QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXX, NUMBER 4
DECEMBER 2012, PAGES 759–772
S 0033-569X(2012)01278-5
Article electronically published on August 27, 2012

THE BLOW-UP PROFILE FOR A NONLOCAL NONLINEAR PARABOLIC EQUATION WITH A NONLOCAL BOUNDARY CONDITION

By

YOUPENG CHEN (*School of Mathematics, Yancheng Normal University, Yancheng 224002, Jiangsu, People's Republic of China*)

AND

LIHUA LIU (*School of Mathematics, Yancheng Normal University, Yancheng 224002, Jiangsu, People's Republic of China, and School of Science, Hehai University, Nanjing 210098, Jiangsu, People's Republic of China*)

Abstract. This paper deals with the blow-up properties of positive solutions to a nonlinear parabolic equation with a nonlocal reaction source and a nonlocal boundary condition. Under certain conditions, the blow-up criteria is established. Furthermore, under two additional conditions, the global blow-up behavior is shown, and when $f(u) = u^p, 0 < p \le 1$, the blow-up rate estimates are also obtained.

1. Introduction. In this paper we consider the following nonlinear parabolic equation with a nonlocal reaction source and a weighted nonlocal boundary condition:

$$\begin{aligned} &u_t = f(u)(\Delta u + a\int_\Omega u(x,t)dx), && x \in \Omega,\ t > 0,\\ &u(x,t) = \int_\Omega g(x,y)u(y,t)dy, && x \in \partial\Omega,\ t > 0, \\ &u(x,0) = u_0(x), && x \in \Omega, \end{aligned} \tag{1.1}$$

where a is a positive constant, and Ω is a bounded domain in $\mathbb{R}^N (N \ge 1)$ with smooth boundary $\partial\Omega$.

Problem (1.1) arises in the study of the flow of a fluid through a porous medium and in the study of population dynamics (see [1, 8, 9, 10, 2, 13]). There is an extensive literature which deals with the properties of solutions to local semilinear parabolic equations

Received February 17, 2011.
2000 *Mathematics Subject Classification.* Primary 35A07, 35B40, 35K55, 35K57, 35K60.
Key words and phrases. Nonlinear parabolic equation, nonlocal reaction source, nonlocal boundary condition, global blow-up, blow-up rate.
This research is supported by the research scheme of the natural science of the universities of Jiangsu province (08KJD110017).
E-mail address: youpengc@yahoo.com.cn
E-mail address: liulihua11111@163.com

or systems of heat equations with homogeneous Dirichlet boundary conditions or with nonlinear boundary conditions (see [22, 14, 16, 19, 11, 21, 18] and the references therein). However, there are some important phenomena formulated as parabolic equations which are coupled with nonlocal boundary conditions in mathematical modeling such as thermoelasticity theory (see [5, 6]). In this case, the solution $u(x,t)$ describes entropy per volume of the material.

Problem (1.1) with the nonlocal boundary condition replaced by a homogeneous Dirichlet boundary condition was discussed by Deng, Li and Xie [9]. It is proved that there exists no global positive solution to this problem if and only if $\int^{\infty} ds/(sf(s)) < +\infty$ and $a\int_\Omega \phi(x)dx > 1$, where $\phi(x)$ is the unique positive solution of the linear elliptic problem $-\Delta\phi(x) = 1, x \in \Omega;\ \phi(x) = 0, x \in \partial\Omega$. Later in [3], Chen and Gao further discussed the above problem with the constant a replaced by the bounded positive function $a(x)$. They again obtained the blow-up condition, and in the special case $f(u) = u^p, 0 < p \le 1$, they also obtained the blow-up set and the blow-up rate estimates.

The problem of nonlocal boundary values for linear parabolic equations of the type

$$\begin{aligned} &u_t - Au = c(x)u, && x \in \Omega, t > 0,\\ &u(x,t) = \int_\Omega K(x,y)u(y,t), && x \in \partial\Omega, t > 0,\\ &u(x,0) = u_0(x), && x \in \Omega \end{aligned} \tag{1.2}$$

with uniformly elliptic operator $A = \sum_{i,j=1}^n a_{ij}(x)\dfrac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^n b_i(x)\dfrac{\partial}{\partial x_i}$ and $c(x) \le 0$ was studied by Friedman [15]. The global existence and monotonic decay of the solution of problem (1.2) were obtained under the condition $\int_\Omega |k(x,y)|dy < 1$ for all $x \in \partial\Omega$. Later the problem (1.2) with Au replaced by Δu and the linear term $c(x)u$ replaced by the nonlinear term $g(x,u)$ was discussed by Deng [7]. The comparison principle and the local existence were established. On the basis of Deng's work, Seo in [20] investigated the above problem with $g(x,u) = g(u)$. By using the upper and lower solutions' technique, he gained the blow-up condition of the positive solution, and in the special case $g(u) = u^p$ or $g(u) = e^u$, he also derived the blow-up rate estimates.

Parabolic equations with both nonlocal sources and nonlocal boundary conditions have been studied as well. For example, problem (1.2) with Au replaced by Δu and the linear reaction source term $c(x)u$ replaced by the nonlocal reaction source $\int_\Omega g(u)dx$ was studied by Lin and Liu [17]. They established local existence, global existence and nonexistence of solutions and discussed the blow-up properties of solutions.

Recently, porous medium equations with local sources or with nonlocal sources subjected to nonlocal boundary conditions were studied by Wang et al. [23] and by Cui et al. [4]. The blow-up conditions and the blow-up rate estimates were obtained.

The above studies show that the growth or decay properties of the solutions to the above problems depend on the growth of the nonlinear reaction term $g(u)$ or of the nonlocal nonlinear reaction term $\int_\Omega g(u)dx$, which is similar to general semilinear equations with homogeneous Dirichlet boundary conditions. On the other hand, due to the appearance of the nonlocal boundary condition, the properties of the solution heavily depend on the weight function $K(x,y)$ as well.

Motivated by the above works, we are interested in the blow-up properties of problem (1.1). The aim of this paper is twofold. Firstly, we establish the global existence and finite time blowup of the solution of problem (1.1). Secondly, we discuss the blow-up profile for the special case of $f(u)$.

Before stating our main results, we make some assumptions on $f(s)$, the weight function $g(x,y)$ and the initial datum $u_0(x)$ as follows:

(H_1) $f(s) \in C([0,\infty)) \cap C^1(0,\infty)$ such that $f(0) \ge 0$ and $f'(s) > 0$ in $(0,\infty)$.

(H_2) $g(x,y)$ is continuous and nonnegative on $\partial\Omega \times \overline{\Omega}$ with $\int_\Omega g(x,y)dy > 0$ for all $x \in \partial\Omega$.

(H_3) $u_0 \in C^{2+\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, $u_0(x) > 0$ in Ω, $u_0(x) = \int_\Omega g(x,y)u_0(y)dy$ on $\partial\Omega$.

For example, we can take $u_0(x) = \varphi(x)+1$, $g(x,y) = k[\varphi(x)+1]\varphi(y)$, where $\varphi(x)$ is the corresponding eigenfunction of the first eigenvalue of the following eigenvalue problem:

$$-\Delta\varphi(x) = \lambda\varphi(x), x \in \Omega;\ \varphi(x) = 0, x \in \partial\Omega,$$

and k is the positive constant such that $k\int_\Omega \varphi(y)(\varphi(y)+1)dy = 1$.

Our main results are given below.

THEOREM 1.1. Suppose that $f(s)$, $g(x,y)$ and $u_0(x)$ satisfy (H_1)-(H_3), that $\int_\Omega g(x,y)\,dy \ge 1$ on $\partial\Omega$ and that $\int_\delta^{+\infty} \frac{ds}{f(s)s} < +\infty$ for some positive constant δ, then the solution $u(x,t)$ of problem (1.1) blows up in finite time.

THEOREM 1.2. Let hypothesis (H_2) hold and assume that $\int_\Omega g(x,y)dy < 1$ on $\partial\Omega$. Then there exists a unique positive solution $\psi(x)$ to the following elliptic problem:

$$-\Delta\psi(x) = 1, x \in \Omega;\ \psi(x) = \int_\Omega g(x,y)\psi(y)dy, x \in \partial\Omega. \tag{1.3}$$

Let $\psi(x)$ be the unique positive solution of problem (1.3), and let $\mu = \int_\Omega \psi(x)dx$. Then we have

THEOREM 1.3. Let hypotheses (H_1)-(H_3) hold, and assume that $\int_\Omega g(x,y)dy < 1$ on $\partial\Omega$. Then all solutions of (1.1) are global under either one of the following two conditions:

(i) $a\mu \le 1$;

(ii) $\int_\delta^{+\infty} \frac{ds}{f(s)s} = +\infty$ for some constant $\delta > 0$.

THEOREM 1.4. Let hypotheses (H_1)-(H_3) hold, and assume that $\int_\Omega g(x,y)dy < 1$ on $\partial\Omega$. Then every solution of problem (1.1) blows up in finite time if $a\mu > 1$ and $\int_\delta^{+\infty} \frac{ds}{f(s)s} < +\infty$ for some $\delta > 0$.

To describe the blow-up profile of the blow-up solutions, we need the following two additional assumptions on the initial datum $u_0(x)$:

(H_4) $\Delta u_0(x) \le 0$ and $\Delta u_0(x) + a\int_\Omega u_0(x)dx \ge 0$ on $\overline{\Omega}$.

Then we have

THEOREM 1.5. Let hypotheses (H_1)-(H_4) hold, and let $\int_\Omega g(x,y)dy < 1$ on $\partial\Omega$. Assume that the solution $u(x,t)$ of problem (1.1) blows up in finite time, and that $f''(s) \le 0$ in

$(0,+\infty)$ and $\int_\delta^{+\infty} ds/f(s) = +\infty$ for some $\delta > 0$. Then the blow-up set of $u(x,t)$ is the whole domain Ω.

THEOREM 1.6. Let hypotheses (H_2)-(H_4) hold, and let $\int_\Omega g(x,y)dy < 1$ on $\partial\Omega$. Assume that $a\mu > 1$ and that $f(u) = u^p, 0 < p \le 1$. If we denote the blow-up time of $u(x,t)$ by T^*, then for the case $0 < p < 1$ there exist three positive constants d_1, D_1 and D_1' such that

$$d_1(T^* - t)^{-\frac{1}{p}} \le \max_{x\in\overline{\Omega}} u(x,t) \le D_1(T^* - t)^{-\frac{1}{p}} + D_1', \tag{1.4}$$

and for the case $p = 1$ there exist three positive constants d_2, D_2 and $r > 1$ such that

$$d_2(T^* - t)^{-1} \le \max_{x\in\overline{\Omega}} u(x,t) \le D_2(T^* - t)^{-r}. \tag{1.5}$$

This paper is organized as follows. In section 2, we show the comparison principle and the local existence. In section 3, some criteria for the positive solution to exist globally or to blow up in finite time are given. In section 4, the global blow-up result and the blow-up rate estimates of blow-up solutions for the special case of $f(s)$ are obtained.

2. The comparison principle and the local existence. In this section we start with the definition of a supersolution and a subsolution of problem (1.1). For convenience, we set $Q_T = \Omega \times (0,T]$, $S_T = \partial\Omega \times (0,T]$, and let $\overline{Q}_T$ be the closure of Q_T.

DEFINITION 2.1. A function $\hat{u}(x,t)$ is called a subsolution of problem (1.1) in Q_T if $\hat{u}(x,t) \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ and satisfies

$$\begin{aligned} &\hat{u}_t \le f(\hat{u})(\Delta\hat{u} + a\int_\Omega \hat{u}(x,t)dx), && (x,t) \in Q_T,\\ &\hat{u}(x,t) \le \int_\Omega g(x,y)\hat{u}(y,t)dy, && (x,t) \in S_T,\\ &\hat{u}(x,0) \le u_0(x), && x \in \Omega. \end{aligned} \tag{2.1}$$

A supersolution $\tilde{u}(x,t)$ of problem (1.1) is defined analogously by the above inequalities with each inequality reversed. A solution of problem (1.1) is a function which is both a subsolution and a supersolution of problem (1.1).

Before studying our problem, we give a comparison lemma.

LEMMA 2.2. Assume that $w(x,t) \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ and satisfies

$$\begin{aligned} &w_t - d(x,t)\Delta w \ge \sum_{i=1}^n b_i(x,t)w_{x_i} + c_1(x,t)w + c_2(x,t)\int_\Omega w(x,t)dx, && (x,t) \in Q_T,\\ &w(x,t) \ge c_3(x,t)\int_\Omega c_4(x,y)w(y,t)dy, && (x,t) \in S_T,\\ &w(x,0) > 0, && x \in \overline{\Omega}, \end{aligned} \tag{2.2}$$

where $d(x,t)$, $b_i(x,t)(i = 1,2,\dots,n)$ and $c_j(x,t)(j = 1,2,3)$ are continuous in Q_T, $c_1(x,t)$, $c_2(x,t)$ are bounded in Q_T, $d(x,t) \ge d_0 > 0, c_2(x,t), c_3(x,t) \ge 0$ in Q_T and $c_4(x,y)$ is nonnegative and continuous on $\partial\Omega \times \Omega$ and is not identically zero. Then $w(x,t) \ge 0$ on $\overline{Q}_T$.

The proof is a trivial modification of that of Theorem 2.1 in [7]. We omit it here.

REMARK 2.3. If $c_3(x,t)\int_\Omega c_4(x,y)dy \le 1$ on S_T and $w(x,t)$ satisfies all inequalities in (2.2) except with the third inequality replaced by $w(x,0)\ge 0$ on $\overline{\Omega}$, then we also have $w(x,t)\ge 0$ on $\overline{Q}_T$.

In order to get the global existence and finite time blow-up results for problem (1.1), we need yet the following comparison principle, which is a direct consequence of Lemma 2.2 and Remark 2.3.

LEMMA 2.4. Assume that $\tilde{u}(x,t)\in C^{2,1}(Q_T)\cap C(\overline{Q}_T)$ is a nonnegative supersolution of problem (1.1) and $\hat{u}(x,t)\in C^{2,1}(Q_T)\cap C(\overline{Q}_T)$ is a nonnegative subsolution of problem (1.1), that there exists a small positive constant η such that $\tilde{u}(x,t)\ge\eta$ on $\overline{Q}_T$ or $\hat{u}(x,t)\ge \eta$ on $\overline{Q}_T$, and that $\tilde{u}(x,0)>\hat{u}(x,0)$ on $\overline{\Omega}$ or $\tilde{u}(x,0)\ge\hat{u}(x,0)$ on $\overline{\Omega}$ if $\int_\Omega g(x,y)dy\le 1$ on $\partial\Omega$. Then $\tilde{u}(x,t)\ge\hat{u}(x,t)$ on $\overline{Q}_T$.

Local-in-time existence of the positive classical solution of problem (1.1) can be obtained by using the fixed point theorem, the representation formula and the contraction mapping principle as in [24, 17]. By the above comparison principle, we can get the uniqueness of the solution to problem (1.1), and then we have

THEOREM 2.5. Let hypotheses (H_1)-(H_3) hold. Then there exist $T^*(0<T^*\le+\infty)$ and $u(x,t)\in C(\overline{\Omega}\times[0,T^*))\cap C^{2,1}(\Omega\times(0,T^*))$, such that $u(x,t)$ is the unique maximal-in-time solution of problem (1.1). If $T^*<+\infty$, then we have $\limsup_{t\to T^*}\sup_{x\in\Omega}u(x,t)=+\infty$.

The proof is more or less standard and is therefore omitted here.

3. The global existence and finite time blowup. In this section we give out the proofs of Theorems 1.1-1.4. Comparing with the usual homogeneous Dirichlet boundary condition, we can find out that the weight function $g(x,y)$ plays an important role in the global existence and global nonexistence for problem (1.1).

Proof of Theorem 1.1. By virtue of hypotheses (H_3) and (H_2), we know that $u_0(x)>0$ on $\overline{\Omega}$. Then we can choose a constant v_0 such that $0<v_0<\min_{x\in\overline{\Omega}}u_0(x)$ and consider the initial value problem,

$$v'(t)=a|\Omega|f(v)v, t>0;\ v(0)=v_0, \tag{3.1}$$

with $|\Omega|$ denoting the Lebesgue measure of the bounded domain Ω. From hypothesis (H_1) and the theory of ordinary differential equations, we know that there exists a unique solution $v(t)$ of problem (3.1) which increases in the time variable t. Since $\int_\delta^{+\infty}\frac{ds}{f(s)s}<+\infty$, $v(t)$ blows up at finite time $T_v^*=\frac{1}{a|\Omega|}\int_{v_0}^{+\infty}\frac{ds}{f(s)s}<+\infty$. Due to the condition $\int_\Omega g(x,y)dy\ge 1$, we can easily verify that the solution $v(t)$ of (3.1) is a subsolution of problem (1.1). Noting that $v(t)\ge v_0>0$ and $u_0(x)>v(0)$ on $\overline{\Omega}$, by using the comparison principle Lemma 2.4, we know that $u(x,t)\ge v(t)$ for $x\in\overline{\Omega}, t\ge 0$, and this shows that the solution $u(x,t)$ of problem (1.1) blows up in finite time. □

From now on, we begin to study problem (1.1) in the case $\int_\Omega g(x,y)dy<1$ on $\partial\Omega$. First, we consider the linear elliptic problem (1.3), that is,

$$-\Delta\psi(x)=1, x\in\Omega; \psi(x)=\int_\Omega g(x,y)\psi(y)dy, x\in\partial\Omega,$$

and give out the proof of Theorem 1.2.

Proof of Theorem 1.2. Choose $\psi_0(x) = u_0(x)$ on $\overline{\Omega}$ and define a sequence $\{\psi_m(x)\}$ inductively as follows: for given $\psi_m(x)$, let $\tilde{\psi}_{m+1}(x) = \int_\Omega g(x,y)\psi_m(y)dy, x \in \partial\Omega$, and let $\psi_{m+1}(x)$ be the solution of the following linear elliptic problem:

$$-\Delta\psi_{m+1}(x) = 1, x \in \Omega;\ \psi_{m+1}(x) = \tilde{\psi}_{m+1}(x), x \in \partial\Omega. \tag{3.2}$$

By the theory of linear elliptic equations, we know that $\psi_{m+1}(x)$ exists and is positive and continuous on $\overline{\Omega}$ provided the same is true for $\psi_m(x)$. Further, by using (3.2), we have

$$-\Delta(\psi_{m+1}(x) - \psi_m(x)) = 0, x \in \Omega. \tag{3.3}$$

Then the maximum principle of elliptic equations [12, Chapter 2] implies that

$$\sup_{x\in\overline{\Omega}} |\psi_{m+1}(x) - \psi_m(x)| = \sup_{x\in\partial\Omega} |\psi_{m+1}(x) - \psi_m(x)|.$$

By virtue of (H_2), $g(x,y)$ is a nonnegative continuous function on $\partial\Omega \times \Omega$ and $\int_\Omega g(x,y)\ dy < 1$, we have $\rho = \max_{x\in\partial\Omega} \int_\Omega g(x,y)dy < 1$. Then by induction, we obtain

$$\sup_{x\in\overline{\Omega}} |\psi_{m+1}(x) - \psi_m(x)| \le C\rho^m, \tag{3.4}$$

where $C = \sup_{x\in\overline{\Omega}} |\psi_1(x) - \psi_0(x)|$, and equation (3.4) shows that $\{\psi_m(x)\}$ is a uniform Cauchy sequence in $C(\overline{\Omega})$. By the standard theory of elliptic equations (see also [12, chapter 2]), we know that $\psi(x) = \lim_{m\to+\infty} \psi_m(x)$ is a solution of problem (1.3) and $\psi(x) \in C(\overline{\Omega}) \cap C^2(\Omega)$.

Finally, if $\varphi(x)$ is another solution of problem (1.3), then we have

$$\begin{aligned} &-\Delta(\psi(x) - \varphi(x)) = 0, && x \in \Omega, \\ &\psi(x) - \varphi(x) = \int_\Omega g(x,y)(\psi(y) - \varphi(y))dy, && x \in \partial\Omega. \end{aligned}$$

Again by the elliptic maximum principle, we have

$$\begin{aligned} \sup_{x\in\overline{\Omega}} |\psi(x) - \varphi(x)| &= \sup_{x\in\partial\Omega} |\psi(x) - \varphi(x)| \\ &= \sup_{x\in\partial\Omega} \left|\int_\Omega g(x,y)(\psi(y) - \varphi(y))dy\right| \le \rho \sup_{x\in\overline{\Omega}} |\psi(x) - \varphi(x)|. \end{aligned} \tag{3.5}$$

Since $\rho < 1$, (3.5) implies that $\varphi(x) \equiv \psi(x)$. Then the uniqueness of the positive solution of problem (1.3) follows. Then we complete the proof of Theorem 1.2. □

Now, we can show the global existence and global nonexistence.

Proof of Theorem 1.3. (i) Let $\psi(x)$ be the unique positive solution of the linear elliptic problem (1.3). From the proof of Theorem 1.2, we know that $\psi(x) \ge 0$ on $\overline{\Omega}$. Then the elliptic maximum principle and hypothesis (H_2) ensure that $\psi(x) > 0$ on $\overline{\Omega}$. Let $\max_{x\in\overline{\Omega}} \psi(x) = K_1$, $\min_{x\in\overline{\Omega}} \psi(x) = K_2$; then $K_1, K_2 > 0$. We define a function $w(x,t)$ as follows:

$$w(x,t) = M\psi(x), \tag{3.6}$$

where M is a constant to be determined later. Noting that $a\int_\Omega \psi(x)dx = a\mu \le 1$, we have for $x \in \Omega, t > 0$,

$$w_t - f(w)(\Delta w + a\int_\Omega w(x,t)dx) = f(M\psi(x))(-M\Delta\psi(x) - aM\int_\Omega \psi(x)dx) \qquad (3.7)$$
$$= f(M\psi(x))M(1 - a\int_\Omega \psi(x)dx) \ge 0.$$

On the other hand, by using the fact that $\psi(x)$ is the solution of problem (1.3), we have for $x \in \partial\Omega$,

$$w(x,t) = M\psi(x) = M\int_\Omega g(x,y)\psi(y)dy = \int_\Omega g(x,y)w(y,t)dy. \qquad (3.8)$$

Choose $M > K_2^{-1}\max_{x\in\overline{\Omega}} u_0(x)$. Then $w(x,0) = M\psi(x) \ge MK_2 > u_0(x)$ on $\overline{\Omega}$. Combining this inequality with (3.7) and (3.8), we know that $w(x,t)$ defined as (3.6) is a supersolution of problem (1.1). Since $w(x,t) \ge MK_2 > 0$, $w(x,0) > u_0(x)$, and $w(x,t)$ exists globally, by Lemma 2.4, we know that $u(x,t) \le w(x,t)$, and $u(x,t)$ exists globally.

(ii) Choose $b > a|\Omega|$ and $z_0 > \max_{x\in\overline{\Omega}} u_0(x)$, and consider the following initial value problem:

$$z'(t) = bf(z(t))z(t),\ t > 0;\ z(0) = z_0. \qquad (3.9)$$

It follows from hypothesis (H_1) and the theory of ordinary differential equations that there exists a unique solution $z(t)$ to problem (3.9) and $z(t)$ is increasing. Noticing the condition $\int_\delta^{+\infty} \frac{ds}{f(s)s} = +\infty$ for some $\delta > 0$, we also know that the solution $z(t)$ of problem (3.9) exists globally. Set $w(x,t) = z(t)$. Then by using the condition $\int_\Omega g(x,y)dy < 1$ on $\partial\Omega$, we obtain

$$w_t - f(w)(\Delta w + a\int_\Omega w(x,t)dx) = z'(t) - f(z(t))(\Delta z(t) + a|\Omega|z(t)) \qquad (3.10)$$
$$= (b - a|\Omega|)f(z(t))z(t) > 0,\ x \in \Omega, t > 0,$$
$$w(x,t) = z(t) > \int_\Omega g(x,y)z(t)dy = \int_\Omega g(x,y)w(x,t)dy,\ x \in \partial\Omega, t > 0,$$
$$w(x,0) = z(0) = z_0 > u_0(x),\ x \in \overline{\Omega}.$$

The above inequalities show that $w(x,t) = z(t)$ is the supersolution of problem (1.1), noting that $w(x,t) = z(t) > z_0 > 0$. Then the comparison principle Lemma 2.4 implies that the solution of problem (1.1), $u(x,t) \le w(x,t)$ and exists globally.

From (i) and (ii), we complete the proof of Theorem 1.3. □

Proof of Theorem 1.4. Since $a\int_\Omega \psi(x)dx = a\mu > 1$, we have $b_1 = K_1^{-1}(a\int_\Omega \psi(x)dx - 1) > 0$, where K_1, K_2 are positive constants representing the maximum and minimum of the solution $\psi(x)$ of problem (1.3) on $\overline{\Omega}$. From the proof of Theorem 1.1 we know that the initial datum $u_0(x) > 0$ on $\overline{\Omega}$. Let $z(t)$ be the solution of the following initial value problem of ordinary differential equations:

$$z'(t) = b_1 f(K_2 z(t))z(t), t > 0;\ z(0) = z_0, \qquad (3.11)$$

where $0 < z_0 < K_1^{-1} \min_{x\in\overline{\Omega}} u_0(x)$. Then $z(t)$ is increasing and $z(t) \geq z_0 > 0$. Due to the condition $\int_\delta^{+\infty} \frac{ds}{f(s)s} < +\infty$ for some $\delta > 0$, we know that the solution $z(t)$ of problem (3.11) blows up in finite time.

Set $w(x,t) = z(t)\psi(x)$. Then for $x \in \Omega, t > 0$, we have

$$\begin{aligned} &w_t - f(w)(\Delta w + a\int_\Omega w(x,t)dx) \\ &\quad = z'(t)\psi(x) - f(z(t)\psi(x))(z(t)\Delta\psi(x) + az(t)\int_\Omega \psi(x)dx) \\ &\quad \leq z'(t)K_1 - f(K_2 z(t))z(t)(a\int_\Omega \psi(x)dx - 1) = 0. \end{aligned} \tag{3.12}$$

On the other hand, for $x \in \partial\Omega, t > 0$, we have

$$w(x,t) = z(t)\psi(x) = z(t)\int_\Omega g(x,y)\psi(y)dy = \int_\Omega g(x,y)w(y,t)dy. \tag{3.13}$$

Also for $x \in \overline{\Omega}$, we have

$$w(x,0) = z(0)\psi(x) = z_0\psi(x) < K_1^{-1}\psi(x)\min_{x\in\overline{\Omega}} u_0(x) \leq u_0(x). \tag{3.14}$$

The inequalities (3.12), (3.13) and (3.14) show that $w(x,t)$ is a subsolution of problem (1.1). Since $w(x,t) = z(t)\psi(x) \geq z_0 K_2 > 0$ and $w(x,t)$ blows up in finite time, Lemma 2.4 implies that the solution $u(x,t)$ of problem (1.1) satisfies $u(x,t) \geq w(x,t)$ and $u(x,t)$ blows up in finite time, and this completes the proof of Theorem 1.4. □

4. Blow-up profile. In this section we give out the proof of Theorems 1.5 and 1.6. Throughout this section we assume that the solution $u(x,t)$ of problem (1.1) blows up in finite time. We denote by T^* the blow-up time of the blow-up solution $u(x,t)$ of problem (1.1) and set

$$h(t) = a\int_\Omega u(x,t)dx,\ H(t) = \int_0^t h(s)ds. \tag{4.1}$$

We divide the proof of Theorem 1.5 into the following two lemmas.

LEMMA 4.1. Let hypotheses (H_1)-(H_4) hold, and let $\int_\Omega g(x,y)dy < 1$ for $x \in \partial\Omega$. Assume that the solution $u(x,t)$ of problem (1.1) blows up in finite time, and that $f''(s) \leq 0$ in $(0,+\infty)$. Then $\Delta u \leq 0$ in $\Omega \times (0,T^*)$.

Proof. Differentiating Equation (1.1) with respect to t, from the condition $\Delta u_0(x) + a\int_\Omega u_0(x)dx \geq 0$ on $\overline{\Omega}$ in (H_4), Lemma 2.2 and Remark 2.3, we can easily obtain $u_t(x,t) \geq 0$ on $\overline{\Omega} \times [0,T^*)$, and then we know that

$$u(x,t) \geq u_0(x) \geq \min_{x\in\overline{\Omega}} u_0(x)(=\eta) > 0. \tag{4.2}$$

Let $w(x,t) = \Delta u(x,t)$. Then it follows from (1.1) that for $(x,t) \in \Omega \times (0,T^*)$,

$$\begin{aligned} w_t &= f(u)\Delta w + 2f'(u)\nabla u\cdot\nabla w + f'(u)(\Delta u + h(t))w + f''(u)(\Delta u + h(t))|\nabla u|^2 \\ &= f(u)\Delta w + 2f'(u)\nabla u\cdot\nabla w + \frac{f'(u)}{f(u)}u_t w + \frac{f''(u)}{f(u)}u_t|\nabla u|^2. \end{aligned}$$

In view of (4.2), hypothesis (H_1), $u_t(x,t) \geq 0$ and $f''(u) \leq 0$ in $\Omega \times (0,T^*)$, we get

$$w_t - f(u)\Delta w - 2f'(u)\nabla u \cdot \nabla w \leq \frac{f'(u)}{f(u)} u_t w, \ (x,t) \in \Omega \times (0,T^*). \tag{4.3}$$

On the other hand, we can show that for $(x,t) \in \partial\Omega \times (0,T^*)$,

$$\int_\Omega g(x,y) f(u(y,t)) dy / f(u) - 1 \leq 0. \tag{4.4}$$

In fact, noting the condition $f''(s) \leq 0$ in $(0,+\infty)$ in this lemma and using Jensen's inequality, we get

$$(\int_\Omega g(x,y)dy)^{-1} \int_\Omega g(x,y) f(u(y,t)) dy \leq f((\int_\Omega g(x,y)dy)^{-1} \int_\Omega g(x,y) u(y,t) dy). \tag{4.5}$$

Utilizing Taylor's formula of second order

$$f(0) = f(s) + f'(s)(0-s) + \frac{1}{2!} f''(\xi)(0-s)^2, s > 0,$$

where ξ is an intermediate value between 0 and s, and noting that $f''(s) \leq 0$ in $(0,+\infty)$ and that $f(0) \geq 0$, we have $f(s) \geq sf'(s)$, $s > 0$. Then

$$(\frac{f(s)}{s})' = \frac{sf'(s) - f(s)}{s^2} \leq 0.$$

Then noting $\int_\Omega g(x,y)dy < 1$ on $\partial\Omega$, we have

$$\frac{f((\int_\Omega g(x,y)dy)^{-1} \int_\Omega g(x,y)u(y,t)dy)}{(\int_\Omega g(x,y)dy)^{-1} \int_\Omega g(x,y)u(y,t)dy} \leq \frac{f(\int_\Omega g(x,y)u(y,t)dy)}{\int_\Omega g(x,y)u(y,t)dy}.$$

Hence by using (4.5), the above inequality and (1.1), we obtain for $(x,t) \in \partial\Omega \times (0,T^*)$,

$$\begin{aligned}
\int_\Omega g(x,y) f(u(y,t)) dy &\leq \int_\Omega g(x,y)dy f((\int_\Omega g(x,y)dy)^{-1} \int_\Omega g(x,y)u(y,t)dy) \\
&\leq \int_\Omega g(x,y)dy \cdot \frac{f(\int_\Omega g(x,y)u(y,t)dy)}{\int_\Omega g(x,y)u(y,t)dy} \cdot (\int_\Omega g(x,y)dy)^{-1} \int_\Omega g(x,y)u(y,t)dy \\
&= f(\int_\Omega g(x,y)u(y,t)dy) = f(u),
\end{aligned}$$

and this shows that (4.4) holds. Then from (1.1) and (4.4), we know that for $(x,t) \in \partial\Omega \times (0,T^*)$,

$$\begin{aligned}
w(x,t) &= u_t(x,t)/f(u) - h(t) = \textstyle\int_\Omega g(x,y)u_t(y,t)dy/f(u) - h(t) \\
&= \int_\Omega g(x,y) f(u(y,t)) w(y,t) dy / f(u) + (\int_\Omega g(x,y) f(u(y,t)) dy / f(u) - 1) h(t) \\
&\leq \textstyle\int_\Omega g(x,y) f(u(y,t)) w(y,t) dy / f(u).
\end{aligned} \tag{4.6}$$

Also the hypothesis (H_4) implies that

$$w(x,0) = \Delta u_0(x) \leq 0, x \in \overline{\Omega}. \tag{4.7}$$

Then from (4.3), (4.6) and (4.7), by using (4.2),(4.4), Lemma 2.2 and Remark 2.3, we know that $w(x,t) = \Delta u(x,t) \leq 0, (x,t) \in \Omega \times (0,T^*)$. This completes the proof. $\square$

LEMMA 4.2. Under the same conditions of Lemma 4.1, if $\int_\delta^{+\infty} ds/f(s) = +\infty$ for some $\delta > 0$, then $\lim_{t\to T^*} h(t) = \lim_{t\to T^*} H(t) = +\infty$, and the blow-up set of the solution $u(x,t)$ of problem (1.1) is the whole domain of Ω.

Proof. From Lemma 4.1, we have

$$u_t(x,t) \le f(u(x,t))h(t),\ (x,t) \in \Omega \times (0,T^*). \tag{4.8}$$

Integrating (4.8) from 0 to t, we get

$$\int_{u_0(x)}^{u(x,t)} \frac{ds}{f(s)} \le H(t),\ (x,t) \in \Omega \times (0,T^*). \tag{4.9}$$

Due to $\lim_{t\to T^*} \sup_{x\in\Omega} u = +\infty$ and $\int_\delta^{+\infty} ds/f(s) = +\infty$ for some $\delta > 0$, (4.9) ensures that $\lim_{t\to T^*} H(t) = +\infty$. Since $T^* < +\infty$, from the above equality we have $\lim_{t\to T^*} h(t) = +\infty$.

To show the second conclusion, let $x_1 \in \Omega$, $R = \mathrm{dist}(x_1, \partial\Omega)$, $\Omega_1 = \{x : |x - x_1| < R\}$, $r = |x - x_1|$ and consider the following problem:

$$\begin{aligned} &v_t = f(v)(\Delta v + h(t)), && x \in \Omega_1, t > 0,\\ &v(x,t) = \eta/2, && x \in \partial\Omega_1, t > 0,\\ &v(x,0) = v_0(x) \le u_0(x), && x \in \Omega_1, \end{aligned} \tag{4.10}$$

where η is the positive constant given by (4.2), $v_0(x) > 0$ for $x \in \Omega_1$, $v_0(x) = \eta/2$ for $x \in \partial\Omega_1$, $v_0(x) = v_0(r)$ and $v_0'(r) \le 0$ for $0 \le r \le R$. Then the solution of problem (4.10) exists and satisfies $v(x,t) = v(r,t)$ and $v_r'(r,t) \le 0$ for $0 \le r \le R, t \ge 0$. It follows from hypothesis (H_1) that $f(v) \ge f(\eta/2) > 0$ for all $x \in \Omega_1, t > 0$.

From the proof of Lemma 4.1 we know that the solution $u(x,t)$ of problem (1.1) satisfies $u(x,t) \ge \eta > 0$. Then the comparison principle, which is the direct consequence of Lemma 2.2, implies that

$$v(x,t) \le u(x,t),\ x \in \Omega_1, t > 0. \tag{4.11}$$

Denote by $\lambda_1 > 0$ and $\varphi(x)$ the first eigenvalue and the corresponding eigenfunction of the following eigenvalue problem:

$$-\Delta\varphi(x) = \lambda_1\varphi(x), x \in \Omega_1;\ \varphi(x) = 0, x \in \partial\Omega_1,$$

such that $\int_{\Omega_1} \varphi(x)dx = 1$.

We rewrite equation (4.10) as follows:

$$v_t/f(v) = \Delta v + h(t), x \in \Omega_1, t > 0. \tag{4.12}$$

Multiplying both sides of (4.12) by $\varphi(x)$ and integrating over $\Omega_1 \times (0,t)$, we get

$$\int_{\Omega_1} \int_{v_0(x)}^{v(x,t)} \frac{ds}{f(s)}\varphi(x)dx = \frac{\eta\lambda_1}{2} - \lambda_1 \int_0^t \int_{\Omega_1} v\varphi dxds + |\Omega_1|H(t),\ t \in (0,T^*). \tag{4.13}$$

By the above equality and $\lim_{t\to T^*} H(t) = \infty$ in the first conclusion, we know that if $\int_0^{T^*} \int_{\Omega_1} v\varphi dxds < +\infty$, then for some $x \in \Omega_1$ we have

$$\limsup_{t\to T^*} \int_{v_0(x)}^{v(x,t)} \frac{ds}{f(s)} = +\infty. \tag{4.14}$$

Noticing that $\int_\delta^{+\infty} \frac{ds}{f(s)} = +\infty$ for some $\delta > 0$ and that $v'_r(r,t) \le 0$ for $0 \le r \le R, t \ge 0$, we have

$$\limsup_{t\to T^*} v(x_1,t) = \infty.$$

On the contrary, if $\int_0^{T^*} \int_{\Omega_1} v\varphi dxds = +\infty$, then we also have $\limsup_{t\to T^*} v(x_1,t) = \infty$. By virtue of (4.11) and the arbitrariness of $x_1 \in \Omega$, we obtain the second conclusion. $\square$

From Lemma 4.2, we have completed the proof of Theorem 1.5.

Now, we will give the proof of Theorem 1.6. Here and after we assume that $f(u) = u^p, 0 < p \le 1$. Then $f(s)$ satisfies (H_1), $f''(s) \le 0$ and $\int_\delta^{\infty} \frac{ds}{f(s)} = +\infty$ for some $\delta > 0$, and then all conclusions in Lemmas 4.1 and 4.2 hold for this case of $f(s)$. We divide the proof of Theorem 1.6 into the following three lemmas. Let $U(t) = \max_{x\in\overline{\Omega}} u(x,t)$. Then we have

LEMMA 4.3. Let hypotheses (H_2)-(H_4) hold, and let $\int_\Omega g(x,y)dy < 1$ for $x \in \partial\Omega$. If the solution $u(x,t)$ of problem (1.1) blows up in finite time T^*, then there exists a positive constant d such that

$$U(t) = \max_{x\in\overline{\Omega}} u(x,t) \ge d(T^* - t)^{-\frac{1}{p}},\ t \in (0,T^*). \tag{4.15}$$

Proof. Using the equation (1.1) and Theorem 4.5 in [14], we get

$$U'(t) \le U^p(t) \cdot a|\Omega|U(t),\ \text{a.e. } t \in (0,T^*).$$

Rewrite it as follows:

$$U^{-p-1}(t)U'(t) \le a|\Omega|,\ \text{a.e. } t \in (0,T^*). \tag{4.16}$$

Integrating (4.16) over (t,T^*) and noting that $\lim_{t\to T^*} U(t) = +\infty$, we get

$$U(t) = \max_{x\in\overline{\Omega}} u(x,t) \ge d(T^* - t)^{-\frac{1}{p}}, t \in (0,T^*),$$

where $d = (ap|\Omega|)^{-\frac{1}{p}}$. $\square$

LEMMA 4.4. Let hypotheses (H_2)-(H_4) hold, and let $\int_\Omega g(x,y)dy < 1$ for $x \in \partial\Omega$. If the solution $u(x,t)$ of problem (1.1) blows up in finite time T^*, then there exist three positive constants C_1, C_2 and C_3 such that

$$\begin{array}{ll} H(t) \le C_1(T^* - t)^{-\frac{1-p}{p}},\ \text{if } 0 < p < 1, \\ H(t) \le -C_2 \ln(T^* - t) + C_3,\ \text{if } p = 1, \end{array}\quad t \in (0,T^*). \tag{4.17}$$

Proof. Let $\psi(x)$ be the solution of problem (1.3). Then from the proof of Theorem 1.3 we know that there exist two positive constants K_1, K_2 such that $K_2 \le \psi(x) \le K_1, x \in \overline{\Omega}$. Set $w(x,t) = A\psi(x)z(t)$, where A is a positive constant to be determined later and $z(t) = H^{\frac{1}{1-p}}(t)$ for the case $0 < p < 1$ and $z(t) = e^{BH(t)}$ for the case $p = 1$, and B is a positive constant to be determined later. From the continuity of the functions $u(x,t)$ and $w(x,t)$ on $\overline{\Omega} \times [0,T^*)$, we know that the set $\{(x,t) \in \overline{\Omega} \times [0,T^*) | u(x,t) < w(x,t)\}$ is open, and we can denote it by $\Omega_2 \times \bigcup_{i\in S}(t_i,t_{i+1})$, where $\Omega_2 \subset \Omega$ is an open set, $S \subset \mathbb{N}$

is a set of natural integers, $(t_i, t_{i+1}) \subset [0, T^*)$ is an open interval ($i \in S$). Then a direct computation yields for the case $0 < p < 1$ and $(x,t) \in \Omega_2 \times (t_i, t_{i+1}), i \in S$,

$$\begin{aligned} & w_t - w^p(\Delta w + a\int_\Omega w(x,t)dx) \\ &= A\psi(x)z'(t) - A^{p+1}\psi^p(x)z^{p+1}(t)(\Delta\psi(x) + a\int_\Omega \psi(x)dx) \\ &\le A\psi^p(x)[K_1^{1-p}z'(t) - A^p(a\int_\Omega \psi(x)dx - 1)z^{p+1}(t)] \\ &= A\psi^p(x)z^p(t)[K_1^{1-p}h(t)/(1-p) - A^p(a\int_\Omega \psi(x)dx - 1)z(t)] \\ &\le A^{p+1}\psi^p(x)z^{p+1}(t)[a(K_1A)^{1-p}\int_\Omega \psi(x)dx - (a\int_\Omega \psi(x)dx - 1)], \end{aligned} \tag{4.18}$$

and for the case $p = 1$ and $(x,t) \in \Omega_2 \times (t_i, t_{i+1}), i \in S$,

$$\begin{aligned} & w_t - w(\Delta w + a\int_\Omega w(x,t)dx) \\ &= A\psi(x)z'(t) - A^2\psi(x)z^2(t)(\Delta\psi(x) + a\int_\Omega \psi(x)dx) \\ &= A\psi(x)[Bz(t)h(t) - A(a\int_\Omega \psi(x)dx - 1)z^2(t)] \\ &\le A^2\psi(x)z^2(t)[aB\int_\Omega \psi(x)dx - (a\int_\Omega \psi(x)dx - 1)]. \end{aligned} \tag{4.19}$$

If we choose the positive constants A and B such that

$$\begin{aligned} & A \le [(a\int_\Omega \psi(x)dx - 1)/(aK_1^{1-p}\int_\Omega \psi(x)dx)]^{\frac{1}{1-p}}, \\ & B \le (a\int_\Omega \psi(x)dx - 1)/(a\int_\Omega \psi(x)dx), \end{aligned} \tag{4.20}$$

then from inequalities (4.18) and (4.19) we obtain for all $0 < p \le 1$,

$$w_t - w^p(\Delta w + a\int_\Omega w(x,t)dx) \le 0, (x,t) \in \Omega_2 \times (t_i, t_{i+1}), i \in S. \tag{4.21}$$

On the other hand, for $(x,t) \in \partial\Omega_2 \times (t_i, t_{i+1}) \cup \Omega_2 \times \{t_i\}, i \in S$, $w(x,t) = u(x,t)$. All these inequalities and the comparison principle, which is a direct consequence of Lemma 2.2, imply that $w(x,t) \le u(x,t)$ on $\overline{\Omega}_2 \times [t_i, t_{i+1}), i \in S$, and then the set $\{(x,t) \in \overline{\Omega} \times [0,T^*) | u(x,t) < w(x,t)\}$ is empty, that is to say, $u(x,t) \ge w(x,t), (x,t) \in \overline{\Omega} \times [0,T^*)$. Then we have

$$h(t) \ge aA\int_\Omega \psi(x)dx \cdot z(t). \tag{4.22}$$

By virtue of the definition of $z(t)$, and by integrating the above inequality over (t, T^*), we obtain the inequalities in (4.17), where $C_1 = [paA\int_\Omega \psi(x)dx/(1-p)]^{-\frac{1-p}{p}}$, $C_2 = \frac{1}{B}$ and $C_3 = -\frac{1}{B}\ln[aAB\int_\Omega \psi(x)dx]$. □

LEMMA 4.5. Let hypotheses (H_2)-(H_4) hold, and let $\int_\Omega g(x,y)dy < 1$ for $x \in \partial\Omega$. If the solution $u(x,t)$ of problem (1.1) blows up in finite time T^*, then there exist four positive constants D_1, D_1', D_2 and $r > 1$ such that

$$\begin{aligned} &U(t) = \max_{x\in\overline{\Omega}} u(x,t) \le D_1(T^*-t)^{-\frac{1}{p}} + D_1', \text{ if } 0<p<1, \\ &U(t) = \max_{x\in\overline{\Omega}} u(x,t) \le D_2(T^*-t)^{-r}, \text{ if } p=1, \end{aligned} \qquad t\in(0,T^*), \tag{4.23}$$

where $D_1 = [2(1-p)C_1]^{\frac{1}{1-p}}$, $D_1' = 2^{\frac{1}{1-p}} \max_{x\in\overline{\Omega}} u_0(x)$, $r = C_2 = \frac{1}{B} > 1$ and $D_2 = \max_{x\in\overline{\Omega}} u_0(x)e^{C_3}$, $C_i, i=1,2,3$ and B are given in Lemma 4.4.

Proof. Using the equation (1.1) and Theorem 4.5 in [14], we get

$$U'(t) \le U^p(t)h(t), \text{ a.e. } t\in(0,T^*).$$

Rewriting it as follows:

$$U^{-p}(t)U'(t) \le h(t), \text{ a.e. } t\in(0,T^*)$$

and integrating it over $(0,t)$, we have

$$\begin{aligned} &\frac{1}{1-p}U^{1-p}(t) - \frac{1}{1-p}U^{1-p}(0) \le H(t), \text{ if } 0<p<1, \\ &\ln U(t) - \ln U(0) \le H(t), \text{ if } p=1, \end{aligned} \qquad t\in(0,T^*).$$

By virtue of the conclusion (4.17) of Lemma 4.4, we get the desired result. □

From Lemmas 4.3-4.5, we complete the proof of Theorem 1.6.

Acknowledgments. The authors would like to thank the anonymous referee for his/her valuable and constructive suggestions which improved this work.

REFERENCES

[1] J.R. Anderson, K. Deng, Global existence for degenerate parabolic equations with a non-local forcing, Math. Methods Appl. Sci. 20(1997) 1069-1087. MR1465394 (98f:35083)

[2] R.S. Cantrell, C. Cosner, Diffusive logistic equation with indefinite weights: population models in disrupted environments II, SIAM J. Math. Anal. 22(1991) 1043-1064. MR1112065 (92h:92015)

[3] Y.J. Chen, H.J. Gao, Asympotic blow-up behavior for a nonlocal degenerate parabolic equation, J. Math. Anal. Appl. 330(2007) 852-863. MR2308412 (2007m:35156)

[4] Z. Cui, Z. Yang, Roles of weight functions to a nonlinear porous medium equation with nonlocal source and nonlocal boundary condition, J. Math. Anal. Appl. 342(2008) 559-570. MR2440821 (2009m:35261)

[5] W.A. Day, Extensions of property of heat equation to linear thermoelasticity and other theories, Quart. Appl. Math. 40(1982) 319-330. MR678203 (84a:73042)

[6] W.A. Day, A decreasing property of solutions of parabolic equations with applications to thermoelasticity, Quart. Appl. Math. 40(1983) 468-475. MR693879 (84h:35089)

[7] K. Deng, Comparison principle for some nonlocal problems, Quart. Appl. Math. 50(1992) 517-522. MR1178431 (93g:35075)

[8] J.I. Diaz, R. Kerser, On a nonlinear degenerate parabolic equation in infiltration or evaporation through a porous medium, J. Differential Equations 69(1987) 368-403. MR903393 (88i:35088)

[9] W. Deng, Y. Li, C. Xie, Existence and nonexistence of global solutions of some nonlocal degenerate parabolic equations, Appl. Math. Lett. 16(2003) 803-808. MR1986054 (2004d:35142)

[10] W. Deng, Y. Li, C. Xie, Global existence and nonexistence for a class of degenerate parabolic systems, Nonlinear Anal. 55(2003) 233-244. MR2007471 (2004i:35148)

[11] M. Escobedo, M.A. Herreo, A semilinear parabolic system in a bounded domain, Ann. Math. Pura Appl. 165(1993) 315-336. MR1271424 (95c:35119)

[12] L.C. Evans, Partial Differential Equations, American Mathematical Society, Providence, Rhode Island, 1998. MR1625845 (99e:35001)

[13] J. Furter, M. Grinfeld, Local vs. nonlocal interactions in population dynamics, J. Math. Biol. 27(1989) 65-80. MR984226 (90g:92061)

[14] A. Friedman, B. McLeod, Blow-up of positive solutions of semilinear heat equations, Indiana Univ. Math. J. 34(1985) 425-447. MR783924 (86j:35089)

[15] A. Friedman, Monotonic decay of solutions of parabolic equations with nonlocal boundary conditions, Quart. Appl. Math. 44(3)(1986) 401-407. MR860893 (88h:35045)

[16] Y. Giga, R.V. Kohn, Asymptotically self-similar blow-up of semilinear heat equations, Comm. Pure Appl. Math. 38 (1985) 297-319. MR784476 (86k:35065)

[17] Z.G. Lin, Y.R. Liu, Uniform blow-up profiles for diffusion equations with nonlocal source and nonlocal boundary, Acta Math. Sci. Ser. B 24(2004) 443-450. MR2073260 (2005d:35122)

[18] Z.G. Lin, C.H. Xie, The blow-up rate for a system of heat equations with nonlinear boundary condition, Nonlinear Anal. 34(1998) 767-778. MR1634742 (2000a:35096)

[19] D.F. Rial, J.D. Rossi, Blow-up results and localization of blow-up points in an n-dimensional smooth domain, Duke Math. J. 88(2)(1997) 391-405 MR1455526 (98i:35079)

[20] S. Seo, Blowup of solutions to heat equations with nonlocal boundary conditions, Kobe J. Math. 13(1996) 123-132. MR1442200 (98g:35116)

[21] M.X. Wang, Blow-up rate estimates for semilinear parabolic systems, J. Differential Equations 170(2001) 317-324. MR1815186 (2001k:35159)

[22] F.B. Weissler, An L^∞ blow-up estimate for a nonlinear heat equation, Comm. Pure Appl. Math. 38(1985) 291-296. MR784475 (86k:35064)

[23] Y.Wang, C.Mu, Z.Xiang, Blow up of solutions to a porous medium equation with nonlocal boundary condition, Appl. Math. Comput. 192(2007) 579-585. MR2385623 (2009b:35232)

[24] Y.F. Yin, On nonlinear parabolic equations with nonlocal boundary condition, J. Math. Anal. Appl. 185(1994) 54-60. MR1283048 (95f:35126)