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LONG-TIME ASYMPTOTICS OF THE ZERO LEVEL SET FOR THE HEAT EQUATION

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Abstract. The zero level set $Z(t) := \{\mathbf{x} \in \mathbf{R}^d : u(\mathbf{x}, t) = 0\}$ of a solution u to the heat equation in $\mathbf{R}^d$ is considered. Under vanishing conditions on moments of the initial data, we will prove that the set $Z(t)$ in a ball of radius $C\sqrt{t}$ for any $C > 0$ converges to a specific graph as $t \to \infty$ when the set is divided by $\sqrt{t}$. Solving a linear combination of the Hermite polynomials gives the graph, and coefficients of the linear combination depend on moments of the initial data. Also the graphs to which the zero level set $Z(t)$ converges are classified in some cases.

1. Introduction. Consider the heat equation in the whole domain $\mathbf{R}^d,\ d \geq 1$, with integrable initial data

$$\begin{aligned} u_t &= \Delta u \quad \text{in } \mathbf{R}^d \times (0, \infty), \\ u(\mathbf{x}, 0) &= u_0(\mathbf{x}) \in L^1(\mathbf{R}^d). \end{aligned} \tag{1.1}$$

The purpose of this paper is to investigate the long-time asymptotics of the zero level set $Z(t) := \{\mathbf{x} \in \mathbf{R}^d : u(\mathbf{x}, t) = 0\}$ of the solution u. Note that the nonzero level set $\{\mathbf{x} \in \mathbf{R}^d : u(\mathbf{x}, t) = c \neq 0\}$ becomes empty in finite time since the solution u goes to zero uniformly as time passes. So we consider the zero level set only. We start with a simple example. Recall the heat kernel of the whole space $\mathbf{R}^d$

$$G(\mathbf{x}, t) := (4\pi t)^{-\frac{d}{2}} \exp\Big(- \frac{|\mathbf{x}|^2}{4t} \Big) \tag{1.2}$$

and define

$$u(\mathbf{x}, t) := G(\mathbf{x}, t) - G(\mathbf{x}, t + T) \quad \text{for some fixed } T > 0.$$

Then the function u is a solution to the heat equation. Let $\mathbf{z}(t)$ be a zero of the function u, i.e., $u(\mathbf{z}(t), t) = 0$ for each time $t > 0$. Then we can show that

$$\frac{|\mathbf{z}(t)|}{\sqrt{t}} \to \sqrt{2d} \quad \text{as } t \to \infty.$$

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This example suggests that the order of $\sqrt{t}$ describes the long-time asymptotics of zeros. It is natural because $\sqrt{t}$ is a self-similar scale which represents the speed of spatial propagation in the diffusion process [4]. In this paper we will show that this is true in general; under vanishing conditions on moments of the initial data, the zero level set $Z(t)$ converges to a specific graph as $t \to \infty$. Also the graphs to which the zero level set $Z(t)$ converges will be classified in some cases.

REMARK 1.1. Consider a parabolic equation

$$v_t = \Delta v + \mathbf{b}(\mathbf{x},t) \cdot \nabla v + c(\mathbf{x},t)v, \quad \mathbf{b}(\mathbf{x},t) = (b_i(\mathbf{x},t)).$$

Then using a transform given by Angenent [2],

$$w(\mathbf{x},t) := \exp\Big\{\frac{1}{2}\sum_{i=1}^{d}\int_0^{x_i} b_i(s,t)\,ds\Big\}\, v(\mathbf{x},t),$$

we have a reaction-diffusion equation

$$w_t = \Delta w + w\Big(\frac{1}{2}\sum_{i=1}^{d}\int_0^{x_i}\frac{\partial b_i}{\partial t}(s,t)\,ds + c - \frac{1}{2}(\nabla\cdot\mathbf{b}) - \frac{1}{4}|\mathbf{b}|^2\Big).$$

Also two functions v and w have exactly the same zeros. Hence when the reaction coefficient of the above equation vanishes, our study on zeros of the heat equation can be applicable to more general parabolic equations without any change.

Study of the zero level set is important in both theoretical and applied aspects. In the theoretical aspects, for solutions u of a *one-dimensional* semilinear parabolic equation, there are important facts about the zero level set $Z(t) = \{x \in \mathbf{R} : u(x,t) = 0\}$. Angenent [2] proved that the set is *discrete* if the initial data is nontrivial. Also Sturm (see [12]) and Matano [16] showed that *the number of zeros does not increase* as time passes. Local behavior and the decrease of the number of zeros near a multiple zero has been studied by Angenent [2] and Chen [6]. Brunovský and Fiedler [5] showed that a solution $u(\cdot,t)$ to a one-dimensional reaction-diffusion equation has only simple zeros for t in an open dense subset of $\mathbf{R}^+$. The nonincreasing property of the number of zeros has many applications including the asymptotic stability of nonlinear parabolic equations. We refer those interested to a book by Galaktionov [12] and a survey by Galaktionov and Harwin [13]. In multi-dimensions, Chen [6] also classified the local behavior of solutions to a system of second-order parabolic inequalities and obtained upper bounds for dimensions of zero level sets. However, *counting the number of zeros is not applicable* to multi-dimensions where the zero level set consists of curves, not points.

In the applied aspects, a topologically complicated curve can be a zero level set of a high-dimensional function. Hence the evolution of the zero level set can represent a moving interface like flame front. A numerical technique based on this idea is the level set method, which was developed by Osher and Sethian [19]. Those interested in the method are referred to a book by Osher and Fedkiw [18]. In this aspect, the long-time asymptotics of the zero level set show us the ultimate shape of a moving interface.

2. Hermite polynomial approximation. Our result is based on an approximation of solutions to the heat equation given by Duoandikoetxea and Zuazua [10]. As a special case of their estimates, we can yield the following result:

THEOREM 2.1 (Duoandikoetxea-Zuazua). Let u be the solution to the heat equation (1.1) with initial data $u_0 \in L^1(\mathbf{R}^d; 1+|\mathbf{x}|^{k+1})$ for some nonnegative integer k. Then there is a positive constant $C(k,d)$ such that

$$\Big\| u(\cdot,t) - \sum_{|\alpha|\le k} \frac{(-1)^{|\alpha|}}{\alpha!} \Big(\int \mathbf{x}^\alpha u_0(\mathbf{x})\, d\mathbf{x} \Big) D^\alpha G(\cdot,t) \Big\|_\infty \le C t^{-(k+d+1)/2} \big\| |\mathbf{x}|^{k+1} u_0(\mathbf{x}) \big\|_1 .$$

Here the function $G(\mathbf{x},t)$ is the heat kernel (1.2) and $D^\alpha G(\mathbf{x},t)$ is its partial derivative with respect to multi-index α.

The theorem states that a linear combination of partial derivatives of the heat kernel can describe the long-time asymptotics of a solution to the heat equation. Partial derivatives of the heat kernel can be written as a product of the Hermite polynomials $H_n(x)$, which are given by (see [1, p. 775])

$$H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

The first few polynomials are

$$\begin{aligned} H_0(x) &= 1, \\ H_1(x) &= 2x, \\ H_2(x) &= 4x^2 - 2, \\ H_3(x) &= 8x^3 - 12x, \\ H_4(x) &= 16x^4 - 48x^2 + 12, \\ H_5(x) &= 32x^5 - 160x^3 + 120x. \end{aligned}$$

By the change of variables $\mathbf{y} = \mathbf{x}/(2\sqrt{t})$, we can get

$$\begin{aligned} D^\alpha_{\mathbf{x}} G(\mathbf{x},t) &= (4\pi t)^{-\frac{d}{2}} D^\alpha_{\mathbf{x}} \exp\Big(- \frac{|\mathbf{x}|^2}{4t} \Big) \\ &= (4\pi t)^{-\frac{d}{2}} D^\alpha_{\mathbf{y}} \exp(-|\mathbf{y}|^2) \frac{1}{(2\sqrt{t})^{|\alpha|}} \\ &= \pi^{-\frac{d}{2}} (4t)^{-\frac{|\alpha|+d}{2}} (-1)^{|\alpha|} \prod_{i=1}^d \big[H_{\alpha_i}(y_i) e^{-y_i^2} \big] \\ &= \pi^{-\frac{d}{2}} (4t)^{-\frac{|\alpha|+d}{2}} (-1)^{|\alpha|} e^{-\frac{|\mathbf{x}|^2}{4t}} \prod_{i=1}^d H_{\alpha_i}\Big(\frac{x_i}{2\sqrt{t}} \Big). \end{aligned}$$

Using this relation we can obtain the following estimate via the Hermite polynomials:

THEOREM 2.2 (Hermite polynomial approximation). Let u be the solution to the heat equation (1.1) with initial data $u_0 \in L^1(\mathbf{R}^d; 1+|\mathbf{x}|^{k+1})$ for some nonnegative integer k.

Then there is a positive constant $C(k,d)$ such that

$$\Big|u(\mathbf{x},t) - \pi^{-\frac{d}{2}} e^{-\frac{|\mathbf{x}|^2}{4t}} \sum_{|\alpha|\le k} \frac{\int \mathbf{x}^\alpha u_0(\mathbf{x})\,d\mathbf{x}}{\alpha!} (4t)^{-\frac{|\alpha|+d}{2}} \prod_{i=1}^{d} H_{\alpha_i}\Big(\frac{x_i}{2\sqrt{t}}\Big)\Big| \le Ct^{-\frac{k+d+1}{2}} \big\| |\mathbf{x}|^{k+1} u_0(\mathbf{x})\big\|_1 \quad \text{for all } \mathbf{x} = (x_i) \in \mathbf{R}^d. \tag{2.1}$$

Remark 2.3. Because the constant C depends on k, the inequality (2.1) does not tell us what happens when $k \to \infty$, i.e., when the number of terms in the sum goes to infinity. Actually Kim and Ni [14] provided a numerical example in which the Hermite polynomial approximation diverges. Also they suggested a linear combination of the heat kernels (not their derivatives) as an approximation to solutions of the heat equation and conjectured it would converge even if the number of terms in the linear combination goes to infinity. Their idea can be applied to find an approximation of solutions to the viscous Burgers equation [7].

Remark 2.4. The Hermite polynomial approximation also can be thought of as an eigenfunction expansion. See Witelski and Bernoff [21].

3. Long-time asymptotics of the zero level set. The Hermite polynomial approximation (2.1) will play a leading role in studying the asymptotics of zeros. Let $\mathbf{z}(t)$ be the zeros of a solution $u(\mathbf{x},t)$ defined for each time $t \in (T,\infty)$ for some $T > 0$, i.e., $u(\mathbf{z}(t),t) = 0$. If $u_0 \in L^1(\mathbf{R}^d; 1+|\mathbf{x}|)$ and the mass of initial data $\int u_0(\mathbf{x})\,d\mathbf{x}$ is nonzero, then the inequality (2.1) with $k = 0$ and $\mathbf{x} = \mathbf{z}(t)$ implies that

$$\Big|(4\pi t)^{-\frac{d}{2}} e^{-\frac{|\mathbf{z}(t)|^2}{4t}} \int u_0(\mathbf{x})\,d\mathbf{x}\Big| \le Ct^{-\frac{d+1}{2}} \big\| |\mathbf{x}| u_0(\mathbf{x})\big\|_1.$$

Assume $|\mathbf{z}(t)|/\sqrt{t}$ is bounded. Then for some positive constant $\tilde{C}$,

$$\Big|\int u_0(\mathbf{x})\,d\mathbf{x}\Big| \le \tilde{C} t^{-\frac{1}{2}} \big\| |\mathbf{x}| u_0(\mathbf{x})\big\|_1$$

and $\int u_0(\mathbf{x})\,d\mathbf{x} \to 0$ as $t \to \infty$. But this contradicts the fact that the mass is nonzero. Hence $|\mathbf{z}(t)|/\sqrt{t}$ cannot be bounded. This result is stated in the following:

Theorem 3.1 (Zeros with nonzero mass). Let u be the solution to the heat equation (1.1) with initial data $u_0 \in L^1(\mathbf{R}^d; 1+|\mathbf{x}|)$. If the mass of initial data $\int u_0(\mathbf{x})\,d\mathbf{x}$ is nonzero, then for any zeros $\mathbf{z}(t)$ of the solution $u(\cdot,t)$, defined for $t \in (T,\infty)$ for some $T > 0$,

$$\limsup_{t\to\infty} \frac{|\mathbf{z}(t)|}{\sqrt{t}} = \infty.$$

Remark 3.2. For the *one-dimensional* heat equation, Mizoguchi [17] proved that any zero level set $Z(t)$ is contained in $[-Ct, Ct]$ for large $t > 0$ with some $C > 0$ if the initial data changes sign a finite number of times. Because $u(x,t) := e\,G(x,t) - G(x-2,t)$ has a unique zero $z(t) = t+1$, Mizoguchi's upper bound t is optimal. Theorem 3.1 suggests a (strictly) lower bound $\sqrt{t}$. But we do not know whether this lower bound is optimal or not.

Another easy consequence we can obtain from the Hermite polynomial approximation (2.1) is about the asymptotic behavior of bounded zeros. Assume $|\mathbf{z}(t)|/\sqrt{t}$ is bounded and $u_0 \in L^1(\mathbf{R}^d; 1 + |\mathbf{x}|^{k+1})$. Then multiplying both sides of the inequality (2.1) with $\mathbf{x} = \mathbf{z}(t) = (z_i(t))$ by $e^{\frac{|\mathbf{z}(t)|^2}{4t}}$, we obtain

$$\Big|\pi^{-\frac{d}{2}} \sum_{|\alpha|\le k} \frac{\int \mathbf{x}^\alpha u_0(\mathbf{x})\, d\mathbf{x}}{\alpha!} (4t)^{-\frac{|\alpha|+d}{2}} \prod_{i=1}^{d} H_{\alpha_i}\Big(\frac{z_i(t)}{2\sqrt{t}}\Big)\Big| \le C e^{\frac{|\mathbf{z}(t)|^2}{4t}}\, t^{-\frac{k+d+1}{2}} \left\| |\mathbf{x}|^{k+1} u_0(\mathbf{x})\right\|_1 .$$

Because $|\mathbf{z}(t)|/\sqrt{t}$ is bounded, the right-hand side is just a constant times $t^{-\frac{k+d+1}{2}}$ and so we proved the following theorem:

Theorem 3.3 (Bounded zeros). Let u be the solution to the heat equation (1.1) with initial data $u_0 \in L^1(\mathbf{R}^d; 1 + |\mathbf{x}|^{k+1})$ and let $\mathbf{z}(t) = (z_i(t))$ be zeros of the solution $u(\cdot, t)$ defined for $t \in (T, \infty)$ for some $T > 0$. If $|\mathbf{z}(t)|/\sqrt{t}$ is bounded, then

$$\Big|\sum_{|\alpha|\le k} \frac{\int \mathbf{x}^\alpha u_0(\mathbf{x})\, d\mathbf{x}}{\alpha!} (4t)^{-\frac{|\alpha|}{2}} \prod_{i=1}^{d} H_{\alpha_i}\Big(\frac{z_i(t)}{2\sqrt{t}}\Big)\Big| = O\big(t^{-\frac{k+1}{2}}\big) \quad \text{as } t \to \infty.$$

If the zeroth moment (mass) is nonzero, Theorem 3.1 tells us that no zeros can be inside of a ball of radius $\sqrt{t}$. If the zeroth moment is zero, then without loss of generality we may assume that the first $k-1$ moments ($k \ge 1$) are zero and the k-th moment is nonzero. Under this vanishing moments condition, the above theorem can be simplified to the following theorem:

Theorem 3.4 (Bounded zeros with vanishing moments). Let u be the solution to the heat equation (1.1) with initial data $u_0 \in L^1(\mathbf{R}^d; 1 + |\mathbf{x}|^{k+1})$ and let $\mathbf{z}(t) = (z_i(t))$ be zeros of the solution $u(\cdot, t)$ defined for $t \in (T, \infty)$ for some $T > 0$. If $|\mathbf{z}(t)|/\sqrt{t}$ is bounded and the moments of the solution u satisfy

$$\int \mathbf{x}^\alpha u_0(\mathbf{x})\, d\mathbf{x} = 0 \quad \text{for all } |\alpha| < k,$$

then

$$\Big|\sum_{|\alpha|= k} \frac{\int \mathbf{x}^\alpha u_0(\mathbf{x})\, d\mathbf{x}}{\alpha!} \prod_{i=1}^{d} H_{\alpha_i}\Big(\frac{z_i(t)}{2\sqrt{t}}\Big)\Big| = O\big(t^{-\frac{1}{2}}\big) \quad \text{as } t \to \infty. \tag{3.1}$$

The theorem tells us that when the propagation speed of zeros is bounded by $\sqrt{t}$, the zeros divided by $\sqrt{t}$ eventually become close to the zeros of a linear combination of the Hermite polynomials. But we can go further; near simple zeros of the polynomial, there is at least one zero of the solution under the same moments condition. To put it concretely, the theorem we will prove is the following:

Theorem 3.5 (Existence of zeros). Let u be the solution to the heat equation (1.1) with initial data $u_0 \in L^1(\mathbf{R}^d; 1+|\mathbf{x}|^{k+1})$, $k \ge 1$. Assume the initial data u_0 has zero moments up to order $k-1$ and nonzero k-th moment. Then for each simple zero $\mathbf{x}_0$ of a polynomial

$\phi_k(\mathbf{x})$, there exist a constant $T > 0$ and zeros $\mathbf{z}(t) = (z_i(t))$ of the solution $u(\cdot, t)$ defined for $t \in (T, \infty)$ such that

$$\frac{\mathbf{z}(t)}{2\sqrt{t}} \to \mathbf{x}_0 \quad \text{as } t \to \infty.$$

Here the polynomial $\phi_k(\mathbf{x})$ is defined by

$$\phi_k(\mathbf{x}) := \sum_{|\alpha|=k} \frac{\int \mathbf{x}^\alpha u_0(\mathbf{x})\, d\mathbf{x}}{\alpha!} \prod_{i=1}^{d} H_{\alpha_i}(x_i), \quad \mathbf{x} = (x_i).$$

The zeros $\mathbf{z}(t)$ in Theorem 3.5 satisfy the boundedness condition of Theorem 3.4 and so the convergence order (3.1) is obtained.

Proof. Let $\mathbf{x}_0$ be a simple zero of the polynomial $\phi_k(\mathbf{x})$. Then there are two points $\mathbf{x}_1$, $\mathbf{x}_2 \in \mathbf{R}^d$ such that

$$\phi_k(\mathbf{x}_1) > 0, \quad \phi_k(\mathbf{x}_2) < 0$$

and on the line segment $I := \{\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 : 0 \le \lambda \le 1\}$ joining $\mathbf{x}_1$ and $\mathbf{x}_2$, the polynomial $\phi_k(\mathbf{x})$ have a unique zero $\mathbf{x}_0$. Now notice that from the Hermite polynomial approximation (2.1) there is a positive constant C such that

$$\left| \pi^{\frac{d}{2}} (4t)^{\frac{k+d}{2}} e^{\frac{|\mathbf{x}|^2}{4t}} u(\mathbf{x}, t) - \phi_k\Big(\frac{\mathbf{x}}{2\sqrt{t}}\Big) \right| \le C e^{\frac{|\mathbf{x}|^2}{4t}} t^{-\frac{1}{2}} \tag{3.2}$$

for any $\mathbf{x} \in \mathbf{R}^d$. Put $\mathbf{x} = 2\sqrt{t}\,\mathbf{x}_1$ and $\mathbf{x} = 2\sqrt{t}\,\mathbf{x}_2$ separately into (3.2) and observe that as $t \to \infty$,

$$\pi^{\frac{d}{2}} (4t)^{\frac{k+d}{2}} e^{|\mathbf{x}_j|^2} u(2\sqrt{t}\,\mathbf{x}_j, t) \to \phi_k(\mathbf{x}_j), \quad j = 1, 2.$$

Hence there exists a constant $T > 0$ such that

$$u(2\sqrt{t}\,\mathbf{x}_1, t) > 0 \quad \text{and} \quad u(2\sqrt{t}\,\mathbf{x}_2, t) < 0 \quad \text{for every } t \ge T.$$

Because the solution $u(\cdot, t)$ is continuous, there is at least one zero $\mathbf{z}(t)$ on the line segment joining $2\sqrt{t}\,\mathbf{x}_1$ and $2\sqrt{t}\,\mathbf{x}_2$, i.e., $\mathbf{z}(t)/(2\sqrt{t}) \in I$. (Of course $\mathbf{z}(t)$ is defined for $t \ge T$.) Therefore $|\mathbf{z}(t)|/\sqrt{t}$ is bounded and we can apply Theorem 3.4. By equation (3.1)

$$\phi_k\Big(\frac{\mathbf{z}(t)}{2\sqrt{t}}\Big) \to 0 \quad \text{as } t \to \infty.$$

But $\mathbf{z}(t)/(2\sqrt{t}) \in I$ and the only zero of the polynomial $\phi_k(\mathbf{x})$ on the line segment I is $\mathbf{x}_0$. Consequently we can conclude that $\mathbf{z}(t)/(2\sqrt{t}) \to \mathbf{x}_0$ as $t \to \infty$. $\square$

In summary, the zero level set $Z(t)$ in a ball of radius $C\sqrt{t}$ for some $C > 0$ converges to the zeros of a linear combination of the Hermite polynomial when the set is divided by $\sqrt{t}$. Hence the asymptotics of the zero level set are closely related to the zeros of the Hermite polynomials. In the following sections we will present more detailed results on the asymptotics for some specific cases.

4. One-dimensional heat equation. In this section we assume the dimension $d = 1$. Hence the zeros can be counted [2] and the number of zeros is nonincreasing [16]. Furthermore we can count the initial number of zeros under the vanishing moments condition as the following lemma gives us:

LEMMA 4.1. Suppose a nonzero continuous function $f : \mathbf{R} \to \mathbf{R}$ has zero moments up to order m. Then the function f changes sign at least $m+1$ times.

Proof. If $m = 0$, the nonzero continuous function f should change sign at least one time because otherwise the mass (zeroth moment) cannot be zero. Now assume $m \geq 1$ and the function f changes sign k times where $1 \leq k < m+1$. Then there are k points $c_1, c_2, \cdots, c_k$ such that

(1) $f(x)$ does not change sign in the interval $[c_i, c_{i+1}]$ and

(2) $f(\xi_i) \cdot f(\xi_{i+1}) < 0$ for some $\xi_i \in (c_i, c_{i+1}),\ \xi_{i+1} \in (c_{i+1}, c_{i+2})$

where $c_0 = -\infty, c_{k+1} = \infty$ and $0 \leq i \leq k-1$. Then $\int (x-c_1)(x-c_2)\cdots(x-c_k)f(x)\,dx$ is a linear combination of moments of order less than or equal to m and should be zero. But it cannot be because a continuous function $(x-c_1)(x-c_2)\cdots(x-c_k)f(x)$ does not change sign and is nonzero. □

On the other hand, Theorem 3.5 can be simplified because the polynomial $\phi_k(\mathbf{x})$ is just

$$\phi_k(x) = \frac{\int x^k u_0(x)\,dx}{k!} H_k(x).$$

Thus if the k-th moment is nonzero, then the zeros of $\phi_k(x)$ and the zeros of $H_k(x)$ are exactly the same. Also we know that every zero of the Hermite polynomials is simple. (Actually all the zeros of the orthogonal polynomials are simple. See [1, p. 787]) Hence a simplified version of Theorem 3.5 can be found:

THEOREM 4.2 (Existence of zeros in one dimension). Let u be the solution to the one-dimensional heat equation (1.1) with initial data $u_0 \in L^1(\mathbf{R}; 1 + |\mathbf{x}|^{k+1})$, $k \geq 1$. If the initial data u_0 has zero moments up to order $k-1$ and nonzero k-th moment, then for each zero x_0 of the Hermite polynomial $H_k(x)$, there exist a constant $T > 0$ and zeros $z(t)$ of the solution $u(\cdot, t)$ defined for $t \in (T, \infty)$ such that

$$\frac{z(t)}{2\sqrt{t}} \to x_0 \quad \text{as } t \to \infty.$$

We may assume that the initial data u_0 is continuous because the solution u becomes continuous immediately and the vanishing moments condition at initial time holds for all time $t > 0$ (see [14]). If the initial data u_0 has zero moments up to order $k-1$, then by Lemma 4.1 the solution u has at least k zeros for all time $t > 0$. Furthermore if the initial data u_0 has k zeros initially, then by the nonincreasing property of the zero set, the solution u has at most k zeros for all time $t > 0$. Therefore there are exactly k zeros for all time $t > 0$ and by Theorem 4.2 we can describe the asymptotics of all those k zeros:

THEOREM 4.3 (Asymptotics of zeros in one dimension). Let u be the solution to the one-dimensional heat equation (1.1) with initial data $u_0 \in L^1(\mathbf{R}; 1 + |\mathbf{x}|^{k+1})$, $k \geq 1$. Assume the initial data u_0 has zero moments up to order $k-1$ and nonzero k-th moment. If the initial data u_0 has k zeros, then there are exactly k continuous curves of zeros $z_i(t),\ i = 1, \cdots, k$, defined for all time $t \in (0, \infty)$ and each zero satisfies

$$\frac{z_i(t)}{2\sqrt{t}} \to x_i \quad \text{as } t \to \infty,$$

where $x_i,\ i=1,\cdots,k$, are different zeros of the Hermite polynomial $H_k(x)$.

The zeros of the first few Hermite polynomials $H_k(x)$ are given below:

k	zeros of $H_k(x)$	approximate values
1	0	0
2	$\pm\sqrt{2}/2$	± 0.707
3	$0,\ \pm\sqrt{6}/2$	$0,\ \pm 1.225$
4	$\pm\sqrt{6-2\sqrt{6}}/2,\ \pm\sqrt{6+2\sqrt{6}}/2$	$\pm 0.525,\ \pm 1.651$
5	$0,\ \pm\sqrt{10-2\sqrt{10}}/2,\ \pm\sqrt{10+2\sqrt{10}}/2$	$0,\ \pm 0.959,\ \pm 2.020$

Now we are ready to explain the example given in the introduction for the one spatial dimension.

Example 4.4. Let the initial data u_0 be the difference of two time-delayed heat kernels:

$$u_0(x) := G(x,t) - G(x,t+T), \quad \text{for some fixed } T>0.$$

Then due to the symmetry, the zeroth and first moments of u_0 are zero and the second moment is nonzero. Also u_0 has only two zeros. Therefore by Theorem 4.3, there are exactly two continuous curves of zeros $z_\pm(t)$ and they satisfy

$$\frac{z_\pm(t)}{\sqrt{t}} \to \pm\sqrt{2} \quad \text{as } t\to\infty.$$

We can verify this fact via computation because the solution u is explicitly given by

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} - \frac{1}{\sqrt{4\pi(t+T)}} e^{-\frac{x^2}{4(t+T)}}.$$

Assume $u(z(t),t)=0$ implies that

$$\begin{aligned}\sqrt{1+\frac{T}{t}} &= \exp\left(-\frac{z^2(t)}{4(t+T)} + \frac{z^2(t)}{4t}\right)\\ &= \exp\left(\frac{T}{4t(t+T)} z^2(t)\right).\end{aligned}$$

Hence $z^2(t)/t = 2(1+t/T)\ln(1+T/t) \to 2$ as $t\to\infty$.

5. Radially symmetric initial data. In this section we assume that the initial data u_0 is radially symmetric, i.e., $u_0(\mathbf{x}) = u_0(r)$ where $r=|\mathbf{x}|$. Then the solution is radially symmetric for all time $t>0$ and the heat equation (1.1) can be rewritten as

$$\begin{aligned} u_t &= u_{rr} + (d-1)\frac{u_r}{r} \quad \text{in } \mathbf{R}\times(0,\infty),\\ u(r,0) &= u_0(r) \quad \text{in } \mathbf{R}.\end{aligned}$$

Therefore the situation is similar to the one-dimensional case; the circles of zeros can be counted and their number is nonincreasing. Also the polynomial $\phi_k(\mathbf{x})$ in Theorem 3.5

can be simplified by observing moments. Assume α_i is *odd*. Then

$$\begin{aligned}\int_{-\infty}^{\infty} x_i^{\alpha_i} u_0(r)\, dx_i &= \int_0^{\infty} x_i^{\alpha_i} u_0(r)\, dx_i + \int_{-\infty}^{0} x_i^{\alpha_i} u_0(r)\, dx_i \\ &= \int_0^{\infty} x_i^{\alpha_i} u_0(r)\, dx_i - \int_0^{\infty} x_i^{\alpha_i} u_0(r)\, dx_i = 0.\end{aligned}$$

Hence for a nonzero moment *every* α_i *is even.* Consequently a moment is zero when $|\alpha|$ is odd.

Now we may assume every α_i is even and $d \geq 2$. Using hyperspherical coordinates

$$\begin{aligned} x_1 &= r\cos(\psi_1), \\ x_2 &= r\sin(\psi_1)\cos(\psi_2), \\ &\ \vdots \\ x_{d-1} &= r\sin(\psi_1)\cdots\sin(\psi_{d-2})\cos(\psi_{d-1}), \\ x_d &= r\sin(\psi_1)\cdots\sin(\psi_{d-2})\sin(\psi_{d-1}), \end{aligned}$$

moments of the initial data can be written as

$$\begin{aligned}\int \mathbf{x}^\alpha u_0(\mathbf{x})\, d\mathbf{x} &= \int_{r=0}^{\infty}\int_{\psi_1=0}^{\pi}\cdots\int_{\psi_{d-2}=0}^{\pi}\int_{\psi_{d-1}=0}^{2\pi} r^{|\alpha|}\big(\cos(\psi_1)\big)^{\alpha_1}\big(\sin(\psi_1)\cos(\psi_2)\big)^{\alpha_2} \\ &\quad \times\cdots\times\big(\sin(\psi_1)\cdots\sin(\psi_{d-2})\cos(\psi_{d-1})\big)^{\alpha_{d-1}}\big(\sin(\psi_1)\cdots\sin(\psi_{d-2})\sin(\psi_{d-1})\big)^{\alpha_d} \\ &\quad \times r^{d-1}\sin^{d-2}(\psi_1)\sin^{d-3}(\psi_2)\cdots\sin(\psi_{d-2})\, u_0(r)\, dr\, d\psi_1\, d\psi_2\cdots d\psi_{d-1} \\ &= \int_0^{\infty} r^{|\alpha|+d-1}u_0(r)\, dr \times \prod_{i=1}^{d-2}\int_0^{\pi}\cos^{\alpha_i}(\psi_i)\sin^{\sum_{j=i+1}^{d}\alpha_j+d-i-1}(\psi_i)\, d\psi_i \\ &\quad \times \int_0^{2\pi}\cos^{\alpha_{d-1}}(\psi_{d-1})\sin^{\alpha_d}(\psi_{d-1})\, d\psi_{d-1}.\end{aligned}$$

Now we use trigonometric integrals

$$\begin{aligned}\frac{1}{n!}\int_0^{\pi}\cos^n(\psi)\sin^m(\psi)\, d\psi &= \frac{\int_0^{\pi}\sin^m(\psi)\, d\psi}{(m+n)(m+n-2)\cdots(m+2)\, 2^{n/2}(n/2)!} \quad \text{if } n \text{ is even} \\ &= \begin{cases} \frac{2((m-1)(m-3)\cdots 2)}{((m+n)(m+n-2)\cdots 1)\, 2^{n/2}(n/2)!} & \text{if } m \text{ is odd,} \\ \frac{\pi((m-1)(m-3)\cdots 1)}{((m+n)(m+n-2)\cdots 2)\, 2^{n/2}(n/2)!} & \text{if } m \text{ is even,} \end{cases}\end{aligned}$$

$$\frac{1}{n!m!}\int_0^{2\pi}\cos^n(\psi)\sin^m(\psi)\, d\psi = \frac{2\pi}{((n+m)(n+m-2)\cdots 2)\, 2^{(n+m)/2}(n/2)!(m/2)!} \quad \text{if } n, m \text{ are even}$$

to conclude that

$$\frac{1}{\alpha!}\int \mathbf{x}^\alpha u_0(\mathbf{x})\, d\mathbf{x} = \frac{C(|\alpha|, d)}{2^{|\alpha|/2}\prod_{i=1}^{d}(\alpha_i/2)!}\int_0^{\infty} r^{|\alpha|+d-1}u_0(r)\, dr$$

where $C(|\alpha|, d)$ is a nonzero constant defined by

$$C(|\alpha|, d) := \begin{cases} \frac{(2\pi)^{d/2}}{(|\alpha|+d-2)(|\alpha|+d-4)\cdots 2} & \text{if } d \text{ is even,} \\ \frac{2\,(2\pi)^{(d-1)/2}}{(|\alpha|+d-2)(|\alpha|+d-4)\cdots 1} & \text{if } d \text{ is odd.} \end{cases}$$

Also we recall the generalized Laguerre polynomials [1, p. 775]

$$L_n^{(a)}(x) := \frac{x^{-a}e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+a})$$

and their properties [1, p. 785, p. 779]

$$L_n^{(a+b+1)}(x+y) = \sum_{i=0}^{n} L_i^{(a)}(x) L_{n-i}^{(b)}(y),$$

$$H_{2n}(x) = (-1)^n\, 2^{2n}\, n!\, L_n^{(-1/2)}(x^2).$$

Then the polynomial $\phi_k(\mathbf{x})$ in Theorem 3.5 becomes

$$\begin{aligned} \phi_k(\mathbf{x})\Big/\int_0^\infty r^{|\alpha|+d-1}u_0(r)\,dr &= \sum_{|\alpha|=k}\frac{\int \mathbf{x}^\alpha u_0(\mathbf{x})\,d\mathbf{x}}{\alpha!}\prod_{i=1}^{d} H_{\alpha_i}(x_i)\Big/\int_0^\infty r^{|\alpha|+d-1}u_0(r)\,dr \\ &= \sum_{|\alpha|=k}\frac{C(|\alpha|,d)}{2^{|\alpha|/2}\prod_{i=1}^{d}(\alpha_i/2)!}\prod_{i=1}^{d} H_{\alpha_i}(x_i) \\ &= (-2)^{k/2}C(k,d)\sum_{|\alpha|=k}\prod_{i=1}^{d} L_{\alpha_i/2}^{(-1/2)}(x_i^2) \\ &= (-2)^{k/2}C(k,d)\, L_{k/2}^{((d-2)/2)}(r^2). \end{aligned}$$

Therefore $\phi_k(\mathbf{x})$ and $L_{k/2}^{((d-2)/2)}(r^2)$ have exactly the same zeros. Hence we can conclude the following theorem:

THEOREM 5.1 (Radial symmetric initial data). Let u be the solution to the heat equation (1.1) with radially symmetric initial data $u_0 \in L^1(\mathbf{R}^d; 1 + |\mathbf{x}|^{k+1})$. Assume that the initial data u_0 has zero moments up to order $k-1$ and a nonzero k-th moment. Then k should be an even number and for each zero r_0 of the generalized Laguerre polynomial $L_{k/2}^{((d-2)/2)}(r^2)$ there exist a constant $T > 0$ and zeros $\mathbf{z}(t) = (z_i(t))$ of the solution $u(\cdot, t)$ defined for $t \in (T, \infty)$ such that

$$\frac{|\mathbf{z}(t)|}{2\sqrt{t}} \to r_0 \quad \text{as } t \to \infty.$$

Hence if the initial data u_0 has exactly k circles of zeros, then there are always exactly k circles of zeros and their radii divided by $2\sqrt{t}$ converges to separate zeros of the generalized Laguerre polynomial $L_{k/2}^{((d-2)/2)}(r^2)$ as $t \to \infty$. From this observation we can generalize Example 4.4 to multi-dimensions.

EXAMPLE 5.2. Let the initial data u_0 be the difference of two time-delayed heat kernels:

$$u_0(\mathbf{x}) := G(\mathbf{x}, t) - G(\mathbf{x}, t+T) \quad \text{for some fixed } T > 0.$$

Then due to the radial symmetry, the zeroth and first moments of u_0 are zero and $\int x_1^2 u_0(\mathbf{x})\,d\mathbf{x}$ is nonzero. Also u_0 has only one circle of zeros. Because

$$L_1^{((d-2)/2)}(x) = x^{-(d-2)/2} e^x \frac{d}{dx}\left(e^{-x} x^{1+(d-2)/2}\right) = -x + \frac{d}{2},$$

the unique positive zero of $L_1^{((d-2)/2)}(r^2)$ is $r = \sqrt{d/2}$ and therefore by Theorem 5.1 and the nonincreasing property of the zeros, there is only one circle of zeros with radius $r(t)$ and it satisfies

$$\frac{r(t)}{2\sqrt{t}} \to \sqrt{\frac{d}{2}} \quad \text{as } t \to \infty.$$

We can verify this fact via computation because the solution u is explicitly given by

$$u(\mathbf{x},t) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|\mathbf{x}|^2}{4t}} - (4\pi(t+T))^{-\frac{d}{2}} e^{-\frac{|\mathbf{x}|^2}{4(t+T)}}.$$

Assume $u(\mathbf{z}(t),t) = 0$ implies that

$$\begin{aligned}\left(1+\frac{T}{t}\right)^{\frac{d}{2}} &= \exp\left(-\frac{|\mathbf{z}(t)|^2}{4(t+T)} + \frac{|\mathbf{z}(t)|^2}{4t}\right) \\ &= \exp\left(\frac{T}{4t(t+T)}|\mathbf{z}(t)|^2\right).\end{aligned} \tag{5.1}$$

Hence $|\mathbf{z}(t)|^2/t = 2d(1+t/T)\ln(1+T/t) \to 2d$ as $t \to \infty$.

It is worth noting that *the asymptotics of the radially symmetric solution is independent of the exact values of the moments*; the "nonzero" moment is enough to describe the asymptotics as in the one-dimensional case. But for general cases, this is not true as we will see in the following section.

6. Two-dimensional heat equation. This section is provided to demonstrate a variety of asymptotic behaviors of zeros for multi-dimensions. We assume $d = 2$ and $k \le 2$ in Theorem 3.5. When $k = 1$, the polynomial $\phi_1(\mathbf{x})$ is

$$\begin{aligned}\phi_1(\mathbf{x}) &= \sum_{|\alpha|=1} \frac{\int \mathbf{x}^\alpha u_0(\mathbf{x})\,d\mathbf{x}}{\alpha!} H_{\alpha_1}(x_1) H_{\alpha_2}(x_2) \\ &= \int x_1 u_0(\mathbf{x})\,d\mathbf{x}\; H_1(x_1)H_0(x_2) + \int x_2 u_0(\mathbf{x})\,d\mathbf{x}\; H_0(x_1)H_1(x_2) \\ &= 2x_1 \int x_1 u_0(\mathbf{x})\,d\mathbf{x} + 2x_2 \int x_2 u_0(\mathbf{x})\,d\mathbf{x}.\end{aligned}$$

Hence the zero level set of $\phi_1(\mathbf{x})$ is a straight line passing through the origin.

When $k = 2$, the polynomial $\phi_2(\mathbf{x})$ is

$$\begin{aligned}\phi_2(\mathbf{x}) &= \int x_1^2 u_0(\mathbf{x})\,d\mathbf{x}\; H_2(x_1)H_0(x_2) + \int x_2^2 u_0(\mathbf{x})\,d\mathbf{x}\; H_0(x_1)H_2(x_2) \\ &\quad + \int x_1 x_2 u_0(\mathbf{x})\,d\mathbf{x}\; H_1(x_1)H_1(x_2) \\ &= (4x_1^2 - 2)A + 4x_1x_2B + (4x_2^2 - 2)C \\ &= 4(Ax_1^2 + Bx_1x_2 + Cx_2^2) - 2(A+C),\end{aligned}$$

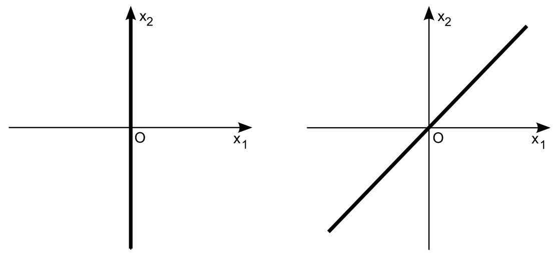


FIG. 1. The zero level set of the polynomial $\phi_1(\mathbf{x})$ when $d = 2$. If $\int x_1 u_0(\mathbf{x})\, d\mathbf{x} = 0$, the zero level set is a straight line $x_2 = 0$ (left). If not, the zero level set is a straight line $x_1 = -\frac{\int x_2 u_0(\mathbf{x})\, d\mathbf{x}}{\int x_1 u_0(\mathbf{x})\, d\mathbf{x}} x_2$ (right).

where

$$A := \int x_1^2 u_0(\mathbf{x})\, d\mathbf{x},\ B := \int x_1 x_2 u_0(\mathbf{x})\, d\mathbf{x} \quad \text{and } C := \int x_2^2 u_0(\mathbf{x})\, d\mathbf{x}.$$

Hence we are looking for the graph of a quadratic equation

$$Ax_1^2 + Bx_1x_2 + Cx_2^2 = \frac{A+C}{2}, \tag{6.1}$$

which is a *conic section*. (Note that second-order moments are not all zero, i.e., $A \neq 0$ or $B \neq 0$ or $C \neq 0$.) The graph can be two lines, one line, one point and the empty set; they are called degenerate curves. If the graph is nondegenerate, by the discriminant classification,

- The graph is an *ellipse* if $B^2 - 4AC < 0$. (The graph is a *circle* if $A = C$ and $B = 0$.)
- The graph is a *parabola* if $B^2 - 4AC = 0$. This shape is *impossible* because the equation (6.1) does not have a first degree term.
- The graph is a *hyperbola* if $B^2 - 4AC > 0$. (The graph is a *rectangular hyperbola* if we also have $A + C = 0$.)

Even if the graph is degenerate, certain cases can be excluded; we can verify that the conic section (6.1) cannot be one line, one point or the empty set. Now assume that the graph represents two lines $ax_1 + bx_2 + c = 0$ and $\tilde{a}x_1 + \tilde{b}x_2 + \tilde{c} = 0$. Then we have

$$\begin{aligned}\phi_2(\mathbf{x}) &= 4(Ax_1^2 + Bx_1x_2 + Cx_2^2) - 2(A+C)\\ &= (ax_1 + bx_2 + c)(\tilde{a}x_1 + \tilde{b}x_2 + \tilde{c})\\ &= a\tilde{a}x_1^2 + (a\tilde{b} + \tilde{a}b)x_1x_2 + b\tilde{b}x_2^2 + (a\tilde{c} + \tilde{a}c)x_1 + (b\tilde{c} + \tilde{b}c)x_2 + c\tilde{c}.\end{aligned}$$

Comparing the zeroth and first degree terms we have

$$a\tilde{c} + \tilde{a}c = b\tilde{c} + \tilde{b}c = 0 \quad \text{and} \quad c\tilde{c} = -\frac{1}{2}(a\tilde{a} + b\tilde{b}).$$

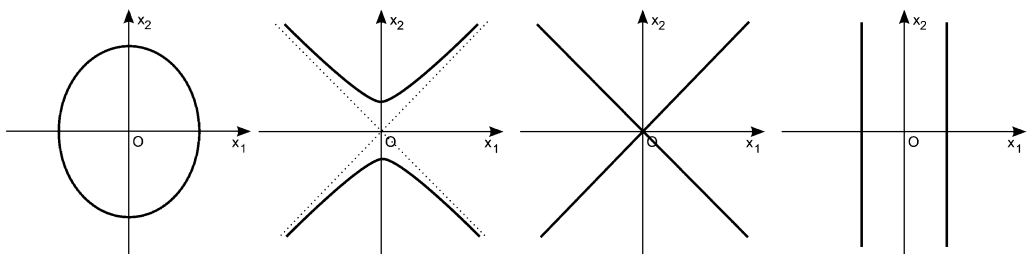
If $c = 0$ and $\tilde{c} \neq 0$, then $a = b = 0$ and the graph is not two lines. If $c = \tilde{c} = 0$, then $A + C = 0$ and the graph is a rectangular hyperbola, or more specifically, two lines intersecting at the origin. Now assume $c \neq 0$ and $\tilde{c} \neq 0$. Then from $ab\tilde{c} = \tilde{a}bc =$

$ab\tilde{c} + a\tilde{b}c = 0$ we have $\tilde{a}bc = a\tilde{b}c$ or $\tilde{a}b = a\tilde{b}$. Therefore *two lines are parallel.* Let $\tilde{a}/a = \tilde{b}/b = \kappa$. Then we have

$$\phi_2(\mathbf{x}) = \kappa\Big\{(ax_1 + bx_2)^2 - (a^2 + b^2)/2\Big\}.$$

Thus the zero level set of $\phi_2(\mathbf{x})$ is not changed by reflection through the origin.

We completed the classification on the graph of the zero level set and the result is summarized in the Figure 2.



(a) An ellipse

(b) A hyperbola

(c) A rectangular hyperbola intersecting at the origin

(d) Two parallel lines invariant under the reflection through the origin

FIG. 2. The zero level set of the polynomial $\phi_2(\mathbf{x})$ when $d = 2$.

7. Numerical examples. Finally we give numerical examples which show that our classification on the graph of the zero level set really works. There are eight figures; four figures on the left are initial data and four figures on the right are their solution at time $t = 1$. We can see the zero level set of the solution from the contour at the bottom of the graph.

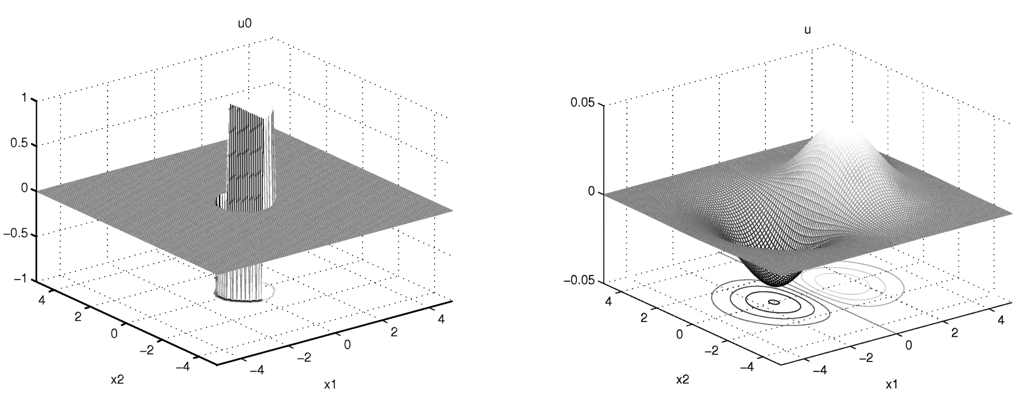


FIG. 3. The zero level set is *a straight line.*

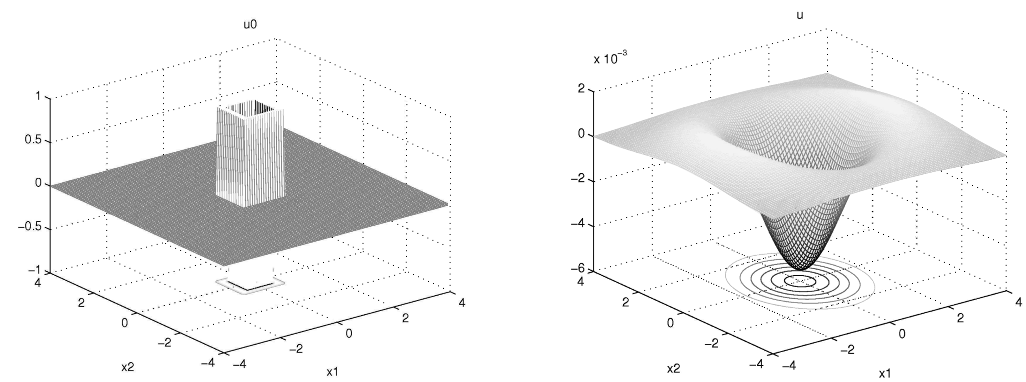


FIG. 4. The zero level set is *an ellipse*.

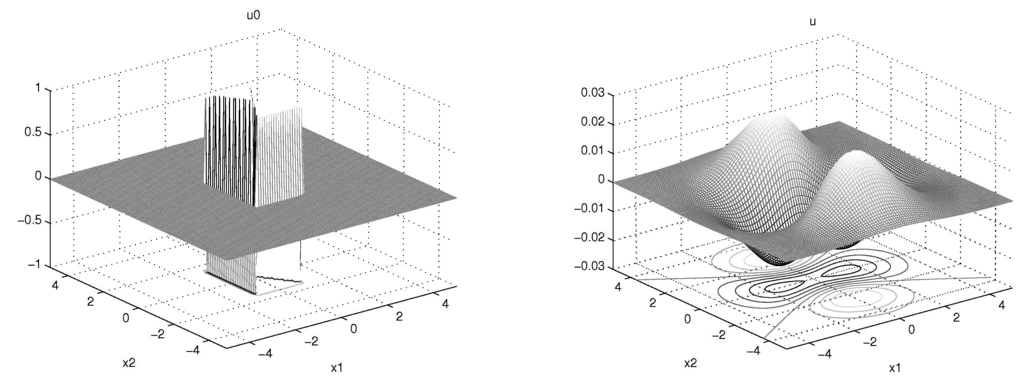


FIG. 5. The zero level set is *a hyperbola*.

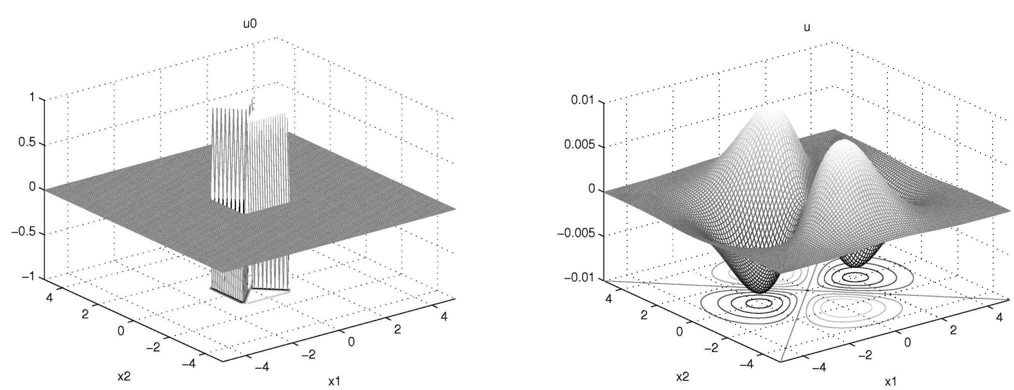


FIG. 6. The zero level set is *a rectangular hyperbola* intersecting at the origin.

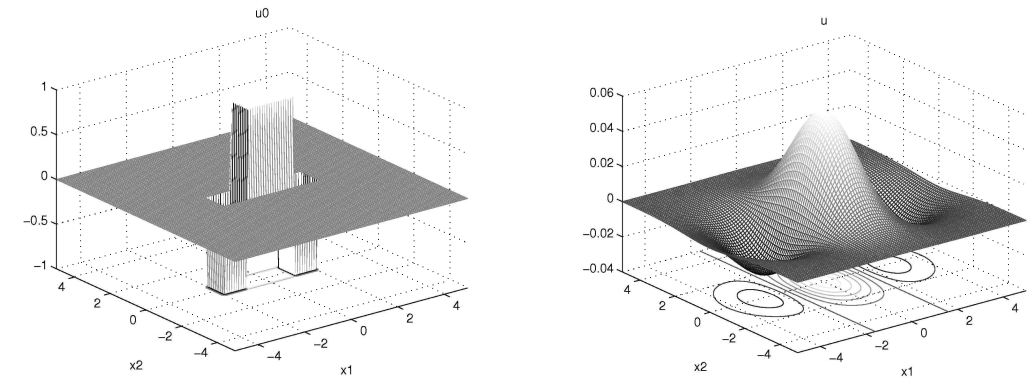


FIG. 7. The zero level set consists of *two parallel lines*. The set is invariant under the reflection through the origin.

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