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ON THE OVERDAMPING PHENOMENON: A GENERAL RESULT AND APPLICATIONS

By

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Abstract. We study the best possible energy decay rates for a class of linear second-order dissipative evolution equations in a Hilbert space. The models we consider are generated by a positive selfadjoint operator A having a bounded inverse. Our discussion applies to important examples such as the classical wave equation, the dynamical wave equation with Wentzell boundary conditions and many others.

1. Introduction. Let

$$\frac{d^2y}{dt^2} + 2a\frac{dy}{dt} + w^2y = 0 \tag{1.1}$$

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describe the motion $y(t)$ of a damped spring with frequency $w > 0$ and friction coefficient $2a > 0$. The general real solution is given by

$$\begin{aligned} y(t) &= (c_1 \cos(\sqrt{w^2 - a^2}\, t) + c_2 \sin(\sqrt{w^2 - a^2}\, t))e^{-at} \quad \text{for } a < w, \\ y(t) &= (c_1 + c_2 t)e^{-at} \quad \text{for } a = w, \\ y(t) &= c_1 e^{-(a-\sqrt{a^2-w^2})t} + c_2 e^{-(a+\sqrt{a^2-w^2})t} \quad \text{for } a > w, \end{aligned}$$

where c_1 and c_2 are arbitrary real constants. For each $a < w$ all nonzero solutions oscillate and decay to zero like e^{-at} as $t \to +\infty$. For $a = w$ the largest solution decays to zero like te^{-at} and no solution oscillates. For $a > w$ no solution oscillates and the largest solution (for which $c_1 > 0$) decays like $e^{-tF(a)}$ as $t \to +\infty$ where $F(a) = a - \sqrt{a^2 - w^2}$. If we fix w and vary "a", then the exponential function determining the slowest decay rate is $e^{-tF(a)}$, where

$$F(a) = \begin{cases} a & \text{for } a \le w, \\ a - \sqrt{a^2 - w^2} & \text{for } a \ge w \end{cases}$$

(with the caveat that a linear factor is included in the decay rate when $a = w$). $F(a)$ increases from zero to w as "a" does, but then decreases to zero for $w < a$ as $a \to \infty$. This phenomenon is called *overdamping* and is a standard topic in the study of simple electrical networks.

Nevertheless, an overdamping result holds in the context of *energy*. Let u satisfy the telegraph equation or the dissipative wave equation

$$u''(t) + 2au'(t) + Au(t) = 0, \quad t \in \mathbb{R}. \tag{1.2}$$

Here A is an injective nonnegative selfadjoint operator on a Hilbert space $\mathcal{H}$. Let

$$K(t) = ||u'(t)||^2, \quad P(t) = ||A^{1/2}u(t)||^2 \tag{1.3}$$

denote, respectively, the *kinetic* and *potential* energies of the solution u at time $t \ge 0$. For a system of "particles", $K(t)$ is the sum (or integral) of $\frac{1}{2}m|v|^2$, where m denotes mass or the mass density of the particles and v is the velocity. The *total energy* is

$$E = K + P.$$

If $u(t)$ is a strong solution of (1.2) we see that

$$E'(t) = -4aK(t) \tag{1.4}$$

because

$$\frac{d}{dt}\langle w(t), w(t)\rangle = 2\mathrm{Re}\langle w(t), w'(t)\rangle$$

for $w = A^{1/2}u, u'$. Thus $E(t)$ is nonincreasing in t for solutions of (1.2). Intuitively, when the coefficient "a" increases, (1.4) suggests that $E(t)$ decreases. When $a = 0$, (1.2) reduces to the *wave equation* and the energy is conserved, i.e., $E(t) = E(0)$ for all $t \ge 0$ (and indeed for all $t \in \mathbb{R}$ but we shall only consider $t \ge 0$). When $a > 0$ then, in some cases, we obtain an exponential decay rate

$$E(t) \le CE(0)e^{-\alpha t} \tag{1.5}$$

for some $\alpha = \alpha(a, A) > 0$, where $C = C(a, A)$ is a fixed positive constant. Of interest in this work is the *optimal exponent* α_0 which cannot be replaced by any $\alpha > \alpha_0$ without some solution violating (1.5). Suppose

$$0 < b = \min \sigma(A),$$

where $\sigma(A)$ denotes the spectrum of A.

Roughly speaking, we obtain, for

$$F(a) = \begin{cases} a \ \text{ if } \ 0 < a < \sqrt{b}, \\ a - \sqrt{a^2 - b} \ \text{ if } \ a > \sqrt{b} \end{cases}$$

the optimal decay

$$E(t) \leq Ce^{-F(a)t},$$

but C is not necessarily a function of $E(0)$. In some sense $C = C(a, A, u(0), u'(0))$. This is like the overdamping result for (1.1) with "b" playing the role of w^2. For $a = \sqrt{b}$ one may take $C = C_1 t$ for large t.

A large number of articles dealing with particular cases of these overdamping situations were published, notably for functional differential equations. See [4], [5], [6] and the references therein.

A precise version of the above description is the main result of this work. It will be stated precisely in Section 2 after some preliminaries. The proof will be given in Section 3. Section 4 contains some examples. The spectral theorem and its associated functional calculus are used heavily in the proof. These topics are reviewed concisely in the Appendix. We refer to the classical references: J.A. Goldstein [10] and M. Reed and B. Simon [11].

2. The dissipative wave equation (alias the telegraph equation). Let A be a nonnegative injective selfadjoint operator on a Hilbert space $\mathcal{H}$. Consider the wave equation

$$u''(t) + Au(t) = 0, \quad t \in \mathbb{R}, \tag{2.1}$$

which is (1.2) with $a = 0$. Associated with (2.1) we have the initial conditions

$$u(0) = f, \quad u'(0) = g. \tag{2.2}$$

By a (*strong*) *solution* we mean a function $u \in C^2(\mathbb{R}; \mathcal{H})$ satisfying (2.1) for all $t \in \mathbb{R}$ and (2.2). Let $A^{1/2}$ be the nonnegative selfadjoint square root of A and define

$$U(t) = \begin{pmatrix} A^{1/2}u(t) \\ u'(t) \end{pmatrix}$$

for $t \in \mathbb{R}$. Then, (2.1) is equivalent to

$$U'(t) = \begin{bmatrix} 0 & A^{1/2} \\ -A^{1/2} & 0 \end{bmatrix} U(t) = GU(t) \tag{2.3}$$

for $t \in \mathbb{R}$ and (2.2) is equivalent to

$$U(0) = \begin{pmatrix} A^{1/2}f \\ g \end{pmatrix} = F. \tag{2.4}$$

For $F_j = \binom{A^{1/2}f_j}{g_j}$ in the direct sum $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$, the inner product of F_1 with F_2 is

$$\langle F_1, F_2\rangle_{\mathcal{K}} = \langle A^{1/2}f_1, A^{1/2}f_2\rangle + \langle g_1, g_2\rangle.$$

Thus, $F \in \mathcal{K}$ if and only if $f \in \mathcal{D}(A^{1/2})$, $g \in \mathcal{H}$.
Moreover, the domain of G is

$$\mathcal{D}(G) = \mathcal{D}(A) \oplus \mathcal{D}(A^{1/2}).$$

Then $G^* = -G$ on $\mathcal{K}$, iG is selfadjoint and e^{itG} is unitary for each real t.

By the spectral theorem (see the Appendix) the energy is

$$E(t) = ||e^{itG}F||^2_{\mathcal{K}} = ||F||^2_{\mathcal{K}}.$$

From now on, we omit the subscript $\mathcal{K}$. In terms of u and $\mathcal{H}$, this says

$$\begin{aligned} E(t) &= ||A^{1/2}u(t)||^2 + ||u'(t)||^2 = E(0) \\ &= ||A^{1/2}f||^2 + ||g||^2 \end{aligned}$$

for all $t \in \mathbb{R}$ as was earlier noted. But this works even when $F \in \mathcal{K}$, i.e., $f \in \mathcal{D}(A^{1/2})$ and $g \in \mathcal{H}$. In this case u'' may not exist and u is not a strong solution, but we will call u (and U as well) a "mild solution" of the problem and it is the unique limit of strong solutions U_n corresponding to $F_n \in \mathcal{D}(G)$ where $||F_n - F||_{\mathcal{K}} \to 0$ as $n \to +\infty$.

Now we put in friction and consider the damped wave equation:

$$u''(t) + 2au'(t) + Au(t) = 0, \quad t \geq 0 \tag{2.5}$$

with initial data (2.2) with U, G and $\mathcal{K}$ as above. Then (2.5) is equivalent to

$$U' = GU + P_0U, \tag{2.6}$$

where

$$P_0 = \begin{bmatrix} 0 & 0 \\ 0 & -2aI \end{bmatrix}$$

and usually we drop the "I" and write $-2a$ for notational simplicity. If we write $U(t)$ as $\binom{u_1(t)}{u_2(t)}$, then

$$E(t) = ||U(t)||^2$$

satisfies

$$\frac{dE(t)}{dt} = -4a||u_2(t)||^2 = -4aK(t)$$

(if $F \in \mathcal{D}(G)$ and U is a strong solution). Consequently $E(t)$ is nonincreasing in t. We next examine the asymptotics of $K(t)$ and $P(t)$. Note that the initial value problem (2.6), (2.4) is solvable for all $t \in \mathbb{R}$, but we restrict ourselves to $t \geq 0$ since our sole interest is in the behavior of $(K(t), P(t), E(t))$ as $t \to +\infty$.

We seek solutions of (2.5) of the form

$$u(t) = e^{tC}h,$$

where $h \in \mathcal{H}$ and $C = q(A)$ is a Borel function of A as explained in the Appendix. Then (2.5) implies that

$$(C^2 + 2aC + A)u(t) = 0$$

for all $t \geq 0$. Solving $C^2 + 2aC + A = 0$ gives $C = C_\pm$, where

$$C_1 = C_+ = -a + (a^2 - A)^{1/2},$$
$$C_2 = C_- = -a - (a^2 - A)^{1/2},$$

where $B^{1/2} = \sqrt{B}$ is *uniquely* defined in the sense explained at the end of the Appendix.

The general mild solution of (2.5) is then

$$u(t) = e^{tC_1} h_1 + e^{tC_2} h_2$$

and for convenience we take $u(t)$ to be a strong solution. Then, using (2.5) and (2.2) we deduce that

$$\begin{aligned} h_1 + h_2 &= f, \\ C_1 h_1 + C_2 h_2 &= g. \end{aligned} \tag{2.7}$$

Solving this system gives

$$h_1 = \frac{1}{2}(f + Q^{-1}(af + g))$$

and

$$h_2 = \frac{1}{2}(f - Q^{-1}(af + g)), \tag{2.8}$$

where

$$Q = (a^2 - A)^{1/2}. \tag{2.9}$$

We *assume* $a^2 \notin \sigma_p(A)$, i.e., that a^2 is not an eigenvalue of A, so that $a^2 - A$ is an injective normal operator with nonempty resolvent set and so Q is well defined.

Thus, (2.8) expresses h_1, h_2 in terms of (f, g) and (2.7) does the converse. Noting that

$$C_1 = -a + Q, \quad C_2 = -a - Q, \tag{2.10}$$

then the solution is

$$u(t) = e^{tC_1} h_1 + e^{tC_2} h_2 = e^{-at}(e^{tQ} h_1 + e^{-tQ} h_2) \tag{2.11}$$

and

$$A^{1/2} u(t) = e^{tC_1} A^{1/2} h_1 + e^{tC_2} A^{1/2} h_2, \tag{2.12}$$
$$u'(t) = e^{tC_1} C_1 h_1 + e^{tC_2} C_2 h_2. \tag{2.13}$$

Next, we will calculate

$$K(t) = ||u'(t)||^2, \quad P(t) = ||A^{1/2} u(t)||^2, \tag{2.14}$$

and study the asymptotic behavior of

$$E(t) = K(t) + P(t) \tag{2.15}$$

as $t \to +\infty$. Our main result can now be stated.

THEOREM 2.1. Let $A = A^* \geq bI$ on a Hilbert space $\mathcal{H}$, where

$$0 < b = \inf \sigma(A). \tag{2.16}$$

Let $u \in C^2(\mathbb{R}^+; \mathcal{H}) \setminus \{0\}$ satisfy

$$\begin{cases} u'' + 2au' + Au = 0, & t \geq 0, \\ u(0) = f \in \mathcal{D}(A), \\ u'(0) = g \in \mathcal{D}(A^{1/2}), \end{cases}$$

where $a > 0$. The energy associated to u is

$$E(t) = K(t) + P(t) = ||u'(t)||^2 + ||A^{1/2}u(t)||^2.$$

Suppose $a^2 \notin \sigma_p(A)$. Then there exists a constant $C = C(a, A, f, g) > 0$ such that

$$E(t) \leq Ce^{-2F(a)t} \tag{2.17}$$

holds for all $t \geq 0$, where

$$F(a) = \begin{cases} a & \text{if } 0 < a < \sqrt{b}, \\ a - \sqrt{a^2 - b} & \text{if } \sqrt{b} < a \end{cases}$$

provided $a^2 \neq b$ and (2.16) holds. The number $F(a)$ in the exponent is best possible; (2.17) will fail to hold for some solutions if $F(a)$ is replaced by $F(a) + \delta$ for some $a \in (0, +\infty) \setminus \{\sqrt{b}\}$ and some $\delta > 0$. The constant C can be made more explicit and (2.17) can be replaced by

$$E(t) \leq C_0(a, A, \varepsilon)E(0)e^{-2F(a)t} \tag{2.18}$$

for all $t \geq 0$, provided the data f and g in (2.2) satisfy

$$af \pm g \in \text{Range}\left(\chi_{[b, a^2-\varepsilon)\cup(a^2+\varepsilon, +\infty)}(A)\right) \equiv \widetilde{\mathcal{H}}_\varepsilon$$

for some $\varepsilon > 0$. The set $\bigcup_{\varepsilon>0} \widetilde{\mathcal{H}}_\varepsilon$ is dense in $\mathcal{H}$. But for $af \pm g \in \mathcal{H} \setminus \bigcup_{\varepsilon>0} \widetilde{\mathcal{H}}_\varepsilon$, (2.18) will not hold and one can only assert (2.17).

3. Proof of Theorem 2.1. The initial data for (2.5) is given by (2.2). We want to work with h_1, h_2 rather than f, g. Thus we must assume (see (2.8)) that

$$af \pm g \in \text{Range}(Q) = \text{Range}((a^2 - A)^{1/2})$$

and, by hypothesis, $h_1 \neq 0$.

We begin with (2.7)–(2.14). Since u is a strong solution, we have $f \in \mathcal{D}(A)$, $g \in \mathcal{D}(A^{1/2})$, which is equivalent to $h_1, h_2 \in \mathcal{D}(A^{1/2})$. By (2.10)–(2.14) and the law of cosines we obtain

$$\begin{aligned} e^{2at}P(t) &= ||e^{tQ}A^{1/2}h_1 + e^{-tQ}A^{1/2}h_2||^2 \\ &= ||e^{tQ}A^{1/2}h_1||^2 + ||e^{-tQ}A^{1/2}h_2||^2 \qquad (3.1) \\ &\quad + 2\text{Re}\langle e^{tQ}A^{1/2}h_1, e^{-tQ}A^{1/2}h_2\rangle \\ &\leq (||e^{tQ}A^{1/2}h_1|| + ||e^{-tQ}A^{1/2}h_2||)^2. \qquad (3.2) \end{aligned}$$

Also

$$\begin{aligned} e^{2at}K(t) &= ||e^{tQ}(Q-a)h_1 + e^{-tQ}(Q+a)h_2||^2 \\ &= ||e^{tQ}(Q-a)h_1||^2 + ||e^{-tQ}(Q+a)h_2||^2 \qquad (3.3) \\ &\quad + 2\mathrm{Re}\langle e^{tQ}(Q-a)h_1, e^{-tQ}(Q+a)h_2\rangle \\ &\le (||e^{tQ}(Q-a)h_1|| + ||e^{-tQ}(Q+a)h_2||)^2. \qquad (3.4) \end{aligned}$$

We shall break the proof into several parts.

Case 1. Suppose $\sigma(A) \subseteq [b,c]$, where $0 < b \le c \le a^2$.

Then $Q = (a^2 - A)^{1/2}$ is selfadjoint, nonnegative and bounded. Furthermore

$$||e^{\sqrt{a^2-c}t}h|| \le ||e^{tQ}h|| \le e^{\sqrt{a^2-b}t}||h|| \qquad (3.5)$$

for all $t \ge 0$ and all $h \in \mathcal{H}$. It follows from (3.2), (3.4) and (3.5) that

$$E(t) \le Ce^{-2(a-\sqrt{a^2-b})t}, \qquad (3.6)$$

where we may take

$$C = 2\sum_{j=1}^{2}(||A^{1/2}h_j||^2 + ||(Q+(-1)^j a)h_j||^2). \qquad (3.7)$$

This bound is crude. For $h_1 \ne 0$ if we let C depend on time we can write

$$C = C(t) = ||A^{1/2}h_1||^2 + ||(Q-a)h_1||^2 + o(1) \qquad (3.8)$$

as $t \to +\infty$ and then take

$$C = ||A^{1/2}h_1||^2 + ||(Q-a)h_1||^2 + \widetilde{C},$$

where $\widetilde{C}$ is obtained by careful analysis of the $o(1)$ term. Or we can simply regard $||A^{1/2}h_1||^2 + ||(Q-a)h_1||^2$ as an asymptotic bound. Now, (3.7) is adequate for our purposes. But it will be convenient to refer to (3.8) to simplify certain proofs.

Since A and Q, C_1, C_2 (see (2.9) and (2.10)) are bounded, using (2.7), (3.6), (3.7) we obtain

$$E(t) \le C_0(a,A)E(0)e^{-2(a-\sqrt{a^2-b})\,t} \qquad (3.9)$$

for all $t \ge 0$. Next we show why (3.9) is a sharp inequality in Case 1. First, suppose $b \in \sigma_p(A)$. Choose $\varphi \ne 0$ such that $A\varphi = b\varphi$. Then

$$u(t) = e^{(-a+\sqrt{a^2-b})t}\varphi$$

is a solution of (2.5) and by (3.1), (3.2) we deduce, for u, using $h_1 = \varphi$, $h_2 = 0$,

$$\begin{aligned} e^{2at}E(t) &= ||e^{tQ}A^{1/2}\varphi||^2 + ||e^{tQ}(Q-a)\varphi||^2 \\ &= e^{2\sqrt{a^2-b}\,t}\left(b + \left(a - \sqrt{a^2-b}\right)^2\right)||\varphi||^2 \end{aligned} \qquad (3.10)$$

since $f = \varphi$, $g = \left(-a + \sqrt{a^2-b}\right)\varphi$ and

$$E(0) = \left[b^2 + \left(a - \sqrt{a^2-b}\right)^2\right]||\varphi||^2.$$

Thus, by (3.10), (3.9) is sharp when $b \in \sigma_p(A)$. Now suppose $b \in \sigma_c(A)$. We recall that $b = \inf \sigma(A)$. Then

$$\mathcal{H}_\varepsilon = \overline{\mathrm{Range}(\chi_{[b,b+\varepsilon]}(A))}$$

is an infinite-dimensional subspace of $\mathcal{H}$ for every $\varepsilon > 0$. Then $L(A)$ is a well-defined operator for all Borel functions L (see the Appendix). We are taking L to be the characteristic (or indicator) function of the interval $[b, b+\varepsilon]$, and choosing initial data f, g so that $h_1, h_2 \in \mathcal{H}_\varepsilon$ and $h_1 \neq 0$. Then, by our previous estimates (3.1), (3.3), (3.5), [the lower bound] and (3.8), we have, for the solution u satisfying (2.2), with $0 < \varepsilon < a^2 - b$,

$$e^{2at}E(t) \geq e^{2\sqrt{a^2-b-\varepsilon}\,t}\Big(||A^{1/2}h_1||^2 + ||(Q-a)h_1||^2 + o(1)\Big)$$

as $t \to +\infty$. Since $\varepsilon > 0$ is arbitrary, we deduce

$$E(t) \geq C_\varepsilon e^{-2(a-\sqrt{a^2-b-\varepsilon})t}$$

for a nonzero solution u_ε with $C_\varepsilon > 0$ for every $\varepsilon > 0$. Thus the exponent in (3.6) is best possible.

We remark that, in (3.6), the main part of the constant C is

$$2\sum_{j=1}^{2}(||A^{1/2}h_j||^2 + ||(Q+(-1)^j a)h_j||^2$$

(cf. (3.7)). This is bounded above by $C_0(a, A)E(0)$ provided the map

$$(A^{1/2}f, g) \mapsto (A^{1/2}h_1, A^{1/2}h_2, (Q-a)h_1, (Q+a)h_2) \tag{3.11}$$

from $\mathcal{H}^2$ to $\mathcal{H}^4$ is bounded. This is automatic by (2.8) if $a^2 \in \rho(A)$, but we are only assuming $a^2 \notin \sigma_p(A)$. But, by the spectral theorem, the map in (3.11) is bounded when f and g are restricted to satisfy

$$af \pm g \in \mathrm{Range}(\chi_{[b,a^2-\varepsilon]\cup[a^2+\varepsilon,+\infty)}(A)) \equiv \widetilde{\mathcal{H}}_\varepsilon \tag{3.12}$$

for some $\varepsilon > 0$. Note that $\bigcup_{\varepsilon>0} \widetilde{\mathcal{H}}_\varepsilon$ is dense in $\mathcal{H}$.

Case 2. Suppose $\sigma(A) \subset [b, +\infty)$, $0 < a^2 < b$. Then $Q = (a^2 - A)^{1/2}$ is skewadjoint and has a bounded inverse. Then, by (3.1)–(3.4),

$$e^{2at}E(t) = \sum_{j=1}^{2}(||e^{tQ}F_j||^2 + ||e^{-tQ}G_j||^2 + 2\mathrm{Re}\langle e^{2tQ}F_j, G_j\rangle), \tag{3.13}$$

where

$$F_1 = (Q-a)h_1, \quad F_2 = A^{1/2}h_1, \quad G_1 = (Q+a)h_2, \quad G_2 = A^{1/2}h_2. \tag{3.14}$$

Since e^{tQ} is unitary it follows from (3.13), (3.14) that

$$E(t) \leq 2\sum_{j=1}^{2}(||F_j||^2 + ||G_j||^2)e^{-2at} \tag{3.15}$$

for all $t \geq 0$. This is the desired bound.

Estimate (3.15) can be replaced by

$$E(t) \leq C_0(a, A)E(0)e^{-2at} \tag{3.16}$$

since $a^2 < b$.

If u is a nonzero solution of (2.5), then the inner product term in (3.13),

$$\mathrm{Re}\langle e^{2tQ}F_j, G_j\rangle = \mathrm{Re}\left\langle \int_{\mathbb{R}} e^{2it\sqrt{\lambda-a^2}} d_\lambda \langle E(\lambda)F_j, G_j\rangle \right\rangle, \tag{3.17}$$

can be interpreted as the Fourier transform of a finite complex measure on $\mathbb{R}$ (see the final part of the Appendix). Thus, we deduce from (3.13) and (3.17) that

$$\limsup_{t\to+\infty} \mathrm{Re}\langle e^{2tQ}F_j, G_j\rangle \geq 0$$

for $j = 1, 2$, and then by (3.13),

$$\limsup_{t\to+\infty} e^{2at}E(t) > 0,$$

proving that inequality (3.15) is best possible (regarding the exponent).

We may replace (3.15) by (3.16) provided that, as before, the map defined by (3.11) is bounded. This holds since the initial data as in (2.2) satisfies

$$af \pm g \in \mathrm{Range}(\chi_{[a^2+\varepsilon,+\infty)}(A)) \equiv \widetilde{J}_\varepsilon$$

for some $\varepsilon > 0$. Also $\bigcup_{\varepsilon>0} \widetilde{J}_\varepsilon$ is dense in $\mathrm{Range}(\chi_{[a^2,+\infty)}(A))$ since $a^2 \notin \sigma_p(A)$.

Case 3. The general case. Let u be the unique solution of (2.5), (2.2). Let $P_a = \chi_{(0,a^2)}(A)$ be the orthogonal projection onto $\mathrm{Range}(\chi_{(0,a^2)}(A))$. Thus $Q_a = I - P_a$ is the orthogonal projection onto $\mathrm{Range}(\chi_{(a^2,+\infty)}(A))$ since $a^2 \notin \sigma_p(A)$. Define f_j, g_j by

$$\begin{aligned} af_1 \pm g_1 &= P_a(af \pm g) = \ell_\pm, \\ af_2 \pm g_2 &= Q_a(af \pm g) = m_\pm; \end{aligned}$$

solving yields

$$\begin{aligned} f_1 &= \frac{\ell_+ + \ell_-}{2a}, \quad g_1 = \frac{\ell_+ - \ell_-}{2}, \\ f_2 &= \frac{m_+ + m_-}{2a}, \quad g_2 = \frac{m_+ - m_-}{2}. \end{aligned}$$

Let $u_j \quad (j = 1, 2)$ be the solution of (2.5) with initial data

$$u_j(0) = f_j, \quad u_j'(0) = g_j$$

$(j = 1, 2)$. Then $u = u_1 + u_2$ and the total energy for u is the sum of the total energies for u_1 and u_2 (which we denote by $E_{u_1}(t)$ and $E_{u_2}(t)$ respectively). Then, by Cases 1 and 2, we have

$$\begin{aligned} E(t) &= E_{u_1}(t) + E_{u_2}(t) \\ &\leq C_1 e^{-2(a-\sqrt{a^2-b})t} + C_2 e^{-2at} \end{aligned}$$

by (3.6), (3.15) and orthogonality. Note that $C_1 > 0$ iff $h_1 \neq 0$, where

$$h_1 = \frac{1}{2}(f_1 + Q^{-1}(af_1 + g_1))$$

by (2.8) (and (2.8) also defines h_2 similarly). Note that for $b \neq a^2$, $C_1 > 0$ can occur iff $a^2 > b$. It follows that

$$E(t) \leq Ce^{-2F(a)t} \tag{3.18}$$

holds for all solutions and all $a > 0$ with $a \neq \sqrt{b}$. Moreover, the exponent $F(a)$ is best possible, that is, it cannot be increased. Let

$$\mathcal{L}_\varepsilon = \chi_{(0,a^2)\cup(a^2+\varepsilon,+\infty)}(A)(\mathcal{H})$$

for $\varepsilon > 0$. Then, by (3.9) and (3.17), we can replace C in (3.18) by $C_0(a, A)E(0)$, provided $af \pm g \in \mathcal{L}_\varepsilon$ for some $\varepsilon > 0$. Thus if $af \pm g$ is in the dense set $\bigcup\limits_{\varepsilon>0} \mathcal{L}_\varepsilon$, then it is in some $\mathcal{L}_\varepsilon$ for some $\varepsilon > 0$ and we can say that

$$E(t) \leq C_0(a, A, \varepsilon)e^{-2F(a)t} \tag{3.19}$$

for all $t \geq 0$. But if $af \pm g \in \mathcal{H} \setminus \bigcup_{\varepsilon>0} \mathcal{L}_\varepsilon$, then (3.19) will not hold and the most we can conclude is that (3.18) holds with $C = C(a, A, f, g) > 0$. There exists a solution u of (2.5), (2.2) with $u \not\equiv 0$ and

$$E(t) \geq C_0 e^{-2F(a)t}$$

for all $t > 0$ and some $C_0 > 0$, provided that either $af \pm g \in \mathrm{Range}(\chi_{[a^2,+\infty)}(A))$ or $b \in \sigma_p(A)$. If $b \notin \sigma_p(A)$, $b = \inf \sigma(A)$ and using our earlier discussion $(af + g, af - g) \neq (0, 0)$, then for every $\varepsilon > 0$ there is a solution u_ε of (2.5), (2.2) satisfying

$$E(t) \geq C_\varepsilon e^{-2[F(a)+\varepsilon]t}$$

for every $\varepsilon > 0$. This completes the proof of Theorem 2.1.

4. Examples. In this section we present some examples to illustrate the conclusions of Theorem 2.1.

EXAMPLE 4.1. Let $\mathcal{H} = \mathbb{C}^n$ and A be an Hermitian $n \times n$ matrix with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Consider the initial value problem for the system of ordinary differential equations

$$\begin{cases} u'' + 2au' + Au = 0, \\ u(0) = u_0, \ u'(0) = u_1, \end{cases}$$

where $a > 0$ and $u_0, u_1 \in \mathbb{C}^n$. The total energy is given by

$$E(t) = \frac{1}{2}|u'(t)|^2 + \frac{1}{2}\langle Au, u\rangle.$$

Using Theorem 2.1, in this case $b = \lambda_1$, therefore

$$E(t) \leq CE(0)e^{-2F(a)\,t} \quad \text{for every} \quad t \geq 0,$$

where

$$F(a) = \begin{cases} a & \text{if} \quad 0 < a < \sqrt{\lambda_1}, \\ a - \sqrt{a^2 - \lambda_1} & \text{if} \quad \sqrt{\lambda_1} < a. \end{cases}$$

EXAMPLE 4.2. Let $\mathcal{H} = L^2(\Omega)$, where $\Omega = \{0 < x < \pi\}$. Consider the operator $A = -\frac{d^2}{dx^2}$ with domain

$$\mathcal{D}(A) = \{u \in H^2(\Omega),\ u(0) = u(\pi) = 0\}.$$

Then, A is selfadjoint, $A\varphi_m = \lambda_m \varphi_m$, where $\varphi_m(x) = \sqrt{\frac{2}{\pi}}\sin(mx)$; this is an orthonormal basis and $\lambda_m = m^2$. We consider the initial boundary value problem for the damped wave equation

$$\begin{cases} u_{tt} + 2au_t + Au = 0, & 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, & \text{for every } \ t \geq 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & 0 < x < \pi \end{cases}$$

for $a > 0$. It is well known that in this case $b = \lambda_1 = 1 = \inf \sigma(A)$; consequently $A = A^* \geq bI = I$ because Poincare's constant is equal to 1.

The total energy is given by

$$E(t) = \frac{1}{2}\int_0^\pi [u_t^2 + u_x^2]\, dx.$$

Thus, using Theorem 2.1 we obtain that

$$E(t) \leq CE(0)e^{-2F(a)\,t} \quad \text{for every } \ t \geq 0,$$

where

$$F(a) = \begin{cases} a & \text{if} \quad 0 < a < 1, \\ a - \sqrt{a^2 - 1} & \text{if} \quad a > 1, \end{cases}$$

where C is a positive constant as in Theorem 2.1.

EXAMPLE 4.3. Let $H = -\Delta - \frac{C}{|x|}$ on $\mathcal{H} = L^2(\mathbb{R}^3)$. This is the quantum mechanical Hamiltonian for a one-electron system with nucleus at the origin. Various constants have been normalized to one, so the positive constant C is proportional to the positive charge Z in the nucleus. The spectrum of H consists of $[0, +\infty)$ together with a known sequence of negative eigenvalues converging to zero (see for instance [11]). The smallest eigenvalue, which corresponds to the ground state, has a one-dimensional eigenspace spanned by the function $\varphi(x) = e^{-\alpha|x|}$. Also

$$H\varphi = -\varphi_{rr} - \frac{2}{r}\,\varphi_r - \frac{C\varphi}{r} = \lambda_1\varphi$$

(where $r = |x|$, $\varphi_r = \frac{x}{r} \bullet \nabla\varphi$, $\varphi_{rr} = \frac{\partial}{\partial r}(\varphi_r)$). Here $\lambda_1 = -\frac{C^2}{4}$ and $\alpha = \frac{C}{2}$. Now, let $A = H + bI$, where $b > \frac{C^2}{4}$. Then, A is selfadjoint, $\sigma(A) = \{b - \frac{C^2}{4}\} \cup \mathcal{O}$, where $\mathcal{O} \subset [b^2 - \frac{C^2}{4} + \lambda_2 - \lambda_1, +\infty)$ and where $\lambda_2 > -\frac{C^2}{4}$ and λ_2 is the smallest eigenvalue of H above λ_1.

Using Theorem 2.1 we conclude the optimal decay rate as $t \to +\infty$ of the energy

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^3} \left[|u_t|^2 + |A^{1/2}u|^2\right]\, dx$$

for the corresponding telegraph equation

$$u_{tt} + 2au_t + Au = 0, \quad x \in \mathbb{R}^3, \quad t \geq 0.$$

EXAMPLE 4.4. The next example is in the context of Wentzell boundary conditions. Let Ω be a bounded domain in $\mathbb{R}^n$ and $A(x) = [a_{ij}(x)]_{n\times n}$ be a real symmetric uniformly positive definite matrix function and assume that each a_{ij} is in $C^2(\overline{\Omega})$. Let $A_0 u = \nabla \cdot (A\nabla u)$ for $u \in C^2(\overline{\Omega})$ and $\partial_\eta^A u(x) = (A\nabla u(x)) \cdot \eta(x)$ be the conormal derivative of u with respect to A at $x \in \partial\Omega$, where $\eta(x)$ denotes the unit outer normal at $x \in \partial\Omega$. Let

$$\mathcal{H} = L^2(\Omega, dx) \oplus L^2\left(\partial\Omega, \frac{d\Gamma}{\beta}\right).$$

Define L_0 as follows:

$$L_0 u = A_0 u \text{ in } \Omega$$

with domain

$$\mathcal{D}(L_0) = \{u \in C^2(\overline{\Omega}), A_0 u|_{\partial\Omega} + \beta\partial_\eta^A u + \gamma u - q\beta\Delta_{LB} u = 0 \text{ on } \partial\Omega\} \tag{4.1}$$

viewed as a subset of $\mathcal{H}$.

Here $\beta > 0$, $\gamma \geq 0$, $\beta, \gamma \in C(\partial\Omega, \mathbb{R})$, $\{x \in \partial\Omega,\ \gamma(x) > 0\} \neq \emptyset$, $0 \leq q < \infty$ and Δ_{LB} is the Laplace-Beltrami operator on the boundary. The closure of L_0, denoted by L, is a negative selfadjoint operator in $\mathcal{H}$ with compact resolvent [7], [8]. Note that for $u \in \mathcal{D}(L_0)$ we identify u with $\binom{u/\Omega}{u/\partial\Omega} \in \mathcal{H}$. We calculate

$$\begin{aligned}
\langle L_0 u, u\rangle_{\mathcal{H}} &= \int_\Omega \operatorname{div}(A\nabla u)\overline{u}dx + \int_{\partial\Omega} (A_0 u)\overline{u}\frac{d\Gamma}{\beta} \\
&= -\int_\Omega (A\nabla u)\cdot\nabla\overline{u}dx + \int_{\partial\Omega}(A_0 u + \beta\partial_\eta^A u)\overline{u}\frac{d\Gamma}{\beta} \\
&\leq -\alpha_0 ||\nabla u||^2_{L^2(\Omega)} - \int_{\partial\Omega}(\gamma u - q\beta\Delta_{LB} u)\overline{u}\frac{d\Gamma}{\beta},
\end{aligned} \tag{4.2}$$

where we use the divergence theorem, the uniform positive definiteness of A and the boundary condition in (4.1). The right-hand side of (4.2) equals

$$\begin{aligned}
-\alpha_0 ||\nabla u||^2_{L^2(\Omega)} - \textstyle\int_{\partial\Omega}\gamma|u|^2\frac{d\Gamma}{\beta} - q\int_{\partial\Omega}|\nabla_\tau u|^2 d\Gamma \\
\leq -C||u||^2_{\mathcal{H}}
\end{aligned}$$

for some positive constant C. In the last step we use the hypothesis on γ and Stokes' theorem. Here $\nabla_\tau u$ is the tangential gradient of u on $\partial\Omega$.

We have $-L\varphi_m = \lambda_m\varphi_m$, where $\{\varphi_m\}$ is an orthonormal basis for $\mathcal{H}$ and

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_m \to +\infty$$

as $m \to +\infty$. In this case λ_1 is a simple eigenvalue.

Now, consider the telegraph equation (2.5) written as

$$u_{tt} - Lu + 2au_t = 0 \quad \text{in} \quad \Omega \times (0, +\infty), \tag{4.3}$$

$$Lu + \beta\partial_\eta^A u + \gamma u - q\beta\Delta_{LB} u = 0 \quad \text{on} \quad \partial\Omega \times (0, +\infty). \tag{4.4}$$

In (4.4) we can replace Lu by $u_{tt} + 2au_t$ so that (4.4) becomes a "dynamic boundary condition". Recall that

$$\left\|\begin{pmatrix} u|\Omega \\ u|\partial\Omega \end{pmatrix}\right\|^2_{\mathcal{H}} = ||u_0||^2_{L^2} + ||u_1||^2_{L^2(\partial\Omega, \frac{d\Gamma}{\beta})}.$$

The energy

$$E(t) = ||u'(t)||^2 + ||(-L)^{1/2}u(t)||^2$$

satisfies (due to Theorem 2.1)

$$E(t) \le C(a, L)\exp(-2at) \quad \text{for } 0 < a < \sqrt{b}$$

for all $t \ge 0$, while

$$E(t) \le \tilde{C}(a, L)\exp\left(2\left(-a + \sqrt{a^2 - b}\right)t\right)$$

for $a > \sqrt{b}$ provided $a^2 \notin \sigma_p(L)$.

In general it is a difficult problem to find $b = \lambda_1$ explicitly. We shall solve this problem for a one-dimensional case. Consider as a special case the operator $L_0 u = -u''$ for $x \in [0, \pi/2]$ with boundary conditions

$$(u'' + (-1)^j \beta_j u' + \gamma_j u)\left(\frac{j\pi}{2}\right) = 0, \quad j = 0, 1. \tag{4.5}$$

The coefficients β_j and γ_j will be chosen later. Let us consider the Hilbert space

$$\mathcal{H} = L^2(0, \pi/2) \oplus (\mathbb{C}_{\beta_0} \times \mathbb{C}_{\beta_1})$$

with inner product

$$\left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_0^{\pi/2} u_1(x)\overline{v_1}(x)dx + \sum_{j=0}^{1} \beta_j^{-1} u_1\left(j\frac{\pi}{2}\right)\overline{v_1}\left(j\frac{\pi}{2}\right),$$

where for $w = u, v$, we have $w = \binom{w_1}{w_2}$,

$$w_1 \in L^2(0, \pi/2), \quad w_2 = \begin{pmatrix} w_2(0) \\ w_2(\pi/2) \end{pmatrix} \in \mathbb{C}^2.$$

The operator L_0 has domain $\{v \in H^2\left(\left[0, \frac{\pi}{2}\right]\right), v \text{ satisfies } (4.5)\}$. Furthermore, L, the closure of L_0 satisfies $L = L^* \ge \varepsilon I$ for some $\varepsilon > 0$.

We consider the function $\varphi(x) = \sin x + \cos x$. Then $-\varphi'' = \varphi$ and $\varphi(x) > 0$ on $[0, \pi/2]$. It follows that $L\varphi = \varphi$. Thus $b = 1$ is the lowest eigenvalue of L. Taking $u = \varphi$ in (4.5) we obtain

$$\begin{aligned} -1 + \beta_0 + \gamma_0 &= 0 \quad (\text{at } x = 0), \\ -1 + \beta_1 + \gamma &= 0 \quad (\text{at } x = \pi/2). \end{aligned} \tag{4.6}$$

There are many different choices for β_j, γ_j $(j = 0, 1)$ in the boundary conditions which make φ the ground state. For example,

a) $\beta_0 = \beta_1 \in (0, 1)$, $\gamma_0 = \gamma_1 = 1 - \beta_0$,
b) $\beta_0, \beta_1 \in (0, 1)$ and $\gamma_j = 1 - \beta_j$, $j = 0, 1$,
c) $\beta_1 = 1$, $\gamma_1 = 1$, $\beta_0 \in (0, 1)$, $\gamma_0 = 1 - \beta_0$.

In cases a), b) and c) we have $\beta_j > 0$, $\gamma_j \ge 0$, $\gamma_0 + \gamma_1 > 0$. Summarizing, the solution $u(x, t)$ of

$$\begin{cases} u_{tt} - u_{xx} + 2au_t = 0, \quad 0 < x < \pi/2, \quad t \ge 0, \\ (u_{xx} + (-1)^j \beta_j u_x + \gamma_j u)\left(\frac{j\pi}{2}\right) = 0, \quad j = 0, 1 \end{cases}$$

has energy

$$E(t) = \int_0^{\pi/2} [|u_t|^2 + |u_x|^2] dx + \sum_{j=0}^{1} \beta_j^{-1} \left[\left| u_t \left(\frac{j\pi}{2} \right) \right|^2 + \left| u_x \left(\frac{j\pi}{2} \right) \right|^2 \right] dx$$

and Theorem 2.1 applies. Thus for any $t \geq 0$,

$$E(t) \leq C(a,b)E(0)\exp(-2at) \quad \text{if } 0 < a < 1,$$
$$E(t) \leq C_1(a,L)E(0)\exp\left(2\left[\sqrt{a^2-1} - a\right]t\right) \quad \text{if } a > 1,$$

and $a^2 \notin \sigma_p(L)$.

The decay rate is the best possible.

A large number of articles studying evolution models with Wentzell boundary conditions are available; see for instance references [1], [2], [7], [8], [9] and [12].

5. Appendix: The spectral theorem. Let $A\colon \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H}$ be a densely defined linear operator on a complex Hilbert space $\mathcal{H} = (\mathcal{H}, \langle\cdot,\cdot\rangle, ||\cdot||)$. The adjoint of A, i.e. A^*, is defined as follows. Let $v \in \mathcal{H}$. Suppose there is a constant $C = C(v)$ such that $|\langle Au, v\rangle| \leq C(v)||u||$ for all $u \in \mathcal{D}(A)$. Then $u \mapsto \langle Au, v\rangle$ defines a bounded linear functional on the dense subspace $\mathcal{D}(A)$ of $\mathcal{H}$. This extends uniquely to a bounded linear functional on all of $\mathcal{H}$ with the same bound. By the Riesz Representation Theorem, there is a unique $w \in \mathcal{H}$ such that this linear functional is given by $u \to \langle u, w\rangle$. In this case one says that $v \in \mathcal{D}(A^*)$ and $A^*v = w$. It follows that

$$\langle Au, v\rangle = \langle u, A^*v\rangle \tag{A.1}$$

for all $u \in \mathcal{D}(A)$ and the above discussion explains how to interpret (A.1) in order to define $\mathcal{D}(A^*)$. Note that A^* is always a closed linear operator.

We say that $A\colon \mathcal{D}(A) \subseteq \mathcal{H} \mapsto \mathcal{H}$ is *symmetric* if A is linear, $\mathcal{D}(A)$ is dense and $\langle Au, v\rangle = \langle u, Av\rangle$ holds for all $u, v \in \mathcal{D}(A)$. It follows that A^* is an extension of A; i.e., $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $A^*v = Av$ for $v \in \mathcal{D}(A)$. We say that A is *selfadjoint* if $A = A^*$. Thus, A is symmetric and $\mathcal{D}(A) = \mathcal{D}(A^*)$ in this case. If A is symmetric, then its graph is given by $G(A) = \{(f,g) : f \in \mathcal{D}(A), g = Af\} \subseteq \mathcal{H} \oplus \mathcal{H}$. In this case the closure $\overline{G(A)}$ satisfies the following: If (f, g_1) and $(f, g_2) \in \overline{G(A)}$, then $g_1 = g_2$; thus $\overline{G(A)}$ is the graph of a (single-valued) closed linear operator $\overline{A}$ called the closure of A, and $\overline{A}$ is symmetric. If $\overline{A}$ is selfadjoint we say that A is essentially selfadjoint. A basic lemma says: A is essentially selfadjoint iff the closure $\overline{A}$ of A is selfadjoint iff A has a unique selfadjoint extension. If A is merely symmetric, then it is possible that A can have no selfadjoint extensions or many selfadjoint extensions in general.

THEOREM A.1 (Spectral Theorem, first version). *Let A be a selfadjoint operator on $\mathcal{H}$. Then, there exists a unitary operator W from $\mathcal{H}$ onto some L^2 space, $L^2(\Omega, \Sigma, \mu)$, such that*

$$A = W^{-1} M_m W$$

(or $M_m = WAW^{-1}$), where $m\colon \Omega \to \mathbb{R}$ is a Σ-measurable function and M_m is the operator of multiplication by m with maximal domain: $f \in \mathcal{D}(M_m)$ iff Wf and $M_m(Wf) = mWf$ are both in $L^2(\Omega, \Sigma, \mu)$. The essential range of m, given by

$$\text{ess range}(m) = \bigcap \{\overline{m(\Omega \setminus N)} \ : \ N \in \Sigma, \ \mu(N) = 0\},$$

is the spectrum of A, $\sigma(A) \subseteq \mathbb{R}$. If $F\colon \sigma(A) \subseteq \mathbb{R} \to \mathbb{C}$ is any Borel measurable function on $\sigma(A)$, then $F(A)$ is defined by

$$F(A) = W^{-1} M_{F(m)} W,$$

where the operator $M_{F(m)}$ (multiplication by $F(m)$) has maximal domain. The mapping

$$F \mapsto F(A)$$

defined in this way is an algebra homomorphism and $F(A)$ is bounded iff

$$||F(A)|| = \text{ess sup}\{|F(x)|, x \in \sigma(A)\} < +\infty.$$

Let $W\colon \mathcal{H} \mapsto L^2(\Omega, \Sigma, \mu)$ be unitary, as above. Let $b\colon \Omega \to \mathbb{C}$, a Σ-measurable function whose essential range is contained in $\mathbb{C} \setminus D$ for some nonempty open set D in $\mathbb{C}$. Consider M_b, the (maximally defined) multiplication operator determined by b. Then we call $B = W^{-1} M_b W$ a *normal operator* on $\mathcal{H}$ with nonempty resolvent set and we write $B \in \mathcal{N}(\mathcal{H})$.

If $A = A^*$ on $\mathcal{H}$ and F_j is a Borel measurable function from $\sigma(A) \subseteq \mathbb{R}$ to $\mathbb{C} \setminus D_j$ for some nonempty disk D_j, then $B_j = F_j(A) \in \mathcal{N}(\mathcal{H})$, $j = 1, 2$. Moreover B_1 and B_2 commute in the sense that

$$[(\lambda_1 I - B_1)^{-1}, (\lambda_2 I - B_2)^{-1}] = 0$$

(where $[\cdot, \cdot]$ means the commutator) for $\lambda_j \in \rho(B_j)$, $j = 1, 2$. Note that $\rho(B_j) \supset D_j$. Examples of such operators B are $B = e^{itA}$, $B = \cos(tA)$, $B = A^{-1}\sin(tA)$ for $t \in \mathbb{R}$. For $t \in \mathbb{R}$, the real-valued function $f(x) = \frac{\sin(tx)}{x}$ (defined for $x \neq 0$) is continuous on $\mathbb{R}$ if we define $f(0) = t$. For $\lambda I - A$ injective, $B = (\lambda I - A)^{-1}$ is another such example and this B is bounded iff $\lambda \in \rho(A)$. For $B = F(A) \in \mathcal{N}(\mathcal{H})$, the essential range of $F(m)$ must be contained in $\mathbb{C} \setminus D$ for some open disk D.

Now let $B_1, B_2, \ldots, B_n$ be finitely many operators in $\mathcal{N}(\mathcal{H})$. We say that $B_1, B_2, \ldots, B_n$ commute if

$$[(\lambda_j I - B_j)^{-1}, (\lambda_k I - B_k)^{-1}] = 0$$

for all $\lambda_\ell \in \rho(B_\ell)$ and all $j, k, \ell \in \{1, 2, \ldots, n\}$.

THEOREM A.2 (Spectral Theorem, second version). *Let $B_1, B_2, \ldots, B_n$ be any finite set of commuting normal operators in $\mathcal{N}(\mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space. Then, there is a unitary operator W from $\mathcal{H}$ onto an L^2 space, $L^2(\Omega, \Sigma, \mu)$, such that*

$$B_j = W^{-1} M_{m_j} W$$

(or $M_{m_j} = WB_jW^{-1}$), where $m_j\colon \Omega \to \mathbb{C}$ is a Σ-measurable function and the corresponding multiplication operator M_m has maximal domain. The essential range of m_j is

the nonempty set $\sigma(B_j) \underset{\neq}{\subset} \mathbb{C}$. *If* $F\colon \sigma(B_1)\times\sigma(B_2)\times\cdots\times\sigma(B_n) \subseteq \mathbb{C}^n \mapsto \mathbb{C}$ *is any Borel measurable function, then* $F(B_1, B_2, \ldots, B_n)$ *is defined by*

$$F(B_1, B_2, \ldots, B_n) = W^{-1} M_{F(m_1, m_2, \ldots, m_n)} W,$$

where $M_{F(m_1,m_2,\ldots,m_n)}$ *is the operator of multiplication by* $F(m_1, m_2, \ldots, m_n)$ *on* $L^2(\Omega^n, \sigma(\Sigma^n), \mu^n)$ *where* $\Omega^n = \Omega\times\Omega\times\cdots\times\Omega$ *(n times),* $\sigma(\Sigma^n)$ *is the* σ*-algebra generated by* $\Sigma\times\Sigma\times\cdots\times\Sigma$ *(n factors) and* μ^n *is the corresponding product measure* $\mu\times\mu\times\cdots\times\mu$ *(n factors). The map*

$$F \to F(B_1, B_2, \ldots, B_n)$$

defined in this way is an algebra homomorphism, and $F(B_1, B_2, \ldots, B_n)$ *is bounded iff*

$$||F(B_1, B_2, \ldots, B_n)|| = \text{ess sup}\{|F(w)| \ : \ w \in \sigma(B_1)\times\cdots\times\sigma(B_n)\} < +\infty.$$

Thus the Spectral Theorem essentially reduces the analysis of one or n commuting selfadjoint operators to a problem on $\mathbb{R}$ or on $\mathbb{R}^n$. For example, suppose that $A = A^*$ on $\mathcal{H}$ and $\sigma(A) \subset (-\infty, \gamma)$, with $\gamma \in \mathbb{R}$. Then we can write $A = W^{-1}M_m W$ with $m\colon \Omega \to (-\infty, \gamma)$ being Σ-measurable and M_m has maximal domain in $L^2(\Omega, \Sigma, \mu)$. The unique solution of

$$\frac{du}{dt} = Au(t), \quad u(0) = f \quad \text{(for } t \geq 0)$$

is given by $u(t) = e^{tA}f$, where $e^{tA}f = W^{-1}M_{e^{tm}}Wf$. This is a continuous function from $[0, +\infty)$ into $\mathcal{H}$. It is of class C^1 in time, provided $f \in \mathcal{D}(A)$, i.e. $m(Wf) \in L^2(\Omega, \Sigma, \mu)$. It is of class $C^\infty((0, +\infty), \mathcal{H})$; in fact, it is an analytic function of t with values in $\mathcal{H}$ for $\operatorname{Re} t > 0$ and for such complex t,

$$||e^{tA}f|| \leq e^{\gamma(\operatorname{Re} t)}||f||.$$

Similar remarks apply to

$$u'' + 2au' = Au, \quad u(0) = f, \quad u'(0) = g$$

for $a \in \mathbb{R}$. If $B = (\lambda I - A)$ is an injection, then $B^{-1} = WM_{(1/(\lambda-m))}W^{-1}$ is a well-defined member of $\mathcal{N}(\mathcal{H})$, and B^{-1} is bounded iff $\lambda \in \rho(A)$.

If $z \in \mathbb{C} \setminus \{0\}$, then z has two square roots $z_j = x_j + iy_j$, $j = 1, 2$. We shall write $z_1 = \sqrt{z}$ provided that $x_1 > x_2$ or if $x_1 = x_2$ and $y_2 > y_1$. Otherwise $z_2 = \sqrt{z}$. Thus $\sqrt{4} = 2$ not -2 and $i = \sqrt{-1}$, not $-i$. Then, if $z \in \mathbb{C}$ and $w^2 = z$ for $w \in \mathbb{C}$, then either $w = \sqrt{z}$ or $w = -\sqrt{z}$ and $\sqrt{z}$ is well defined.

Now let $B \in \mathcal{N}(\mathcal{H})$. By the spectral theorem we can write $B = W^{-1}M_m W$ for a suitable $m\colon \Omega \to \mathbb{C}$. Define $B^{1/2} = W^{-1}M_{\sqrt{m}}W$, where $\sqrt{m}(x) = \sqrt{m(x)}$ for $x \in \Omega$. We call $B^{1/2}$ the "usual square root" of B. (If $\dim \mathcal{H} = +\infty$, then any injective B (even I) has uncountably many square roots. They are of the form $c(x)$ where $c(x) \in \{-\sqrt{m(x)}, \sqrt{m(x)}\}$ for each $x \in \Omega$.)

Given a selfadjoint operator A, we define

$$g(A)f = \int_{\sigma(A)} g(\lambda)dE(\lambda)f \tag{A.2}$$

and we say that $f \in \mathcal{D}(g(A))$ iff $g \in L^1(\mathbb{R}; \mu)$, where the Borel measure μ on $\mathbb{R}$ is the unique one determined by the nondecreasing function $\lambda \to ||E(\lambda)f||^2$ on $\mathbb{R}$. This

definition of $g(A)$ agrees with our previous definition of $g(A)$, and the domain of $g(A)$ is given by (A.2). In particular $Af = \int_{\sigma(A)} \lambda E(d\lambda)f$ for all $f \in \mathcal{D}(A)$. Let $A = A^*$ on $\mathcal{H}$ and let for each $\lambda \in \mathbb{R}$, $E(\lambda) = \chi_{(-\infty,\lambda]}(A)$. Then, $E(\lambda) = E(\lambda)^*$ is a selfadjoint (orthogonal) projection, Range $E(\lambda) \supset$ Range $E(\mu)$ if $\lambda > \mu$, and $\lim_{\lambda\to-\infty} E(\lambda)f = 0$, $\lim_{\lambda\to+\infty} E(\lambda)f = f$, $\lim_{\varepsilon\to 0^+} E(\alpha+\varepsilon)f = E(\alpha)f$ for all $f \in \mathcal{H}$ and $\alpha \in \mathbb{R}$. Thus, the vector-valued Lebesgue-Stieltjes integral

$$\int_{\sigma(A)} g(\lambda)E(d\lambda)f = \int_{\sigma(A)} g(\lambda)dE(\lambda)f$$

can be defined for all Borel functions g on $\mathbb{R}$ and suitable $f \in \mathcal{H}$.

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