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THE SONIC LINE AS A FREE BOUNDARY

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Abstract. We consider the steady transonic small disturbance equations on a domain and with data that lead to a solution that depends on a single variable. After writing down the solution, we show that it can also be found by using a hodograph transformation followed by a partial Fourier transform. This motivates considering perturbed problems that can be solved with the same technique. We identify a class of such problems.

1. Introduction. There are many contexts in which free boundaries arise in systems of conservation laws that change type when they are formulated as steady or quasi-steady problems. The free boundary may take the form of a transonic shock. In that case, one is often presented with a situation where the flow is completely known (typically it is a constant state) on one side of the free boundary. On the free boundary itself, a set of boundary conditions is given by the Rankine-Hugoniot conditions. This could be

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called a 'one-way free boundary'. It has been studied recently by a number of authors, for example [1, 3, 4], as it arises naturally in looking at multidimensional self-similar problems.

A second type of transonic free boundary problem occurs when the transition between supersonic and subsonic flow is continuous. Then there is no natural set of boundary conditions except that the solution be continuous and that it form a weak solution to the conservation law system. It would not even be particularly natural to formulate this as a free boundary problem if techniques were available, say, to solve the mixed-type system with only the natural boundary conditions in a larger domain. An example where this was studied numerically is given in [12]; in that paper we coined the term 'two-way free boundary' to describe this coupled problem. We succeeded in formulating boundary conditions that sufficed for computation, but without any proof that they were correct. The present paper provides a partial answer to the question of appropriate boundary conditions that might lead to a well-posed problem.

If a 'two-way' problem is viewed as two separate problems (supersonic and subsonic), then the solution is not known a priori in either the supersonic or the subsonic region. The states are coupled through the condition of continuity at the boundary. A linear version of the problem is given by boundary value problems for the Tricomi equation formulated by Cathleen Morawetz [9] and solved by her in the plane, as well as by Lax and Phillips [8] and others. An important difference is that the linear problem does not have a free boundary.

The linear Tricomi problem is a representation of a fluid dynamics problem for ideal compressible gas flow in the hodograph plane, that is, in phase space. By contrast, the quasi-linear problem formulated here describes a situation in physical space, either in the spatial plane (for two-dimensional steady flow) or in a two-dimensional similarity space, where the variables represent x/t and y/t. In the latter case, there is a clear notion of dynamics, with the forward time direction represented by motion towards the origin. For steady flow, the underlying direction of motion typically determines the timelike direction.

The quasilinear problems that occur in self-similar reductions of two-dimensional gas dynamics are more complicated than the problem considered in this paper. For example, the continuous part of the sonic line in the problem of diverging rarefaction waves [12], and the continuous side of the supersonic bubble in Guderley Mach reflection [11, 13, 14], both develop into shocks downstream, and in both problems the subsonic region is noncompact. A principal motivation for the work in this paper is the hope that the solution of the simplified problem considered here will be helpful in devising solutions for these more natural (and more complex) problems.

As well as involving more intricate global geometry, the physical problems that occur in gas dynamics also involve more than just an acoustic transition from supersonic to subsonic flow. Linear characteristic families are present in general and, as a result, the transition, in PDE terms, is between a hyperbolic and a mixed-type system. Analysis conducted so far seems to show that the basic problem can be understood by looking at a model problem where no linear waves are present. Incorporating the linear waves is far from straightforward, but it can be seen as a generalization imposed on the acoustic problem.

Thus, for this paper we consider the simplest nonlinear context: the steady transonic small-disturbance equation (TSDE). This system has the advantage of transforming naturally to a second-order equation, and of having a transparent characteristic structure where it is hyperbolic. The nature of the nonlinearity and the change of type appear to be generic. (An alternative model, the transonic full potential equation, is also appealing. As well as being of greater physical generality, it has the feature of respecting the Euclidean symmetry of space. The standard procedure for the full-potential system is to work with the velocity potential. We have not chosen that approach, partly on the grounds that we want to move towards the consideration of nonpotential flows. However, a full-potential version of the problem considered here would be interesting.)

We begin with a particularly simple boundary value problem for the TSDE. Assuming a domain and boundary conditions that are invariant in one direction, we formulate a problem that possesses a closed-form solution. This, in turn, is a springboard for a perturbed problem that involves the transverse variable in a nontrivial way.

The structure of this paper is as follows. In Section 2 we pose the problem illustrated in Figure 1 and write down the explicit solution. In Section 3 we rewrite the problem in hodograph coordinates, and in Section 4 we set up a Fourier transform procedure to solve problems in the hodograph plane, and we recover the one-dimensional solution via Fourier transformation. Section 5 determines a class of perturbed problems that may be solved by the same technique, and in Section 6 we prove that these solutions actually exist, under certain conditions. In the concluding section, 7, we interpret our solutions.

2. The one-dimensional problem. A convenient formulation of the TSDE is

$$\begin{aligned} uu_x + v_y &= 0, \\ v_x - u_y &= 0. \end{aligned} \tag{2.1}$$

It is hyperbolic when $u < 0$, elliptic when $u > 0$. We examine problems for which we expect to find $u = 0$ on a regular curve, called the *sonic line*. We recall that this equation is derived from steady potential flow in the more complete compressible gas dynamics system by assuming an underlying flow in the negative x-direction that is close to sonic. Then u and v are perturbation velocity components (with positive u measuring a decrease in speed). In addition, u is proportional to density (and also pressure) differences from the ambient sonic flow. (For steady irrotational flows, Bernoulli's law implies that density and speed vary inversely with each other.)

In order not to be unduly influenced by the anisotropic nature of the equations, we consider a strip, Ω_P, in general position, bounded by two lines

$$\alpha x + \beta y = \pm 1, \qquad \beta \neq 0, \tag{2.2}$$

on which we impose boundary conditions that will ensure change of type in the interior. See Figure 1. The condition $\beta \neq 0$ is necessary for the one-dimensional problem to have any solutions. Then, without loss of generality except for the possibility of rescaling the variables by a dilation, we may choose $\beta = 1$, and let $z = \alpha x + y$. We assume now that

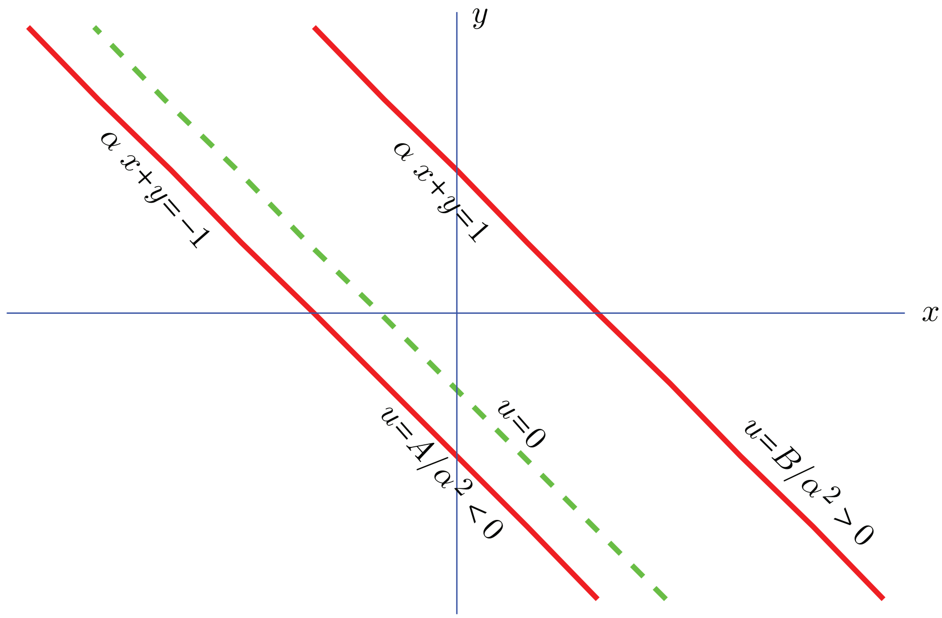

FIG. 1. The domain for the problem

$\alpha \neq 0$ as well. There is a solution for $\alpha = 0$, but it is found by a different method. The one-dimensional boundary conditions for $u(z)$ are

$$\begin{aligned} u(-1) &= A/\alpha^2 < 0, \\ u(1) &= B/\alpha^2 > 0, \end{aligned} \tag{2.3}$$

where A and B are constant. Because of the physical interpretation above, the basic flow is expansive. (We could alternatively formulate a problem for a compression wave.)

We deduce from (2.1) a second-order equation for u:

$$(uu_x)_x + u_{yy} = 0\,.$$

The one-dimensional problem becomes an ordinary differential equation for $u(z)$:

$$\alpha^2(uu')' + u'' = 0\,.$$

The solution is

$$u(z) = \frac{1}{\alpha^2}\big(-1 \pm \sqrt{1 + Cz + D}\big)\,,$$

where C and D are constants of integration. To fix ideas, we assume from now on that $\alpha > 0$, and we select the $+$ sign in front of the square root so that we may have $u(-1) < 0 < u(1)$. We derive expressions for C and D in terms of the boundary conditions:

$$\begin{aligned} C &= (B - A)\left(\frac{A+B}{2} + 1\right), \\ D &= \frac{A^2 + B^2}{2} + A + B\,. \end{aligned}$$

We want a solution with $u(-1) < 0 < u(1)$ (that is, $A < 0 < B$), and hence we need $C > 0$. (Note that we have scaled all the constants, including the boundary values themselves, by α^2.)

Now the solution of the one-dimensional boundary value problem (2.1), (2.3), on the strip

$$-1 \leq \alpha x + y \leq 1, \tag{2.4}$$

is given by

$$\begin{aligned} u(x,y) &= \frac{1}{\alpha^2}\Big(-1 + \sqrt{1 + C(\alpha x + y) + D}\Big), \\ v(x,y) &= \frac{\sqrt{1 + C(\alpha x + y) + D}}{\alpha^3} - \frac{C}{2\alpha^3} y\,. \end{aligned} \tag{2.5}$$

Here we have recovered v from (2.1) in the standard way. The boundary conditions, and the solution as given, are incomplete, as v is determined, from (2.1), only up to an additive constant (which we have taken to be zero in (2.5)). To fix that constant, it suffices to impose a boundary condition on v at a single point on the boundary, say

$$v(0,-1) = E\,.$$

We note that v is not a function of $\alpha x + y$ alone, and we will return to the correct specification of the boundary conditions for v when we consider the two-dimensional problem.

We shall refer to the solution (2.5) as (u^0, v^0) in the next section.

We note the domain of the (constant) boundary values. From

$$\begin{aligned} A &= \alpha^2 u(-1) = -1 + \sqrt{1 - C + D}, \\ B &= \alpha^2 u(1) = -1 + \sqrt{1 + C + D}, \end{aligned}$$

we require $1 - C + D > 0$ and $1 + C + D > 0$ for a real solution which is differentiable up to the boundaries, and $1 - C + D < 1$ and $1 + C + D > 1$ for a transonic solution. One of these conditions is redundant, and so we find that C and D lie in the interior of a half-strip, $0 < C - D < 1$ and $C + D > 0$, corresponding to $-1 < A < 0$, $B > 0$. See Figure 2.

From the solution we can now recover the sonic line, given by $u = 0$ for this model system. It is

$$\alpha x + y = z = -\frac{D}{C} = \frac{A^2 + B^2 + 2(A + B)}{A^2 - B^2 + 2(A - B)}\,. \tag{2.6}$$

Note that, as $C > 0$, the sonic line lies to the right or left of the origin according as $D < 0$ or $D > 0$.

For completeness, we include the case $\alpha = 0$. Now we seek $u = u(y)$ solving the second-order equation, and the solution, for $u(-1) = A$, $u(1) = B$, is

$$\begin{aligned} u(x,y) &= \frac{B - A}{2} y + \frac{A + B}{2}, \\ v(x,y) &= \frac{B - A}{2} x\,. \end{aligned}$$

In this case, there is no restriction on the relative sizes of A and B.

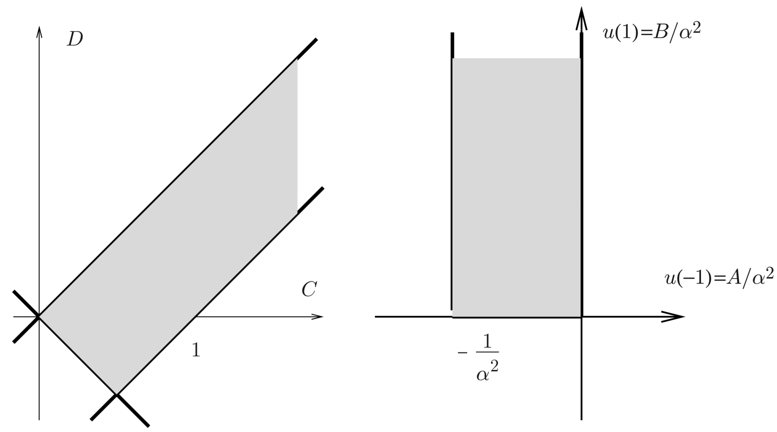

FIG. 2. Normalized domain of the boundary values

For the remainder of this paper, we assume $\alpha \neq 0$. As far as we know, this simple explicit solution has not appeared in the literature, though it may be known to researchers in transonic flow. Our interest in it, explored in this paper, is as an avenue for finding possible boundary conditions for two-dimensional self-similar problems. In the current paper, we exploit the fact that this geometry and solution are amenable to treatment with a hodograph transform and, further, with the Fourier transform.

3. Formulation in the hodograph plane. The point of departure for the hodograph transform is the observation that the functions $u(x,y)$, $v(x,y)$ can be considered as a mapping $T:\mathbb{R}^2 \to \mathbb{R}^2$,

$$T : (x,y) \mapsto (u,v)\,,$$

and this mapping is locally invertible if its Jacobian determinant,

$$\det(dT) = u_x v_y - v_x u_y, \tag{3.1}$$

is nonzero. For the one-dimensional solution, (2.5), this is the case, as

$$\det(dT) = \frac{C\alpha}{2\alpha^2 S}\left(-\frac{C\alpha}{2\alpha^2}\right) = -\frac{C^2}{4\alpha^4 S}\,. \tag{3.2}$$

Here $S = S(x,y) = \sqrt{1 + C(\alpha x + y) + D}$ is an abbreviation for the quantity that appears repeatedly in the formulas for u and v; it is nonzero throughout. The mapping T for the one-dimensional solution is in fact globally invertible, with inverse given by (3.5) below. From (3.1) and (3.2) we see that if we work with solutions that are small perturbations (uniformly in C^1) of the one-dimensional solution, then we can use the hodograph plane to describe solutions.

The transformed variables satisfy

$$(dT)^{-1} = \frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}$$

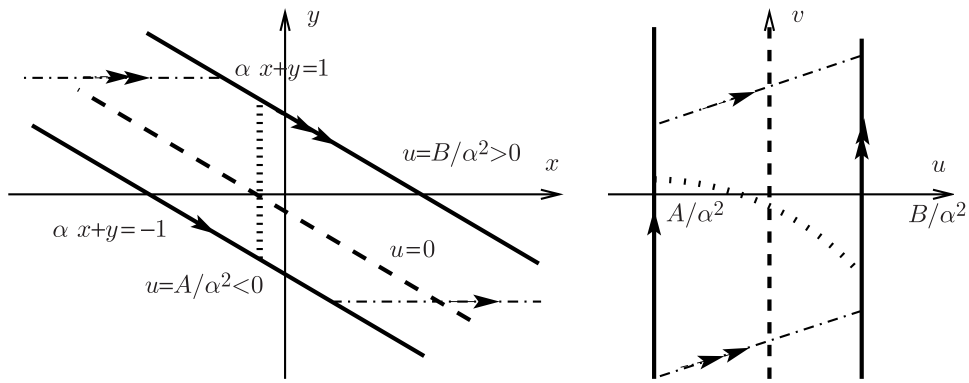

FIG. 3. The hodograph mapping

or

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \frac{1}{\det(dT)} \begin{pmatrix} v_y & -u_y \\ -v_x & u_x \end{pmatrix} .$$

Thus the hodograph system for the small-disturbance equations (after dividing by $\det(dT)$) is

$$\begin{aligned} x_u + u y_v &= 0, \\ y_u - x_v &= 0 . \end{aligned} \tag{3.3}$$

This can also be written as a second-order equation (the Tricomi equation):

$$y_{uu} + u y_{vv} = 0 . \tag{3.4}$$

We again note that $u < 0$ corresponds to supersonic states (hyperbolic equation) and $u > 0$ to subsonic (elliptic) regions.

We will use solution techniques for (3.4) to formulate solvable boundary-value problems for (2.1). We work with perturbations of the one-dimensional problem solved in Section 2. For the one-dimensional solution (2.5), the mapping T is clearly globally invertible. The mapping is orientation-reversing (with our choice of parameters) and maps the infinite strip Ω_P to an infinite vertical strip Ω_H; see Figure 3. Solving (2.5) for the inverse mapping yields

$$\begin{aligned} x &= \frac{1}{C}\left(\alpha^3 u^2 + 2\alpha^2 v - \frac{1}{\alpha}\big(2 + D\big)\right), \\ y &= \frac{2}{C}\big(\alpha^2 u - \alpha^3 v + 1\big) . \end{aligned} \tag{3.5}$$

Again, we note that the Jacobian,

$$\frac{1}{\det(dT)} = -\frac{4}{C^2}\alpha^4(\alpha^2 u + 1) ,$$

is bounded away from zero (because $\alpha^2 u \geq A > -1$), with a bound depending only on the domain and the boundary conditions (α, A and B).

4. Application of the Fourier transform. The discussion above motivates our trying to solve problems for hodograph variables $\bigl(x(u,v), y(u,v)\bigr)$ with data given on the vertical sides of Ω_H. In particular, we will identify a set of hodograph data that is larger than the specific linear data that give rise to the one-dimensional solution (3.5), but that still admit an inverse hodograph transform (and hence correspond to a flow in physical coordinates).

A natural way to solve (3.3) or (3.4) on the infinite strip is to take a partial Fourier transform (in v). To fix ideas, we use the following normalization:

$$\hat{y}(u,\eta) = \int_{-\infty}^{\infty} e^{-iv\eta} y(u,v)\,dv \equiv \mathcal{F}_v\bigl(y(u,\cdot)\bigr)\,. \tag{4.1}$$

In the motivating one-dimensional problem, y has linear growth in v, and so its Fourier transform is a tempered distribution (element of $\mathcal{S}'$) and is defined as a limit of integrals.

Because $y(v)$ (or $v(y)$) in the motivating problem is invertible on both sides of Ω_H, we choose to prescribe boundary conditions for $y(v)$.

From (3.5), the one-dimensional data on the vertical sides of the domain are

$$y_L^0(v) = \frac{2}{C}\bigl(\sqrt{1-C+D} - \alpha^3 v\bigr), \qquad y_R^0(v) = \frac{2}{C}\bigl(\sqrt{1+C+D} - \alpha^3 v\bigr)\,. \tag{4.2}$$

The Fourier transforms of these functions are also tempered distributions. A Fourier transform applied to equation (3.4) results in

$$\hat{y}_{uu} + u\widehat{y_{vv}} = 0\,.$$

From properties of the Fourier transform in $\mathcal{S}'$ we have

$$\widehat{y_{vv}} = -\eta^2 \hat{y}\,.$$

Thus, (3.4) transforms to

$$\hat{y}_{uu} - \eta^2 u \hat{y} = 0\,. \tag{4.3}$$

Now, consider η as a parameter, and note that (4.3) is a scaling of the Airy equation,

$$Y_{tt} - tY = 0\,, \tag{4.4}$$

with $t = \lambda u = \eta^{2/3} u$. Thus, we may use standard Airy functions to solve (4.3), and further apply the Fourier transforms of the boundary conditions (4.2) at $u = A/\alpha^2$ and $u = B/\alpha^2$ to recover $\hat{y}$. For perturbations of the data (4.2), in Section 5 we will similarly solve the perturbed problem for $\hat{y}$. In either case, we integrate (3.3) to obtain x. Our intention is to produce a solution to the problem for (u,v) close to the original one-dimensional solution. First, we display the mechanics for calculating the solution.

Let A_1 and A_2 be a fundamental set of solutions to the Airy equation. The data $\hat{y_L}(\eta)$, $\hat{y_R}(\eta)$ give a two-point boundary-value problem (in u) for $\hat{y}(u,\eta)$, at each value of η. We write

$$\hat{y}(u,\eta) = c_1(\eta) A_1(\eta^{2/3} u) + c_2(\eta) A_2(\eta^{2/3} u)\,.$$

Boundary conditions are imposed on $\hat{y}$ at $u = A/\alpha^2 < 0$ and at $u = B/\alpha^2 > 0$. If the solution is continuous in u, we have

$$\hat{y}\left(\frac{A}{\alpha^2}, \eta\right) = \hat{y}_L(\eta), \tag{4.5}$$

$$\hat{y}\left(\frac{B}{\alpha^2}, \eta\right) = \hat{y}_R(\eta)\,. \tag{4.6}$$

Thus we obtain a system of equations for $c_1(\eta)$ and $c_2(\eta)$:

$$\begin{pmatrix} A_1\left(\frac{\eta^{2/3}A}{\alpha^2}\right) & A_2\left(\frac{\eta^{2/3}A}{\alpha^2}\right) \\ A_1\left(\frac{\eta^{2/3}B}{\alpha^2}\right) & A_2\left(\frac{\eta^{2/3}B}{\alpha^2}\right) \end{pmatrix} \begin{pmatrix} c_1(\eta) \\ c_2(\eta) \end{pmatrix} = \begin{pmatrix} \hat{y}_L(\eta) \\ \hat{y}_R(\eta) \end{pmatrix}. \tag{4.7}$$

Letting $M = M(\eta)$ stand for the matrix in (4.7), we write (4.7) in vector notation as

$$M\mathbf{c} = \hat{\mathbf{y}}\,. \tag{4.8}$$

4.1. *Recovery of the one-dimensional solution via Fourier transform.* To fix ideas, we solve the one-dimensional problem with data (y_L^0, y_R^0) from (4.2), to recover $y^0(u,v) = 2(\alpha^2 u - \alpha^3 v + 1)/C$ as in (3.5).

The transforms of y_L^0 and y_R^0 are straightforward:

$$\begin{aligned} \hat{y}_L^0(\eta) &= \frac{2}{C}\sqrt{1 - C + D}\cdot 2\pi\delta(\eta) - \frac{2}{C}\alpha^3 \cdot 2\pi i \delta'(\eta), \\ \hat{y}_R^0(\eta) &= \frac{2}{C}\sqrt{1 + C + D}\cdot 2\pi\delta(\eta) - \frac{2}{C}\alpha^3 \cdot 2\pi i \delta'(\eta)\,. \end{aligned} \tag{4.9}$$

Now, a general problem that arises in applying this method is that the matrix M in (4.7) may not be invertible. In fact, the support of the data in (4.9) is concentrated at the origin, and precisely at that point (among many points, as we shall see) M is singular. The resolution of this is to regard M^{-1} as a complex function, analytic except at a countable set of values, but containing singularities in the form of poles and (as here at the origin) also algebraic singularities. Then we interpret in the sense of distributions products of terms in M^{-1} with distributions. The following calculus produces the correct result. We take for A_1 and A_2 a convenient pair:

$$A_1(t) = 1 + \mathcal{O}(t^2)\,, \qquad A_2(t) = t + \mathcal{O}(t^2)\,,$$

for $t \approx 0$ by choosing an appropriate fundamental set of solutions to (4.4), and examine M and M^{-1}, for uniformly small values of η:

$$M = \begin{pmatrix} 1 + O_1 & \dfrac{\eta^{2/3}A}{\alpha^2} + O_2 \\ & \\ 1 + O_3 & \dfrac{\eta^{2/3}B}{\alpha^2} + O_4 \end{pmatrix}, \tag{4.10}$$

where $O_j = \mathcal{O}(\eta^{4/3})$ for $j = 1, \ldots, 4$. Then if we write M^{-1} in the form

$$M^{-1} = \frac{1}{B - A}\begin{pmatrix} B + a_1 & -A + a_2 \\ & \\ -\dfrac{\alpha^2}{\eta^{2/3}} + a_3 & \dfrac{\alpha^2}{\eta^{2/3}} + a_4 \end{pmatrix} \tag{4.11}$$

we will have

$$\begin{aligned}
a_1 &= \mathcal{O}(\eta^{4/3}),\\
a_2 &= \mathcal{O}(\eta^{4/3}),\\
a_3 &= \eta^{-2/3}\alpha^2(O_4 - O_2) + \mathcal{O}(\eta^{4/3}),\\
a_4 &= -\eta^{-2/3}\alpha^2(O_4 - O_2) + \mathcal{O}(\eta^{4/3})\,.
\end{aligned}$$

Now we can compute, formally,

$$\begin{pmatrix} c_1(\eta) \\ c_2(\eta) \end{pmatrix} = M^{-1} \begin{pmatrix} \hat{y}^0_L(\eta) \\ \hat{y}^0_R(\eta) \end{pmatrix} = \begin{pmatrix} k_1\delta(\eta) + k_2\delta'(\eta) \\ k_3\eta^{-2/3}\delta(\eta) \end{pmatrix}\,, \tag{4.12}$$

with

$$\begin{aligned}
k_1 &= \frac{4\pi}{C}\left(\frac{B\sqrt{1-C+D} - A\sqrt{1+C+D}}{B-A}\right) = \frac{4\pi}{C},\\
k_2 &= -\frac{4\pi i\alpha^3}{C},\\
k_3 &= \frac{4\pi\alpha^2}{C(B-A)}\left(\sqrt{1-C+D} - \sqrt{1+C+D}\right) = \frac{4\pi\alpha^2}{C}\,,
\end{aligned}$$

using the fact that in (4.9) the coefficients of δ' are the same in $\hat{y}^0_L$ and $\hat{y}^0_R$, from which it follows that multiplication with the a_j does not give any additional terms. Because the support of δ is concentrated at $\eta = 0$, the expressions for c_1 and c_2 are exact. The formula for c_2 shows that it cannot stand alone as a distribution. However, the expression c_2A_2 is well-defined, as we can use the approximations for A_1 and A_2 again to compute

$$\begin{aligned}
\hat{y}(u,\eta) &= c_1(\eta)A_1(\eta^{2/3}u) + c_2(\eta)A_2(\eta^{2/3}u)\\
&= k_1\delta(\eta) + k_2\delta'(\eta) + k_3\eta^{-2/3}\delta(\eta)\big(\eta^{2/3}u\big)\\
&= k_1\delta(\eta) + k_2\delta'(\eta) + k_3\delta(\eta)u\,.
\end{aligned}$$

Inverting the transform, we now find

$$y(u,v) = \frac{k_1}{2\pi} + k_2\left(\frac{-i}{2\pi}\right)v + \frac{k_3}{2\pi}u = \frac{2}{C}\left(1 - \alpha^3 v + \alpha^2 u\right),$$

the y-component of the one-dimensional solution in (3.5).

In this one-dimensional case the behavior of M and M^{-1} is important only near $\eta = 0$, and M^{-1} has an integrable singularity at $\eta = 0$.

5. Perturbation of the one-dimensional problem. Sections 3 and 4 have expounded a technique. We now exploit the technique on some transonic problems. Consider perturbing the hodograph data y^0_L and y^0_R. Let

$$y_L(v) = y^0_L(v) + \varepsilon y^1_L(v), \quad y_R(v) = y^0_R(v) + \varepsilon y^1_R(v)\,. \tag{5.1}$$

As the boundary-value problem for (3.4) and (5.1) is linear, we have

$$y(u,v) = y^0(u,v) + \varepsilon y^1(u,v)$$

and

$$\hat{y}^1(u,\eta) = c^1_1(\eta)A_1(\eta^{2/3}u) + c^1_2(\eta)A_2(\eta^{2/3}u)\,,$$

with

$$M\begin{pmatrix} c_1^1(\eta) \\ c_2^1(\eta) \end{pmatrix} = \begin{pmatrix} \hat{y}_L^1(\eta) \\ \hat{y}_R^1(\eta) \end{pmatrix} . \tag{5.2}$$

Finally,

$$y^1 = \mathcal{F}^{-1}\big(c_1^1(\eta)A_1(\eta^{2/3}u) + c_2^1(\eta)A_2(\eta^{2/3}u)\big) .$$

The contribution at $\eta = 0$ can be handled as before, assuming that $\hat{y}_L^1$ and $\hat{y}_R^1$ are tempered distributions. However, finding M^{-1} presents other difficulties, as $\det M$ is zero for a countable set of values of η. (See Section 6.1.) A standard calculation shows that $d(\det M)/d\eta \neq 0$ at any point η_0 where $\det M = 0$. Thus

$$M^{-1} = \frac{1}{\eta - \eta_0} M_r ,$$

where M_r is a bounded matrix. That is, M^{-1}, considered as a distribution, contains terms that have the form of a Cauchy principal value. (See, for example, [5, pages 100–101].) Finally, using equation (5.2), $\mathbf{c}^1 = M^{-1}\hat{\mathbf{y}}^1$, in the notation of equation (4.8), is well-defined as long as $\hat{\mathbf{y}}^1$ is a multiplier (see [5, pages 102-103]). We obtain y^1 from

$$y^1 = \mathcal{F}^{-1}(c_1^1) * \mathcal{F}^{-1}(A_1) + \mathcal{F}^{-1}(c_2^1) * \mathcal{F}^{-1}(A_2) , \tag{5.3}$$

where $*$ represents convolution. The operations in (5.3) are well-defined since A_1 and A_2, scaled solutions to Airy's equation, are analytic functions (except at $\eta = 0$), and their inverse transforms are smooth and compactly supported. As a result y^1 is smooth.

We now turn to verifying the hypotheses under which (5.3) was constructed, and to interpreting the solution.

If we regard (5.1) as coming from a perturbation of the data (4.2), then a linear perturbation in ε is unnatural, as the functions in (4.2) are the inverses of the physical data. If we write the physical data as

$$v_j(y, \varepsilon) = v_j^0(y) + \mathcal{O}(\varepsilon), \quad j = L, R$$

with

$$\frac{dv_j}{dy} \neq 0 \quad \text{and} \quad \frac{dv_j}{dy}(y, \varepsilon) = \frac{dv_j^0}{dy}(y) + \mathcal{O}(\varepsilon) ,$$

then when we invert we obtain

$$\begin{aligned} y_j(v) = v_j^{-1} &= y_j^0(v) + \mathcal{O}(\varepsilon) \\ &= y_j^0(v) + \varepsilon y_j^1(v, \varepsilon) \end{aligned}$$

with

$$\sup_v \big(|y_j^1(v,\varepsilon)| + |\partial_v y_j^1(v,\varepsilon)|\big) < 1, \quad j = L, R, \tag{5.4}$$

say. The constant 1 is arbitrary but serves to prescribe a range of ε for which the function $y_j(v)$ is invertible.

The problem in Ω_H consists of (3.4) in Ω_H with boundary conditions

$$\begin{aligned} y(A/\alpha^2, v) &= y_L(v), \\ y(B/\alpha^2, v) &= y_R(v) . \end{aligned} \tag{5.5}$$

As the problem is linear, we find that $y(u,v) = y^0(u,v) + \varepsilon y^1(u,v,\varepsilon)$, based on the two terms in the boundary conditions. The contribution y^0 is as before. The y^1-portion may contain ε as a parameter. (We can ignore this additional dependence on ε.)

Using the partial Fourier transform in the v-variable, we obtain (5.3). As stated before, this method gives a solution if the perturbations y^1_j are restricted to be inverse transforms of Fourier multipliers.

6. Details of the construction. In this section, we take up three questions in turn:

(1) The singularities of M.
(2) The substitution of boundary conditions in v for the original problem.
(3) The nature of the perturbation problems we have solved.

6.1. *The matrix M.* We examine M and (4.7) for each η. As noted, M is singular at $\eta = 0$, with $M^{-1} = \mathcal{O}(\eta^{-2/3})$. Although y^1 is determined by (5.3) for a wide set of data, it is desirable, since we want to invert the hodograph transform, for y^1 to be differentiable. To control the behavior of $\hat{y}^1$ at $\eta = 0$ it is sufficient to impose the condition (5.4), since this forces on $\hat{y}_j$ a behavior leading to singularities in c_1 and c_2 no worse than in (4.12).

However, we expect in addition a countable set of singularities for M at points $\eta_k \neq 0$. For, consider the eigenvalue problem for Airy's equation:

$$x''(t) - \lambda^3 t x(t) = 0, \quad x(a) = 0 = x(b)\,, \tag{6.1}$$

or

$$X''(\tau) - \tau X(\tau) = 0, \quad X(\lambda a) = 0 = X(\lambda b) \tag{6.2}$$

with $\tau = \lambda t$ and $x(t) = X(\lambda t)$. Now, in our problem $a < 0 < b$, leading to a weight term t of indefinite sign. The problem (6.1) has a countable set of eigenvalues $\lambda_1 < \lambda_2 < \ldots$ and corresponding eigenfunctions $x_k(t) = Ai_k(\lambda_k t)$, where each Ai_k is a solution of Airy's equation with $Ai_k(\lambda_k a) = 0 = Ai_k(\lambda_k b)$. Existence of these eigenvalues is a classical result; see Ince [7].

Proposition 6.1. The matrix M is singular at precisely the values $\eta_k = \lambda_k^{3/2}$, where λ_k are the eigenvalues of (6.1) with $a = A/\alpha^2$, $b = B/\alpha^2$.

Proof. With the notation in the statement of the proposition, if λ_k is an eigenvalue of (6.1), then at $\eta = \lambda_k^{3/2}$,

$$M(\eta) = \begin{pmatrix} A_1(\lambda_k a) & A_2(\lambda_k a) \\ A_1(\lambda_k b) & A_2(\lambda_k b) \end{pmatrix}\,,$$

where $\{A_1(\cdot), A_2(\cdot)\}$ form a fundamental set of solutions to Airy's equation. It does not affect the singularity in M if we choose A_1 so that $A_1(\lambda_k a) = 0$. In that case, $A_1(\lambda_k t)$ satisfies (6.2) which means that $A_1(\lambda_k b) = 0$ and M is singular.

Conversely, if M is singular for a given value of $\eta \neq 0$, then define $\lambda = \eta^{2/3}$, and again, without loss of generality, take $A_1(\lambda a) = 0$. As A_1 and A_2 are linearly independent, we have $A_2(\lambda a) \neq 0$. Thus

$$\det M \equiv A_1(\lambda a)A_2(\lambda b) - A_1(\lambda b)A_2(\lambda a) = 0$$

implies $A_1(\lambda b) = 0$, whence $A_1(\lambda \cdot)$ satisfies (6.2). $\square$

As explained in Section 5, these singularities in M do not affect the existence or smoothness of y^1 (provided always that $\hat{\mathbf{y}}^1$ is a multiplier). It would be interesting to learn whether they have any physical significance.

6.2. *Boundary conditions: v versus u.* Our point of departure in Section 4 was to solve by Fourier transforms a boundary value problem for $y(u,v)$. That meant substituting boundary conditions in v for those in u in the original problem (2.1) and (2.3). It is natural to ask whether the problems are equivalent.

If $dy_j(v)/dv \neq 0$ for $j = L$ and R, then there are unique functions $v_j(y)$ that are perturbations of the one-dimensional boundary data for v, which, from (2.5), are

$$v_L^0(y) = \frac{1}{\alpha^3}\big(\sqrt{1-C+D} - \frac{C}{2}y\big), \quad v_R^0(y) = \frac{1}{\alpha^3}\big(\sqrt{1+C+D} - \frac{C}{2}y\big) \tag{6.3}$$

at $\alpha x + y = -1$ and $\alpha x + y = 1$, respectively. This gives an alternate formulation for the one-dimensional problem and its perturbations. In support of the equivalence between these boundary conditions we have

PROPOSITION 6.2. For any values C and D in the admissible range, the problem consisting of (2.1) in Ω_P with data (6.3) on $\partial\Omega_P$ has a unique solution in $\mathcal{S}'$ in the class of hodograph-invertible mappings. The solution coincides with the one-dimensional solution found in Section 2.

We note that it appears to be difficult to prove a uniqueness result without restricting the class of solutions to those to which the hodograph transformation can be applied. An equivalent formulation is to state that this is the only solution in a $\mathcal{C}^1$ neighborhood of the one-dimensional solution. With this hypothesis, the proof is simple and is omitted.

6.3. *The nature of the perturbation problems.* Finally, we examine restrictions on the set of problems that can be solved by this method.

Upon inverting the perturbed solution $(x(u,v), y(u,v))$, we obtain a solution $(u(x,y), v(x,y))$ in a domain $\Omega \subset \mathbb{R}^2$ bounded by the curves

$$\begin{aligned}(x_L(v), y_L(v)) &= (x(A/\alpha^2, v), y_L(v)),\\ (x_R(v), y_R(v)) &= (x(B/\alpha^2, v), y_R(v))\,.\end{aligned} \tag{6.4}$$

Here $x(u,v)$ is found by integrating (3.3) and is determined up to an additive constant. In order to preserve invertibility, the perturbed boundary condition must be chosen so that all components of dT are uniformly small perturbations of the one-dimensional problem. From standard properties of convolution operators, we obtain bounds in the C^1-norm (as in equation (5.4)).

PROPOSITION 6.3. There exists a constant C, independent of y_L^1 and y_R^1, such that for the function y^1 satisfying (5.3), with $\mathbf{y}^1 = (y_L^1, y_R^1)$,

$$\|y^1\|_{C^1(\Omega_H)} \leq C\|\mathbf{y}^1\|_{C^1(\mathbb{R})}\,.$$

Proof. We cite L^p and Sobolev estimates on convolutions (see for example Hörmander [6, Section 4.5]), noting that since y^1 is smooth we can differentiate (5.3) as many times as needed to obtain the desired estimate. □

Nonetheless, the form of the perturbation is peculiar. The original domain Ω_P is replaced by a domain Ω that is a priori unknown, but in which the flow satisfies $u \equiv A/\alpha^2$, $v = v_L(y)$ on the left, and $u \equiv B/\alpha^2$, $v = v_R(y)$ on the right. The solution itself determines the boundaries of the domain given by (6.4).

This unorthodox perturbation was forced on us by the choice of technique. Unless we keep u constant on the boundaries, the hodograph domain will not be suitable for the partial Fourier transformation we applied. Even if we are willing to abandon the Fourier transform, it is not possible to fix boundaries simultaneously in Ω_P and Ω_H (unless the solution is already known).

In addition, as there is no apparent compactness mechanism, it also seems infeasible to use the perturbations we have constructed in the hodograph plane to approximate a desired perturbation in Ω_P.

7. Conclusions. The study of multidimensional conservation laws, a subject of great current interest, has prompted new investigations of steady systems, such as the TSDE. The change-of-type phenomena exhibited in these simpler problems have already motivated techniques useful for the study of self-similar two-dimensional systems. For example, the work of Čanić, Keyfitz and Lieberman, [2], established the method that was expanded in the works already cited, [1, 3, 4].

A situation that occurs in self-similar two-dimensional systems (the occurrence of a sonic line across which the flow is continuous) has similarly motivated the excursion in the present paper. This phenomenon has been displayed in our research in two contexts: as a continuous two-way free boundary in a problem involving rarefactions and the unsteady TSDE [12], and in the more complicated supersonic patch in Guderley Mach reflection [11, 13, 14]. (We have suggested the former as a simple model for the latter.) At the moment, we have only numerical evidence for the behavior of the sonic line in these problems. If we adopted the technique from the earlier work cited of solving self-similar problems separately in the supersonic and subsonic regions, then the sonic line would appear as a free boundary in both a degenerate hyperbolic system and in a degenerate elliptic or mixed system. This is a somewhat intimidating prospect. The review paper by Morawetz [10], describing the state of the art of about seven years ago, points out the advances in linear problems, but notes that for the nonlinear system the only analytical results are those for a uniformly positive viscosity. The current paper takes a different, though rather specialized, approach.

One of the contributions of this paper is to display a simple but (to our knowledge) new solution to TSDE, the solution (2.5) to the problem (2.1) and (2.3).

We found this solution by elementary means, but we also recovered it via a partial Fourier transform in the hodograph plane. We also showed that Fourier transformation can be used to solve a collection of problems in hodograph variables. It is noteworthy that this is a method that does not take account of the type of the equation locally; one works on transonic problems and recovers the sonic line as a part of the solution.

By transforming back to physical variables, we have defined a type of boundary value problem in the physical domain that has a solution. That problem consists of giving the same data, (2.3), but on a region with boundaries defined by (6.4), with y_L and y_R

almost linear. In the boundary value problem we have solved, the domain boundaries are now free boundaries.

The family of problems we have solved appears to offer some insight into the continuous sonic line problems mentioned above, particularly the 'diverging rarefactions' problem posed in [12]. We note in particular the apparent symmetry in the boundary conditions that gives rise to a well-posed problem. There is one boundary condition on each side (y_L and y_R, or v_L and v_R) even though the equation is hyperbolic on one side and elliptic on the other. This can be justified, of course, by the fact that of the two characteristic families in the hyperbolic region one is outgoing and one incoming at the boundary. Thus, in this example, the intuition that one should pose a single boundary condition on each boundary is confirmed.

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