QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXXI, NUMBER 2
JUNE 2013, PAGES 339–368
S 0033-569X(2012)01287-7
Article electronically published on October 22, 2012

# SIMULTANEOUS TEMPERATURE AND FLUX CONTROLLABILITY FOR HEAT EQUATIONS WITH MEMORY

By

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**Abstract.** It is known that, in the case of the heat equation with memory, temperature can be controlled to an arbitrary square integrable target provided that the system evolves for a sufficiently long time. The control is the temperature on the boundary. In this paper we consider heat equations with memory (one-dimensional space variable) and we first show that when the control is square integrable, then the flux is square integrable too. Then we prove that both temperature and flux can be simultaneously controlled to a pair of independent targets, both square integrable. This solves a problem first raised by Renardy.

The method of proof relies on moment theory, and one of the contributions of this paper is the identification of $\mathcal{L}$-bases and Riesz bases especially suited to heat equations with memory, which so appear to be endowed with a very rich bases structure.

**1. Introduction.** The derivation of heat equations depends on conservation of energy (and temperature being a measure of energy):

$$\frac{\partial}{\partial t}e(x,t) = -\nabla \cdot q(x,t)\,, \qquad \frac{\partial}{\partial t}e(x,t) = \gamma\frac{\partial}{\partial t}\theta(x,t)\,. \tag{1.1}$$

The quantity denoted $q(x,t)$ is the (density of) heat flux at time $t$ at the position $x$ while $e(x,t)$ and $\theta(x,t)$ denote energy and temperature. This is just a balance law. To

Received July 27, 2011.
2010 *Mathematics Subject Classification.* Primary 76A10, 93C05, 47N70.
Supported in part by the National Science Foundation, grant ARC 0724860. This paper was partly written while the first author visited the Dipartimento di Matematica, Politecnico di Torino, as a visiting professor supported by GNAMPA-INDAM.
Supported in part by Italian MURST and by the project "Groupement de Recherche en Contrôle des EDP entre la France et l'Italie (CONEDP)". This paper fits into the research programs of GNAMPA-INDAM.
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obtain the heat equation we have to combine (1.1) with a constitutive equation. The usual choice is *Fourier law*, which for homogeneous materials is

$$q(x,t) = -\nabla\theta(x,t)\,. \tag{1.2}$$

Combining the formulas (1.1) and (1.2) we get the usual form of the heat equation (we put $\gamma = 1$):

$$\theta_t = \Delta\theta\,. \tag{1.3}$$

Equation (1.3) holds on a suitable region $\Omega$ which in this paper is taken to be an interval $(0,l)$ and equipped with suitable boundary conditions described below.

Equation (1.2) shows that the flux is uniquely identified by the temperature $\theta$.

Note that we are considering (1.3) as the heat equation, but it describes equally well the diffusion of a solute in a solvent when the Fick law holds.

Both when interpreted as the heat or diffusion equations, (1.3) has been criticized on several grounds (the first move in this direction was made by J.C. Maxwell; see [21]). An alternative to (1.3) is

$$\frac{\partial}{\partial t}\theta(x,t) = 2\alpha\theta(x,t) + \int_0^t N(t-s)\Delta\theta(x,s)\,\mathrm{d}s\,. \tag{1.4}$$

The kernel $N(t)$ has suitable properties which depend on the material we are working with. *Standing assumptions in this paper: we assume* $N(t) \in W^{4,2}_{\mathrm{loc}}(0,\infty)$ *and* $N(0) = N_0 > 0$. *In order to simplify the formulas, we shall put* $N(0) = 1$ *and* $l = \pi$. *The results we prove hold in general, as discussed in Appendix* 5.5.

In the models encountered usually in the literature (see [11, 21]) the coefficient $\alpha$ is often put equal to 0. The origin of a coefficient $\alpha \neq 0$ in our problem is explained in Section 1.2.

The boundary conditions we associate to (1.4) are

$$\theta(0,t) = u(t)\,, \qquad \theta(\pi,t) = 0\,. \tag{1.5}$$

The function $u(t) \in L^2_{\mathrm{loc}}(0,\infty)$ is *the control.*

The heat flux is (see [21])

$$q(x,t) = -\frac{\partial}{\partial x}\int_0^t N(t-s)\theta(x,s)\,\mathrm{d}s\,. \tag{1.6}$$

So, now the relation of temperature and flux is not so direct, and we wonder to what extent their values are related. This problem was already raised in [9, p. 98]. We shall clarify in this paper that the relation is quite weak, since temperature and flux can be simultaneously controlled (from the boundary) to an arbitrary target in $L^2(0,\pi) \times L^2(0,\pi)$. This makes sense thanks to the following result:

THEOREM 1.1. Let $\theta(x,0) = 0$ and $u(\cdot) \in L^2_{\mathrm{loc}}(0,\infty)$. Then we have

$$\theta(x,t) \in C(0,\infty; L^2(0,\pi))\,, \qquad q(x,t) \in C(0,\infty; L^2(0,\pi))\,.$$

The result concerning $\theta(x,t)$ is already known (see [19] where also the dependence of $\theta(t,x)$ and of $\theta_x(t,x)$ on the initial conditions $\theta(x,0)$ is studied). So, we must prove only the property of the flux. This is done in Section 3 and Appendix 5.2.

The exact controllability theorem is:

THEOREM 1.2. Let $\theta(\cdot,0)=0$ and $T\geq 2\pi$. We have

$$\big\{(\theta(\cdot,T),q(\cdot,T)) \ : \quad u(\cdot)\in L^2(0,T)\big\}=L^2(0,\pi)\times L^2(0,\pi)\,.$$

Of course, the length $\pi$ of the space interval and the condition $N(0)=1$ have no special role. If $x\in(0,l)$ and $N(0)>0$, possibly not equal to 1, then our results still hold, with the minimal controllability time equal to $2l/\sqrt{N(0)}$; see Appendix 5.5.

REMARK 1.3. When $N(t)\equiv 1$, (1.4) is the integrated form of the wave equation

$$\theta_{tt}=\Delta\theta\,.$$

The analogue of the flux in this case is the stress, $\nabla\theta$, which is identified once $\theta$ is known. But, in the case under study here, the variable $q$ is the *integral* of the stress,

$$q(x,t)=-\int_0^t \nabla\theta(x,s)\,\mathrm{d}s\,.$$

Theorem 1.2 implies controllability of the pair $(\theta,q)$. This result seems new even in the simple case of the wave equation.

The organization of the paper is as follows:

In Section 1.2 we present a preliminary transformation used to simplify the computations. We shall see that we can assume $N'(0)=0$ without restriction.

In Section 2 we present preliminary information on bases in Hilbert spaces, and we introduce a special series representation for the solution of (1.4)-(1.5).

In Section 3 we prove Theorem 1.1. A special $\mathcal{L}$-basis encountered in this proof will be put in use in Section 4 where we prove Theorem 1.2.

Technical computations are relegated to appendices.

In this paper we need asymptotic estimates of sequences. We are interested only in estimates for *large values of the index* $n$. So, in order to simplify the exposition, we write $f_n\asymp g_n$ (for $n\to+\infty$) when the inequalities $M_1||g_n||\leq||f_n||\leq M_2||g_n||$ hold with $M_1>0$ *and* $n$ *large enough.*

The sequences $\{f_n\}$ and $\{g_n\}$ may be sequences of numbers or of functions in suitable normed spaces.

1.1. *Comments on the literature.* We refer to the review paper [21] for an overview of the origin and multiple applications of equations of the form (1.4). As we said it represents heat equations in the presence of memory effects, but also diffusion of solutes in solvents, when Fick law does not hold, i.e. in the presence of high concentration or complex molecular structures. Furthermore, taking the derivatives in time of both sides, we get a second-order equation with memory used in viscoelasticity.

Equations of this type have been studied from many different points of view (qualitative properties of the solutions, stability, etc.). Here we are interested in the study of controllability. G. Leugering seems to have been the first author to study controllability for viscoelastic systems (see [12, 13]). Further controllability results are in [8, 10, 16, 17].

The first papers to identify a special $\mathcal{L}$-basis, adapted to the structure of the heat equation with memory are [18], which relies on previous controllability properties, and [19]. These papers study controllability properties of the temperature $\theta(\cdot,t)$ while the paper [15] studies controllability of both $\theta(\cdot,t)$ and $\theta_t(\cdot,t)$ (for a special class of kernels).

We mention also that similar $\mathcal{L}$-bases of functions quadratically close to exponentials arise in the study of controllability of hyperbolic equations with time-dependent coefficients [2, 3, 4, 5].

The problem whether the pair of temperature and flux can be controlled was raised in [22], where a special case was studied. So, the present paper can be considered as a positive answer to the problem raised in [22], and also as a continuation of [19]. In fact, we shall use the special $\mathcal{L}$-basis identified in [19] and, using the ideas presented in this paper, we shall identify further Riesz bases especially suited to the study of heat equations with memory, which appear to be endowed by a very rich structure, from the point of view of the bases which can be associated to these equations.

In [7] the question of temperature and flux dependence/independence is considered from the viewpoint of their controllability.

Finally, we cite [20]. In this paper an application of the controllability results to an inverse problem has been presented.

The present paper is based on basis theory and moment problems. See [6, 14, 25] for background material.

1.2. *A preliminary transformation.* We perform a transformation which is used to change Equation (1.4) into a new but equivalent equation with $N'(0) = 0$. We define $\tilde{\theta}(x,t) = e^{-N'(0)t}\theta(x,t)$. It is easily computed that

$$\tilde{\theta}_t(x,t) = [2\alpha - N'(0)]\tilde{\theta}(x,t) + \int_0^t e^{-N'(0)(t-s)}N(t-s)\Delta\tilde{\theta}(x,s)\,\mathrm{d}s\,.$$

Let $\tilde{q}(x,t) = e^{-N'(0)t}q(x,t)$. We see that

$$\tilde{q}(x,t) = \frac{\partial}{\partial x}\int_0^t e^{-N'(0)(t-s)}N(t-s)\tilde{\theta}(x,s)\,\mathrm{d}s\,.$$

The new kernel

$$\tilde{N}(t) = e^{-N'(0)t}N(t)$$

satisfies the equality $\tilde{N}'(0) = 0$, and the properties of our interest in this paper are not changed. So, from now on, we disregard the $\tilde{}$ notation and assume that the transformation has been performed. The nonrestrictive property $N'(0) = 0$ will simplify the computations below.

Note that this transformation introduces a coefficient $\alpha \neq 0$ in the case that the original equation has $\alpha = 0$.

**2. Preliminary information.** In this section we present preliminaries on the abstract theory of moments in Hilbert spaces, bases and representation of the solutions of (1.4) and the flux. Then, we reduce controllability to a moment problem.

It is convenient to denote by $\mathcal{A}$ the following operator in $L^2(0,\pi)$:

$$\operatorname{dom}\mathcal{A} = H^2(0,\pi)\cap H_0^1(0,\pi)\,, \qquad (\mathcal{A}\phi)(x) = \Delta\phi(x) = \phi''(x)\,.$$

Its eigenvalues are $-n^2, n = 1,2,\ldots$. The normalized eigenvector corresponding to $-n^2$ is

$$\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin nx\,. \tag{2.1}$$

Note that $\phi_n'(0) = n\sqrt{2/\pi}$.

2.1. *Moment problems and bases.* The results in this paper will be derived as consequences of the theory of moments and bases in Hilbert spaces. We shortly recall the crucial information. We refer to [6, 25] for the general theory and to [18, 19] for a more detailed overview.

Here we consider sequences indexed by natural numbers, but the arguments are easily adapted to more general index sets, in particular to $\mathbb{Z}$ and $\mathbb{Z}' = \mathbb{Z} \setminus \{0\}$.

We are interested in $\mathcal{L}$-bases and Riesz bases: a sequence $\{z_n\}$ in a Hilbert space $H$ is a Riesz basis if it is the image of an orthonormal basis under a linear bounded and boundedly invertible transformation. A sequence $\{z_n\}$ is an $\mathcal{L}$-basis if it is a Riesz basis in its closed span.

We shall use the following two tests of basicity: the first one is a consequence of the Paley-Wiener theorem (see, e.g., [25, p. 32 and p. 35]):

THEOREM 2.1. Let $\{y_n\}$ be an $\mathcal{L}$-basis in a Hilbert space $H$ and let $\{z_n\}$ be a sequence such that

$$\sum_{n=1}^{+\infty} \|z_n - y_n\|^2 < +\infty\,. \tag{2.2}$$

Then, there exists $N$ such that the sequence $\{z_n\}_{n>N}$ is an $\mathcal{L}$-basis in $H$.

So, we also have:

THEOREM 2.2. Let the conditions of Theorem 2.1 hold and let the series

$$\sum_{n=1}^{+\infty} \alpha_n z_n$$

converge in $H$. Then $\{\alpha_n\} \in l^2$.

We need a condition which, when coupled with (2.2), implies that $\{z_n\}$ is itself an $\mathcal{L}$-basis. This is provided by the Bari theorem; see, e.g., [25, p. 38]:

THEOREM 2.3. Let the conditions of Theorem 2.1 hold and let the sequence $\{z_n\}$ be $\omega$-independent, which means

$$\sum_{n=1}^{+\infty} \alpha_n z_n = 0 \text{ in } H \Longrightarrow \{\alpha_n\} = 0\,.$$

Then $\{z_n\}$ is an $\mathcal{L}$-basis. If furthermore $\{y_n\}$ is a Riesz basis, then $\{z_n\}$ is a Riesz basis too.

Now we define a moment problem with respect to a sequence $\{z_n\}$ in a Hilbert space $H$. This is the problem of solving the sequence of equations

$$\langle z_n, u\rangle = c_n,\ n = 1, 2, \ldots$$

for the unknown $u$. The sequence $\{c_n\}$ is given. The result we are interested in is as follows (see, e.g., [6, Theorem I.2.1]):

THEOREM 2.4. The transformation $u \to \{\langle z_n, u\rangle\}$ is bounded with bounded inverse from $H$ to $l^2$ if and only if $\{z_n\}$ is a Riesz basis. If $\{z_n\}$ is an $\mathcal{L}$-basis, then the transformation $u \to \{\langle z_n, u\rangle\}$ is bounded with bounded inverse from $X$ to $l^2$, where $X$ is the closed subspace of $H$ generated by $\{z_n\}$.

The relation of the problem (1.4)-(1.5) to a moment problem is seen as in [18]. We consider the eigenvectors $\phi_n$ of the operator $\mathcal{A}$ and represent the solution of (1.4)-(1.5) as

$$\theta(x,t) = \sum \phi_n(x) a_n(t),$$

where

$$a_n(t) = \int_0^\pi \theta(x,t)\phi_n(x)\,\mathrm{d}x\,.$$

After suitable integration by parts, we see that

$$a_n'(t) = 2\alpha a_n(t) - n^2 \int_0^t N(t-s)a_n(s)\,\mathrm{d}s + \phi_n'(0)\int_0^t N(t-s)u(s)\,\mathrm{d}s\,,$$
$$a_n(0) = 0\,.$$

This formula suggests the introduction of the functions $z_n(t)$ which solve

$$z_n'(t) = 2\alpha z_n - n^2 \int_0^t N(t-s)z_n(s)\,\mathrm{d}s\,, \qquad z_n(0) = 1\,, \tag{2.3}$$

so also

$$z_n'' = 2\alpha z_n' - n^2 z_n - n^2 \int_0^t N'(t-s)z_n(s)\,\mathrm{d}s\,, \qquad z_n(0) = 1\,, \quad z_n'(0) = 2\alpha\,. \tag{2.4}$$

The following integral equation for $z_n(t)$ is valid (see [19, formula 19]). It holds when $\beta_n$ (defined below) is different from 0:

$$z_n(t) = g_n(t) + \frac{n^2}{\beta_n^2}\left\{-\int_0^t N'(t-r)z_n(r)\,\mathrm{d}r \right.$$
$$\left. + \int_0^t e^{\alpha s}[\cos\beta_n s]\left[\int_0^{t-s} N_0(t-s-r)z_n(r)\,\mathrm{d}r\right]\,\mathrm{d}s\right\}\,, \tag{2.5}$$

where

$$\begin{cases} N_0(t) &= N''(t) - \alpha N'(t), \\ g_n(t) &= e^{\alpha t}\left[\cos\beta_n t + \frac{\alpha}{\beta_n}\sin\beta_n t\right], \\ \beta_n &= \sqrt{n^2-\alpha^2}\,. \end{cases} \tag{2.6}$$

Note that $\beta_n$ is positive real for large $n$.

Equality (2.5) is easily found. In fact, let

$$z_n'' = 2\alpha z_n' - n^2 z_n + f(t)\,.$$

Then

$$z_n(t) = A\cos\beta_n t + B\sin\beta_n t + \frac{1}{\beta_n}\int_0^t e^{\alpha(t-s)}\sin\beta_n(t-s)f(s)\,\mathrm{d}s\,.$$

We take into account the initial conditions in (2.4) and we substitute

$$f(t) = -n^2\int_0^t N'(t-s)z_n(s)\,\mathrm{d}s\,.$$

We then use $\sin\beta_n(t-s) = \frac{1}{\beta_n}(\,\mathrm{d}/\,\mathrm{d}s)\cos\beta_n(t-s)$ in order to integrate by parts and we get (2.5).

It is possible to have $\beta_n = 0$ for at most one value of the index $n$. If $\beta_{n_0} = 0$, then the formula for $z_{n_0}(t)$ is

$$z_{n_0}(t) = (1+\alpha t)e^{\alpha t} - \alpha^2 \int_0^t K_{n_0}(t-s) z_{n_0}(s)\,\mathrm{d}s\,,$$
$$K_{n_0}(t) = \int_0^t (t-r)e^{\alpha(t-r)} N'(r)\,\mathrm{d}r\,.$$

For simplicity of presentation, in this paper we confine ourselves to the generic case $\beta_n \neq 0$ for every $n$. If instead there exists $n_0$ such that $\beta_{n_0} = 0$, then we have to include $z_{n_0}(t)$ in each of the series we encounter below, as done in [19]. This will not make any essential change in the proofs.

We recall from [19]:

THEOREM 2.5. Let $T > \pi$. Then the sequences

$$\{z_n\}\,, \qquad \left\{\phi_n'(0)\int_0^t N(t-s)z_n(s)\,\mathrm{d}s\right\}$$

are $\mathcal{L}$-bases in $L^2(0,T)$.

Appendix 5.1 shows that the result holds also with $T = \pi$ but the inequality suffices to prove convergence of the series used in the representation of $\theta(x,t)$.

With $\theta(x,0) = 0$ we have the representation

$$\theta(x,t) = \sum_{n=1}^{+\infty} \phi_n(x)\phi_n'(0)\int_0^t z_n(t-s)v(s)\,\mathrm{d}s\,, \qquad v(t) = \int_0^t N(t-s)u(s)\,\mathrm{d}s\,. \tag{2.7}$$

As proved in [19], the series in (2.7) converges in $C(0,T;L^2(0,\pi))$ thanks to the property of the second sequence in Theorem 2.5. So, $\theta_x(x,t)$ can be computed termwise, and the corresponding series converges in $C(0,T;H^{-1}(0,\pi))$. We can conclude from (1.6) that $q(\cdot,t) \in W^{1,2}\big(0,T;H^{-1}(0,\pi)\big)$. In fact, we shall see below that this property of the flux can be improved.

2.2. *Controllability and moment problem.* In this section we reduce our control problem to a suitable moment problem. We noted that the condition $q(\cdot,t) \in H^{-1}(0,\pi)$ can be improved. But, for the moment, we consider the flux as a distribution.

We insert (2.7) in the expression of the flux and we find

$$\begin{aligned} q(x,t) &= -\int_0^t N(t-s)\theta_x(x,s)\,\mathrm{d}s \\ &= -\sum_{n=1}^{+\infty} \phi_n'(0)\phi_n'(x)\int_0^t N(s)\int_0^{t-s} z_n(t-s-r)v(r)\,\mathrm{d}r \\ &= \sum_{n=1}^{+\infty} \phi_n'(x)\int_0^t v(t-r)\left\{\phi_n'(0)\int_0^r N(r-s)z_n(s)\,\mathrm{d}s\right\}\,\mathrm{d}r\,. \end{aligned} \tag{2.8}$$

We observe that $\{\phi_n(x)\}_{n>0}$ is an orthonormal basis in $L^2(0,\pi)$ while $\{\phi_n'(x)\}_{n>0}$ is a Riesz basis of $H^{-1}(0,\pi)$. So, every target $(\xi,\eta) \in L^2(0,\pi)\times H^{-1}(0,\pi)$ can be represented

as

$$\xi = \sum_{n=1}^{+\infty} \xi_n \phi_n(x)\,, \qquad \eta = \sum_{n=1}^{+\infty} \eta_n \phi_n'(x)\,. \tag{2.9}$$

Both the sequences $\{\xi_n\}$ and $\{\eta_n\}$ belong to $l^2$.

So, the target is reachable at a certain time $T$ if and only if we can find a control $u \in L^2(0,T)$ such that the following moment problem can be solved (the function $v(t)$ is given in (2.7)):

$$\begin{cases} \displaystyle\int_0^T z_n(r)v(T-r)\,\mathrm{d}r = \frac{\xi_n}{\phi_n'(0)}, \\ \displaystyle\int_0^T \left\{\phi_n'(0)\int_0^r N(r-s)z_n(s)\,\mathrm{d}s\right\} v(T-r)\,\mathrm{d}r = \eta_n\,. \end{cases} \tag{2.10}$$

Now we observe:

$$\begin{aligned} \phi_n'(0)\int_0^r N(r-s)z_n(s)\,\mathrm{d}s &= \frac{\phi_n'(0)}{n^2}\left[n^2\int_0^r N(r-s)z_n(s)\,\mathrm{d}s\right] \\ &= \frac{\phi_n'(0)}{n^2}\left[2\alpha z_n(r) - z_n'(r)\right]\,. \end{aligned}$$

So, the second line in (2.10) can be written as follows:

$$\begin{aligned} \eta_n &= \frac{\phi_n'(0)}{n^2}\int_0^T \left[2\alpha z_n(r) - z_n'(r)\right] v(T-r)\,\mathrm{d}r \\ &= 2\alpha\frac{\phi_n'(0)}{n^2}\int_0^T z_n(r)v(T-r)\,\mathrm{d}r - \frac{\phi_n'(0)}{n}\left[\frac{1}{n}\int_0^T z_n'(r)v(T-r)\,\mathrm{d}r\right]\,. \end{aligned}$$

Taking into account the first line in (2.10), equalities (2.10) can be written as

$$\begin{cases} \displaystyle\int_0^T z_n(r)v(T-r)\,\mathrm{d}r = \frac{\xi_n}{\phi_n'(0)}\,, \\ \displaystyle\int_0^T \frac{1}{n}z_n'(r)v(T-r)\,\mathrm{d}r = \tilde{\eta}_n\,, \qquad \tilde{\eta}_n = \frac{n}{\phi_n'(0)}\left\{2\alpha\frac{\xi_n}{n^2} - \eta_n\right\}\,. \end{cases}$$

The pair $(\{\xi_n\},\{\tilde{\eta}_n\})$ is an arbitrary pair in $l^2 \times l^2$ since $\frac{n}{\phi_n'(0)} = \sqrt{\pi/2}$. Upon renaming $n\xi_n/\phi_n'(0)$ and $\tilde{\eta}_n$ as, respectively, $\xi_n$ and $\eta_n$, we can study the problem

$$\begin{cases} \displaystyle\int_0^T z_n(r)v(T-r)\,\mathrm{d}r = \frac{\xi_n}{n}, \\ \displaystyle\int_0^T \frac{1}{n}z_n'(r)v(T-r)\,\mathrm{d}r = \eta_n\,, \end{cases} \tag{2.11}$$

where $(\{\xi_n\},\{\eta_n\}) \in l^2 \times l^2$ is arbitrary.

REMARK 2.6. We observe:

- The coefficients $\tilde{\eta}_n$ are computed as Fourier coefficients of an element of $H^{-1}$ with respect to the basis $\{\phi_n'\}$. If it happens that $\eta \in H^{-1}$ is smoother, then the sequence $\{\eta_n\}$ has additional properties. We shall use the following observation: $\eta$ is a regular distribution identified by a square integrable function if and only if $\{n\eta_n\} \in l^2$. This fact is easily proved since $\{\phi_n'(x)/n\}$ is an orthonormal sequence in $L^2(0,\pi)$.

- The Fourier coefficients of $\theta(x,T)$ and $q(x,T)$ at time $T$ have the form

$$\sqrt{\frac{2}{\pi}}\xi_n\,, \qquad \sqrt{\frac{2}{\pi}}\left[2\alpha\frac{\xi_n}{n^2}-\eta_n\right]\,, \tag{2.12}$$

  where $\{\xi_n\}$ and $\{\eta_n\}$ are the sequences in (2.11).
- Temperature and flux are given respectively by the series (2.7) and (2.8), but these series in fact give special extensions to $\mathbb{R}$ of $x\to\theta(x,t)$ and $x\to q(x,t)$. So, the objects of our interest are not these series, but their restrictions to $(0,\pi)$.

Problem (2.11) can be written in one line, as

$$\int_0^T\left[z_n(s)-i\frac{1}{n}z_n'(s)\right]v(T-s)\,\mathrm{d}s=\left[\frac{\xi_n}{n}-i\eta_n\right]\,, \tag{2.13}$$

and the sequence

$$\left\{\frac{\xi_n}{n}-i\eta_n\right\}$$

belongs to a dense subspace of $l^2(\mathbb{C})$. We set

$$\hat{\zeta}_n(t)=z_n(t)-i\frac{1}{n}z_n'(t)\,. \tag{2.14}$$

With this notation, problem (2.11) takes the form

$$\int_0^T\hat{\zeta}_n(r)v(T-r)\,\mathrm{d}r=c_n\,, \qquad n>0\,, \tag{2.15}$$

where $\{c_n\}$ is a sequence in $l^2$ of the special form

$$c_n=\frac{\xi_n}{n}-i\eta_n\,, \qquad \{\xi_n\}\,,\ \{\eta_n\}\in l^2$$

(since $n\phi_n'(0)$ is constant).

In fact, a solution $v(\cdot)$ of problem (2.11) clearly solves (2.13) and also the following problem:

$$\int_0^T\left[z_n(s)+i\frac{1}{n}z_n'(s)\right]v(T-s)\,\mathrm{d}s=\left[\frac{\xi_n}{\phi_n'(0)}+i\eta_n\right]\,.$$

Conversely, we obtain a real solution $v(\cdot)$ of (2.13) if we require the conditions

$$\int_0^T\hat{\zeta}_n(r)v(T-r)\,\mathrm{d}r=c_n\,, \qquad n\in\mathbb{Z}\setminus\{0\}=\mathbb{Z}'\,, \tag{2.16}$$

where we define for $n>0$:

$$\hat{\zeta}_{-n}(t)=\overline{\hat{\zeta}_n(t)}=z_n(t)+i\frac{1}{n}z_n'(t)\,, \qquad c_{-n}=\bar{c}_n$$

(since $z_n(t)$ is real.) So, we can use (2.14) as the definition of $\hat{\zeta}_n(t)$, for every $n\in\mathbb{Z}\setminus\{0\}$, provided that we also put

$$z_{-n}(t)=z_n(t)\,.$$

We sum up: the moment problem to be studied is the problem (2.16).

Using (2.3) and (2.4), we see that $\hat{\zeta}_n(t)$ solves the following initial value problem:

$$\begin{aligned}\hat{\zeta}_n(0) &= 1-(2\alpha/n)i\,,\\ \hat{\zeta}_n'(t) &= 2\alpha\hat{\zeta}_n(t)+inz_n(t)\\ &\quad -n^2\int_0^t N(t-s)z_n(s)\,\mathrm{d}s+in\int_0^t N'(t-s)z_n(s)\,\mathrm{d}s\,. \end{aligned}\tag{2.17}$$

Integration by parts shows the equality:

$$\begin{aligned}\hat{\zeta}_n'(t) &= 2\alpha\hat{\zeta}_n(t)+inz_n(t)\\ &= 2\alpha\hat{\zeta}_n(t)-n^2\int_0^t N(t-s)\hat{\zeta}_n(s)\,\mathrm{d}s+inN(t)\,. \end{aligned}\tag{2.18}$$

These equalities have been obtained with $n>0$. They hold for every $n\in\mathbb{Z}'$ since we defined $z_{-n}(t)=z_n(t)$.

We compute the derivatives of both sides of (2.17) (recall that $N'(0)=0$). We get

$$\begin{aligned}\hat{\zeta}_n'' &= -n^2\hat{\zeta}_n+2\alpha\hat{\zeta}_n'+in\int_0^t N''(t-s)z_n(s)\,\mathrm{d}s\\ &\quad -n^2\int_0^t N'(t-s)z_n(s)\,\mathrm{d}s\,,\\ \hat{\zeta}_n(0) &= 1-(2\alpha/n)i\,,\qquad \hat{\zeta}_n'(0)=2\alpha+in\left(1-\frac{4\alpha^2}{n^2}\right)\,. \end{aligned}\tag{2.19}$$

So, $\hat{\zeta}_n(t)$ solves also the following integral equation:

$$\begin{aligned}\hat{\zeta}_n(t) &= e^{(\alpha+i\beta_n)t}\\ &\quad +\frac{e^{\alpha t}}{2\beta_n}\left\{e^{-i\beta_n t}\left[\beta_n-n+\frac{2\alpha^2}{n}+\alpha\frac{n-2\beta_n}{n}i\right]\right.\\ &\quad \left.-e^{i\beta_n t}\left[\beta_n-n+\frac{2\alpha^2}{n}+\alpha\frac{n+2\beta_n}{n}i\right]\right\}\\ &\quad +\frac{1}{\beta_n}\int_0^t e^{\alpha(t-s)}\sin\beta_n(t-s)\left[in\int_0^s N''(s-r)z_n(r)\,\mathrm{d}r\right.\\ &\quad \left.-n^2\int_0^s N'(s-r)z_n(r)\,\mathrm{d}r\right]\,\mathrm{d}s\,,\\ \beta_n &= \sqrt{n^2-\alpha^2}\qquad \text{so that } \beta_n\asymp n,\text{ and } \tfrac{n-\beta_n}{\beta_n}\asymp\tfrac{1}{\beta_n^2}\asymp\tfrac{1}{n^2}\,. \end{aligned}\tag{2.20}$$

REMARK 2.7. We observe:

- As we already noted, for simplicity we assume $\beta_n\neq 0$ for every $n$.
- The computation which led to formula (2.20) has been performed with $n>0$, but our moment problem has to be studied also with $n<0$. Using the definitions of $\hat{\zeta}_{-n}(t)$, $c_{-n}$, $z_{-n}(t)$ we see that formula (2.20) holds for $\hat{\zeta}_{-n}(t)$, where $n>0$, provided we add the definition

$$\beta_{-n}=-\beta_n\,.$$

- The following inequality is easily seen from (2.20), using boundedness of $\{z_n(t)\}$ on every fixed interval $[0,T]$:

$$\left|\hat{\zeta}_n(t) - e^{(\alpha+i\beta_n)t}\right| \le M\frac{1}{|\beta_n|}\,, \qquad M = M(T)\,. \tag{2.21}$$

Our goal is to prove that $\{\hat{\zeta}_n\}_{n\in\mathbb{Z}'}$ is an $\mathcal{L}$-basis in $L^2(0,T)$, for a suitable number $T$, or, equivalently, to prove the same property for the sequence $\{e^{-\alpha t}\hat{\zeta}_n(t)\}$. Let

$$\zeta_n(t) = e^{-\alpha t}\hat{\zeta}_n(t)\,.$$

Using (2.18) we see that

$$\zeta_n'(t) = \alpha\zeta_n(t) - n^2\int_0^t M(t-s)\zeta_n(s)\,\mathrm{d}s + inM(t)\,, \qquad M(t) = e^{-\alpha t}N(t)\,. \tag{2.22}$$

From (2.20) we have:

$$\begin{aligned}
\zeta_n(t) &= e^{i\beta_n t} + \frac{1}{2\beta_n}\left\{e^{-i\beta_n t}\left[\beta_n - n + \frac{2\alpha^2}{n} + \alpha\frac{n-2\beta_n}{n}i\right]\right.\\
&\left.-e^{i\beta_n t}\left[\beta_n - n + \frac{2\alpha^2}{n} + \alpha\frac{n+2\beta_n}{n}i\right]\right\}\\
&+i\frac{n}{\beta_n}\int_0^t [\sin\beta_n(t-s)]\,e^{-\alpha s}\int_0^s N''(s-r)z_n(r)\,\mathrm{d}r\;\mathrm{d}s\\
&-\frac{n^2}{\beta_n^2}\int_0^t\left[\frac{\mathrm{d}}{\mathrm{d}s}\cos\beta_n(t-s)\right]e^{-\alpha s}\int_0^s N'(s-r)z_n(r)\,\mathrm{d}r\;\mathrm{d}s\,.
\end{aligned} \tag{2.23}$$

Integrating by parts the last integral we see that it is equal to

$$\left\{e^{-\alpha t}\int_0^t N'(t-r)z_n(r)\,\mathrm{d}r \right. \tag{2.24}$$

$$\left. -\int_0^t \cos\beta_n(t-s)e^{-\alpha s}\int_0^s N_0(s-r)z_n(r)\,\mathrm{d}r\;\mathrm{d}s\right\}\,. \tag{2.25}$$

From (2.21) we get:

Lemma 2.8. Let $T \ge 0$. There exists a number $M = M_T$ such that for every $n \in \mathbb{Z}'$ we have

$$|\zeta_n(t)| \le M\,, \qquad 0 \le t \le T\,.$$

(We recall the notation $\mathbb{Z}' = \mathbb{Z}\setminus\{0\}$.) Now we compare the two sequences $\{\zeta_n(t)\}_{n\in\mathbb{Z}'}$ and

$$\left\{e^{i\beta_n t}\right\}_{n\in\mathbb{Z}'}\,, \tag{2.26}$$

which is an $\mathcal{L}$-basis in $L^2(0,T)$ for every $T \ge 2\pi$; see Appendix 5.1. For this, we describe further the term (2.24), using the following result from [19]:

Lemma 2.9. For every $T$ there exists a number $M$ such that

$$|z_n(t) - e^{\alpha t}\cos\beta_n t| < \frac{M}{\beta_n}\,.$$

Using this fact, the integral in (2.24) is rewritten as follows:

$$\int_0^t N'(t-r)z_n(r)\,\mathrm{d}r = \int_0^t N'(t-r)\left[z_n(r) - e^{\alpha r}\cos\beta_n r\right]\,\mathrm{d}r$$
$$+\int_0^t N'(t-r)e^{\alpha r}\cos\beta_n r \asymp \frac{1}{\beta_n}\,,$$

as is seen with an integration by parts of the last integral. The integral in (2.25) can be integrated by parts and we get

$$-\frac{1}{\beta_n}\int_0^t e^{\alpha s}\sin\beta_n(t-s)\left[\int_0^s \left(N_0'(s-r) - \alpha N(s-r)\right)z_n(r)\,\mathrm{d}r\right]\,\mathrm{d}s \asymp \frac{1}{\beta_n}\,.$$

We use $(n-\beta_n)\asymp 1/\beta_n$, which implies that the second and third terms in (2.23) are dominated by (const)$/\beta_n$ and we get:

THEOREM 2.10. For every $T>0$ there exists a constant $M$ such that

$$\left|\zeta_n(t) - e^{i\beta_n t}\right| < \frac{M}{\beta_n} \tag{2.27}$$

so that

$$\sum_{n\in\mathbb{Z}'} \|\zeta_n(t) - e^{i\beta_n t}\|^2_{L^2(0,T)} < +\infty\,.$$

Using Theorem 2.1, we also have:

THEOREM 2.11. Let $T\geq 2\pi$. There exists $N=N_T$ such that the sequence $\{\zeta_n(t)\}_{|n|\geq N}$ is an $\mathcal{L}$-basis in $L^2(0,T)$.

**3. Hidden regularity of the flux.** Up to now, we used that the flux belongs to $H^{-1}(0,\pi)$. In this section we prove that the flux, i.e. the restriction to $(0,\pi)$ of (2.8), is smoother: $q(\cdot,t)\in L^2(0,\pi)$. In order to understand the argument, we examine first a simple example.

EXAMPLE 3.1. We consider the time interval $[0,\pi]$ and the kernel $N(t)\equiv 1$. Furthermore, we let the initial condition be 0. So, our equation

$$\theta_t(x,t) = \int_0^t \theta_{xx}(x,s)\,\mathrm{d}s\,, \qquad \theta(0,t)=u(t)\,, \quad \theta(\pi,t)=0$$

is nothing else than the wave equation

$$\theta_{tt}(x,t) = \theta_{xx}(x,t)\,, \qquad \theta(0,t)=u(t)\,, \quad \theta(\pi,t)=0$$

with zero initial conditions. For $t\leq\pi$ there is no reflection from the right end of the interval, and we know that

$$\theta(x,t) = u(t-x)H(t-x)\,, \tag{3.1}$$

where $H(t)$ is the Heaviside function. Note that if $u(t)$ is smooth, then this function has a smooth extension to a left neighborhood of $x=0$ for every $t>0$. It is easily seen, using this formula, that if $t<\pi$, then

$$q(x,t) = \theta(x,t)\,.$$

By the way, this shows that the pair (temperature/flux) can't be simultaneously controlled in time $t < \pi$. Let us now look at the series expansion (2.7) for $\theta(x,t)$ and the corresponding expansion (2.8) for the flux. We introduce the function

$$f^t(x) = u(t-x)H(t-x)\,.$$

We write down explicitly the series for this example and we get the equalities:

$$\theta(x,t) = \sum_{n=1}^{+\infty}\left[\frac{2}{\pi}\int_0^t f^t(s)\sin ns\ \mathrm{d}s\right]\sin nx\,, \tag{3.2}$$

$$\theta_x(x,t) = \frac{2}{\pi}\sum_{n=1}^{+\infty}\left[\int_0^t n\sin n(t-s)f^t(s)\ \mathrm{d}s\right]\cos nx\,, \tag{3.3}$$

$$q(x,t) = -\frac{2}{\pi}\sum_{n=1}^{+\infty}\left[\int_0^t\int_0^r n\sin n(r-s)f^t(s)\ \mathrm{d}s\ \mathrm{d}r\right]\cos nx\,. \tag{3.4}$$

Equation (3.2) gives an odd extension (with respect to $x$) of $u(t-x)H(t-x)$. So, it has a jump of $2f^t(0) = 2u(t)$ at $x = 0$ even if $u(t)$ is smooth. Hence, expressions (3.2) and (3.1) have the same restriction at the interval $(0,\pi)$ (since the series in (3.2) is a Fourier sine series of $f^t(s) = u(t-x)H(t-x)$), but they are different functions on larger intervals, and have different distributional derivatives. In particular, the jump at $x = 0$ is reflected in the fact that the series in the expression of $q(x,t)$ does not converge in $L^2(0,\pi)$. It converges in the sense of distributions and identifies a distribution with a singular part. We can extract the singular part of the distribution (3.4) if we recall the expansions

$$\frac{\pi - x}{2} = \sum_{1}^{+\infty}\frac{1}{n}\sin nx\,, \qquad \sum_{n=1}^{+\infty}\cos nx = \pi\delta(x) - \frac{1}{2}\,. \tag{3.5}$$

Then, we can proceed as follows:

$$\begin{aligned}
q(x,t) &= -\frac{2}{\pi}\sum_{n=1}^{+\infty}\cos nx\left[\int_0^t\int_0^r n\sin n(r-s)f^t(s)ds\,dr\right]\\
&= -\frac{2}{\pi}\left[\int_0^t f^t(s)ds\right]\sum_{n=1}^{+\infty}\cos nx\\
&\quad + \sum_{n=1}^{+\infty}\frac{2}{\pi}\cos nx\int_0^t f^t(t-s)\cos ns\,ds\\
&= -\frac{2}{\pi}\left[\int_0^t f^t(s)ds\right]\left(\pi\delta(x) - \frac{1}{2}\right) + \sum_{n=1}^{+\infty}\frac{2}{\pi}\cos nx\int_0^t f^t(t-s)\cos ns\,ds\\
&= -2\delta(x)\int_0^t f^t(s)\,ds\\
&\quad + \left\{\frac{1}{\pi}\int_0^\pi f^t(s)\,ds + \frac{2}{\pi}\sum_{n=1}^{+\infty}\cos nx\int_0^\pi f^t(s)\cos ns\,ds\right\}\,.
\end{aligned} \tag{3.6}$$

The brace is a cosine expansion and it is equal to $f^t(x)$ for $x \in (0,\pi)$. So, we see that $q(x,t)$ is a distribution with singular support $x = 0$ plus a regular part, which is an even

extension of $f(t-x)$: as an element of $H^{-1}(0,\pi)$, the delta function supported at 0 has no effect, and we see that the distribution $q(x,t)$ is identified by a square integrable function for every $t$: the two expressions $q(x,t) = f^t(x)$ and (3.4), i.e. (3.6), taken for $t \in (0,\pi)$, identify the same element of $H^{-1}(0,\pi)$.

We will demonstrate that the procedure used in the previous example is completely general and that it can be repeated on every interval $(0,T)$, even if $T > \pi$. In fact we prove:

THEOREM 3.2. Let $N(t) \in W^{4,2}_{\mathrm{loc}}(0,\infty)$. Then, for every $t \in [0,T]$, $q(\cdot,t)$ is a regular element of $H^{-1}(0,\pi)$, identified by a square integrable function.

The proof consists of extracting several parts in the expressions of $q(x,t)$, which correspond to an $L^2$-function, and to show that the remaining parts sum up to $\delta$, the Dirac delta at 0, as in the example. This technical computation can be found in Appendix 5.2.

**4. $L^2(0,\pi)$-Exact controllability of the temperature and flux pair.** In this section we prove Theorem 1.2; i.e., we prove that any element of $L^2(0,\pi)\times L^2(0,\pi)$ can be reached by the heat/flux pair. Furthermore, we shall see that the minimal controllability time is $2\pi$.

We recall that the sequence $\{\eta_n\}$ is

$$\langle \eta, \phi_n' \rangle_{H^{-1}(0,\pi)}$$

so that $\eta \in L^2(0,\pi)$ if and only if $\{n\eta_n\} \in l^2$; see also Remark 2.6.

We consider Theorem 2.11. The sequence (2.26) forms an $\mathcal{L}$-basis in $L^2(0,T)$ for $T \geq 2\pi$ and we will show that the sequence $\{\zeta_n(t)\}$ is an $\mathcal{L}$-basis in $L^2(0,T)$ too. This is an intermediate result, which can be found in Appendix 5.4 where we prove:

THEOREM 4.1. The sequences $\{\zeta_n(t)\}_{n\in\mathbb{Z}'}$ and $\{\hat{\zeta}_n(t)\}_{n\in\mathbb{Z}'}$ are $\mathcal{L}$-bases in the same space $L^2(0,T)$ as the sequence $\{e^{i\beta_n t}\}_{n\in\mathbb{Z}'}$.

Here, we use this result in order to prove the exact controllability of the temperature/flux pair in $L^2(0,\pi)\times L^2(0,\pi)$, i.e. Theorem 1.2.

More precisely, we will demonstrate that the moment problem (2.16) is solvable for a suitable subset of sequences $\{c_n\} \in l^2$. In fact we prove:

THEOREM 4.2. Let the sequence $\{c_n\}_{n\in\mathbb{Z}'}$ which appears in (2.16) have the special form

$$c_n = \frac{d_n}{n} - i\frac{\gamma}{n}\,, \qquad n \in \mathbb{Z}'\,, \qquad \{d_n\}_{n\in\mathbb{Z}'} \in l^2\,, \qquad \gamma \in \mathbb{R}\,.$$

Then, the moment problem (2.16) is solvable in any time $T \geq 2\pi$.

Before proving this result, we show that it implies Theorem 1.2. We represent

$$d_n = \xi_n - i\chi_n\,,$$

where $\{\xi_n\}$ and $\{\chi_n\}$ are real-valued and

$$\sum_{n=1}^{+\infty} |\xi_n|^2 < \infty\,, \qquad \sum_{n=1}^{+\infty} |\chi_n|^2 < \infty\,.$$

Theorem 4.2 asserts that we can assign the Fourier coefficients of $\theta(x,t)$ in (2.7) and $q(x,t)$ in (2.8). Keeping in mind equalities (2.12) we have

$$\begin{cases} \theta(x,T) &= \displaystyle\sum_{n=1}^{+\infty} \xi_n \phi_n(x)\,, \\ q(x,T) &= \displaystyle\sum_{n=1}^{+\infty} k_n \cos nx + \gamma \sum_{n=1}^{+\infty} \cos nx\,, \end{cases}$$

where

$$k_n = \chi_n + 2\alpha \frac{\xi_n}{n}\,.$$

So, $\{k_n\}$ is an arbitrary sequence in $l^2$, independent of $\{\xi_n\}$. Also the coefficient $\gamma$ can be arbitrarily assigned (of course, provided that $T \geq 2\pi$).

Using the second equality in (3.5) we see that

$$q(x,t) = \left\{ \frac{\gamma}{2} + \sum_{n=1}^{+\infty} k_n \cos nx \right\} + \gamma\pi\delta(x)\,.$$

The coefficients $\xi_n$, $\gamma$ and $\eta_n$ can be arbitrarily assigned, provided that $T \geq 2\pi$.

So, $q(x,T)$ is a distribution, sum of a delta function plus a regular part represented by a square integrable function. If $T \geq 2\pi$ both $\theta(x,T)$ and the regular part of $q(x,T)$ can be arbitrarily assigned in $L^2(0,\pi)$. This is Theorem 1.2.

As a by-product, we found also that the series of $q(x,T)$ converges in the sense of distributions to an element of the subspace of $L^2(0,\pi) \oplus \{r\delta(x)\,,\ r \in \mathbb{R}\}$. The regular part and the coefficient of $\delta(x)$ are related through the integral mean of $q(x,T)$.

In order to prove Theorem 4.2, we need a few preliminaries and a lemma. First of all we recall the definitions

$$c_{-n} = \bar{c}_n\,, \qquad \beta_{-n} = -\beta_n\,, \qquad z_{-n}(t) = z_n(t)$$

so that equality (2.14) holds for every $n \in \mathbb{Z}'$. Then, we recall that the moment problem takes the form (see (2.16))

$$c_n = \frac{d_n - i\gamma}{n} = \int_0^T u(T-t) \left[ \int_0^t N(t-s)\hat{\zeta}_n(s)\,\mathrm{d}s \right] \mathrm{d}t\,. \tag{4.1}$$

We recall the definition (2.14) of $\hat{\zeta}_n(s)$, i.e.

$$\hat{\zeta}_n(t) = z_n(t) - (i/n)z_n'(t)\,.$$

Integration by parts shows that we must solve the problem

$$d_n - i\gamma = \int_0^T u(T-t) \left\{ n \int_0^t H_n(t-s) z_n(s)\,\mathrm{d}s - i\,[z_n(t) - N(t)] \right\} \mathrm{d}t\,. \tag{4.2}$$

Here, $n \in \mathbb{Z}'$, $\{d_n\}$ is an *arbitrary* $l^2(\mathbb{Z}')$ sequence and

$$H_n(t) = N(t) - i\frac{1}{n}N'(t)\,.$$

The next lemma is based on the fact that $z_n(t) \sim \cos\beta_n t$ (as seen from (2.27)) so that its integral should behave like $(\sin\beta_n t)/\beta_n$.

Note that the index $n$ as used up to now takes values in $\mathbb{Z}'$. Instead, in the next lemma, we use $n \in \mathbb{Z}$. It is convenient to define $\beta_0 = 0$ and we denote by $T$ any number such that $\left\{e^{i\beta_n t}\right\}_{n\in\mathbb{Z}}$ is an $\mathcal{L}$-basis; i.e. $T \geq 2\pi$ (see Appendix 5.1).

LEMMA 4.3. Let the index $n$ take values in $\mathbb{Z}$. We define

$$R_0(t) = iN(t)\,, \qquad R_n(t) = n\int_0^t H_n(t-s)z_n(s)\,\mathrm{d}s - iz_n(t)\,,\ n \in \mathbb{Z}'\,.$$

The sequence $\{R_n(t)\}_{n\in\mathbb{Z}}$ is quadratically close to the sequence $\left\{-ie^{(\alpha+i\beta_n)t}\right\}_{n\in\mathbb{Z}}$ in $L^2(0,T)$ for every $T$ and it is $\omega$-independent in $L^2(0,T)$ if $T \geq 2\pi$.

*Proof.* We state the following property: there exists a number $M$ such that for every $n > 0$ we have

$$\begin{cases} |z_n(t) - e^{\alpha t}\cos\beta_n t| < \frac{M}{\beta_n}\,, \\ \left|\left[n\int_0^t H_n(t-r)z_n(r)\,\mathrm{d}r\right] - e^{\alpha t}\sin\beta_n t\right| < \frac{M}{\beta_n}\,. \end{cases} \tag{4.3}$$

The inequality in the first line is Lemma 2.9 while the proof of the second inequality (based on Lemma 5.3) is similar to the corresponding proof in [19].

These inequalities hold for $n \in \mathbb{Z}'$.

The first statement of the lemma follows from here. Now we prove $\omega$-independence in $L^2(0,T)$, $T \geq 2\pi$. Let $\{\alpha_n\}_{n\in\mathbb{Z}} \in l^2$ be such that

$$\sum_{n\in\mathbb{Z}} \alpha_n R_n(t) = 0\,.$$

So,

$$0 = i\alpha_0 N(t) + \sum_{n\in\mathbb{Z}'} \alpha_n \left(n\int_0^t H_n(t-s)z_n(s)\,\mathrm{d}s - iz_n(t)\right) \tag{4.4}$$

$$= i\alpha_0 N(t) + \sum_{n\in\mathbb{Z}'} \alpha_n \left(n\int_0^t H_n(t-s)z_n(s)\,\mathrm{d}s - e^{\alpha t}\sin\beta_n t\right) \tag{4.5}$$

$$- i\sum_{n\in\mathbb{Z}'} \alpha_n \left[z_n(t) - e^{\alpha t}\cos\beta_n t\right] - i\Phi(t)\,, \tag{4.6}$$

where

$$\Phi(t) = e^{\alpha t}\left[\sum_{n\in\mathbb{Z}'} \alpha_n e^{i\beta_n t}\right]\,. \tag{4.7}$$

We now demonstrate that the function $\Phi(t)$ belongs to $W^{1,2}(0,T)$ for every $T > 0$. It is sufficient to prove this property for the series at the lines (4.5)-(4.6). For this, we use representation (2.5) and the following weaker form of Lemma 5.3 in Appendix 5.2:

$$z_n(t) = e^{\alpha t}\left[\cos\beta_n t + \frac{N''(0)}{2\beta_n} t\sin\beta_n t\right] + \frac{b_n(t)}{\beta_n^2} \quad \text{and} \quad |b_n(t)| < M\,. \tag{4.8}$$

First we consider the series at the line (4.6):

$$\sum_{n\in\mathbb{Z}'} \alpha_n \left[z_n(t) - e^{\alpha t}\cos\beta_n t\right] = \alpha e^{\alpha t} \sum_{n\in\mathbb{Z}'} \frac{\alpha_n}{\beta_n}\sin\beta_n t \tag{4.9}$$

$$-\sum_{n\in\mathbb{Z}'} \frac{\alpha_n n^2}{\beta_n^2}\int_0^t N'(t-s)z_n(s)\,\mathrm{d}s \tag{4.10}$$

$$+\sum_{n\in\mathbb{Z}'} \frac{\alpha_n n^2}{\beta_n^2}\left\{\int_0^t e^{\alpha s}\cos\beta_n s\left[\int_0^{t-s} N_0(t-s-r)z_n(r)\,\mathrm{d}r\right]\,\mathrm{d}s\right\}. \tag{4.11}$$

The series on the right-hand side of (4.9) converges uniformly and it can be differentiated termwise, since the series of the derivatives is $L^2$-convergent thanks to the fact that both $\{\sin\beta_n t\}$ and $\{\cos\beta_n t\}$ are $\mathcal{L}$-bases in $L^2(0,T)$ for sufficiently large $T$. So, it defines a $W^{1,2}$ function.

The series in (4.10) can be interchanged with the integral, since we know that $\{z_n(t)\}$ is an $\mathcal{L}$-basis in $L^2(0,T)$ for sufficiently large $T$ (see Theorem 2.5). So, it defines a $W^{1,2}$ function.

The terms of the series in (4.11) are first integrated by parts. We gain a factor $1/\beta_n$ and this implies uniform convergence. So, this series can also be interchanged with the integral. It follows that the line (4.11) defines a $C^1$ function.

So, the series at the line (4.6) defines a $W^{1,2}(0,T)$ function.

Now we prove that the series at the line (4.5) defines a $W^{1,2}(0,T)$ function too. We replace $z_n(t)$ with its expression in (4.8) and we get

$$\sum_{n\in\mathbb{Z}'} \alpha_n \left(n\int_0^t H_n(t-s)z_n(s)\,\mathrm{d}s - e^{\alpha t}\sin\beta_n t\right)$$

$$= \sum_{n\in\mathbb{Z}'} \frac{\alpha_n n}{\beta_n^2}\int_0^t H_n(t-s)b_n(s)\,\mathrm{d}s \tag{4.12}$$

$$+\frac{1}{2}N''(0)\sum_{n\in\mathbb{Z}'} \frac{\alpha_n n}{\beta_n}\int_0^t [se^{\alpha s}H_n(t-s)]\sin\beta_n s\,\mathrm{d}s \tag{4.13}$$

$$+\sum_{n\in\mathbb{Z}'} \alpha_n\left\{n\int_0^t H_n(t-s)e^{\alpha s}\cos\beta_n s\,\mathrm{d}s - e^{\alpha t}\sin\beta_n t\right\}. \tag{4.14}$$

Boundedness of the sequence $\{b_n(t)\}$ (see (4.8)) shows that the series

$$\sum_{n\in\mathbb{Z}'} \frac{\alpha_n n}{\beta_n^2} H_n(t-s)b_n(s)$$

is uniformly convergent. So, the series can be interchanged with the integral and line (4.12) defines a $W^{1,2}$ function.

Taking into account the definition of $H_n(t)$, we can integrate by parts each integral in the series (4.13). We gain a further factor $1/\beta_n$ and, as above, we see that (4.13) defines a $C^1$ function too.

In order to see that line (4.14) defines a $W^{1,2}$ function, we first integrate by parts the integral. We get

$$e^{\alpha t}\sum_{n\in\mathbb{Z}'}\alpha_n\left(\frac{n}{\beta_n}-1\right)\sin\beta_n t-\sum_{n\in\mathbb{Z}'}\frac{\alpha_n n}{\beta_n}\int_0^t\sin\beta_n s\frac{\mathrm{d}}{\mathrm{d}s}\left[e^{\alpha s}H_n(t-s)\right]\,\mathrm{d}s\,.$$

The result follows, since the integrals can be integrated by parts again and

$$\left(\frac{n}{\beta_n}-1\right)\asymp\frac{1}{\beta_n^2}\,.$$

This proves the regularity of $\Phi(t)$: it is a $W^{1,2}(0,T)$ function for every $T$, and the function

$$\Psi(t)=e^{-\alpha t}\Phi(t)$$

belongs to the space $X$ which is the closure of the linear span of $\left\{e^{i\beta_n t}\right\}_{n\in\mathbb{Z}}$.

Now, $\Psi'(t)$ is the $L^2$-limit of the incremental quotient, and

$$\frac{\Psi(t+h)-\Psi(t)}{h}\in X\,.$$

So, $\Psi'(t)\in X$ too. It follows that

$$\Psi'(t)=\sum_{n\in\mathbb{Z}'}\tilde{\sigma}_n e^{i\beta_n t}\,,\qquad \Psi(t)-\Psi(0)=-i\sum_{n\in\mathbb{Z}'}\frac{\tilde{\sigma}_n}{\beta_n}e^{i\beta_n t}+i\sum_{n\in\mathbb{Z}'}\frac{\tilde{\sigma}_n}{\beta_n}\,.$$

Now we recall: we confined ourselves to work on an interval $[0,T]$ over which $\left\{e^{i\beta_n t}\right\}_{n\in\mathbb{Z}}$ (here $\beta_0=0$) is an $\mathcal{L}$-basis. So, the expansion of $\Psi(t)$ in terms of $\left\{e^{i\beta_n t}\right\}_{n\in\mathbb{Z}}$ is unique. Comparing with (4.7) we see that

$$\alpha_n=\frac{\sigma_n}{\beta_n}\,,\qquad \sigma_n=i\tilde{\sigma}_n\,,\qquad \{\sigma_n\}\in l^2\,.$$

Furthermore, we also see that

$$\alpha_0=\Phi(0)=\Psi(0)=-i\sum_{n\in\mathbb{Z}'}\frac{\tilde{\sigma}_n}{\beta_n}=-\sum_{n\in\mathbb{Z}'}\frac{\sigma_n}{\beta_n}\,. \tag{4.15}$$

Now we go back to the original equality (4.4) which takes the form

$$i\alpha_0 N(t)+\sum_{n\in\mathbb{Z}'}\frac{\sigma_n}{\beta_n}\left\{n\int_0^t H_n(t-s)z_n(s)\,\mathrm{d}s-iz_n(t)\right\}=0\,. \tag{4.16}$$

The estimates in (4.3) show uniform convergence of the series. Now we observe that

$$n\int_0^t H_n(t-s)z_n(s)\,\mathrm{d}s-iz_n(t)=n\int_0^t N(t-s)\hat{\zeta}_n(s)\,\mathrm{d}s-iN(t)\,. \tag{4.17}$$

Substituting equality (4.17) into (4.16) we get (using (4.15))

$$2\alpha_0 iN(t)+\sum_{n\in\mathbb{Z}'}\frac{\sigma_n}{\beta_n}n\int_0^t N(t-s)\hat{\zeta}_n(s)\,\mathrm{d}s=0\,.$$

The sequence $\{\hat{\zeta}_n(t)\}$ is an $\mathcal{L}$-basis in $L^2(0,T)$ and $\{\sigma_n\beta_n/n\} \in l^2$ (see Theorem 4.1). So, the series can be exchanged with the integral:

$$\sum_{n\in\mathbb{Z}'} \frac{\sigma_n}{\beta_n} n \int_0^t N(t-s)\hat{\zeta}_n(s)\,\mathrm{d}s = \int_0^t \left[\sum_{n\in\mathbb{Z}'} \frac{\sigma_n}{\beta_n} nN(t-s)\right] \hat{\zeta}_n(s)\,\mathrm{d}s\,.$$

The right-hand side is a continuous function which is zero at $t = 0$. Hence we have $\alpha_0 = 0$ and we get

$$\int_0^t N(t-s)\left[\sum_{n\in\mathbb{Z}'} \frac{\sigma_n}{\beta_n} n\right] \hat{\zeta}_n(s)\,\mathrm{d}s = 0$$

for every $t$. This shows that

$$\sum_{n\in\mathbb{Z}'} \left[\frac{\sigma_n}{\beta_n} n\right] \hat{\zeta}_n(t) = 0$$

and this implies $\sigma_n = 0$ i.e. $\alpha_n = 0$, for every $n$, since we already know that $\{\hat{\zeta}_n(t)\}$ is an $\mathcal{L}$-basis; see Theorem 4.1. This ends the proof of $\omega$-independence of the sequence $\{R_n(t)\}_{n\in\mathbb{Z}}$. □

Now we go back to problem (4.2). Using the sequence $\{R_n(t)\}_{n\geq 0}$, the problem is rewritten as

$$d_n - \left\{ i\gamma + \int_0^T R_0(t)u(T-t)\,\mathrm{d}t \right\} = \int_0^T R_n(t)u(T-t)\,\mathrm{d}t\,. \tag{4.18}$$

This is not a standard form of a moment problem, but we can easily reduce it to a standard form thanks to the fact that $\{R_n(t)\}$ is an $\mathcal{L}$-basis in $L^2(0,T)$. We add a first term, with index $n = 0$, to the sequence $\{d_n\}$. So, we introduce the sequence $\{\tilde{d}_n\}$

$$\tilde{d}_0 = -i\gamma\,, \qquad \tilde{d}_n = d_n \quad \text{for } n \in \mathbb{Z}'.$$

We search for a function $u(T-t)$ which solves the moment problem

$$\tilde{d}_n = \int_0^T R_n(t)u(T-t)\,\mathrm{d}t\,, \qquad n \in \mathbb{Z}\,.$$

This function exists, since $\{R_n(t)\}$ is an $\mathcal{L}$-basis, and also solves the required problem (4.18).

*This ends the proof of Theorem* 4.2 *and hence, of Theorem* 1.2.

**5. Appendices.** We collect here technical proofs omitted from the text.

5.1. *Appendix: Riesz-basis properties of the sequence* $\{e^{i\beta_n t}\}$.

LEMMA 5.1. (i) If $\beta_n \neq 0$ for all $n \in \mathbb{N}$, then the sequence $\{1\} \cup \left\{e^{i\beta_n t}\right\}_{n\in\mathbb{Z}'}$ forms a Riesz basis in $L^2(0,2\pi)$.

(ii) If, for some $n \in \mathbb{N}$, $\beta_n = 0$, then the sequence $\{1\} \cup \{t\} \cup \left\{t^2\right\} \cup \left\{e^{i\beta_n t}\right\}_{n\in\mathbb{Z}':\beta_n\neq 0}$ forms a Riesz basis in $L^2(0,2\pi)$.

*Proof.* Since $\beta_n - n \asymp 1/\beta_n$, the statement (i) follows from the "1/4 in the mean" theorem [1] (see also [6, Proposition II.4.6]).

Using the same asymptotic estimates of $\beta_n$ and [1, Theorem 2], one can easily check that the generating function of the family $\{1\} \cup \{t\} \cup \left\{t^2\right\} \cup \left\{e^{i\beta_n t}\right\}_{n\in\mathbb{Z}':\beta_n\neq 0}$ satisfies

the Muckenhoupt condition (see, e.g., [6, Section II.4]). The statement (ii) follows then from [23] (see also [6, Theorem II.4.22]). □

This lemma implies the following result, which we also use in the paper.

THEOREM 5.2. Let $T \geq 2\pi$ and $\beta_n \neq 0$ for all $n \in \mathbb{N}$. The sequence $\left\{e^{i\beta_n t}\right\}_{n\in\mathbb{Z}'}$ is an $\mathcal{L}$-basis in $L^2(0,T)$. If, for one $n \in \mathbb{N}$, $\beta_n = 0$, the same statement is valid for the sequence $\{1\} \cup \{t\} \cup \left\{e^{i\beta_n t}\right\}_{n\in\mathbb{Z}':\beta_n\neq 0}$.

5.2. *Appendix: the regularity of the flux.* For the proof of Theorem 3.2, hence of Theorem 1.1, we need an improved version of [18, Lemma 4.1]. Namely, we need the following result, whose proof is in Appendix 5.3:

LEMMA 5.3. Let $\mu_n = n^2/[\beta_n^2]$ and $T > 0$ be fixed. There exists a bounded sequence $\{M_n(t)\}$ of square integrable functions and two differentiable functions $F(t)$ and $G(t)$ such that

$$z_n(t) = e^{\alpha t}\cos\beta_n t + \frac{\mu_n}{2\beta_n}\left(\frac{2\alpha}{\mu_n} + N''(0)t\right)e^{\alpha t}\sin\beta_n t + \frac{\mu_n}{\beta_n^2}G(t)\cos\beta_n t + \frac{\mu_n}{\beta_n^2}F(t) + \frac{M_n(t)}{\beta_n^3}\,. \tag{5.1}$$

Moreover, we need

$$\frac{1}{n}\phi_n'(x) = \sqrt{2/\pi}\cos nx\,. \tag{5.2}$$

Using (2.8) and the definition of $v(t)$ we see that

$$\begin{aligned} &q(x,t)\\ &= \sum_{n=1}^{+\infty}\phi_n'(x)\phi_n'(0)\int_0^t u(r)\left\{\int_0^{t-r} N(t-r-s)\left[\int_0^s N(s-\nu)z_n(\nu)\,\mathrm{d}\nu\right]\mathrm{d}s\right\}\mathrm{d}r\,. \end{aligned} \tag{5.3}$$

For any fixed $t$, this series converges in $H^{-1}(0,\pi)$ and we will prove that it is a *regular* element of $H^{-1}(0,\pi)$. More precisely, we will prove that it is the sum of a square integrable function and a multiple of $\delta(x)$, the Dirac delta (which identifies the null element of $H^{-1}(0,\pi)$).

We shall ignore inessential multiplicative constants. So we shall use

$$\phi_n'(x) = n\cos nx\,, \qquad \phi_n'(0) = n\,, \qquad \left\{\frac{1}{n}\phi_n'(x)\right\} = \{\cos nx\}\,.$$

We substitute in (5.3) in turn every addendum on the right-hand side of (5.1) and study the resulting series. In the following computations, $\{M_n(t)\}$ will denote a uniformly bounded sequence of functions on $[0,T]$, not the same at every occurrence.

When substituting $M_n(t)/\beta_n^3$ we get an $L^2(0,T)$-convergent series, which contributes a regular distribution to the expression of $q(\cdot,t)$ for every $t$.

When substituting the term $(\mu_n/\beta_n^2)G(t)\cos\beta_n t$ we can integrate by parts and we get a term of the order $1/\beta_n^3$, so again the series converges to an $L^2$-function.

Now we substitute $e^{\alpha t}\cos\beta_n t$. Integrating by parts twice we get the following three series (inessential constants are ignored). The first of them, given by formula (5.4), collects the contribution of all the terms which, integrated again by parts, give an integrand

of the form $M_n(t)/\beta_n^3$, with a bounded sequence $\{M_n(t)\}$:

$$\sum_{n=1}^{+\infty} \frac{\phi_n'(x)\phi_n'(0)}{\beta_n^3} \int_0^t M_n(t-r)u(r)\,\mathrm{d}r\,, \tag{5.4}$$

$$\sum_{n=1}^{+\infty} \frac{\phi_n'(x)\phi_n'(0)}{\beta_n^2} \int_0^t u(r)e^{\alpha(t-r)}\cos\beta_n(t-r)\,\mathrm{d}r\,, \tag{5.5}$$

$$H(t)\left[\sum_{n=1}^{+\infty} \frac{\phi_n'(x)\phi_n'(0)}{\beta_n^2}\right]. \tag{5.6}$$

Here

$$H(t) = \int_0^t u(r)\left[N(t-r) - \int_0^{t-r} N(t-r-s)\left(N'(s) - \alpha N(s)\right)\,\mathrm{d}s\right]\,\mathrm{d}r\,.$$

Both the series (5.4) and (5.5) are $L^2$-convergent. This is clear for (5.4) while convergence of (5.5) follows since the Parseval identity shows that

$$\sum_{n=1}^{+\infty}\left|\int_0^t u(r)e^{\alpha(t-r)}\cos\beta_n(t-r)\,\mathrm{d}r\right|^2 \le M\int_0^T e^{-2\alpha r}|u(r)|^2\,\mathrm{d}r\,.$$

Convergence of the series in (5.6) is in the sense of distributions. Using (5.2), we see that the series (5.6) is the sum of the following ones:

$$\sum_{n=1}^{+\infty}\left[\frac{n^2}{\beta_n^2} - 1\right]\cos nx\,, \qquad \sum_{n=1}^{+\infty}\cos nx\,.$$

The first series converges uniformly, since

$$\left[\frac{n^2}{\beta_n^2} - 1\right] \asymp \frac{1}{n^2}\,.$$

Using (3.5), we see that the second series converges to a multiple of $\delta(x)$, plus a constant.

When substituting the terms with $(\sin\beta_n t)/\beta_n$ we get a similar computation. This is left to the reader.

Substituting $F(t)/\beta_n^2$ we get a term such as (5.6).

This completes the proof of Theorem 1.1.

5.3. *Appendix: the proof of Lemma* 5.3. In this appendix we prove Lemma 5.3. We fix an interval $[0,T]$ and use $\{M_n(t)\}$ to denote a sequence of bounded functions on $[0,T]$ (not the same at every occurrence.) Put

$$g_n(t) = e^{\alpha t}\cos\beta_n t + \frac{\alpha}{\beta_n}e^{\alpha t}\sin\beta_n t\,, \qquad e_n(t) = z_n(t) - g_n(t)\,.$$

As a starting point, we use Formula (2.5), i.e., we use the definition of $\mu_n$ in Lemma 5.3 and $N_0(t) = N''(t) - \alpha N'(t)$:

$$\begin{aligned} e_n(t) = -\mu_n \int_0^t N'(t-r) e_n(r)\,\mathrm{d}r \\ + \mu_n \int_0^t e^{\alpha s} \cos\beta_n s \left[\int_0^{t-s} N_0(t-s-r) e_n(r)\,\mathrm{d}r\right] \mathrm{d}s - \mu_n \int_0^t N'(t-r) g_n(r)\,\mathrm{d}r \\ + \mu_n \int_0^t e^{\alpha s} \cos\beta_n s \left[\int_0^{t-s} N_0(t-s-r) g_n(r)\,\mathrm{d}r\right] \mathrm{d}s\,. \end{aligned} \tag{5.7}$$

We substitute the expression of $g_n(t)$ into these integrals and integrate by parts as many times as possible to get as many factors $1/\beta_n$ as we can, and examine the resulting terms. We use $N'(0) = 0$, $N \in W^{4,2}(0,T)$ and the integrals

$$\int_0^t \sin\beta_n s \cos\beta_n(t-s)\,\mathrm{d}s = \frac{1}{2} t \sin\beta_n t\,,$$
$$\int_0^t \sin\beta_n s \sin\beta_n(t-s)\,\mathrm{d}s = \frac{1}{2\beta_n}\sin\beta_n t - \frac{1}{2} t\cos\beta_n t\,.$$

We end up with the following integral equation for $e_n(t)$. Here $f(t)$ and $g(t)$ (not to be confused with $g_n(t)$) are differentiable functions, which can be explicitly computed, but we are not interested in their expressions. The important fact is that they do not depend on $n$:

$$\begin{aligned} e_n(t) + N''(0)\frac{\mu_n}{2\beta_n} e^{\alpha t} t \sin\beta_n t \;&= \frac{M_n(t)}{\beta_n^3} + \mu_n \frac{f(t)}{\beta_n^2} + \mu_n \frac{g(t)}{\beta_n^2}\cos\beta_n t \\ &- \int_0^t N'(t-r) e_n(r)\,\mathrm{d}r + \int_0^t N'(t-r)\left[1-\mu_n\right] e_n(r)\,\mathrm{d}r \\ &- \frac{1-\mu_n}{\beta_n}\int_0^t \left[e^{\alpha(t-r)}\sin\beta_n(t-r) - \int_0^{t-r} N_1(t-r-s) e^{\alpha s}\sin\beta_n s\,\mathrm{d}s\right] e_n(r)\,\mathrm{d}r\,. \end{aligned}$$

Note that

$$\left[1-\mu_n\right] = \frac{1}{\beta_n}\left[\beta_n\left(1-\mu_n\right)\right] \asymp \frac{1}{\beta_n}\,.$$

We introduce

$$\epsilon_n(t) = e_n(t) - N''(0)\frac{\mu_n}{2\beta_n} e^{\alpha t} t \sin\beta_n t\,.$$

A few manipulations give

$$\begin{aligned} \epsilon_n(t) = \frac{M_n(t)}{\beta_n^3} + \mu_n \frac{f(t)}{\beta_n^2} + \mu_n \frac{g(t)}{\beta_n^2}\cos\beta_n t \\ - \int_0^t N'(t-r)\epsilon_n(r)\,\mathrm{d}r + \frac{1}{\beta_n}\int_0^t G_n(t-r)\epsilon_n(r)\,\mathrm{d}r\,, \end{aligned}$$

where

$$G_n(t) = N'(t)\beta_n\left[1-\mu_n\right] - \mu_n \int_0^t N_1(t-s) e^{\alpha s}\sin\beta_n s\,\mathrm{d}s - \mu_n N''(0) e^{\alpha t}\sin\beta_n t$$

is bounded.

We recapitulate:

$$\epsilon_n(t) = \frac{M_n(t)}{\beta_n^3} + \mu_n \frac{f(t)}{\beta_n^2} + \mu_n \frac{g(t)}{\beta_n^2} \cos \beta_n t \\ - \int_0^t N'(t-r)\epsilon_n(r)\, \mathrm{d}r + \frac{1}{\beta_n} \int_0^t G_n(t-r)\epsilon_n(r)\, \mathrm{d}r\,. \quad (5.8)$$

Let $R(t)$ be the resolvent kernel of $-N'(t)$. Then, the resolvent kernel of the integral equation (5.8) has the form

$$R(t) + \frac{1}{\beta_n} F_n(t)\,,$$

where $\{F_n(t)\}$ is bounded with bounded derivatives.

Using the resolvent formula and further integration by parts we get

$$\epsilon_n(t) = \frac{M_n(t)}{\beta_n^3} + \mu_n \frac{f(t)}{\beta_n^2} + \frac{\mu_n g(t)}{\beta_n^2} \cos \beta_n t + \frac{\mu_n}{\beta_n^2} \int_0^t R(t-s) f(s)\, \mathrm{d}s\,,$$

which is the required formula with

$$F(t) = f(t) + \int_0^t R(t-s) f(s)\, \mathrm{d}s\,, \qquad G(t) = g(t)\,.$$

5.4. *Appendix: Riesz basis properties of the sequences* $\{\zeta_n(t)\}$ *and* $\{\hat{\zeta}_n(t)\}$. The sequence $\{\zeta_n(t)\}$ is quadratically close to an $\mathcal{L}$-basis and in this appendix we prove that it is an $\mathcal{L}$-basis too. For that we use the Bari Theorem. So, it is sufficient to prove the following result.

LEMMA 5.4. Let $T \geq 2\pi$. The sequence $\{\zeta_n(t)\}$ is $\omega$-independent in $L^2(0,T)$.

*Proof.* We adapt a similar proof in [19]. We recall that we are assuming that $\beta_n \neq 0$ for every $n$. If there exists $n_0$ such that $\beta_{n_0} = 0$, then the proof has to be modified, as in [19].

We use the fact that the sequence $\left\{1, \{e^{i\beta_n t}\}_{n\in\mathbb{Z}'}\right\}$ is an $\mathcal{L}$-basis in $L^2(0,T)$ for $T \geq 2\pi$.

Let us assume the existence of $\{\alpha_n\} \in l^2(\mathbb{C})$ such that

$$\sum_{n\in\mathbb{Z}'} \alpha_n \zeta_n(t) = 0 \qquad \text{in } L^2(0,T). \quad (5.9)$$

We define

$$\phi(t) = -\sum_{n\in\mathbb{Z}'} \alpha_n e^{i\beta_n t} = \sum_{n\in\mathbb{Z}'} \alpha_n \left[\zeta_n(t) - e^{i\beta_n t}\right]\,.$$

Note that

$$\phi \in X = \operatorname{cl}\operatorname{span}\left\{1, e^{i\beta_n t}\right\}_{n\in\mathbb{Z}'}\,. \quad (5.10)$$

The function $\phi(t)$ is square integrable, and we will prove that it is smoother: it belongs to $W^{2,2}(0,T)$. This will imply that if condition (5.9) holds, then the sequence $\{\alpha_n\}$ belongs to a weighted $l^2$ space.

Using (2.23)-(2.25) we see that

$$
\begin{aligned}
\phi(t) &= \sum_{n\in\mathbb{Z}'} \alpha_n \left[\beta_n - n + \frac{2\alpha^2}{n} + \alpha\frac{n-2\beta_n}{n} i\right] \frac{e^{-i\beta_n t}}{2\beta_n} \\
&- \sum_{n\in\mathbb{Z}'} \alpha_n \left[\beta_n - n + \frac{2\alpha^2}{n} + \alpha\frac{n+2\beta_n}{n} i\right] \frac{e^{i\beta_n t}}{2\beta_n} \\
&+ i \sum_{n\in\mathbb{Z}'} \frac{\alpha_n n}{\beta_n} \int_0^t e^{-\alpha s} \sin\beta_n(t-s) \int_0^s N''(s-r) z_n(r)\, \mathrm{d}r\ \mathrm{d}s \\
&- \sum_{n\in\mathbb{Z}'} \alpha_n \frac{n^2}{\beta_n^2} e^{-\alpha t} \int_0^t N'(t-r) z_n(r)\, \mathrm{d}r \\
&+ \sum_{n\in\mathbb{Z}'} \alpha_n \frac{n^2}{\beta_n^2} \int_0^t \cos\beta_n(t-s) e^{-\alpha s} \int_0^s N_0(s-r) z_n(r)\, \mathrm{d}r\ \mathrm{d}s \\
&= \boxed{\mathbf{1}} + \boxed{\mathbf{2}} + \boxed{\mathbf{3}} + \boxed{\mathbf{4}} + \boxed{\mathbf{5}}.
\end{aligned}
$$

We use the notation $\xrightarrow{D}$ to denote the action of computing the derivative termwise and we ignore inessential constants.

We prove that each series can be differentiated termwise. This is clear for the series $\boxed{\mathbf{1}}$ and $\boxed{\mathbf{2}}$, thanks to the factors $1/\beta_n$, because we know that $\{\alpha_n\} \in l^2$ and, as we noted,

$$
\beta_n \asymp n\,, \qquad n - \beta_n \asymp \frac{1}{\beta_n}\,,
$$

$$
\begin{aligned}
\boxed{\mathbf{3}} &\xrightarrow{D} \sum_{n\in\mathbb{Z}'} \alpha_n n \int_0^t e^{-\alpha s} \cos\beta_n(t-s) \int_0^s N''(s-r) z_n(r)\, \mathrm{d}r\ \mathrm{d}s \\
&= N''(0) \sum_{n\in\mathbb{Z}'} \frac{\alpha_n n}{\beta_n} \int_0^t e^{-\alpha s} \sin\beta_n(t-s) z_n(s)\, \mathrm{d}s \\
&+ \sum_{n\in\mathbb{Z}'} \frac{\alpha_n n}{\beta_n} \int_0^t e^{-\alpha s} \sin\beta_n(t-s) \left[\int_0^s \left(N'''(s-r) - \alpha N''(s-r)\right) z_n(r)\, \mathrm{d}r\right] \mathrm{d}s \\
&= N''(0) \sum_{n\in\mathbb{Z}'} \frac{\alpha_n n}{\beta_n} \int_0^t e^{-\alpha s} \sin\beta_n(t-s) [z_n(s) - e^{\alpha s}\cos\beta_n s]\, \mathrm{d}s \\
&+ N''(0) \sum_{n\in\mathbb{Z}'} \frac{\alpha_n n}{\beta_n} \int_0^t \sin\beta_n(t-s) \cos\beta_n s\, \mathrm{d}s \\
&+ \sum_{n\in\mathbb{Z}'} \frac{\alpha_n n}{\beta_n} \int_0^t e^{-\alpha s} \sin\beta_n(t-s) \left[\int_0^s \left(N'''(s-r) - \alpha N''(s-r)\right) z_n(r)\, \mathrm{d}r\right] \mathrm{d}s \\
&= \mathbf{S_1} + \mathbf{S_2} + \mathbf{S_3}\,.
\end{aligned}
$$

The series $\mathbf{S_1}$ converges uniformly (since its terms are of the order $\alpha_n/\beta_n$). Using $N(t) \in W^{4,2}$, we integrate by parts the integrals in the series $\mathbf{S_3}$ and we again get uniform convergence of the series.

The series $\mathbf{S_2}$ converges in $L^2(0,T)$ for every $T>0$. We have convergence, because

$$\sum_{n\in\mathbb{Z}'} \frac{\alpha_n n}{\beta_n} \int_0^t \sin\beta_n(t-s)\cos\beta_n s \,\mathrm{d}s$$
$$= \sum_{n\in\mathbb{Z}'} \frac{\alpha_n n}{2\beta_n} \left[ t\sin\beta_n t + \int_0^t \sin\beta_n(t-2s)\,\mathrm{d}s \right],$$
$$\boxed{4} \xrightarrow{D} e^{-\alpha t} \sum_{n\in\mathbb{Z}'} \frac{\alpha_n n^2}{\beta_n^2} \int_0^t \left[N''(t-r)-\alpha N'(t-r)\right] z_n(r)\,\mathrm{d}r$$
$$= e^{-\alpha t} \sum_{n\in\mathbb{Z}'} \frac{\alpha_n n^2}{\beta_n^2} \int_0^t \left[N''(t-r)-\alpha N'(t-r)\right] \left(z_n(r)-e^{\alpha r}\cos\beta_n r\right)\,\mathrm{d}r$$
$$+\, e^{-\alpha t} \sum_{n\in\mathbb{Z}'} \frac{\alpha_n n^2}{\beta_n^2} \int_0^t \left[N''(t-r)-\alpha N'(t-r)\right] e^{\alpha r}\cos\beta_n r\,\mathrm{d}r$$

and both the series converge uniformly (integrate the last series by parts).

In order to simplify the formulas below we put

$$N_1(t) = N_0'(t) - \alpha N_0(t)\,.$$

We compute:

$$\boxed{5} = \sum_{n\in\mathbb{Z}'} \frac{\alpha_n n^2}{\beta_n^2} \int_0^t e^{-\alpha(t-s)}\cos\beta_n s \int_0^{t-s} N_0(t-s-r) z_n(r)\,\mathrm{d}r\,\mathrm{d}s \xrightarrow{D}$$
$$\sum_{n\in\mathbb{Z}'} \frac{\alpha_n n^2}{\beta_n^2} \int_0^t e^{-\alpha(t-s)}\cos\beta_n s \int_0^{t-s} N_1(t-s-r) z_n(r)\,\mathrm{d}r\,\mathrm{d}s$$
$$+N''(0) \sum_{n\in\mathbb{Z}'} \frac{\alpha_n n^2}{\beta_n^2} \int_0^t e^{-\alpha s}\cos\beta_n(t-s)\left[z_n(s)-e^{\alpha s}\cos\beta_n s\right]\,\mathrm{d}s$$
$$+N''(0) \sum_{n\in\mathbb{Z}'} \frac{\alpha_n n^2}{\beta_n^2} \int_0^t \cos\beta_n s\cos\beta_n(t-s)\,\mathrm{d}s\,.$$

Note that the last integral can be computed explicitly:

$$N''(0) \sum_{n\in\mathbb{Z}'} \frac{\alpha_n n^2}{\beta_n^2} \int_0^t \cos\beta_n s\cos\beta_n(t-s)\,\mathrm{d}s$$
$$= N''(0) \sum_{n\in\mathbb{Z}'} \frac{\alpha_n n^2}{2\beta_n^2} \left[ t\cos\beta_n t + \frac{1}{\beta_n}\sin\beta_n t \right].$$

This shows the convergence of the series at the last line while the convergence of the series at the previous lines can be proved as in the previous cases.

In conclusion, the function $\phi$ belongs to $W^{1,2}(0,T)$ for every $T$. Its incremental quotient is

$$\frac{\phi(t+h)-\phi(t)}{h} = \frac{1}{h}\left[\sum_{n\in\mathbb{Z}'} \alpha_n e^{i\beta_n t}\left[e^{i\beta_n h}-1\right]\right] \in X$$

so that also $\phi' \in X$ (the space $X$ is defined in (5.10)):

$$\phi'(t) = \sum_{n\in\mathbb{Z}'} \tilde{\gamma}_n e^{i\beta_n t}\,, \qquad \phi(t) = -\sum_{n\in\mathbb{Z}'} \frac{i\tilde{\gamma}_n}{\beta_n} e^{i\beta_n t} + \phi(0) + \sum_{n\in\mathbb{Z}'} \frac{i\tilde{\gamma}_n}{\beta_n}\,.$$

Our choice of $T$ implies that the representation of $\phi$ is unique. As $\phi \in X$, it follows that

$$\alpha_n = \frac{\gamma_n}{\beta_n}\,, \quad \phi(0) = -\sum_{n\in\mathbb{Z}'} \frac{\gamma_n}{\beta_n}, \qquad \gamma_n = i\tilde{\gamma}_n\,,$$

and

$$\{\gamma_n\} \in l^2\,.$$

Now we insert $\alpha_n$ of this form to (5.9) and use the computation of the first derivatives in order to see that $\phi$ is of the class $W^{2,2}$.

We don't need to collect all the terms, so we confine ourselves to inserting this representation of $\alpha_n$ into the series $\boxed{\mathbf{1}}$–$\boxed{\mathbf{5}}$. We demonstrate that the second derivative of each one of these series can be computed termwise, and it is $L^2$-convergent. In fact, this is clear for the series $\boxed{\mathbf{1}}$ and $\boxed{\mathbf{2}}$. We consider now the series $\boxed{\mathbf{3}}$ whose first derivative is the sum of the three series $\mathbf{S_1}$, $\mathbf{S_2}$ and $\mathbf{S_3}$. We compute the derivative of each one:

$$\mathbf{S_1} \xrightarrow{D} \sum_{n\in\mathbb{Z}'} \gamma_n \frac{n}{\beta_n} \int_0^t e^{-\alpha s} \cos\beta_n(t-s)[z_n(s) - e^{\alpha s}\cos\beta_n s]\,\mathrm{d}s$$

converges uniformly, and

$$\begin{aligned}\mathbf{S_2} &\xrightarrow{D} \sum_{n\in\mathbb{Z}'} \gamma_n \frac{n}{\beta_n} \int_0^t \cos\beta_n(t-s)\cos\beta_n s\,\mathrm{d}s \\ &= \sum_{n\in\mathbb{Z}'} \gamma_n \frac{n}{2\beta_n} \left\{ t\cos\beta_n t + \int_0^t \cos\beta_n(t-2s)\,\mathrm{d}s \right\}\,.\end{aligned}$$

This series converges in $L^2$.

Finally,

$$\mathbf{S_3} \xrightarrow{D} \sum_{n\in\mathbb{Z}'} \gamma_n \frac{n}{\beta_n} \int_0^t e^{-\alpha s}\cos\beta_n(t-s) \int_0^s N_0'(s-r) z_n(r)\,\mathrm{d}r\ \mathrm{d}s\,.$$

Using $N \in W^{4,2}$, integration by parts shows that this series converges uniformly.

This shows that the second derivative of $\boxed{\mathbf{3}}$ is square integrable.

Ignoring the factors $e^{-\alpha t}$, we consider the second derivative of $\boxed{\mathbf{4}}$. We get

$$N_0(0) \sum_{n\in\mathbb{Z}'} \frac{\gamma_n n^2}{\beta_n^3} z_n(t) + \int_0^t N_0'(t-r) \sum_{n\in\mathbb{Z}'} \frac{\gamma_n n^2}{\beta_n^3} z_n(r)\,\mathrm{d}r\,.$$

Both of the series in this expression converge uniformly.

The second derivative of $\boxed{5}$ is computed analogously. It gives several uniformly convergent series (due to the assumption $N \in W^{4,2}$) and the following one which is $L^2$-convergent:

$$\sum_{n\in\mathbb{Z}'} \gamma_n \frac{n^2}{\beta_n^2} \int_0^t \cos\beta_n s \sin\beta_n(t-s)\,\mathrm{d}s\,.$$

Analogously to the above, we see that $\phi''(t) \in X$ and

$$\alpha_n = \frac{\sigma_n}{\beta_n^2}\,, \qquad \{\sigma_n\} \in l^2(\mathbb{C})\,.$$

With this last piece of information we consider again the equality

$$0 = \sum_{n\in\mathbb{Z}'} \alpha_n \zeta_n(t) = \sum_{n\in\mathbb{Z}'} \frac{\sigma_n}{\beta_n^2}\zeta_n(t)\,. \tag{5.11}$$

Using (2.22), we see that we can compute the derivative of this series termwise and get

$$0 = 2\alpha \sum_{n\in\mathbb{Z}'} \frac{\sigma_n}{\beta_n^2}\zeta_n(t) - \left[\int_0^t M(t-s) \sum_{n\in\mathbb{Z}'} \frac{\sigma_n}{\beta_n^2} n^2 \zeta_n(s)\,\mathrm{d}s - iM(t) \sum_{n\in\mathbb{Z}'} n\frac{\sigma_n}{\beta_n^2}\right],$$

so that

$$-\int_0^t M(t-s)\left[\sum_{n\in\mathbb{Z}'} \frac{\sigma_n}{\beta_n^2} n^2 \zeta_n(s)\right]\mathrm{d}s = iM(t) \sum_{n\in\mathbb{Z}'} \frac{\sigma_n}{\beta_n^2} n\,.$$

Note that the exchange of the series and integral is justified, since $\{\zeta_n(t)\}_{n>N}$ is an $\mathcal{L}$-basis if $N$ is large enough and the numerical series on the right-hand side converges. So, the previous equality holds termwise and, computing at $t = 0$, we see that

$$\sum_{n\in\mathbb{Z}'} \frac{\sigma_n}{\beta_n^2} n = 0\,.$$

In fact, $M(0) = 1$; see (2.22).

Hence we also have the equality

$$\sum_{n\in\mathbb{Z}'} \frac{\sigma_n}{\beta_n^2} n^2 \zeta_n(t) = 0\,.$$

Comparing with (5.11), we see that for a suitable sequence $\{\alpha_n^{(1)}\} \in l^2(\mathbb{C})$ we have

$$\sum_{n\in\mathbb{Z}, n\neq 1} \alpha_n^{(1)} \zeta_n(t) = 0\,,$$

and for $n \neq 1$ we have $\alpha_n^{(1)} = 0$ if and only if $\alpha_n = 0$. So, we can iterate this process, as in [19], and we see that only finitely many coefficients $\alpha_n$ can possibly be different from zero. In fact, as we noted, $\{\zeta_n(t)\}_{|n|>N}$ is an $\mathcal{L}$-basis if $N$ is large enough. We conclude that the sum in (5.9) is finite:

$$\sum_{n=-N}^{N} \alpha_n \zeta_n(t) = 0\,.$$

As we shall see in the next lemma, the sequence $\{\zeta_n(t)\}$ is linearly independent and this proves that *every* coefficient $\alpha_n$ must be zero, as wanted. □

So, in order to complete the proof of Lemma 5.4, we now prove:

LEMMA 5.5. For any $N \in \mathbb{N}$, the sequence $\{\zeta_n(t)\}_{|n|\leq N}$ is linearly independent.

*Proof.* If this sequence is linearly dependent, then there exists a minimal $K$ such that $\zeta_K(t)$ is a linear combination of the previous $\zeta_n(t)$ so that

$$\sum_{n=-N}^{K} \alpha_n \zeta_n(t) = 0\,, \qquad \alpha_K \neq 0\,. \tag{5.12}$$

From (2.22) it follows that

$$\int_0^t M(t-s) \left[ \sum_{n=-N}^{K} \alpha_n n^2 \zeta_n(s) \right] \mathrm{d}s = i \left( \sum_{n=-N}^{K} \alpha_n n \right) M(t)\,.$$

We conclude that

$$\sum_{n=-N}^{K} \alpha_n n = 0$$

and so also

$$\sum_{n=-N}^{K} \alpha_n n^2 \zeta_n(t) = 0\,.$$

Comparing this equality with (5.12) we see that

$$\sum_{n=-N}^{K-1} (K^2 - n^2) \alpha_n \zeta_n(t) = 0\,,$$

and this contradicts the definition of $K$. □

This ends the proof of Lemma 5.4.

Lemma 5.4, Lemma 2.10 and the Bari Theorem imply Theorem 4.1, as wanted.

5.5. *Appendix: a more general case.* Now we consider the general case

$$\begin{aligned} &\theta_t(x,t) = 2\alpha + \int_0^t N(t-s)\theta_{xx}(x,s)\,\mathrm{d}s\,, \\ &x \in (0,l)\,, \qquad \theta(0,t) = u(t)\,, \quad \theta(l,t) = 0\,. \end{aligned}$$

We don't assume $N(0) = 1$ or $l = \pi$.

The results we have proved in this paper also hold in this more general setting, with the simple modifications we explain now. First we transform the interval $[0,l]$ to $[0,\pi]$. This is done by the substitution

$$x \to \frac{l}{\pi} x\,.$$

This substitution transforms $N(t)$ to $\pi^2 N(t)/l^2 > 0$.

In the general case we shall have $N(0) > 0$, and this condition is preserved by the substitution. The transformation which reduces to $N'(0) = 0$ can now be made. All the computations in this paper can be repeated, inserting the multiplicative constants $l^2 N(0)/\pi^2$ at the appropriate places in the formulas taken from [19]. In fact, (2.4) is replaced by

$$z_n''(t) = 2\alpha z_n'(t) - n^2 \frac{\pi^2}{l^2} N(0) z_n(t) - n^2 \frac{\pi^2}{l^2} \int_0^t N'(t-s) z_n(s)\,\mathrm{d}s\,.$$

So, now

$$\beta_n = \sqrt{N(0)\frac{\pi^2}{l^2}n^2 - \alpha^2} \asymp \left[\frac{\pi}{l}\sqrt{N(0)}\right] n\,.$$

The sequence $\left\{e^{\pm i\beta_n t}\right\}$ is an $\mathcal{L}$-basis in $L^2(0,T)$, $T \geq 2l/\sqrt{N(0)}$, which is the minimal controllability time.

**Acknowledgment.** We thank M. Grasselli for bringing paper [22] to our attention.

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