

## ON THE STANDARD CANONICAL FORM OF TIME-VARYING LINEAR DAEs

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**Abstract.** We introduce a solution theory for time-varying linear differential-algebraic equations (DAEs)  $E(t)\dot{x} = A(t)x$  which can be transformed into standard canonical form (SCF); i.e., the DAE is decoupled into an ODE  $\dot{z}_1 = J(t)z_1$  and a pure DAE  $N(t)\dot{z}_2 = z_2$ , where  $N$  is pointwise strictly lower triangular. This class is a time-varying generalization of time-invariant DAEs where the corresponding matrix pencil is regular. It will be shown in which sense the SCF is a *canonical* form, that it allows for a transition matrix similar to the one for ODEs, and how this can be exploited to derive a variation of constants formula. Furthermore, we show in which sense the class of systems transferable into SCF is equivalent to DAEs which are analytically solvable, and relate SCF to the derivative array approach, differentiation index and strangeness index. Finally, an algorithm is presented which determines the transformation matrices which put a DAE into SCF.

**1. Introduction.** We study time-varying linear differential-algebraic equations (DAEs) of the form

$$E(t)\dot{x} = A(t)x, \tag{1.1}$$

where  $(E, A) \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n \times n})^2$  for  $n \in \mathbb{N}$  and, throughout the paper,  $\mathcal{I} \subseteq \mathbb{R}$  denotes an open interval. For brevity, the tuple  $(E, A)$  is identified with the DAE (1.1). A function  $x : \mathcal{J} \rightarrow \mathbb{R}^n$  is called a *solution* of  $(E, A)$  if, and only if,  $x$  is a continuously differentiable function on the open interval  $\mathcal{J} \subseteq \mathcal{I}$  and solves (1.1) for all  $t \in \mathcal{J}$ ; it is called a *global*

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Received March 2, 2011 and, in revised form, May 26, 2011.

2010 *Mathematics Subject Classification.* Primary 34A09, 65L80.

*Key words and phrases.* Time-varying linear differential algebraic equations, standard canonical form, analytically solvable, generalized transition matrix.

Supported by DFG grant IL 25/9.

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solution if, and only if,  $\mathcal{J} = \mathcal{I}$ .

If  $(S, T) \in \mathcal{C}(\mathcal{I}; \mathbf{GL}_n(\mathbb{R})) \times \mathcal{C}^1(\mathcal{I}; \mathbf{GL}_n(\mathbb{R}))$ , then it is well known that  $x : \mathcal{J} \rightarrow \mathbb{R}^n$  solves (1.1) if, and only if,  $z(\cdot) := T(\cdot)^{-1}x(\cdot)$  solves

$$S(t)E(t)T(t)\dot{z} = \left[ S(t)A(t)T(t) - S(t)E(t)\dot{T}(t) \right] z.$$

Therefore, we recall the following equivalence relation.

DEFINITION 1.1 (Equivalence of DAEs [KM06, Def. 3.3]). The DAEs  $(E_1, A_1), (E_2, A_2) \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n \times n})^2$  are called *equivalent* if, and only if, there exists  $(S, T) \in \mathcal{C}(\mathcal{I}; \mathbf{GL}_n(\mathbb{R})) \times \mathcal{C}^1(\mathcal{I}; \mathbf{GL}_n(\mathbb{R}))$  such that

$$E_2 = SE_1T, \quad A_2 = SA_1T - SE_1\dot{T}; \quad \text{we write } (E_1, A_1) \stackrel{S,T}{\sim} (E_2, A_2). \quad (1.2)$$

That equivalence of DAEs is in fact an equivalence relation (see e.g. [KM06, Lem. 3.4]) follows easily by exploiting  $\frac{d}{dt}(T^{-1}) = -T^{-1}\dot{T}T^{-1}$ ; the latter follows from differentiation of the identity  $I = T^{-1}T$ .

We now make precise the system class studied in the present paper: Time-varying DAEs transferable into standard canonical form.

DEFINITION 1.2 (Standard canonical form (SCF) [Cam83, CP83]). The DAE  $(E, A) \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n \times n})^2$  is called *transferable into standard canonical form (SCF)* if, and only if, there exist  $(S, T) \in \mathcal{C}(\mathcal{I}; \mathbf{GL}_n(\mathbb{R})) \times \mathcal{C}^1(\mathcal{I}; \mathbf{GL}_n(\mathbb{R}))$  and  $n_1, n_2 \in \mathbb{N}_0$  such that

$$(E, A) \stackrel{S,T}{\sim} \left( \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} \right), \quad (1.3)$$

where  $N : \mathcal{I} \rightarrow \mathbb{R}^{n_2 \times n_2}$  is pointwise strictly lower triangular and  $J : \mathcal{I} \rightarrow \mathbb{R}^{n_1 \times n_1}$ ; a matrix  $N$  is called *pointwise strictly lower triangular* if, and only if, all entries of  $N(t)$  on the diagonal and above are zero for all  $t \in \mathcal{I}$ .  $\diamond$

Systems transferable into SCF have been introduced by Campbell [Cam83] almost 30 years ago. In the meantime, many other approaches to time-varying linear DAEs have been suggested: Campbell and Petzold [CP83] consider *analytically solvable* systems (see Definition 4.1) and prove (see Theorem 4.4) that any analytically solvable DAE with analytic coefficients is transferable into SCF; we show in Theorem 4.4 the converse under weaker conditions; i.e., every system transferable into SCF with  $\mathcal{C}^n$ -coefficients is analytically solvable. Later, Campbell [Cam87] shows equivalence of analytically solvable systems to a form [Cam87, (2.6)] and derives calculable necessary and sufficient criteria for analytic solvability. However, the form [Cam87, (2.6)] has not been proved to be a normal form; it is in general complex-valued even if the given DAE  $(E, A)$  is real-valued, and the two equations in [Cam87, (2.6)] are not completely decoupled.

Other approaches are the *differentiation index*, which is based on deriving an “underlying” ODE of the DAE such that any solution of the DAE also solves this ODE, and the *derivative array* approach by Kunkel and Mehrmann [KM06, KM07], which is also based on deriving a reduced system such that the solutions are in one-to-one correspondence and the differential and algebraic parts contained in the given DAE are separated. Both

approaches are explained in Section 4 in more detail. However, the respective reduced systems are not equivalent to the original DAE.

Rabier and Rheinboldt [RR96] consider DAEs with analytic coefficients and introduce a reduction procedure for a subclass called *regular* [RR96, Def. 3.1]. This reduction procedure is, as differentiation index and derivative array, based on deriving an ODE such that the solutions of the DAE can be obtained from the solutions of this ODE. They call  $(E, A)$  *completely regular* if the reduction procedure can be performed and show that any analytically solvable  $(E, A)$  is completely regular. However, there are completely regular systems which are not analytically solvable; systems with isolated singular points can also be treated by this approach. Similar results, except for the existence and uniqueness properties of solutions, are derived for sufficiently smooth coefficients provided a constant rank condition [RR96, (7.1)] holds. However, the reduced system is also not equivalent to the original DAE.

März considers DAEs with *properly stated leading term* and defines certain sequences of matrix functions; see for example [Mär02, (2.2)]. If these sequences terminate after finitely many steps and satisfy certain properties (see [Mär02, Def. 2.4]), then the DAE is called *regular* and the number of the step is called the *tractability index*. The approach derives a so-called *inherent regular ODE* [Mär02, (3.4)]; however, the derived system is not equivalent to the original DAE. Furthermore, there are systems with properly stated leading term which are not transferable into SCF and vice versa.

Recently, the approaches of a derivative array, differentiation index and analytic solvability have been proved to be equivalent in some sense; see Theorem 4.8. Since the class of systems transferable into SCF is, in general, a subclass of all analytically solvable systems, this class has, to the authors' best knowledge, not been investigated over many years. However, this subclass may be very interesting since it allows for:

- The SCF of the DAE  $(E, A)$  is (in some sense) a normal form within the equivalence class of systems transferable into SCF with respect to the relation  $\overset{S,T}{\sim}$ .
- All system entries within an equivalence class  $\overset{S,T}{\sim}$  remain real-valued.
- The required smoothness conditions are fairly weak: continuous or  $\mathcal{C}^n$ , and the latter may be replaced by  $\mathcal{C}^\nu$  if the (differentiation) index  $\nu$  is known.
- Several new results are stated as generalizations of known ODE results; they are proved directly without any derivative array.
- The transformation matrices leading to the SCF can be determined by an algorithm if the DAE has analytic coefficients.
- The results are the basis for the subsequent results [BI10] on stability theory: explicit solutions to Lyapunov equations and existence and uniqueness results. This is possible since the SCF approach investigates an equivalence class as opposed to the approaches of derivative array, differentiation index, tractability index and Rabier and Rheinboldt, where the derived system is not equivalent to the original system but only the solutions are in one-to-one correspondence.

The paper is organized as follows: In Section 2 we show in which sense the SCF is a canonical form, and that transferability into SCF is, for time-invariant DAEs, equivalent to regularity of the corresponding matrix pencil. In Section 3, the concept of SCF is used

to define a unique generalized transition matrix which has similar semi-group properties as the transition matrix for ODEs. Moreover, the generalized transition matrix is exploited to derive a variation of constants formula for inhomogeneous DAEs. In Section 4, transferability into SCF is shown to be “almost” equivalent to other concepts such as analytic solvability, the derivative array approach and the differentiation index. Finally, in Section 5 we present an algorithm to determine the transformation matrices for the SCF.

### Nomenclature

$\mathbb{N}, \mathbb{N}_0$	the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\ker A, \operatorname{im} A$	the kernel, image, of the matrix $A \in \mathbb{R}^{m \times n}$ , resp.
$M^* := \overline{M}^\top$	the Hermitian conjugate of $M \in \mathbb{C}^{m \times n}$
$\mathbf{GL}_n(\mathbb{R})$	general linear group of degree $n$ , i.e. set of all invertible $n \times n$ matrices over $\mathbb{R}$
$\mathcal{C}^k(\mathcal{I}; \mathcal{S})$	the set of $k$ -times continuously differentiable functions $f : \mathcal{I} \rightarrow \mathcal{S}$ from an open set $\mathcal{I} \subseteq \mathbb{R}$ to a vector space $\mathcal{S}$
$\operatorname{dom} f$	the domain of the function $f$
$f _{\mathcal{M}}$	the restriction of the function $f$ on a set $\mathcal{M} \subseteq \operatorname{dom} f$

**2. Standard canonical form (SCF).** We show that the SCF in (1.3) is unique in the sense that the dimensions of the ODE and the pure DAE are unique, and that the ODE and the pure DAE are unique up to some equivalence as in (1.2).

**THEOREM 2.1 (Uniqueness of SCF).** Let  $n_1, n_2, \tilde{n}_1, \tilde{n}_2 \in \mathbb{N}_0$ ,  $J_1 \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n_1 \times n_1})$ ,  $J_2 \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{\tilde{n}_1 \times \tilde{n}_1})$  and pointwise strictly lower triangular  $N_1 \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n_2 \times n_2})$ ,  $N_2 \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{\tilde{n}_2 \times \tilde{n}_2})$ . If, for some  $S \in \mathcal{C}(\mathcal{I}; \mathbf{GL}_n(\mathbb{R}))$ ,  $T \in \mathcal{C}^1(\mathcal{I}; \mathbf{GL}_n(\mathbb{R}))$ ,

$$\left( \begin{bmatrix} I_{n_1} & 0 \\ 0 & N_1 \end{bmatrix}, \begin{bmatrix} J_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \right) \stackrel{S, T}{\sim} \left( \begin{bmatrix} I_{\tilde{n}_1} & 0 \\ 0 & N_2 \end{bmatrix}, \begin{bmatrix} J_2 & 0 \\ 0 & I_{\tilde{n}_2} \end{bmatrix} \right),$$

then

- (i)  $n_1 = \tilde{n}_1, n_2 = \tilde{n}_2$ ,
- (ii)  $S = \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix}$ ,  $T = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix}$ ,  $T_{11} = S_{11}^{-1}$ ,
- (iii)  $(I_{n_1}, J_1) \stackrel{T_{11}^{-1}, T_{11}}{\sim} (I_{n_1}, J_2)$ ,  $(N_1, I_{n_2}) \stackrel{S_{22}, T_{22}}{\sim} (N_2, I_{n_2})$ . ◇

The proof of Theorem 2.1 requires a lemma on the solution of a *pure* DAE, i.e.  $n_1 = 0$  in (1.3).

**LEMMA 2.2 (Solutions of pure DAEs).** Let  $N \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n \times n})$  be pointwise strictly lower triangular. Then  $x(\cdot) = 0$  is the unique global solution of the pure DAE  $N(t)\dot{x} = x$ , and every (local) solution  $z : \mathcal{J} \rightarrow \mathbb{R}^n$  of the pure DAE satisfies  $z(t) = 0$  for all  $t \in \mathcal{J}$ .

*Proof.* Considering  $N(t)\dot{x} = x$  row-wise and invoking that  $N$  is pointwise strictly lower triangular immediately yields the assertion. Note that higher smoothness of  $N$  is not required.  $\square$

*Proof of Theorem 2.1.*

STEP 1. Assume, without loss of generality, that  $n_1 \geq \tilde{n}_1$ . In view of  $\frac{d}{dt}(T^{-1}) = -T^{-1}\dot{T}T^{-1}$  (which follows from differentiation of  $I = T^{-1}T$ ), we have  $T^{-1} \in \mathcal{C}^1(\mathcal{I}; \mathbf{Gl}_n(\mathbb{R}))$  and therefore we may write

$$T^{-1} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \quad T_{11} \in \mathcal{C}^1(\mathcal{I}; \mathbb{R}^{n_1 \times n_1}), \quad T_{22} \in \mathcal{C}^1(\mathcal{I}; \mathbb{R}^{n_2 \times n_2}), \quad T_{12}, T_{21} \text{ appropriate.}$$

We show that

$$\forall t \in \mathcal{I} : T_{21}(t) = 0 \quad \wedge \quad \det T_{11}(t) \neq 0 \quad \wedge \quad \det T_{22}(t) \neq 0.$$

Let  $(t^0, x^1) \in \mathcal{I} \times \mathbb{R}^{n_1}$ . Then  $x : \mathcal{I} \rightarrow \mathbb{R}^n$ ,  $t \mapsto \begin{bmatrix} \Phi_{J_1}(t, t^0)x^1 \\ 0 \end{bmatrix}$ , where  $\Phi_{J_1}(\cdot, \cdot)$  denotes the transition matrix of  $\dot{z} = J_1(t)z$ , solves  $\begin{bmatrix} I_{n_1} & 0 \\ 0 & N_1(t) \end{bmatrix} \dot{x} = \begin{bmatrix} J_1(t) & 0 \\ 0 & I_{n_2} \end{bmatrix} x$ . Then  $y(\cdot) := T(\cdot)^{-1}x(\cdot)$  solves  $\begin{bmatrix} I_{\tilde{n}_1} & 0 \\ 0 & N_2(t) \end{bmatrix} \dot{y} = \begin{bmatrix} J_2(t) & 0 \\ 0 & I_{\tilde{n}_2} \end{bmatrix} y$ , and it follows from Lemma 2.2 that  $y(\cdot) = [y_1(\cdot)^\top, 0]^\top$  for some  $y_1 \in \mathcal{C}^1(\mathcal{I}; \mathbb{R}^{\tilde{n}_1})$ . Hence

$$\begin{bmatrix} T_{11}(t^0)x^1 \\ T_{21}(t^0)x^1 \end{bmatrix} = T(t^0)^{-1}x(t^0) = y(t^0) = \begin{bmatrix} y_1(t^0) \\ 0 \end{bmatrix}. \quad (2.1)$$

Since  $n_2 \leq \tilde{n}_2$  it follows that  $T_{21}(t^0)x^1 = 0$  and, since  $x^1 \in \mathbb{R}^{n_1}$  is arbitrary, we conclude that  $T_{21}(t^0) = 0$ . Thus  $\det T_{11}(t^0) \cdot \det T_{22}(t^0) = \det T(t^0)^{-1}$ , and invertibility of  $T(t^0)$  yields invertibility of  $T_{11}(t^0)$  and  $T_{22}(t^0)$ .

STEP 2. We prove (i). Assume that  $n_1 > \tilde{n}_1$ . Let  $\alpha$  be the last row of  $T_{11}(t^0)$ ,  $\alpha^\top \in \mathbb{R}^{n_1}$ . Then (2.1) and  $n_1 > \tilde{n}_1$  yield  $\alpha x^1 = 0$ , and, since  $x^1$  is arbitrary, it follows that  $\alpha = 0$ , which contradicts  $\det T_{11}(t^0) \neq 0$ .

STEP 3. We prove (ii) and (iii). Write

$$S^{-1} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad S_{11} \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n_1 \times n_1}), \quad S_{22} \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n_2 \times n_2}), \quad S_{12}, S_{21} \text{ appropriate.}$$

Then

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & N_1 \end{bmatrix} = S^{-1} \begin{bmatrix} I_{n_1} & 0 \\ 0 & N_2 \end{bmatrix} T^{-1} = \begin{bmatrix} S_{11}T_{11} + S_{12}N_2T_{21}, & S_{11}T_{12} + S_{12}N_2T_{22} \\ S_{21}T_{11} + S_{22}N_2T_{21}, & S_{21}T_{12} + S_{22}N_2T_{22} \end{bmatrix}, \quad (2.2)$$

and

$$\begin{aligned} & \begin{bmatrix} J_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \\ & = \begin{bmatrix} S_{11}J_2T_{11} + S_{12}T_{21} - S_{11}\dot{T}_{11} - S_{12}N_2\dot{T}_{21}, & S_{11}J_2T_{12} + S_{12}T_{22} - S_{11}\dot{T}_{12} - S_{12}N_2\dot{T}_{22} \\ S_{21}J_2T_{11} + S_{22}T_{21} - S_{21}\dot{T}_{11} - S_{22}N_2\dot{T}_{21}, & S_{21}J_2T_{12} + S_{22}T_{22} - S_{21}\dot{T}_{12} - S_{22}N_2\dot{T}_{22} \end{bmatrix}. \end{aligned} \quad (2.3)$$

Step 1 and the equations in the first  $n_1$  columns in (2.2) yield

$$\forall t \in \mathcal{I} : S_{11}(t)^{-1} = T_{11}(t) \quad \wedge \quad S_{21}(t) = 0 \quad \wedge \quad \det S_{22}(t) \neq 0,$$

and therefore, by (2.2),

$$N_1 = S_{22}N_2T_{22} \quad (2.4)$$

and, by the lower right block in (2.3),

$$I_{n_2} = S_{22}T_{22} - S_{22}N_2\dot{T}_{22}. \quad (2.5)$$

Now suppose we have shown that  $T_{12} = S_{12} = 0$ . Then (ii) holds true and (2.4) together with (2.5) shows the second claim in (iii). The upper left block in (2.3) yields  $J_1 = S_{11}J_2T_{11} - S_{11}\dot{T}_{11}$ , and invoking  $S_{11} = T_{11}^{-1}$ , we find  $J_1 = T_{11}^{-1}J_2T_{11} - T_{11}^{-1}\dot{T}_{11}$ , which shows the first claim in (iii).

STEP 4. It remains to prove  $T_{12} = S_{12} = 0$ . It follows from (2.5) that  $S_{22}^{-1} = T_{22} - N_2\dot{T}_{22}$ . Observe that the upper right block in (2.3) yields  $0 = S_{11}(J_2T_{12} - \dot{T}_{12}) + S_{12}(T_{22} - N_2\dot{T}_{22})$  and thus

$$S_{12} = -S_{11}(J_2T_{12} - \dot{T}_{12})S_{22}. \quad (2.6)$$

Next, the upper right block in (2.2) gives

$$T_{12} = -S_{11}^{-1}S_{12}N_2T_{22} \stackrel{(2.6)}{=} (J_2T_{12} - \dot{T}_{12})S_{22}N_2T_{22} \stackrel{(2.4)}{=} (J_2T_{12} - \dot{T}_{12})N_1. \quad (2.7)$$

Therefore

$$T_{12}e_{n_2} \stackrel{(2.7)}{=} (J_2T_{12} - \dot{T}_{12})N_1e_{n_2} = (J_2T_{12} - \dot{T}_{12}) \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = 0 \quad (2.8)$$

and so

$$T_{12}e_{n_2-1} \stackrel{(2.7)}{=} (J_2T_{12} - \dot{T}_{12})N_1e_{n_2-1} = (J_2T_{12} - \dot{T}_{12})[0, \dots, 0, *]^T \stackrel{(2.8)}{=} 0.$$

Proceeding in this way gives  $T_{12} = 0$  and, invoking (2.6), we find  $S_{12} = 0$ . This completes the proof of the theorem.  $\square$

In the following proposition we show that transferability into SCF is, in fact, for *time-invariant* DAEs  $(E, A) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$ , a generalization of *regularity* of the matrix pencil  $sE - A \in \mathbb{R}^{n \times n}[s]$ , i.e.  $0 \neq \det(sE - A) \in \mathbb{R}[s]$ .

PROPOSITION 2.3 (Time-invariant DAEs: SCF  $\triangleq$  regularity). For  $(E, A) \in (\mathbb{R}^{n \times n})^2$  we have

$$(E, A) \text{ is transferable into SCF} \iff (E, A) \text{ is regular.}$$

*Proof.* “ $\Leftarrow$ ”: This follows from the Weierstraß canonical form (see, e.g., [Gan59, Thm. XII.3]).

“ $\Rightarrow$ ”: If  $(E, A)$  is transferable into SCF by time-varying  $(S, T) \in \mathcal{C}(\mathcal{I}; \mathbf{GL}_n(\mathbb{R})) \times \mathcal{C}^1(\mathcal{I}; \mathbf{GL}_n(\mathbb{R}))$  as in (1.3) and  $sE - A$  is not regular, then the latter implies (see e.g. [KM06, Thm. 2.14]) that there exists a nontrivial solution  $x(\cdot)$  to the initial value problem (1.1),  $x(0) = 0$ . Now it follows from the theory of ordinary differential equations and Lemma 2.2 that  $T(\cdot)^{-1}x(\cdot) = 0$ ; this contradicts the fact that  $x(\cdot)$  is nontrivial.  $\square$

**3. Transition matrix and variation of constants.** In this section we exploit uniqueness of the SCF to introduce a generalized transition matrix for  $(E, A)$  as a generalization of time-varying ordinary differential equations. This is used to characterize the set of consistent initial values of  $(E, A)$  and to derive a variation of constants formula for inhomogeneous DAEs.

DEFINITION 3.1 (Consistent initial values [KM06, Def. 1.1]). The set of all pairs of consistent initial values of  $(E, A) \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n \times n})^2$  is denoted by

$$\mathcal{V}_{E,A} := \{(t^0, x^0) \in \mathbb{R} \times \mathbb{R}^n \mid \exists (\text{local}) \text{ sln. } x(\cdot) \text{ of (1.1)} : t^0 \in \text{dom } x(\cdot), x(t^0) = x^0\}$$

and the linear subspace of *initial values which are consistent at time*  $t^0 \in \mathcal{I}$  is denoted by

$$\mathcal{V}_{E,A}(t^0) := \{x^0 \in \mathbb{R}^n \mid (t^0, x^0) \in \mathcal{V}_{E,A}\}. \quad \diamond$$

Note that if  $x : \mathcal{J} \rightarrow \mathbb{R}^n$  is a solution of (1.1), then  $x(t) \in \mathcal{V}_{E,A}(t)$  for all  $t \in \mathcal{J}$ .

Now we are in a position to characterize, for DAEs transferable into SCF, the set of consistent initial values and to derive a formula for the solution.

PROPOSITION 3.2 (Solutions of homogeneous DAEs). Suppose that the DAE  $(E, A) \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n \times n})^2$  is transferable into SCF as in (1.3). Then

(i)

$$(t^0, x^0) \in \mathcal{V}_{E,A} \iff x^0 \in \text{im } T(t^0) \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}. \quad (3.1)$$

(ii) Any solution of the initial value problem (1.1),  $x(t^0) = x^0$ , where  $(t^0, x^0) \in \mathcal{V}_{E,A}$ , extends uniquely to a global solution  $x(\cdot)$ , and this solution satisfies

$$x(t) = U(t, t^0)x^0, \quad U(t, t^0) := T(t) \begin{bmatrix} \Phi_J(t, t^0) & 0 \\ 0 & 0 \end{bmatrix} T(t^0)^{-1}, \quad t \in \mathcal{I}, \quad (3.2)$$

where  $\Phi_J(\cdot, \cdot)$  denotes the transition matrix of  $\dot{z} = J(t)z$ .

*Proof.* Let throughout  $x(\cdot)$  be given as in (3.2).

STEP 1. Simple calculations using (1.3) show that  $x(\cdot)$  solves (1.1) for all  $t \in \mathcal{I}$ .

STEP 2. We show that  $x(t^0) = x^0$  if, and only if,  $x^0 \in \text{im } T(t^0)[I_{n_1}, 0]^\top$ . For  $[\alpha^\top, \beta^\top]^\top := T(t^0)^{-1}x^0$ , where  $\alpha \in \mathbb{R}^{n_1}$ ,  $\beta \in \mathbb{R}^{n_2}$ , we have

$$x(t^0) = T(t^0) \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} T(t^0)^{-1}x^0 = T(t^0) \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = x^0 - T(t^0) \begin{bmatrix} 0 \\ \beta \end{bmatrix},$$

and hence  $x(t^0) = x^0$  if, and only if,  $\beta = 0$  or, equivalently,  $x^0 \in \text{im } T(t^0)[I_{n_1}, 0]^\top$ .

STEP 3. We show that every solution  $z : \mathcal{J} \rightarrow \mathbb{R}^n$  of (1.1) such that  $z(t^0) = x^0$ ,  $(t^0, x^0) \in \mathcal{V}$ , fulfills  $z = x|_{\mathcal{J}}$ . Clearly,  $(z - x) : \mathcal{J} \rightarrow \mathbb{R}^n$  solves  $E(t)\frac{d}{dt}(z - x)(t) = A(t)(z - x)(t)$  for all  $t \in \mathcal{J}$ . Then  $[y_1^\top, y_2^\top]^\top = y := T^{-1}(z - x)$  solves  $\dot{y}_1 = J(t)y_1$ ,  $N(t)y_2 = y_2$ , and by Lemma 2.2 it follows that  $y_2(t) = 0$  for all  $t \in \mathcal{J}$ . An application of  $y(t^0) = T(t^0)^{-1}(x^0 - x(t^0))$  gives

$$\begin{aligned} 0 &= \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} y(t^0) = \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} T(t^0)^{-1} \left( x^0 - T(t^0) \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} T(t^0)^{-1}x^0 \right) \\ &= \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} T(t^0)^{-1}x^0 = T(t^0)^{-1}(x^0 - x(t^0)) = y(t^0). \end{aligned}$$

Hence  $y_1(t) = 0$  for all  $t \in \mathcal{J}$  and therefore  $z = x|_{\mathcal{J}}$ . This completes the proof.  $\square$

Next it is shown that the operator  $U(\cdot, \cdot)$  defined in (3.2) is unique.

PROPOSITION 3.3 (Uniqueness of  $U$ ). Suppose  $(E, A) \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n \times n})^2$  is transferable into SCF. Then  $U(\cdot, \cdot)$  defined in (3.2) is independent of the choice of  $(S, T)$  in (1.3).

*Proof.* The assertion follows directly from Theorem 2.1 and [HP05, (3.3.26)].  $\square$

Now Proposition 3.3 ensures that the following is well defined.

DEFINITION 3.4 (Generalized transition matrix). Suppose  $(E, A) \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n \times n})^2$  is transferable into SCF as in (1.3) for some  $(S, T) \in \mathcal{C}(\mathcal{I}; \mathbf{GL}_n(\mathbb{R})) \times \mathcal{C}^1(\mathcal{I}; \mathbf{GL}_n(\mathbb{R}))$ . Then the *generalized transition matrix*  $U(\cdot, \cdot)$  of system (1.1) is defined by

$$U(t, s) := T(t) \begin{bmatrix} \Phi_J(t, s) & 0 \\ 0 & 0 \end{bmatrix} T(s)^{-1}, \quad t, s \in \mathcal{I}. \quad \diamond$$

Semi-group properties of the generalized transition matrix hold similarly to those of the transition matrix for ODEs:

PROPOSITION 3.5 (Properties of  $U(\cdot, \cdot)$ ). Let  $(E, A) \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n \times n})^2$  be transferable into SCF with generalized transition matrix  $U(\cdot, \cdot)$ . Then we have, for all  $t, r, s \in \mathcal{I}$ ,

- (i)  $E(t) \frac{d}{dt} U(t, s) = A(t) U(t, s)$ ,
- (ii)  $\text{im } U(t, s) = \mathcal{V}_{E, A}(t)$ ,
- (iii)  $U(t, r) U(r, s) = U(t, s)$ ,
- (iv)  $U(t, t)^2 = U(t, t)$ ,
- (v)  $\forall x \in \mathcal{V}_{E, A}(t) : U(t, t)x = x$ .

*Proof.* Property (i) is proved similarly to Step 1 of the proof of Proposition 3.2. The proofs of Properties (ii) and (iii) follow easily from the definition of  $U(\cdot, \cdot)$ . Property (iv) follows from (iii) and to see (v), let  $x \in \mathcal{V}(t)$ . Then (ii) gives  $x \in \text{im } U(t, t)$ , and hence there exists  $y \in \mathbb{R}^n$  such that  $U(t, t)y = x$ . Therefore,  $U(t, t)x = U(t, t)^2 y \stackrel{\text{(iv)}}{=} U(t, t)y = x$ . This completes the proof of the proposition.  $\square$

The concept of generalized transition matrix sets us in a position to derive, similar to ODEs, a vector space isomorphism between  $\mathcal{V}_{E, A}(t^0)$  (this is  $\mathbb{R}^n$  for ODEs) and the set of all global solutions of (1.1).

THEOREM 3.6 (Vector space isomorphism). If  $(E, A) \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n \times n})^2$  is transferable into SCF and  $t^0 \in \mathcal{I}$ , then the linear map

$$\varphi : \mathcal{V}_{E, A}(t^0) \rightarrow \{x : \mathcal{I} \rightarrow \mathbb{R}^{n \times n} \mid x(\cdot) \text{ is a global solution of (1.1)}\}, \quad x^0 \mapsto U(\cdot, t^0)x^0$$

is a vector space isomorphism.

*Proof.* Set

$$\mathcal{B}_{E, A} := \{x : \mathcal{I} \rightarrow \mathbb{R}^{n \times n} \mid x(\cdot) \text{ is a global solution of (1.1)}\}.$$

Since  $U(\cdot, \cdot)$  is well defined and Proposition 3.2 gives  $\forall x^0 \in \mathcal{V}_{E, A}(t^0) : (\mathcal{I} \ni t \mapsto U(t, t^0)x^0) \in \mathcal{B}_{E, A}$ ,  $\varphi(\cdot)$  is well defined.

We show that  $\varphi(\cdot)$  is surjective: Let  $x(\cdot) \in \mathcal{B}_{E, A}$ . Then  $x(t^0) \in \mathcal{V}_{E, A}(t^0)$  and from Proposition 3.2 (ii) it follows that  $\forall t \in \mathcal{I} : x(t) = U(t, t^0)x(t^0)$ , and therefore  $\varphi(x(t^0))(\cdot) = x(\cdot)$ .

We show that  $\varphi(\cdot)$  is injective: Let  $x^1, x^2 \in \mathcal{V}_{E, A}(t^0)$  such that  $\varphi(x^1)(\cdot) = \varphi(x^2)(\cdot)$ . Then

$$x^1 \stackrel{\text{Prop. 3.5 (v)}}{=} U(t^0, t^0)x^1 = \varphi(x^1)(t^0) = \varphi(x^2)(t^0) = U(t^0, t^0)x^2 \stackrel{\text{Prop. 3.5 (v)}}{=} x^2. \quad \square$$



As an immediate consequence of Theorem 3.6 we record:

**COROLLARY 3.7** (Constant dimension of  $\mathcal{V}_{E,A}(\cdot)$ ).

$\dim \mathcal{V}_{E,A}(\cdot)$  is constant if  $(E, A) \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n \times n})^2$  is transferable into SCF.  $\diamond$

Corollary 3.7 does, in general, not hold true for DAEs which are not transferable into SCF; this follows from the following example.

**EXAMPLE 3.8.** Consider the initial value problem

$$t\dot{x} = (1-t)x, \quad x(t^0) = x^0, \quad t \in \mathbb{R}, \quad (3.3)$$

for  $(t^0, x^0) \in \mathbb{R}^2$ . In passing, note that  $t \mapsto (E(t), A(t)) = (t, t-1)$  is real analytic. For  $t^0 \neq 0$ ,  $x^0 \in \mathbb{R}$ , the unique global solution  $x(\cdot)$  of (3.3) is

$$x : \mathbb{R} \rightarrow \mathbb{R}, \quad t \mapsto e^{-(t-t^0)} (t/t^0) x^0.$$

For  $t^0 = x^0 = 0$  the problem (3.3) has infinitely many global solutions and every (local) solution  $x : \mathcal{J} \rightarrow \mathbb{R}$  can be uniquely extended to a global solution

$$x_c : \mathbb{R} \rightarrow \mathbb{R}, \quad t \mapsto cte^{-t}, \quad \text{where } c = (e^\tau/\tau) x(\tau) \text{ for some } \tau \in \mathcal{J} \setminus \{0\}.$$

The solutions  $x_c(\cdot)$  are the only global solutions of the initial value problem (3.3),  $t^0 = x^0 = 0$ . Furthermore, any initial value problem (3.3),  $t^0 = 0$ ,  $x^0 \neq 0$  does not have a solution. Therefore, we have

$$\mathcal{V}_{E,A}(t) = \begin{cases} \mathbb{R}, & t \neq 0 \\ \{0\}, & t = 0. \end{cases} \quad \diamond$$

We conclude this section with a variation of constants formula for inhomogeneous time-varying linear differential-algebraic initial value problems

$$E(t)\dot{x} = A(t)x + f(t), \quad x(t^0) = x^0, \quad (3.4)$$

where  $(t^0, x^0) \in \mathbb{R} \times \mathbb{R}^n$  and  $f \in \mathcal{C}(\mathcal{I}; \mathbb{R}^n)$ . In general, it cannot be expected that a variation of constants formula exists if a (generalized) transition matrix for the homogeneous system is present; we come back to this point in Example 4.3.

**THEOREM 3.9** (Solutions of inhomogeneous DAEs). Suppose that the DAE  $(E, A) \in \mathcal{C}^n(\mathcal{I}; \mathbb{R}^{n \times n})^2$  is transferable into SCF by some  $(S, T) \in \mathcal{C}^n(\mathcal{I}; \mathbf{G}\mathbf{1}_n(\mathbb{R}))^2$ . Then the following statements hold for  $f \in \mathcal{C}^{n_2}(\mathcal{I}; \mathbb{R}^n)$ :

(i) The initial value problem (3.4) has a solution if, and only if,

$$x^0 + T(t^0) \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \left( \sum_{k=0}^{n_2-1} (N(\cdot) \frac{d}{dt})^k [0, I_{n_2}] S(\cdot) f(\cdot) \right) \Big|_{t=t^0} \in \text{im } T(t^0) \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}. \quad (3.5)$$

(ii) Any solution of (3.4) such that (3.5) holds can be uniquely extended to a global solution  $x(\cdot)$ , and this solution satisfies, for the generalized transition matrix  $U(\cdot, \cdot)$  of  $(E, A)$  and all  $t \in \mathcal{I}$ ,

$$x(t) = U(t, t^0) x^0 + \int_{t^0}^t U(t, s) T(s) S(s) f(s) ds - T(t) \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \sum_{k=0}^{n_2-1} (N(t) \frac{d}{dt})^k [0, I_{n_2}] S(t) f(t) \quad (3.6)$$

*Proof.* First note that since  $E, A, S, T$  are  $n$ -times continuously differentiable, we have  $N \in \mathcal{C}^n(\mathcal{I}; \mathbb{R}^{n_2 \times n_2})$ .

STEP 1. We have that, for any  $g \in \mathcal{C}^{n_2}(\mathcal{J}; \mathbb{R}^{n_2})$ ,  $\mathcal{J} \subseteq \mathcal{I}$ ,

$$\forall t \in \mathcal{J} : \left(N(t) \frac{d}{dt}\right)^{n_2} g(t) = 0. \quad (3.7)$$

This is due to  $\left(N(\cdot) \frac{d}{dt}\right)^{n_2} = 0$ , which holds since  $N$  is pointwise strictly lower triangular.

STEP 2.  $x(\cdot)$  as in (3.6) solves (3.4) for all  $t \in \mathcal{I}$ . The appendant calculations are involved, but elementary. We leave the details to the reader and just note that (1.3), (3.6), (3.7) and Proposition 3.5 (i) must be used.

STEP 3. We show that  $x(t^0) = x^0$  for  $x(\cdot)$  as in (3.6) if, and only if, (3.5) holds. Set

$$\eta := T(t^0) \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \left( \sum_{k=0}^{n_2-1} \left(N(\cdot) \frac{d}{dt}\right)^k [0, I_{n_2}] S(\cdot) f(\cdot) \right) \Big|_{t=t^0}, \quad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} := T(t^0)^{-1} (x^0 + \eta) \quad (3.8)$$

for  $\alpha \in \mathbb{R}^{n_1}$ ,  $\beta \in \mathbb{R}^{n_2}$ . Then

$$x(t^0) = T(t^0) \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} T(t^0)^{-1} x^0 - \eta = T(t^0) \begin{bmatrix} \alpha \\ 0 \end{bmatrix} - \eta = x^0 - T(t^0) \begin{bmatrix} 0 \\ \beta \end{bmatrix},$$

and hence  $x(t^0) = x^0$  if, and only if,  $\beta = 0$  or, equivalently, (3.5) holds.

STEP 4. Let  $(t^0, x^0) \in \mathcal{I} \times \mathbb{R}^n$  such that (3.4) has a solution. We show that every solution  $z : \mathcal{J} \rightarrow \mathbb{R}^n$  of (3.4),  $z(t^0) = x^0$  fulfills  $z = x|_{\mathcal{J}}$  for  $x(\cdot)$  as in (3.6). Clearly,  $(z - x) : \mathcal{J} \rightarrow \mathbb{R}^n$  solves  $E(t) \frac{d}{dt} (z - x)(t) = A(t) (z - x)(t)$  for all  $t \in \mathcal{J}$ . Then Proposition 3.2 gives  $(z - x)(t^0) \in \text{im } T(t^0) [I_{n_1}, 0]^\top$ , and since, by Step 2,  $x^0 - x(t^0) = T(t^0) [0, \beta^\top]^\top \in \text{im } T(t^0) [0, I_{n_2}]^\top$ , we conclude that

$$z(t^0) - x(t^0) \in \text{im } T(t^0) \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} \cap \text{im } T(t^0) \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} = \{0\}.$$

Therefore, a repeated application of Proposition 3.2 yields  $z = x|_{\mathcal{J}}$ . This concludes the proof.  $\square$

A consequence of Theorem 3.9 is the following corollary which treats a characterization of consistent initial values and a variation of constants analogue for pure DAEs.

**COROLLARY 3.10** (Solutions of inhomogeneous pure DAEs). Let  $N \in \mathcal{C}^n(\mathcal{I}; \mathbb{R}^{n \times n})$  be pointwise strictly lower triangular,  $f \in \mathcal{C}^n(\mathcal{I}; \mathbb{R}^n)$  and  $(t^0, x^0) \in \mathcal{I} \times \mathbb{R}^n$ . Then the initial value problem

$$N(t) \dot{x} = x + f(t), \quad x(t^0) = x^0, \quad (3.9)$$

has a solution if, and only if,

$$- \sum_{k=0}^{n-1} \left(N(\cdot) \frac{d}{dt}\right)^k f(\cdot) \Big|_{t=t^0} = x^0. \quad (3.10)$$

Any solution of (3.9) can be uniquely extended to a global solution  $x(\cdot)$ , and this solution satisfies

$$x(t) = - \sum_{k=0}^{n-1} \left(N(t) \frac{d}{dt}\right)^k f(t), \quad t \in \mathcal{I}. \quad (3.11)$$

*Proof.* Put  $n_1 = 0$ ,  $n_2 = n$ , and  $T = S = I$  in (3.5) and (3.6).  $\square$

REMARK 3.11 (Consistent initial values for inhomogeneous DAEs).

- (i) Note that a consequence of Corollary 3.10 is that the only initial value consistent at time  $t^0 \in \mathcal{I}$  of a pure homogeneous initial value problem (3.9), i.e.  $f = 0$ , is  $x^0 = 0$ .
- (ii) For  $\eta$  as in (3.8), condition (3.5) reads  $x^0 + \eta \in \mathcal{V}_{E,A}(t^0)$ . Hence the *set of initial values which are consistent at time  $t^0$*  of (3.4) is the affine subspace

$$-\eta + \mathcal{V}_{E,A}(t^0) = -T(t^0) \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \left( \sum_{k=0}^{n_2-1} \left( N(\cdot) \frac{d}{dt} \right)^k [0, I_{n_2}] S(\cdot) f(\cdot) \right) \Big|_{t=t^0} + \mathcal{V}_{E,A}(t^0). \quad \diamond$$

We conclude this section with a remark on the index of the system.

REMARK 3.12 (Index). If  $(E, A) \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n \times n})^2$  is transferable into SCF and has well-defined differentiation index  $\nu \in \mathbb{N}_0$  (see Definition 4.6 and [KM06, Def. 3.37]), then  $(N(\cdot) \frac{d}{dt})^\nu = 0$  and hence the smoothness  $n$  in Theorem 3.9 can be weakened to  $\nu$ ; to be precise:  $(E, A) \in \mathcal{C}^{\nu-1}(\mathcal{I}; \mathbb{R}^{n \times n})^2$ ,  $(S, T) \in \mathcal{C}^\nu(\mathcal{I}; \mathbf{GL}_n(\mathbb{R}))^2$  and  $f \in \mathcal{C}^{\nu-1}(\mathcal{I}; \mathbb{R}^n)$ . Likewise, in Corollary 3.10 we may assume  $N \in \mathcal{C}^{\nu-1}(\mathcal{I}; \mathbb{R}^{n \times n})$  and  $f \in \mathcal{C}^{\nu-1}(\mathcal{I}; \mathbb{R}^n)$ .  $\diamond$

#### 4. Analytic solvability, derivative array approach and differentiation index.

In this section we study the relationship of DAEs transferable into SCF to that of other subclasses of time-varying DAEs. Such concepts as analytic solvability, the derivative array approach, differentiation index and strangeness index will be investigated.

DEFINITION 4.1 (Analytic solvability [CP83]). Let  $(E, A) \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n \times n})^2$ . Then the DAE (3.4) is called *analytically solvable* if, and only if, we have, for all  $f \in \mathcal{C}^n(\mathcal{I}; \mathbb{R}^n)$ :

- (i)  $\exists$  a solution to (3.4),
- (ii)  $\forall$  solutions  $y : \mathcal{J} \rightarrow \mathbb{R}^n$  of (3.4) :  $\exists$  a global solution  $x(\cdot)$  of (3.4) with  $x|_{\mathcal{J}} = y$ ,
- (iii)  $\forall$  global solutions  $x_1(\cdot), x_2(\cdot)$  of (3.4) :  $[\exists t^0 \in \mathcal{I} : x_1(t^0) \neq x_2(t^0)] \Rightarrow [\forall t \in \mathcal{I} : x_1(t) \neq x_2(t)]$ .

REMARK 4.2.

- (a) Roughly speaking, system (3.4) is analytically solvable if, and only if, for any inhomogeneity  $f \in \mathcal{C}^n(\mathcal{I}; \mathbb{R}^n)$  there exist solutions to (3.4), and solutions, if they exist, can be extended to all of  $\mathcal{I}$  and are uniquely determined by their value at any  $t^0 \in \mathcal{I}$ .
- (b) Conditions (i) and (ii) in Definition 4.1 do not imply (iii). This follows from Example 3.8, which shows that an initial value problem (3.4) may have infinitely many global solutions and every local solution can be uniquely extended to one of the global solutions (i.e. there do not exist further solutions with finite escape time or other singular behavior).  $\diamond$

EXAMPLE 4.3 (Analytic solvability  $\stackrel{i.g.}{\not\Rightarrow}$  transferable into SCF). We work out the example

$$E(t)\dot{x} = -x + f(t), \quad t \in \mathcal{I} = (-\infty, 1), \quad (4.1)$$

where

$$E(t) := t^3 \begin{bmatrix} \sin(t^{-1}) \\ \cos(t^{-1}) \end{bmatrix} [\cos(t^{-1}), -\sin(t^{-1})], \quad E(0) = 0,$$

provided by [CP83, Ex. 2] to show the following: (i) system  $(E, A)$  has  $\mathcal{C}^1$ -coefficients on  $\mathcal{I}$ , (ii) it is analytically solvable, (iii) it is not transferable into SCF, and (iv) a variation of constant formula does not exist.

(i): This follows since  $E \in \mathcal{C}^1(\mathcal{I}; \mathbb{R}^{2 \times 2})$ .

(ii): It is easily verified that

$$E^2 \equiv 0, \quad \dot{E}(0) = 0 \quad \text{and} \quad E(t)\dot{E}(t) = -tE(t) \quad \text{for all } t \in \mathcal{I} \setminus \{0\}.$$

This yields that, for any  $f \in \mathcal{C}^2(\mathcal{I}; \mathbb{R}^2)$ ,

$$x : \mathcal{I} \rightarrow \mathbb{R}^2, \quad t \mapsto f(t) + (t-1)^{-1}E(t)\dot{f}(t)$$

is continuously differentiable and the unique global solution of (4.1). Furthermore, any local solution of (4.1) can be uniquely extended to  $x(\cdot)$ , and therefore the system (4.1) is analytically solvable.

(iii): Assume that (4.1) is transferable into SCF by  $(S, T) \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{2 \times 2}) \times \mathcal{C}^1(\mathcal{I}; \mathbb{R}^{2 \times 2})$ . Since  $E(0) = 0$ , equation (1.3) together with Theorem 2.1 yields that  $n_1 = 0$  and therefore, for all  $t \in \mathcal{I}$ ,  $S(t)E(t)T(t) = N(t) = \begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix}$  and

$$\forall t \in \mathcal{I} \setminus \{0\} : t^3 \sin(t^{-1})[\cos(t^{-1}), -\sin(t^{-1})] \begin{pmatrix} T_{12}(t) \\ T_{22}(t) \end{pmatrix} = (1, 0)S(t)^{-1} \begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0. \quad (4.2)$$

This gives  $t_k^3 (-1) T_{22}(t_k) \stackrel{(4.2)}{=} 0$  for  $t_k := (2k\pi + \pi/2)^{-1}$ ,  $k \in \mathbb{N}$ , and so, by continuity of  $T(\cdot)$ , we may conclude that  $T_{22}(0) = \lim_{k \rightarrow \infty} T_{22}(t_k) = 0$ . Applying (4.2) again, we have  $s_k^3 \sin(\pi/4) \cos(\pi/4) T_{12}(s_k) \stackrel{(4.2)}{=} 0$ , where  $s_k := (2k\pi + \pi/4)^{-1}$ ,  $k \in \mathbb{N}$ , and again by continuity of  $T(\cdot)$ , we may conclude that  $T_{12}(0) = \lim_{k \rightarrow \infty} T_{12}(s_k) = 0$ ; this contradicts invertibility of  $T(0)$ .

(iv): Let  $\mathcal{I} = \mathbb{R}$  in (4.1). Then the homogeneous system ( $f = 0$ ) has the (generalized) transition matrix  $U(t, s) = 0$ . However, a variation of constants formula does not exist, since e.g. for  $f(t) = t$  the inhomogeneous equation has no global solution (there is a pole at  $t = 1$ ).  $\diamond$

We now show that transferability into SCF and analytic solvability are equivalent for real analytic  $(E, A)$ .

**THEOREM 4.4** ( $(E, A)$  real analytic: SCF  $\triangleq$  analytic solvability). Suppose  $E, A : \mathcal{I} \rightarrow \mathbb{R}^{n \times n}$  are real analytic. Then

$$(3.4) \text{ is analytically solvable} \iff (1.1) \text{ is transferable into SCF.}$$

For “ $\Leftarrow$ ”, it suffices to assume  $E, A \in \mathcal{C}^n(\mathcal{I}; \mathbb{R}^n)$  and  $S, T \in \mathcal{C}^n(\mathcal{I}; \mathbb{R}^n)$  so that (1.3) holds. For “ $\Rightarrow$ ”, real analyticity of  $E$  and  $A$  can, in general, not be dispensed.

*Proof.* “ $\Leftarrow$ ” follows immediately from Theorem 3.9. Note that it is sufficient that  $E, A, S, T$  are  $n$ -times continuously differentiable.

“ $\Rightarrow$ ”: In [CP83, Thm. 2] it is shown that

$$(E, A) \stackrel{S, T}{\sim} \left( \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} \right) \quad \text{for some real analytic } S, T : \mathcal{I} \rightarrow \mathbf{GL}_n(\mathbb{R}),$$

where  $N : \mathcal{I} \rightarrow \mathbb{R}^{n_2 \times n_2}$  is real analytic and pointwise strictly *upper* triangular and  $J : \mathcal{I} \rightarrow \mathbb{R}^{n_1 \times n_1}$  is real analytic. Therefore,

$$(E, A) \stackrel{\tilde{S}^{-1} S, T \tilde{T}^{-1}}{\sim} \left( \begin{bmatrix} I_{n_1} & 0 \\ 0 & \tilde{N} \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} \right) \quad \text{for} \quad \tilde{S} = \tilde{T} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \begin{bmatrix} 1 & \\ & \ddots & \\ & & 1 \end{bmatrix} \end{bmatrix},$$

where  $\tilde{N} : \mathcal{I} \rightarrow \mathbb{R}^{n_2 \times n_2}$  is real analytic and pointwise strictly lower triangular, and the claim is proved.  $\square$

Example 4.3 shows that “ $\Rightarrow$ ” does not hold in general if  $(E, A)$  are not real analytic.

In the remainder of this section we compare the concept of SCF with that of the *differentiation index* and the *derivative array*. We now allow for complex-valued  $E, A \in \mathcal{C}^\infty(\mathcal{I}; \mathbb{C}^{n \times n})$  since this is treated in the literature. To avoid technicalities, we assume that the functions involved are infinitely many times differentiable.

We first state a technical definition on matrices.

DEFINITION 4.5 (1-fullness [KM06, Def. 3.35]). Let  $k, \ell, n \in \mathbb{N}$  and  $M \in \mathcal{C}(\mathcal{I}; \mathbb{C}^{kn \times \ell n})$ . Then  $M$  is called *smoothly 1-full w.r.t.  $n$*  if, and only if,

$$\exists R \in \mathcal{C}(\mathcal{I}; \mathbf{G}\mathbf{l}_{kn}(\mathbb{C})) : RM = \begin{bmatrix} I_n & 0 \\ 0 & * \end{bmatrix}. \quad \diamond$$

DEFINITION 4.6 (Derivative array [KM06, (3.28)-(3.30)] and differentiation index [KM06, Def. 3.37]). For  $E, A \in \mathcal{C}^\infty(\mathcal{I}; \mathbb{C}^{n \times n})$ , the *derivative array* is defined as the sequence of matrix functions  $M_\ell \in \mathcal{C}^\infty(\mathcal{I}; \mathbb{C}^{(\ell+1)n \times (\ell+1)n})$ ,  $N_\ell \in \mathcal{C}^\infty(\mathcal{I}; \mathbb{C}^{(\ell+1)n \times (\ell+1)n})$  given by

$$\begin{aligned} (M_\ell)_{i,j} &= \binom{i}{j} E^{(i-j)} - \binom{i}{j+1} A^{(i-j-1)}, \quad i, j = 0, \dots, \ell, \\ (N_\ell)_{i,j} &= \begin{cases} A^{(i)} & \text{for } i = 0, \dots, \ell, j = 0, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \quad (4.3)$$

and the *differentiation index* of  $(E, A)$  is the smallest number  $\nu \in \mathbb{N}_0$  (if it exists) for which  $M_\nu$  is smoothly 1-full w.r.t.  $n$  and has constant rank.  $\diamond$

The notions of 1-fullness and derivative array go back to [Cam85] and [Cam87], resp.

If the differentiation index  $\nu$  is well-defined for  $(E, A)$ , then one may construct an underlying ODE of the given DAE (3.4) as follows; cf. [KM06, p. 97-98]: By 1-fullness of  $M_\nu$  there exists  $R \in \mathcal{C}(\mathcal{I}; \mathbf{G}\mathbf{l}_{(\nu+1)n}(\mathbb{C}))$  such that

$$RM_\nu = \begin{bmatrix} I_n & 0 \\ 0 & * \end{bmatrix}.$$

Define  $z_j := x^{(j)}$ ,  $g_j := f^{(j)}$  for  $j = 0, \dots, \nu$ . Then

$$M_\nu(t) \dot{z} = N_\nu(t) z + g(t), \quad t \in \mathcal{I},$$

and we obtain the ODE

$$\dot{x} = [I_n, 0] R(t) M_\nu(t) \dot{z} = [I_n, 0] R(t) N_\nu(t) [I_n, 0]^\top x + [I_n, 0] R(t) g(t),$$

which is the so-called *underlying ordinary differential equation*. Here  $x$  is the same variable as in (3.4) and hence any solution of (3.4) is also a solution of this ODE. Therefore,

solving the DAE can be reduced to solving an ODE.

Next we introduce a hypothesis of a certain finite reduction procedure. This hypothesis guarantees that the reduction procedure of the *derivative array approach* presented in [KM06, Sec. 3.2] can be carried out and no consistency conditions for the inhomogeneity or free solution components are present.

**HYPOTHESIS 4.7** ([KM06, Hypothesis 3.48]). There exist  $\mu, a, d \in \mathbb{N}_0$  such that  $(M_\mu, N_\mu)$  defined in Definition 4.6 has the following properties:

- (i)  $\forall t \in \mathcal{I} : \text{rk } M_\mu(t) = (\mu + 1)n - a$ ; choose  $Z_2 \in \mathcal{C}^\infty(\mathcal{I}; \mathbb{C}^{(\mu+1)n \times a})$  with pointwise maximal rank and  $Z_2^* M_\mu = 0$ .
- (ii)  $\forall t \in \mathcal{I} : \text{rk } A_2(t) = a$ , where  $A_2 := Z_2^* N_\mu [I_n, 0, \dots, 0]^*$ ; choose  $T_2 \in \mathcal{C}^\infty(\mathcal{I}; \mathbb{C}^{n \times d})$ ,  $d = n - a$ , with pointwise maximal rank and  $A_2 T_2 = 0$ .
- (iii)  $\forall t \in \mathcal{I} : \text{rk } E(t) T_2(t) = d$ ; choose  $Z_1 \in \mathcal{C}^\infty(\mathcal{I}; \mathbb{C}^{n \times d})$  with pointwise maximal rank and  $\text{rk } E_1 T_2 = d$ ,  $E_1 = Z_1^* E$ .  $\diamond$

If Hypothesis 4.7 holds true, then [KM06, p. 109] have shown that a solution  $x$  of the DAE (3.4) is also a solution of

$$\begin{bmatrix} E_1 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} x + \hat{f}(t), \quad \text{where } E_1 = Z_1^* E, \quad A_1 = Z_1^* A, \quad A_2 = Z_2^* N_\mu [I_n, 0, \dots, 0]^* \quad (4.4)$$

and  $\hat{f}$  is determined by  $f$  and its derivatives. The derived system (4.4) is a so-called *strangeness free* DAE (cf. [KM06, Def. 3.15 and p. 93]).

The following theorem shows in particular that if  $(E, A)$  is real analytic, then transferability into SCF is equivalent to Hypothesis 4.7 and to a well-defined differentiation index.

**THEOREM 4.8.** Let  $E, A \in \mathcal{C}^\infty(\mathcal{I}; \mathbb{C}^{n \times n})$  and consider system (3.4). Then the following conditions are equivalent:

- (i) (3.4) is analytically solvable.
- (ii) The differentiation index  $\nu$  is well-defined for  $(E, A)$ .
- (iii)  $(E, A)$  satisfies Hypothesis 4.7.

*Proof.* The assumptions in [KM06, Thm. 3.39] are equivalent to analytic solvability of (3.4). (Note that [KM06, Thm. 3.39] requires in addition that the solutions depend smoothly on the inhomogeneities and the initial conditions, but this is not needed in the proof; see [Cam87, Thm. 2.1].) Then it follows from [KM06, Thm. 3.45] that (i) $\Rightarrow$ (ii) holds true. The conclusion (ii) $\Rightarrow$ (iii) is identical to [KM06, Thm. 3.50] and finally (iii) $\Rightarrow$ (i) follows from [KM06, pp. 111-112]. This completes the proof.  $\square$

We finalize this section with a remark on the strangeness index as developed in [KM06, Sec. 3.1].

**REMARK 4.9.** The existence of a well-defined strangeness index (see [KM06, Def. 3.15]) guarantees the equivalence of  $(E, A)$  to a DAE in a certain canonical form presented in [KM06, Thm. 3.21]. It turns out that there exist systems with a well-defined strangeness index which are not transferable into SCF (see [KM06, Ex. 3.23]); there also exist

systems which are transferable into SCF and have no well-defined strangeness index (see [KM06, Ex. 3.54]).  $\diamond$

**5. Computing SCF.** In this section we present an algorithm in “quasi-MATLAB code” for computing the transformation matrices as well as the SCF for real analytic DAEs  $(E, A)$ ; the algorithm also determines whether  $(E, A)$  is transferable in SCF or not. This algorithm is indicated by some comments in [CP83]; here we make it precise.

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**Algorithm 5.1** Function transfSCF

---

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1: function  $[S, T, N, J] = \mathbf{transfSCF}(E, A)$ 
2: reachedSCF := 0; % initial value for global variable
3:  $[S_1, T_1, N_1, J_1] := \mathbf{getSCF}(E, A)$ ;
4:  $r := \mathbf{size}(J)$ ;
5:  $S := \begin{bmatrix} I_r & 0 \\ 0 & \begin{bmatrix} 1 & \\ & \ddots \\ & & 1 \end{bmatrix} \end{bmatrix} S_1$ ;  $T := T_1 \begin{bmatrix} I_r & 0 \\ 0 & \begin{bmatrix} 1 & \\ & \ddots \\ & & 1 \end{bmatrix} \end{bmatrix}$ ;
6:  $N := \begin{bmatrix} & & \\ & & 1 \\ & & & \ddots \\ & & & & 1 \end{bmatrix} N_1 \begin{bmatrix} & & \\ & & 1 \\ & & & \ddots \\ & & & & 1 \end{bmatrix}$ ;  $J := J_1$ ;

```

---

PROPOSITION 5.4. Suppose  $E, A : \mathcal{I} \rightarrow \mathbb{R}^{n \times n}$  are real analytic. Then Algorithm 5.1 either terminates after finitely many steps with “not transferable into SCF!” or returns real analytic transformation matrices  $S, T : \mathcal{I} \rightarrow \mathbf{GL}_n(\mathbb{R})$ ,  $J : \mathcal{I} \rightarrow \mathbb{R}^{n_1 \times n_1}$  and  $N : \mathcal{I} \rightarrow \mathbb{R}^{n_2 \times n_2}$  such that  $N$  is pointwise strictly lower triangular and

$$(E, A) \stackrel{S, T}{\sim} \left( \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} \right). \quad (5.1)$$

*Proof.* We consider two cases.

CASE 1. Suppose (1.1) is transferable into SCF. Then, in view of Theorem 4.4, the DAE (3.4) is analytically solvable. Therefore, the tests in lines 5 and 10 of Algorithm 5.3 will always fail for  $(E, A)$  and every reduced pair  $(E_1, A_1)$  (cf. lines 2 and 4 of Algorithm 5.2); see the proof of [CP83, Thm. 2]. Note also that  $(E_1, A_1)$  is again analytically solvable. Hence the algorithm does not stop in line 6 or 11 of Algorithm 5.3 with “not transferable into SCF!”, and therefore the reduction procedure continues until the test in line 2 of Algorithm 5.3 succeeds at some point. Since the reduction procedure reduces the dimension of  $(E, A)$  by at least 1 in each step, we must arrive at this point after at most  $n$  reduction steps. Then the SCF for the pair at lowest level (absolutely reduced) is calculated in lines 6 and 8 of Algorithm 5.2; and a simple calculation shows that the SCF of a DAE at a given level is calculated in lines 10–15 of Algorithm 5.2 provided that the SCF for the reduced pair is given; see also the proof of [CP83, Thm. 2]. The feasibility of lines 8 and 13 of Algorithm 5.3 is due to [SB70, Thm. 1] and also shown in the proof of [CP83, Thm. 2]. The invertibility of  $G$  in line 13 of Algorithm 5.3 follows from

$$n - r = \mathbf{rk}[\hat{A}_{21}(t), \hat{A}_{22}(t)] = \mathbf{rk}[\hat{A}_{21}(t), \hat{A}_{22}(t)]Q = \mathbf{rk} G.$$

**Algorithm 5.2** Function getSCF

---

```

1: function  $[S, T, N, J] = \mathbf{getSCF}(E, A)$ 
2:  $[E_1, E_2, A_1, A_2, G, P, Q] := \mathbf{reduce}(E, A)$ ;
3: if reachedSCF = 0 then
4:    $[S_1, T_1, N_1, J_1] := \mathbf{getSCF}(E_1, A_1)$ ;
5: else if  $E \equiv 0$  then
6:    $N_1 := 0$ ;  $J_1 := \emptyset$ ;  $S_1 := T_1 := I$ ;           % set  $M := \emptyset$  if the matrix  $M$  is absent
7: else
8:    $N_1 := \emptyset$ ;  $J_1 := E_1^{-1}A_1$ ;  $S_1 := E_1^{-1}$ ;  $T_1 := I$ ;
9: end if
10:  $r_1 := \mathbf{size}(J_1)$ ;  $r_2 := \mathbf{size}(N_1)$ ;           % the size of an empty matrix is 0
11:  $\begin{bmatrix} \tilde{E}_1 \\ \tilde{E}_2 \end{bmatrix} := S_1 E_2$  s.t.  $\tilde{E}_i$  has  $r_i$  rows,  $i = 1, 2$ ;
12:  $\begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix} := S_1 A_2$  s.t.  $\tilde{A}_i$  has  $r_i$  rows,  $i = 1, 2$ ;
13:  $S := \begin{bmatrix} I_{r_1} & 0 & \frac{d}{dt}\tilde{E}_1 + J_1\tilde{E}_1 - \tilde{A}_1 \\ 0 & I_{r_2} & 0 \\ 0 & 0 & I_{\mathbf{size}(E)-r_1-r_2} \end{bmatrix} \begin{bmatrix} I_{r_1} & 0 & 0 \\ 0 & I_{r_2} & -\tilde{A}_2 G^{-1} \\ 0 & 0 & G^{-1} \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & I_{\mathbf{size}(E)-r_1-r_2} \end{bmatrix} P$ ;
14:  $T := Q \begin{bmatrix} T_1 & 0 \\ 0 & I_{\mathbf{size}(E)-r_1-r_2} \end{bmatrix} \begin{bmatrix} I_{r_1} & 0 & -\tilde{E}_1 \\ 0 & I_{r_2} & 0 \\ 0 & 0 & I_{\mathbf{size}(E)-r_1-r_2} \end{bmatrix}$ ;
15:  $J := J_1$ ;  $N := \begin{bmatrix} N_1 & \tilde{E}_2 \\ 0 & 0 \end{bmatrix}$  s.t.  $\mathbf{size}(N) + \mathbf{size}(J) = \mathbf{size}(E)$ ;

```

---

So the algorithm stops and returns  $S, T$  and  $J, N$  of the SCF such that (5.1) holds. Since  $N$  constructed by Algorithms 5.2 and 5.3 is strictly upper triangular, the transformation in lines 4–6 of Algorithm 5.1 finally assures that  $N$  is strictly lower triangular.

CASE 2. Suppose (1.1) is not transferable into SCF. Assume that the tests in lines 5 and 10 of Algorithm 5.3 will always fail for  $(E, A)$  and every reduced pair  $(E_1, A_1)$ . Then, in view of Case 1, the algorithm stops and returns  $S, T$  and the matrices  $J, N$  of the SCF,  $N$  strictly lower triangular, such that (5.1) holds. Hence (1.1) would be transferable into SCF, a contradiction. Therefore, one of the tests must fail at some point and the algorithm stops with “not transferable into SCF!”.  $\square$

REMARK 5.5.

- (i) Algorithm 5.3 shows constructively how to transform a system into SCF; it relates the effort of algebraic transformations to that of a derivative array (cf. Section 4).
- (ii) In practice, it is not easy to implement Algorithm 5.1 for the whole class of real analytic functions. The main problem is to find  $P, Q$  such that the conditions in lines 8 and 13 of Algorithm 5.3 are fulfilled. However, if  $(E, A)$  has polynomial entries, then there are efficient (actually, polynomial time) algorithms which solve this problem; see [QV95, Sec. 5].



**Algorithm 5.3** Function reduce

---

```

1: function  $[E_1, E_2, A_1, A_2, G, P, Q] = \text{reduce}(E, A)$ 
2: if  $E \equiv 0$  or  $(\forall t \in \mathcal{I} : \det E(t) \neq 0)$  then
3:    $E_1 := E; A_1 := A; E_2 := A_2 := G := \emptyset; P := Q := I;$ 
4:    $\text{reachedSCF} := 1;$ 
5: else if  $\text{not}(\forall t \in \mathcal{I} : \det E(t) = 0)$  then
6:   print “not transferable into SCF!” STOP
7: else
8:   determine (minimal)  $r < n := \text{size}(E)$  s.t.  $\text{rk } E(t) \leq r < n$  for all  $t \in \mathcal{I}$  and
      $P : \mathcal{I} \rightarrow \mathbb{R}^{n \times n}$  real analytic and pointwise singular s.t.  $PE = \begin{bmatrix} \hat{E}_1 & \hat{E}_2 \\ 0 & 0 \end{bmatrix}$ , where
      $\hat{E}_1(t) \in \mathbb{R}^{r \times r};$ 
9:    $\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} := PA$ , where  $\hat{A}_{11}(t) \in \mathbb{R}^{r \times r};$ 
10:  if  $\text{not}(\forall t \in \mathcal{I} : \text{rk}[\hat{A}_{21}(t), \hat{A}_{22}(t)] = n - r = \max)$  then
11:    print “not transferable into SCF!” STOP
12:  else
13:    choose real analytic, pointwise nonsingular  $Q : \mathcal{I} \rightarrow \mathbb{R}^{n \times n}$  s.t.  $[\hat{A}_{21}, \hat{A}_{22}]Q =$ 
      $[0_{(n-r) \times r}, G], \det G(t) \neq 0 \forall t \in \mathcal{I};$ 
14:     $[E_1, E_2] := [\hat{E}_1, \hat{E}_2]Q;$ 
15:     $[A_1, A_2] := [\hat{A}_{11}, \hat{A}_{12}]Q - [\hat{E}_1, \hat{E}_2]\dot{Q};$ 
16:  end if
17: end if

```

---

(iii) A numerically verifiable algorithm for testing analytic solvability is given in [Cam87]. Due to Theorem 4.4, this algorithm also tests transferability into SCF for real analytic  $(E, A)$ . However, this algorithm does not compute the transformation matrices.  $\diamond$

EXAMPLE 5.6. We illustrate Algorithm 5.1 by

$$E(t) = \begin{bmatrix} \sin t & \cos t & 0 \\ 0 & 0 & 0 \\ -\cos t \sin t & \sin^2 t & 0 \end{bmatrix}, \quad A(t) = \begin{bmatrix} \sin t - \cos t & \cos t + \sin t & 0 \\ -\cos t & \sin t & 0 \\ -\sin^2 t & -\sin t \cos t & t^2 + 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Note that  $E$  does not have constant rank. We show that  $(E, A)$  is transferable into SCF by applying Algorithm 5.1:

$\rightarrow \text{transfSCF}(E, A):$

$\text{reachedSCF} := 0, [S_1^1, T_1^1, N_1^1, J_1^1] = \text{getSCF}(E, A)$

$\rightarrow \text{getSCF}(E, A)$  (first instance):

$[E_1^1, E_2^1, A_1^1, A_2^1, G^1, P^1, Q^1] = \text{reduce}(E, A)$

$\rightarrow \text{reduce}(E, A):$

conditions in lines 2 and 5 of Alg. 5.3 not fulfilled, go to lines 8 to 16

$$r := 2, P^1(t) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \hat{E}_1(t) = \begin{bmatrix} \sin t & \cos t \\ -\cos t \sin t & \sin^2 t \end{bmatrix}, \hat{E}_2(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \hat{A}_{11}(t) & \hat{A}_{12}(t) \\ \hat{A}_{21}(t) & \hat{A}_{22}(t) \end{bmatrix} = P^1(t)A(t) = \begin{bmatrix} \sin t - \cos t & \cos t + \sin t & 0 \\ -\sin^2 t & -\sin t \cos t & t^2 + 1 \\ -\cos t & \sin t & 0 \end{bmatrix}$$

condition in line 10 of Alg. 5.3 not fulfilled, go to lines 13 to 15

$$Q^1(t) := \begin{bmatrix} 0 & \sin t & -\cos t \\ 0 & \cos t & \sin t \\ 1 & 0 & 0 \end{bmatrix}, [\hat{A}_{21}(t), \hat{A}_{22}(t)]Q^1(t) = [0, 0, 1] =: [0, 0, G^1(t)]$$

$$[E_1^1(t), E_2^1(t)] = \left[ \begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & \sin t \end{array} \right]$$

$$[A_1^1(t), A_2^1(t)] = \left[ \begin{array}{cc|c} 0 & 1 & 0 \\ t^2 + 1 & 0 & 0 \end{array} \right]$$

condition in line 3 of Alg. 5.2 fulfilled, go to line 4

$$[S_1^2, T_1^2, N_1^2, J_1^2] = \text{getSCF}(E_1^1, A_1^1)$$

getSCF( $E_1^1, A_1^1$ ) (second instance):

$$[E_1^2, E_2^2, A_1^2, A_2^2, G^2, P^2, Q^2] = \text{reduce}(E_1^1, A_1^1)$$

$$\rightarrow \text{reduce}(E_1^1, A_1^1):$$

conditions in lines 2 and 5 of Alg. 5.3 not fulfilled, go to lines 8 to 16

$$r := 1, P^2(t) := I_2, [\hat{E}_1(t), \hat{E}_2(t)] = [0, 1]$$

$$\begin{bmatrix} \hat{A}_{11}(t) & \hat{A}_{12}(t) \\ \hat{A}_{21}(t) & \hat{A}_{22}(t) \end{bmatrix} = P^2 A_1^1(t) = \begin{bmatrix} 0 & 1 \\ t^2 + 1 & 0 \end{bmatrix}$$

condition in line 10 of Alg. 5.3 not fulfilled, go to lines 13 to 15

$$Q^2(t) := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, [\hat{A}_{21}(t), \hat{A}_{22}(t)]Q^2(t) = [0, t^2 + 1] =: [0, G^2(t)]$$

$$[E_1^2(t), E_2^2(t)] = [1 \mid 0]$$

$$[A_1^1(t), A_2^1(t)] = [1 \mid 0]$$

condition in line 3 of Alg. 5.2 fulfilled, go to line 4

$$[S_1^3, T_1^3, N_1^3, J_1^3] = \text{getSCF}(E_1^2, A_1^2)$$

getSCF( $E_1^2, A_1^2$ ) (third instance):

$$[E_1^3, E_2^3, A_1^3, A_2^3, G^3, P^3, Q^3] = \text{reduce}(E_1^2, A_1^2)$$

$$\rightarrow \text{reduce}(E_1^2, A_1^2):$$

condition in line 2 of Alg. 5.3 fulfilled, go to lines 3 and 4

$$E_1^3 = E_1^2, A_1^3 = A_1^2, E_2^3 = A_2^3 = G^3 = \emptyset, P^3 = Q^3 = I, \text{reachedSCF} = 1$$

conditions in line 3 and 5 of Alg. 5.2 not fulfilled, go to line 8

$$N_1^4 = \emptyset, J_1^4 = 1, S_1^4 = 1, T_1^4 = 1$$

$$\text{(third instance:)} \quad r_1 = 1, r_2 = 0 \quad \tilde{E}_1 = \tilde{E}_2 = \tilde{A} - 1 = \tilde{A}_2 = \emptyset$$

$$S_1^3 = S_1^4 P^3 = 1, T_1^3 = Q^3 T_1^4 = 1, J_1^3 = 1, N_1^3 = \emptyset$$

(second instance:)  $r_1 = 1, r_2 = 0, \tilde{E}_1 = 0, \tilde{A}_1 = 0, \tilde{E}_2 = \tilde{A}_2 = \emptyset$

$$S_1^2(t) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{t^2+1} \end{bmatrix}, T_1^2(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, J_1^2 = 1, N_1^2 = 0$$

(first instance:)  $r_1 = 1, r_2 = 1, \tilde{E}_1(t) = 0, \tilde{E}_2(t) = \frac{\sin t}{t^2+1}, \tilde{A}_1(t) = \tilde{A}_2(t) = 0$

$$S_1^1(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{t^2+1} \\ 0 & 1 & 0 \end{bmatrix}, T_1^1(t) = \begin{bmatrix} \sin t & 0 & -\cos t \\ \cos t & 0 & \sin t \\ 0 & 1 & 0 \end{bmatrix}, J_1^1(t) = 1, N_1^1(t) = \begin{bmatrix} 0 & \frac{\sin t}{t^2+1} \\ 0 & 0 \end{bmatrix}$$

back in transfSCF:  $r = 1$

$$S(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{t^2+1} \end{bmatrix}, T(t) = \begin{bmatrix} \sin t & -\cos t & 0 \\ \cot t & \sin t & 0 \\ 0 & 0 & 1 \end{bmatrix}, N(t) = \begin{bmatrix} 0 & 0 \\ \frac{\sin t}{t^2+1} & 0 \end{bmatrix}, J(t) = 1$$

◇

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