathcal{M}^\varepsilon_{\mathrm{res}} := ((0,T)\times\Omega) - \mathcal{M}^\varepsilon_{\mathrm{ess}}.
\end{aligned}$$

Since ϱ^γ and $\eta\ln\eta$ are clearly strongly convex on $\mathcal{M}^\varepsilon_{\mathrm{ess}}$,

$$\mathcal{H}(\rho_\varepsilon,\eta_\varepsilon) := E_F(\varrho_\varepsilon) + E_P(\eta_\varepsilon) \geq C(|\varrho-\overline{\varrho}|^2 + |\eta-\overline{\eta}|^2) \text{ on } \mathcal{M}^\varepsilon_{\mathrm{ess}}.$$

By the properties of E_F, E_P mentioned above,

$$E_F(\varrho) \geq E_F(\overline{\varrho}/2) > 0 \text{ for } \varrho < \overline{\varrho}/2 \text{ and } E_P(\eta) \geq E_P(2\overline{\eta}) > 0 \text{ for } \eta > 2\overline{\eta}.$$

Thus, on $\mathcal{M}^\varepsilon_{\mathrm{res}}$, $\mathcal{H}(\varrho,\eta) \geq c > 0$ for some constant c. It also becomes clear that the right-hand side of (7.53) is uniformly bounded by some finite, positive constant.

Using the coercivity of E_F, E_P and the boundedness of (7.53), it can be shown that the measures of the residual sets $\mathcal{M}^\varepsilon_{\mathrm{res}}[t] := \{x\Omega|(t,x)\in\mathcal{M}^\varepsilon_{\mathrm{res}}\}$ go as ε^2. Indeed, using the set $\{\rho_\varepsilon(t) \leq \overline{\rho}/2\}$ as an example,

$$|\{\rho_\varepsilon(t)\leq\overline{\rho}/2\}| \leq c_1\int_\Omega 1_{\{\rho_\varepsilon(t)\leq\overline{\rho}/2\}}\mathcal{H}(\rho_\varepsilon,\eta_\varepsilon)\,\mathrm{d}x \leq \int_\Omega \mathcal{H}(\rho_\varepsilon,\eta_\varepsilon)\,\mathrm{d}x \leq \varepsilon^2 c_2.$$

Thus, using the coercivity of $\mathcal{H}$ and (7.53), the following bounds can be obtained:

$$\operatorname{ess\,sup}_{t\in(0,T)}|\mathcal{M}^{\varepsilon}_{\mathrm{res}}[t]| \le \varepsilon^2 c, \tag{7.54}$$

$$\left\{\left[\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\right]_{\mathrm{ess}}\right\} \in_b L^\infty(0,T;L^2(\Omega)), \tag{7.55}$$

$$\left\{\left[\frac{\eta_\varepsilon-\overline{\eta}}{\varepsilon}\right]_{\mathrm{ess}}\right\} \in_b L^\infty(0,T;L^2(\Omega)), \tag{7.56}$$

$$\{\mathbf{u}_\varepsilon\} \in_b L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)), \tag{7.57}$$

$$\{\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\} \in_b L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{7.58}$$

$$\left\{\frac{1}{\varepsilon}\left(\frac{2D\nabla_x\sqrt{\eta}}{\sqrt{\zeta}}+\varepsilon\sqrt{\zeta\eta}\nabla_x\Phi\right)\right\} \in_b L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{7.59}$$

$$\{[\varrho_\varepsilon]\mathrm{res}]\} \in_b L^\infty(0,T;L^\gamma(\Omega)). \tag{7.60}$$

7.2. *Convergence.* From the uniform bounds in (7.54)–(7.60), the following convergences are easily obtained:

- There exists $\varrho^{(1)}$ such that
$$\left[\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\right]_{\mathrm{ess}} \to \varrho^{(1)}$$
weakly in $L^2(0,T;L^2(\Omega))$.
- Also,
$$\left[\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\right]_{\mathrm{res}} \to 0$$
weakly-$*$ in $L^\infty(0,T;L^\gamma(\Omega))$.
- There exists $\eta^{(1)}$ such that
$$\left[\frac{\eta_\varepsilon-\eta}{\varepsilon}\right]_{\mathrm{ess}} \to \eta^{(1)}$$
weakly in $L^2(0,T;L^2(\Omega))$.
- There exists $\overline{\mathbf{u}}$ such that $\mathbf{u}_\varepsilon \to \overline{\mathbf{u}}$ weakly in $L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$.

By (7.59),
$$\left[\frac{\eta_\varepsilon-\overline{\eta}}{\varepsilon}\right]_{\mathrm{res}} \to 0$$
weakly in $L^2(0,T;L^2(\Omega))$. Therefore, letting $q := \min\{2,\gamma\}$,

$$\varrho_\varepsilon \to \overline{\varrho} \quad \text{weakly in} \quad L^2(0,T;L^q(\Omega)), \tag{7.61}$$

$$\eta_\varepsilon \to \overline{\eta} \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)), \tag{7.62}$$

$$\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon} \to \varrho^{(1)} \quad \text{weakly in} \quad L^2(0,T;L^q(\Omega)), \tag{7.63}$$

$$\frac{\eta_\varepsilon-\overline{\eta}}{\varepsilon} \to \eta^{(1)} \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)). \tag{7.64}$$

Using these convergence results and taking $b(z)=0$ and $B(1)=1$ in the renormalized continuity equation, ε can be taken to zero to yield that

$$\int_0^T \int_\Omega \overline{\mathbf{u}}\cdot\nabla_x\phi dx = 0, \tag{7.65}$$

that is, $\overline{\mathbf{u}}$ is weakly divergence-free.

To complete the proof of Theorem 6.1, the convergence of the momentum equation must be shown. The first thing to note is that by using the uniform bounds and the embedding of $W^{1,2}(\Omega;\mathbb{R}^3)$ into $L^6(\Omega;\mathbb{R}^3)$,

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \overline{\varrho\mathbf{u}} \tag{7.66}$$

weakly in $L^2(0,T;L^{6q/q+6}(\Omega;\mathbb{R}^3))$ and weakly-$*$ in $L^\infty(0,T;L^{2q/q+1}(\Omega;\mathbb{R}^3))$. Thus

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \overline{\varrho\mathbf{u}\otimes\mathbf{u}}$$

weakly in $L^2(0,T;L^{6q/4q+3}(\Omega;\mathbb{R}^{3\times 3}))$. So taking the limit as $\varepsilon \to 0$ in the momentum equation yields

$$\begin{aligned} &\int_0^T \int_\Omega \overline{\varrho\mathbf{u}} \cdot \partial_t \mathbf{v} + \overline{\varrho\mathbf{u}\otimes\mathbf{u}} : \nabla_x \mathbf{v} \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_\Omega \mu \nabla_x \overline{\mathbf{u}} : \nabla_x \mathbf{v} - (\beta \varrho^{(1)} + \eta^{(1)}) \nabla_x \Phi \cdot \mathbf{v} \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \overline{\varrho\mathbf{u}}_0 \cdot \mathbf{v}\,\mathrm{d}x \end{aligned} \tag{7.67}$$

for all divergence-free $\mathbf{v} \in C_c^\infty([0,T)\times\Omega;\mathbb{R}^3)$.

At this point, the original momentum equation (5.38) can be divided by ε and taking $\varepsilon \to 0$, with the aid of the uniform estimates, a relation for the quantities $\varrho^{(1)}$ and $\eta^{(1)}$ can be obtained as

$$\int_0^T \int_\Omega \left(a\gamma \overline{\varrho}^{\gamma-1} \varrho^{(1)} + \frac{D}{\zeta} \eta^{(1)} \right) \mathrm{div}_x \mathbf{w} \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_\Omega (\beta\overline{\varrho} + \overline{\eta}) \nabla_x \Phi \cdot \mathbf{w}\,\mathrm{d}x \tag{7.68}$$

for any test function (not necessarily divergence-free) $\mathbf{w}$. Thus, at least weakly,

$$\varrho^{(1)} = -\frac{1}{a\gamma\overline{\varrho}^{\gamma-1}} \left[(\beta\overline{\varrho} + \overline{\eta})\Phi + \frac{D}{\zeta}\eta^{(1)} \right].$$

7.3. *Convective term.* All that is left to do to prove Theorem 6.1 is to show that the divergence of $\overline{\varrho\mathbf{u}} \otimes \overline{\mathbf{u}} - \overline{\varrho\mathbf{u}\otimes\mathbf{u}}$ converges weakly to a gradient. To do this, the standard Helmholtz decomposition is employed to decompose the quantity into a divergence-free and a gradient part. Here, $\mathbf{H}[\mathbf{v}]$ will denote the divergence-free (solenoidal) part and $\mathbf{H}^\perp[\mathbf{v}]$ will denote the gradient part of the vector $\mathbf{v}$. Thus, the convective term can be rewritten as

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon = \mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{u}_\varepsilon + \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}[\mathbf{u}_\varepsilon] + \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^\perp[\mathbf{u}_\varepsilon].$$

By the convergence results and the continuity of the Helmholtz decomposition,

$$\mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \to \mathbf{H}[\overline{\varrho\mathbf{u}}] = \overline{\varrho\mathbf{u}}$$

in $C_{\mathrm{weak}}([0,T];L^{2q/q+1}(\Omega;\mathbb{R}^3))$. Since

$$\overline{\varrho}\mathbf{H}[\mathbf{u}_\varepsilon] \cdot \mathbf{u}_\varepsilon = \left(\varepsilon \mathbf{H}\left[\frac{\overline{\varrho} - \varrho_\varepsilon}{\varepsilon} \mathbf{u}_\varepsilon \right] + \mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \right) \cdot \mathbf{u}_\varepsilon \to \overline{\varrho} |\overline{\mathbf{u}}|^2$$

weakly in $L^1(\Omega)$, it follows that $\mathbf{H}[\mathbf{u}_\varepsilon] \to \overline{\mathbf{u}}$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$. Therefore,

$$\mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{u}_\varepsilon \to \overline{\varrho\mathbf{u}} \otimes \overline{\mathbf{u}}, \tag{7.69}$$

$$\mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}[\mathbf{u}_\varepsilon] \to 0, \tag{7.70}$$

weakly in $L^2(0,T;L^{6q/4q+3}(\Omega;\mathbb{R}^{3\times 3}))$. Thus, it remains to show that the singular term $\mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon]\otimes\mathbf{H}^\perp[\mathbf{u}_\varepsilon]$ converges weakly to a gradient so that it can be absorbed into the term Π in the limit.

Noting that $\overline{\varrho}$ and $\overline{\eta}$ are constant, the scaled weak formulation of the Navier-Stokes-Smoluchowski system can be rewritten as

$$\int_0^T\int_\Omega \varepsilon\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\partial_t\phi+\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x\phi\,\mathrm{d}x\,\mathrm{d}t=0, \tag{7.71}$$

$$\begin{aligned}&\int_0^T\int_\Omega \varepsilon\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t\mathbf{v}+\left[\frac{[p(\varrho_\varepsilon,\eta_\varepsilon)]_{\mathrm{ess}}-p(\overline{\varrho},\overline{\eta})}{\varepsilon}+(\beta\overline{\varrho}+\overline{\eta})\Phi\right]\mathrm{div}_x\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_0^T\int_\Omega[\beta(\overline{\varrho}-\varrho_\varepsilon)+(\overline{\eta}-\eta)]\nabla_x\Phi\cdot\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^T\int_\Omega\left[\varepsilon\mathbb{S}(\nabla_x\mathbf{u}_\varepsilon)-\varepsilon\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon-\frac{[p(\varrho_\varepsilon,\eta_\varepsilon)]_{\mathrm{res}}}{\varepsilon}\mathbb{I}\right]:\nabla_x\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{7.72}$$

$$\int_0^T\int_\Omega \varepsilon\frac{\eta_\varepsilon-\overline{\eta}}{\varepsilon}\partial_t\phi+[\eta\mathbf{u}_\varepsilon-\varepsilon\zeta\eta_\varepsilon\nabla_x\Phi-D\nabla_x\eta_\varepsilon]\cdot\phi\,\mathrm{d}x\,\mathrm{d}t=0 \tag{7.73}$$

for test functions $\phi\in C_c^\infty((0,T)\times\Omega),\mathbf{v}\in C_c^\infty((0,T)\times\Omega;\mathbb{R}^3)$. Thus, defining the following quantities,

$$\begin{aligned}&\varrho_\varepsilon^{(1)}:=\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon},\\ &\eta_\varepsilon^{(1)}:=\frac{\eta_\varepsilon-\overline{\eta}}{\varepsilon},\\ &r_\varepsilon:=\varrho_\varepsilon^{(1)}+\frac{D}{a\gamma\overline{\varrho}^{\gamma-1}\zeta}\eta_\varepsilon^{(1)}+\frac{(\beta\overline{\varrho}+\overline{\eta})\Phi}{a\gamma\overline{\varrho}^{\gamma-1}},\\ &\omega:=a\gamma\overline{\varrho}^{\gamma-1},\\ &\mathbf{V}_\varepsilon:=\varrho_\varepsilon\mathbf{u}_\varepsilon,\\ &h_\varepsilon^1:=\varepsilon\mathbb{S}(\nabla_x\mathbf{u}_\varepsilon)-\varepsilon\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon-\frac{[p(\varrho_\varepsilon,\eta_\varepsilon)]_{\mathrm{res}}}{\varepsilon}\mathbb{I},\\ &\mathbf{h}_\varepsilon^2:=\frac{1}{\omega}\left[\varepsilon D\eta_\varepsilon\nabla_x\Phi+\frac{D^2}{\zeta}\nabla_x\eta_\varepsilon-\frac{D}{\zeta}\eta_\varepsilon\mathbf{u}_\varepsilon\right],\\ &h_\varepsilon^3:=\frac{[p(\varrho_\varepsilon,\eta_\varepsilon)]_{\mathrm{ess}}-p(\overline{\varrho},\overline{\eta})}{\varepsilon}-p_F'(\overline{\varrho})\varrho_\varepsilon^{(1)}+p_P'(\overline{\eta})\eta_\varepsilon^{(1)},\end{aligned}$$

the system (7.71)–(7.73) becomes, after some algebra,

$$\int_0^T\int_\Omega\varepsilon r_\varepsilon\partial_t\phi+\mathbf{V}_\varepsilon\cdot\nabla_x\phi\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_\Omega\mathbf{h}_\varepsilon^2\cdot\nabla_x\phi\,\mathrm{d}x\,\mathrm{d}t, \tag{7.74}$$

$$\begin{aligned}&\int_0^T\int_\Omega\varepsilon\mathbf{V}_\varepsilon\cdot\partial_t\mathbf{v}+\omega r_\varepsilon\mathrm{div}_x\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_0^T\int_\Omega[\beta(\overline{\varrho}-\varrho_\varepsilon)+(\overline{\eta}-\eta_\varepsilon)]\nabla_x\Phi\cdot\mathbf{v}+h_\varepsilon^1:\nabla_x\mathbf{v}-h_\varepsilon^3\mathrm{div}_x\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{7.75}$$

for test functions $\phi \in C_c^\infty((0,T)\times\Omega), \mathbf{v} \in C_c^\infty((0,T)\times\Omega;\mathbb{R}^3)$. By the uniform bounds and convergence results, it is clear that

$$\|h_\varepsilon^1\|_{L^s(0,T;L^1(\Omega;\mathbb{R}^{3\times3}))} \leq \varepsilon c$$

for some $s > 1$ and

$$\mathbf{h}_\varepsilon^2 \to 0$$

weakly in the appropriate Lebesgue space. By the following lemma (adapted from Proposition 5.2 in [16]), $h_\varepsilon^3 \to 0$ weakly-$*$ in $L^\infty(0,T;L^1(\Omega))$:

LEMMA 7.1. Let $\{\varrho_\varepsilon\}_{\varepsilon>0}, \{\eta_\varepsilon\}_{\varepsilon>0}$ be sequences of nonnegative measurable functions such that $[\varrho_\varepsilon^{(1)}]_{\rm ess} \to \varrho^{(1)}, [\eta_\varepsilon^{(1)}]_{\rm ess} \to \eta^{(1)}$ weakly-$*$ in $L^\infty(0,T;L^2(\Omega))$. Suppose

$$\operatorname{ess\,sup}_{t\in(0,T)} |\mathcal{M}_{\rm res}^\varepsilon[t]| \leq \varepsilon^2 c$$

and that $p \in C^2(\overline{\mathcal{O}_{\rm ess}})$. Then defining h_ε^3 as above,

$$h_\varepsilon^3 \to 0 \text{ weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)).$$

Also, from the section on convergence and the properties of Φ, it is clear that

$$[\beta(\overline{\varrho} - \varrho_\varepsilon) + (\overline{\eta} - \eta_\varepsilon)]\nabla_x\Phi$$

converges to zero weakly in the appropriate Lebesgue space. Thus (7.74) and (7.75) represent a system of wave equations for which the right sides converge to zero. Now the associated eigenvalue problem for the left sides of (7.74) and (7.75) are considered:

$$\begin{aligned} \operatorname{div}_x \mathbf{w} &= \lambda q, \\ \omega \nabla_x q &= \lambda \mathbf{w}, \\ \mathbf{w}\cdot\mathbf{n}|_{\partial\Omega} &= 0, \end{aligned}$$

which can easily be reformulated as

$$\begin{aligned} -\Delta_x q &= \Lambda q, \\ \nabla_x q \cdot \mathbf{n}|_{\partial\Omega} &= 0, \\ -\Lambda &= \frac{\lambda^2}{\omega} \end{aligned} \tag{7.76}$$

(note that λ here is unrelated to the λ from the stress tensor). As is well known (cf. [16]), the system in (7.76) admits, in view of the assumption (H), a countable system of eigenvalues

$$0 = \Lambda_0 < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq ...$$

with associated eigenfunctions $\{q_n\}_{n=0}^\infty$ which form an orthogonal basis of $L^2(\Omega)$. Then, corresponding eigenfunctions $\mathbf{w}_{\pm n}$ are defined as

$$\mathbf{w}_{\pm n} := \pm i\sqrt{\frac{\omega}{\Lambda_n}}\nabla_x q_n$$

for each positive n. Also, the space $L^2(\Omega;\mathbb{R}^3)$ can be composed orthogonally into

$$L^2(\Omega;\mathbb{R}^3) = L_\sigma^2(\Omega;\mathbb{R}^3) \oplus L_g^2(\Omega;\mathbb{R}^3),$$

where

$$L^2_g(\Omega;\mathbb{R}^3) := \text{closure}_{L^2}\left\{\text{span}\left\{\frac{-i}{\sqrt{\omega}}\mathbf{w}_n\right\}_{n=1}^{\infty}\right\},$$

representing the closure of the gradient functions and

$$L^2_\sigma(\Omega;\mathbb{R}^3) := \text{closure}_{L^2}\{\mathbf{v}\in C_c^\infty(\Omega;\mathbb{R}^3)|\text{div}_x\mathbf{v}=0\},$$

representing the space of divergence-free functions.

With these spaces defined, the following projection can be defined:

$$\mathbf{P}_M : L^2(\Omega;\mathbb{R}^3) \to \text{span}\left\{\frac{-i}{\sqrt{\omega}}\mathbf{w}_n\right\}_{n\le M}$$

for each $M\in\mathbb{N}$. Noting that $\mathbf{P}_M$ and $\mathbf{H}^\perp$ commute, for the sake of notational simplicity, the operator $\mathbf{H}^\perp_M$ will be defined by

$$\mathbf{H}^\perp_M[\mathbf{v}] := \mathbf{P}_M\mathbf{H}^\perp[\mathbf{v}] = \mathbf{H}^\perp[\mathbf{P}_M\mathbf{v}].$$

Returning to the singular term, it is noted that

$$\begin{aligned}
&\int_0^T\int_\Omega \mathbf{H}^\perp[\varrho_\varepsilon\mathbf{u}_\varepsilon]\otimes\mathbf{H}^\perp[\mathbf{u}_\varepsilon] : \nabla_x\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^T\int_\Omega \mathbf{H}^\perp[\varrho_\varepsilon\mathbf{u}_\varepsilon]\otimes\mathbf{H}^\perp_M[\mathbf{u}_\varepsilon] : \nabla_x\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int_\Omega \mathbf{H}^\perp[\varrho_\varepsilon\mathbf{u}_\varepsilon]\otimes(\mathbf{H}^\perp[\mathbf{u}_\varepsilon]-\mathbf{H}^\perp_M[\mathbf{u}_\varepsilon]) : \nabla_x\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}\tag{7.77}$$

and by estimates shown in Section 5.4.6 of [16] and since $\varrho_\varepsilon\mathbf{u}_\varepsilon \to \overline{\varrho\mathbf{u}}$ weakly-$*$ in $L^\infty(0,T;L^{2q/q+1}(\Omega;\mathbb{R}^3))$,

$$\left|\int_0^T\int_\Omega \mathbf{H}^\perp[\varrho_\varepsilon\mathbf{u}_\varepsilon]\otimes(\mathbf{H}^\perp[\mathbf{u}_\varepsilon]-\mathbf{H}^\perp_M[\mathbf{u}_\varepsilon]) : \nabla_x\mathbf{v}dxdt\right|\to 0$$

uniformly in ε as $M\to\infty$. Also, since for fixed $\mathbf{v}\in[W^{1,2}(\Omega;\mathbb{R}^3)]*$ defined by the standard Reisz formula

$$\|\mathbf{H}^\perp[\mathbf{v}]-\mathbf{H}^\perp_M[\mathbf{v}]\|^2_{[W^{1,2}(\Omega;\mathbb{R}^3)]*}\to 0$$

uniformly in ε as $M\to\infty$, the problem of showing the weak convergence of $\mathbf{H}^\perp[\varrho_\varepsilon\mathbf{u}_\varepsilon]\otimes\mathbf{H}^\perp[\mathbf{u}_\varepsilon]$ to a gradient reduces to showing that for any fixed $M\in\mathbb{N}$,

$$\int_0^T\int_\Omega \mathbf{H}^\perp_M[\varrho_\varepsilon\mathbf{u}_\varepsilon]\otimes\mathbf{H}^\perp_M[\mathbf{u}_\varepsilon] : \nabla_x\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t\to 0$$

or by (7.61),

$$\int_0^T\int_\Omega \mathbf{H}^\perp_M[\varrho_\varepsilon\mathbf{u}_\varepsilon]\otimes\mathbf{H}^\perp_M[\varrho_\varepsilon\mathbf{u}_\varepsilon] : \nabla_x\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t\to 0$$

as $\varepsilon\to 0$ for any divergence-free $\mathbf{v}\in C_c^\infty((0,T)\times\Omega;\mathbb{R}^3)$ with $\mathbf{v}\cdot\mathbf{n}|_{\partial\Omega}=0$.

In order to handle this term, the test functions ϕ and $\mathbf{v}$ are defined as

$$\phi(t,x)=\psi(t)q_n(x)$$

and

$$\mathbf{v}(t,x) = \psi(t)\frac{1}{\sqrt{\Lambda_n}}\nabla_x q_n(x),$$

where $\psi \in C_c^\infty(0,T)$ and q_n is the corresponding eigenfunction from above. After some basic manipulations, the system (7.74) and (7.75) becomes

$$\varepsilon\partial_t b_n[r_\varepsilon] - \sqrt{\Lambda_n}\; a_n[\mathbf{V}_\varepsilon] = \chi^1_{\varepsilon,n}, \tag{7.78}$$

$$\varepsilon\partial_t a_n[\mathbf{V}_\varepsilon] + \omega\sqrt{\Lambda_n}\; b_n[r_\varepsilon] = \chi^2_{\varepsilon,n}, \tag{7.79}$$

where

$$a_n[\mathbf{V}_\varepsilon] := \frac{1}{\Lambda_n}\int_\Omega \mathbf{V}_\varepsilon \cdot \nabla_x q_n \,\mathrm{d}x,$$

$$b_n[\varrho_\varepsilon^{(1)}] := \int_\Omega \varrho_\varepsilon^{(1)} q_n \,\mathrm{d}x$$

are the appropriate Fourier coefficients, and $\chi^1_{\varepsilon,n}, \chi^2_{\varepsilon,n}$ are defined appropriately. It is easily seen that for each n, $\chi^1_{\varepsilon,n}, \chi^2_{\varepsilon,n}$ converge to zero in $L^1(\Omega)$ from the bounds on the remainder terms h^i_ε. Rewriting (7.78)–(7.79) in terms of the Helmholtz projectors, the system becomes

$$\varepsilon\partial_t[r_\varepsilon]_M + \mathrm{div}_x(\mathbf{H}^\perp_M[\varrho_\varepsilon\mathbf{u}_\varepsilon]) = \chi^3_{\varepsilon,M}, \tag{7.80}$$

$$\varepsilon\partial_t\mathbf{H}^\perp_M[\varrho_\varepsilon\mathbf{u}_\varepsilon] + \omega\nabla_x[r_\varepsilon]_M = \chi^4_{\varepsilon,M}, \tag{7.81}$$

where

$$[\varrho_\varepsilon^{(1)}]_M = \sum_{n=1}^M b_n[r_\varepsilon]q_n$$

and $\chi^3_{\varepsilon,M}, \chi^4_{\varepsilon,M}$ both converge to zero in $L^1(\Omega)$. Note also that $[r_\varepsilon]_M$ and $\mathbf{H}^\perp_M[\varrho_\varepsilon\mathbf{u}_\varepsilon]$ are both twice spatially differentiable and absolutely continuous in time. Thus the system (7.80)–(7.81) is defined and the potential $\Psi_{\varepsilon,M}$ can be defined such that

$$\Psi_{\varepsilon,M} = \mathbf{H}^\perp_M[\varrho_\varepsilon\mathbf{u}_\varepsilon], \int_\Omega \Psi_{\varepsilon,M}dx = 0.$$

Thus,

$$\int_0^T\int_\Omega \mathbf{H}^\perp_M[\varrho_\varepsilon\mathbf{u}_\varepsilon]\otimes\mathbf{H}^\perp_M[\varrho_\varepsilon\mathbf{u}_\varepsilon] : \nabla_x\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\int_\Omega \Delta_x\Psi_{\varepsilon,M}\nabla_x\Psi_{\varepsilon,M}\cdot\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t$$

for any test function $\mathbf{v}$ which has zero normal trace and is divergence free. Rewriting the right side of this equation as

$$\begin{aligned}&\int_0^T\int_\Omega \Delta_x\Psi_{\varepsilon,M}\nabla_x\Psi_{\varepsilon,M}\cdot\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t\\ &= \varepsilon\int_0^T\int_\Omega [r_\varepsilon]_M\nabla_x\Psi_{\varepsilon,M}\cdot\partial_t\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^T\int_\Omega \chi^3_{\varepsilon,M}\mathbf{H}^\perp_M[\varrho_\varepsilon\mathbf{u}_\varepsilon]\cdot\mathbf{v} + [r_\varepsilon]_M\chi^4_{\varepsilon,M}\cdot\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{7.82}$$

by using (7.80) and (7.81), it is clear from the convergences of $\chi^3_{\varepsilon,M}, \chi^4_{\varepsilon,M}$ that the right side of (7.82) converges to zero for any fixed zero normal trace, divergence free $\mathbf{v}$ as ε

goes to zero. Thus, it has been shown that $\mathbf{H}^{\perp}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^{\perp}[\mathbf{u}_\varepsilon]$ converges weakly to a gradient, completing the proof of Theorem 6.1.

Remark. The interested reader will notice that while the work in this section follows the outline of the work of Feireisl and Novotný, no time lifting is performed in the current paper. This is due to the fact that the Navier-Stokes-Fourier system investigated in [16] contains an entropy production term that behaves as a measure instead of an integral over the time domain. This complication does not arise in the Navier-Stokes-Smoluchowski system investigated here.

7.4. Ω *unbounded domain in* $\mathbb{R}^3$*.* In this section we will follow the framework of Feireisl [15] based on a Kato's result [23], Theorem 7.1.

Theorem 7.1. Let A be a closed densely defined linear operator and H a selfadjoint densely defined operator in a Hilbert space M. For $\lambda \in \mathbb{C} - \mathbb{R}$, let $R_H[\lambda] = (H - \lambda \mathrm{Id})^{-1}$ denote the resolvent of H. Suppose that

$$\Gamma := \sup_{\lambda \in \mathbb{C}-\mathbb{R}, v \in \mathcal{D}(A^*), \|v\|=1} \|A \circ R_H[\lambda] \circ A^*[v]\|_X < \infty. \tag{7.83}$$

Then

$$\sup_{w \in X, \|w\|_X = 1} \frac{\pi}{2} \int_{-\infty}^{\infty} \|A \exp(-\mathrm{i}tH)[w]\|_X^2 \mathrm{d}t \le \Gamma^2. \tag{7.84}$$

The proof of Theorem 6.2 is obtained using Lemmas 2.1, 2.2, Theorem 7.1 and following a similar line of argument as in [25].

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