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ON THE ILL-POSEDNESS FOR A NONLINEAR SCHRÖDINGER-AIRY EQUATION

By

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Abstract. Using ideas of Kenig, Ponce and Vega and an explicit solution with two parameters, we prove that the solution map of the initial value problem for a particular nonlinear Schrödinger-Airy equation fails to be uniformly continuous.

Also, we will approximate the solution to the nonlinear Schrödinger-Airy equation by the solution to the cubic nonlinear Schrödinger equation and prove ill-posedness in a more general case than above. This method was originally introduced by Christ, Colliander and Tao for the modified Korteweg-de Vries equation.

Finally, we consider the general case and we prove ill-posedness for all values of the parameters in the equation.

1. Introduction. In this paper we will describe some results on ill-posedness for solutions of the initial value problem (IVP)

$$\begin{cases} \partial_t u + ia\,\partial_x^2 u + b\,\partial_x^3 u + ic\,|u|^2 u + d\,|u|^2 \partial_x u + e\,u^2 \partial_x \bar{u} = 0, \quad x, t \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where u is a complex-valued function and a, b, c, d and e are real parameters.

This model was proposed by Hasegawa and Kodama in [15, 20] to describe the nonlinear propagation of pulses in optical fibers. In the literature, this model is referred to as a higher-order nonlinear Schrödinger equation or also as an Airy-Schrödinger equation.

We consider the following gauge transformation:

$$v(x,t) = \exp\Big(i\lambda\, x + i(a\,\lambda^2 - 2b\,\lambda^3)t\Big)\, u(x + (2a\,\lambda - 3b\lambda^2)t, t). \tag{1.2}$$

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Then, u solves (1.1) if and only if v satisfies the IVP

$$\begin{cases} \partial_t v + i(a - 3\lambda\, b)\partial_x^2 v + b\,\partial_x^3 v + i(c - \lambda(d-e))\,|v|^2 v + d\,|v|^2\partial_x v + e\, v^2 \partial_x \bar{v} = 0, \\ v(x,0) = \exp(i\lambda\, x)\, u(x,0). \end{cases} \tag{1.3}$$

Thus, if we take $\lambda = a/(3b)$ in (1.2) and $c = (d-e)a/(3b)$, the function

$$v(x,t) = \exp\left(i\frac{a}{3b}\,x + i\frac{a^3}{27b^2}\,t\right) u\left(x + \frac{a^2}{3b}\,t, t\right) \tag{1.4}$$

satisfies the complex modified Korteweg-de Vries type equation (complex mKdV)

$$\begin{cases} \partial_t v + b\,\partial_x^3 v + d\,|v|^2\partial_x v + e\, v^2\partial_x \bar{v} = 0, \\ v(x,0) = \exp(ia\,x/3b)\, u(x,0). \end{cases} \tag{1.5}$$

We say that the IVP (1.1) is locally well-posed in X (Banach space) if the solution uniquely exists in a certain time interval $[-T,T]$ (unique existence), the solution describes a continuous curve in X in the interval $[-T,T]$ whenever the initial data belong to X (persistence), and the solution varies continuously depending upon the initial data (continuous dependence); i.e., we have continuity of application $u_0 \mapsto u(t)$ from X to $\mathcal{C}([-T,T];X)$. We say that the IVP (1.1) is globally well-posed in X if the same properties hold for all time $T > 0$. If some hypotheses in the definition of locally well-posed fail, we say that the IVP is ill-posed.

Particular cases of (1.1) are the following:

• Cubic nonlinear Schrödinger equation (NLS) ($a = \mp 1$, $b = 0$, $c = -1$, $d = e = 0$):

$$iu_t \pm u_{xx} + |u|^2 u = 0, \quad x,t \in \mathbb{R}. \tag{1.6}$$

The best known local result for the IVP associated to (1.6) in $H^s(\mathbb{R})$ was obtained by Tsutsumi [33] for $s \geq 0$. Using the fact that the L^2-norm of the solution of (1.6) is preserved, he has that (1.6) is globally well-posed in $H^s(\mathbb{R})$, $s \geq 0$.

• Nonlinear Schrödinger equation with derivative ($a = -1$, $b = 0$, $c = 0$, $d = 2e$):

$$iu_t + u_{xx} + i\lambda(|u|^2 u)_x = 0, \quad x,t \in \mathbb{R}. \tag{1.7}$$

The best known local result for the IVP associated to (1.7) in $H^s(\mathbb{R})$, was obtained by Takaoka [32] for $s \geq 1/2$. Colliander et al. [12] proved that (1.7) is globally well-posed in $H^s(\mathbb{R})$, $s > 1/2$.

• A particular case of the complex mKdV equation (1.5) ($a = 0$, $b = 1$, $c = 0$, $d = 1$, $e = 0$):

$$u_t + u_{xxx} + |u|^2 u_x = 0, \quad x,t \in \mathbb{R}. \tag{1.8}$$

If u is real, (1.8) is the usual mKdV equation. Kenig et al. [17] proved that the IVP associated to it is locally well-posed in $H^s(\mathbb{R})$, $s \geq 1/4$, and Colliander et al. [13] proved that (1.8) is globally well-posed in $H^s(\mathbb{R})$, $s > 1/4$.

• When $a \neq 0$ and $b = 0$, we obtain a particular case of the well-known mixed nonlinear Schrödinger equation:

$$u_t = iau_{xx} + \lambda(|u|^2)_x u + g(u), \quad x,t \in \mathbb{R}, \tag{1.9}$$

where g satisfies some appropriate conditions and $\lambda \in \mathbb{R}$ is a constant. Ozawa and Tsutsumi in [24] proved that for any $\rho > 0$, there is a positive constant $T(\rho)$ depending only on ρ and g, such that the IVP (1.9) is locally well-posed in $H^{1/2}(\mathbb{R})$, whenever the initial data satisfies

$$\|u_0\|_{\mathrm{H}^{1/2}} \leq \rho.$$

There are other dispersive models similar to (1.1); see for instance [1, 11, 25, 26, 29] and the references therein.

Regarding the IVP (1.1) with $be \neq 0$ or $bd \neq 0$, Laurey in [21] showed that the IVP is locally well-posed in $H^s(\mathbb{R})$ with $s > 3/4$, and using the quantities conserved she obtained the global well-posedness in $H^s(\mathbb{R})$ with $s \geq 1$. In [28], Staffilani established the local well-posedness in $H^s(\mathbb{R})$ with $s \geq 1/4$, improving Laurey's result. Recently, H. Wang [35] proved global well-posedness in $H^s(\mathbb{R})$ for $s > 6/7$. Carvajal in [6] obtained a sharp global well-posedness result in $H^s(\mathbb{R})$ with $s > 1/4$, for the IVP (1.1), with the condition $c = a(d-e)/(3b)$. It was proved using the theory of almost conserved energies and the "I-method" developed by Colliander, Keel, Staffilani, Takaoka and Tao.

In [7], Carvajal and Linares considered the IVP (1.1) when a, b are real functions of t and they proved the local well-posedness in $H^s(\mathbb{R})$, $s \geq 1/4$. Also, Carvajal and Panthee in [8, 9] gave a positive answer for the unique continuation property for the solution of (1.1).

Below we describe the results concerning the ill-posedness for the equation (1.1). Firstly we will use an explicit solution of the equation (1.1) (see [6]) for particular choices of the parameters to show that in fact the local well-posedness result obtained in [28] is the best possible one in Sobolev spaces.

Secondly some extensions of previous work done by Christ, Colliander and Tao are given, the method is very similar to those of Christ et al., and the technique is to approximate the complex mKdV solution by the cubic NLS solution.

Finally we will establish a weaker result for solutions of the IVP (1.1) for a greater range of parameters. This result is in the same spirit of Bourgain [4], refined by Takaoka [32] and Tzvetkov [22] for the IVPs associated to the Korteweg-de Vries, derivative nonlinear Schrödinger and Benjamin Onno equations, respectively.

THEOREM 1.1. If we let $e = 0$, $b \cdot d > 0$, and $c = ad/(3b)$ in (1.1) the map data–solution associated to the IVP (1.1) is not uniformly continuous for any data $u_0 \in H^s(\mathbb{R})$, $s < 1/4$.

The proof of this result follows the ideas introduced by Kenig, Ponce and Vega in [18] (see also [2]).

THEOREM 1.2. Let $-1/4 < s < 1/4$. The solution map of the initial value problem (1.1) with $c = a(d-e)/(3b)$, $be \neq 0$ or $db \neq 0$ fails to be uniformly continuous; more precisely, for $0 < \delta \ll \epsilon \ll 1$ and $T > 0$ arbitrary, there are two solutions u, v to (1.1) such that

$$\|u(0)\|_{H^s}, \|v(0)\|_{H^s} \lesssim \epsilon, \tag{1.10}$$

$$\|u(0) - v(0)\|_{H^s} \lesssim \delta, \tag{1.11}$$

$$\sup_{0 \leq t \leq T} \|u(t) - v(t)\|_{H^s} \gtrsim \epsilon. \tag{1.12}$$

The method used here is very similar to theirs in [10] (see also [27]).

A further new result can be obtained for a large range of parameters if we ask the map data–solution to be at least of class C^3.

Our result says:

THEOREM 1.3. Let $T > 0$ and V be a bounded neighborhood of zero in $H^s(\mathbb{R})$. Then for the IVP (1.1), the map data–solution

$$\mathbb{F}: \; u_0 \to u(t), \qquad t \in [-T, T]$$

is not C^3 at zero from $H^s(\mathbb{R})$ into $C([-T,T], H^s(\mathbb{R}))$ in the following cases:

(i) $bd \neq 0$ or $be \neq 0$ and $s < 1/4$.

(ii) $b = 0$, $a \neq 0$, $d = e = 0$, $c \neq 0$ and $s < 0$.

(iii) $b = 0$, $a \neq 0$, $d \neq 0$ or $e \neq 0$ and $s < 1/2$.

(iv) $b \neq 0$, $d = e = 0$, $c \neq 0$ and $s < -1/4$.

REMARK 1.4. The case (i) contains the complex mKdV equation ($a = c = 0$). The IVP associated to this equation was proved to be locally well-posed in H^s, $s \geq 1/4$, and ill-posed otherwise by Kenig, Ponce and Vega in [17], [18]. A particular case of (ii) is the cubic nonlinear Schrödinger equation. Stronger results were obtained for the focusing case by Kenig, Ponce and Vega [18] and the defocusing one by Christ, Colliander and Tao [10]. The derivative nonlinear Schrödinger equation ($b = 0$, $d = 2$, $e = 1$) is a particular case of (iii). The result for this equation agrees with the stronger one given in [2].

In the case (iv), local well-posedness was proved in $H^s(\mathbb{R}), s > -1/4$; see [5].

2. Sharpness of the local results for the mKdV equation. We begin this section with the result due to Staffilani [28] regarding local well-posedness for the IVP (1.1). It reads as follows:

THEOREM 2.1. Let $u_0 \in H^s(\mathbb{R})$, $s \geq 1/4$, and $a, b \in \mathbb{R}$, $be \neq 0$ or $bd \neq 0$, $c, d, e \in \mathbb{C}$. Then there exist

$$T = C \min\left\{\|u_0\|_{H^{1/4}}^{-4}, \|u_0\|_{H^{1/4}}^{-8/3}\right\} \tag{2.13}$$

and a unique solution of the IVP (1.1), such that

$$\begin{gathered} u \in C([0,T] : H^s(\mathbb{R})), \\ \|\partial_x u\|_{L_x^\infty L_T^2} + \|D_x^s \partial_x u\|_{L_x^\infty L_T^2} < \infty, \\ \|D_x^{s-1/4} \partial_x u\|_{L_x^{20} L_T^{5/2}} < \infty, \\ \|u\|_{L_x^5 L_T^{10}} + \|D_x^s u\|_{L_x^5 L_T^{10}} < \infty, \\ \|u\|_{L_x^4 L_T^\infty} < \infty, \end{gathered} \tag{2.14}$$

and

$$\|u\|_{L_x^8 L_T^8} + \|D_x^s u\|_{L_x^8 L_T^8} < \infty. \tag{2.15}$$

Moreover, for any $T' \in [0,T]$ there exists a neighborhood $\mathcal{V}$ of $u_0 \in H^s(\mathbb{R})$, $s \geq 1/4$, such that the map $\widetilde{u}_0 \to \widetilde{u}(t)$, from $\mathcal{V}$ into the class defined by (2.14)–(2.15), with T' instead of T is smooth.

REMARK 2.2. Observe that we have added the norms in (2.15) to the original set of norms in [28] to avoid further difficulties.

In this section we will discuss the sharpness of the local results obtained in Theorem 2.1.

The scaling argument does not work in the IVP (1.1), in this case due to the inhomogeneous character of the symbol associated to the linear problem. However, the dominant structure of the equation similar to the mKdV equation leads us to think that the best result is the one given in [28].

In this regard, we will first show that the map data–solution for particular choices of the constants in the equation (1.1) fails to be uniformly continuous for data in $H^s(\mathbb{R})$, $s < 1/4$.

2.1. *Proof of Theorem* 1.1 *(Kenig, Ponce and Vega Method).* The idea of the proof is similar to the one introduced in [18]. More precisely, the equation in (1.1) with the restrictions

$$e = 0, \quad d \neq 0, \quad bd > 0 \text{ and } c = \left(\frac{ad}{3b}\right) \tag{2.16}$$

has a two-parameter family of solutions (see [6]) given by

$$u_{\eta,\omega}(x,t) = f_\eta(x + \psi(\eta,\omega)t)\, \exp\{i(\omega x + \phi(\eta,\omega)t)\}, \tag{2.17}$$

where

$$f_\eta(x) = \eta\, f(\eta x) \quad \text{with} \quad f(x) = (A\cosh x)^{-1},$$
$$\psi(\eta,\omega) = 2a\omega + 3b\omega^2 - \eta^2 b \quad \text{and} \quad \phi(\eta,\omega) = a\omega^2 + b\omega^3 - 3b\eta^2\omega - a\eta^2,$$

and $A = \sqrt{d/(6b)}$.

Then a suitable choice of the parameters η and ω will allow us to exhibit data that remain close in the H^s–norm but the difference of the corresponding solutions will fall apart at any time $T > 0$.

Let $N > 0$ be sufficiently large, $\eta = N^{-2s}$, $\omega_1 = N_1 \sim N$, $\omega_2 = N_2 \sim N$, $\delta > 0$ and $-1 < 2s < 1/2$. The Fourier transform of $u_{\eta,\, N_j}$ is given by

$$\widehat{u}_{\eta,N_j}(\xi,t) = \Theta(\eta,N_j,t)\, \widehat{f_\eta}(\xi - N_j)\, \exp(i\psi(\eta,t)t\,\xi), \quad j = 1,2,$$

where $\Theta(\eta,N_j,t) = \exp\big(i\phi(\eta,N_j)t - i\psi(\eta,N_j)tN_j\big)$.

Notice that $\widehat{f_\eta}(\xi - N_j) = \widehat{f}((\xi - N_j)\eta^{-1})$ is concentrated in the ball $B_\eta(N_j)$. Thus for $\xi \in B_\eta(N_j)$ we have $|\xi| \sim N$ since $s > -1/2$. Hence

$$\begin{aligned} \|u_{\eta,N_1}(0) - u_{\eta,N_2}(0)\|_{H^s}^2 &\lesssim N^{2s} \int_{\mathbb{R}} |\widehat{f_\eta}(\xi - N_1) - \widehat{f_\eta}(\xi - N_2)|^2\, d\xi \\ &\lesssim \frac{N^{2s}|N_1 - N_2|^2}{\eta} \\ &= (N^{2s}(N_1 - N_2))^2. \end{aligned} \tag{2.18}$$

On the other hand, at any time T, we have

$$\|u_{\eta,N_1}(t) - u_{\eta,N_2}(t)\|_{H^s}^2 \sim N^{2s}\|u_{\eta,N_1}(t) - u_{\eta,N_2}(t)\|_{L^2}^2.$$

The solutions $u_{\eta,N_j}(t)$, $j=1,2$, are concentrated in the ball $B_{\eta^{-1}}(-\psi(\eta,N_j)t)$, where $\psi(\eta,N)=2aN+3b\,N^2-\eta^2\,b$. Hence, if

$$|\psi(\eta,N_1)t-\psi(\eta,N_2)t|\gg\eta^{-1},$$

then

$$\|u_{\eta,N_1}(t)-u_{\eta,N_2}(t)\|_{L^2}^2\gtrsim\|u_{\eta,N_1}(t)\|_{L^2}^2+\|u_{\eta,N_2}(t)\|_{L^2}^2\sim N^{-2s}. \tag{2.19}$$

But

$$\begin{aligned}|\psi(\eta,N_1)-\psi(\eta,N_2)|&=|N_1-N_2|\,|2a+3b(N_1+N_2)|\\&\gtrsim\;|N_1-N_2|\,N.\end{aligned} \tag{2.20}$$

Thus to have (2.19) it is enough to satisfy

$$T\,|N_1-N_2|\,N\gg\eta^{-1}=N^{2s}. \tag{2.21}$$

Therefore choosing $N_1=N$ and $N_2=N-\delta\eta$, it follows from (2.18) that

$$\|u_{\eta,N_1}(0)-u_{\eta,N_2}(0)\|_{H^s}\lesssim\delta,$$

since $-1<2s<1/2$. We also obtain (2.21) if $T\gg N^{-1+4s}$ and so (2.19) follows. This completes the proof.

2.2. *Christ, Colliander and Tao Method.* Now in order to prove Theorem 1.2, as $c=a(d-e)/(3b)$, using the gauge transformation (1.3) we will consider the problem

$$\begin{cases}\partial_t u+b\,\partial_x^3u+d\,|u|^2\partial_x u+e\,u^2\partial_x\bar{u}=0, & x,t\in\mathbb{R},\\ u(x,0)=u_0(x).\end{cases} \tag{2.22}$$

For simplicity we pretend the nonlinear term is

$$F(u)=e\,u^2\partial_x\bar{u}.$$

The general case $F(u)=d\,|u|^2\partial_x u+e\,u^2\partial_x\bar{u}$ follows in the same manner. Our method is very close with [10]. For the sake of completeness we present it here; the method is to approximate the complex mKdV solution by the cubic NLS solution

$$\partial_t u+i\,\partial_x^2u+i\,\mathrm{signal}\,\{e\}\,|u|^2u=0.$$

We suppose $b=e=1$; the general case is similar.

Let $u(t,x)$ be the solution to the linear problem $\partial_t u+\,\partial_x^3u=0$ with $u(0)=u_0$.

In a similar way as in [10, 27] we obtain that

$$u(t,x)\approx\frac{e^{itN^3+iNx}}{\sqrt{3N}}v(t,\frac{x}{\sqrt{3N}}+\sqrt{3N}Nt),$$

where $v(t,x)$ is a solution to the linear Schrödinger equation $\partial_t v+i\,\partial_x^2v=0$.

In order to prove Theorem 1.2 we need the following results proved in [10].

THEOREM 2.3. Let $s<0$. The solution map of the initial value problem of the cubic NLS

$$\partial_t u+i\,\partial_x^2u=0 \tag{2.23}$$

fails to be uniformly continuous. More precisely, for $0 < \delta \ll \epsilon \ll 1$ and $T > 0$ arbitrary, there are two solutions u_1, u_2 to (2.23) satisfying (1.10), (1.11) and (1.12). Moreover, for any fixed $K \geq 1$, we can find such solutions to satisfy

$$\sup_{0 \leq t \leq \infty} \|u_j\|_{H^K} \lesssim \epsilon,$$

for $j = 1, 2$.

LEMMA 2.4 ([3, Lemma 2.1]). Let $-1/2 < s$, $\sigma \in \mathbb{R}^+$ and $u \in H^\sigma(\mathbb{R})$. For any $M \geq 1$, $\tau \in \mathbb{R}^+$, $x_0 \in \mathbb{R}$, and $A > 0$ let

$$v(x) = Ae^{iMx} u((x - x_0)/\tau).$$

(i) Suppose $s \geq 0$. Then there exists a constant $C_1 < \infty$, depending only on s, such that whenever $M.\tau \geq 1$,

$$\|v\|_{H^s} \leq C_1 |A| \tau^{1/2} M^s \|u\|_{H^s}$$

for all u, A, x_0.
(ii) Suppose that $s < 0$ and that $\sigma \geq |s|$. Then there exists a constant $C_1 < \infty$, depending only on s and on σ, such that whenever $M^{1+s/\sigma}.\tau \geq 1$,

$$\|v\|_{H^s} \leq C_1 |A| \tau^{1/2} M^s \|u\|_{H^\sigma}$$

for all u, A, x_0.
(iii) There exists $c_1 > 0$ such that for each u there exist $C_u < \infty$ such that

$$\|v\|_{H^s} \geq c_1 |A| \tau^{1/2} M^s \|u\|_{L^2}$$

whenever $\tau.M \geq C_u$.

Let $u(s, y)$ be a solution of the NLS equation (2.23). Using the change of variables

$$(s, y) = (t, \frac{x}{\sqrt{3N}} + \sqrt{3N} N t),$$

we define the approximate solution

$$U_{ap}(t, x) := \frac{2}{\sqrt{N}} \operatorname{Re} e^{itN^3 + iNx} u(s, y). \tag{2.24}$$

Differentiating we have

$$(\partial_t + \partial_x^3) U_{ap} = \frac{2}{\sqrt{N}} \operatorname{Re} e^{itN^3 + iNx} (u_s(s, y) + i u_{yy}(s, y) + \frac{1}{3N\sqrt{3N}} u_{yyy}(s, y))$$

and

$$U_{ap}^2 \partial_x \overline{U_{ap}} = \frac{1}{4} \left(\frac{2}{\sqrt{N}} \right)^3 \operatorname{Re} e^{3iN^3 t} e^{3iNx} \left(3iNu^3 + \frac{1}{\sqrt{3N}} \partial_y (u^3) \right) \tag{2.25}$$

$$+ \frac{3}{4} \left(\frac{2}{\sqrt{N}} \right)^3 \operatorname{Re} e^{iN^3 t} e^{iNx} \left(iN|u|^2 u + \frac{1}{\sqrt{3N}} \partial_y (|u|^2 u) \right), \tag{2.26}$$

where we used that $U_{ap}^2 \partial_x \overline{U_{ap}} = (1/3) \partial_x (U_{ap}^3)$ (U_{ap} is real) and the identity $\left(\operatorname{Re} e^{i\theta} u\right)^3 = (1/4) \operatorname{Re} e^{3i\theta} u^3 + (3/4) \operatorname{Re} e^{i\theta} |u|^2 u$.

Thus

$$\{\partial_t + \partial_{xxx}\} U_{ap} + U_{ap}^2 \partial_x \overline{U_{ap}} = \operatorname{Re} E_1 + \operatorname{Re} E_2 + \operatorname{Re} E_3 + \operatorname{Re} E_4 \tag{2.27}$$

and

$$E_1=\frac{2}{\sqrt{N}}\,e^{itN^3+iNx}\frac{1}{3N\sqrt{3N}}u_{yyy},\quad E_2=\frac{1}{4}\left(\frac{2}{\sqrt{N}}\right)^3\,e^{3iN^3t}e^{3iNx}\frac{1}{\sqrt{3N}}\partial_y(u^3),\tag{2.28}$$

$$E_3=\frac{3}{4}\left(\frac{2}{\sqrt{N}}\right)^3\,e^{iN^3t}e^{iNx}\frac{1}{\sqrt{3N}}\partial_y(|u|^2u),\quad E_4=\frac{1}{4}\left(\frac{2}{\sqrt{N}}\right)^3\,e^{3iN^3t}e^{3iNx}3iNu^3.\tag{2.29}$$

Observe that the constant coefficient in (2.24) is chosen in order to obtain a cancellation of the term with $|u|^2u$. We define the usual Bourgain space $X^{s,b}:=X^{s,b}(\mathbb{R}\times\mathbb{R})$ as the closure of the Schwartz functions $S(\mathbb{R}\times\mathbb{R})$ under the norm

$$\|u\|_{X^{s,b}}:=\|\left\langle\xi\right\rangle^s\left\langle\tau-\xi^3\right\rangle\widetilde{u}(\tau,\xi)\|_{L^2_{\tau,\xi}(\mathbb{R}\times\mathbb{R})},$$

where $\widetilde{\ }$ means the Fourier transform in both variables.

LEMMA 2.5. For each $j=1,\dots,4$, let e_j be the solution to the problem

$$(\partial_t+\partial_{xxx})e_j=E_j,\quad e_j(0)=0.$$

Let $\eta(t)$ be a smooth time cutoff function taking 1 on $[0,1]$ and compactly supported. Then

$$\|\eta(t)e_j\|_{X^{3/4,b}}\lesssim\epsilon N^{-3/2}$$

for arbitrarily small $\delta>0$.

Proof. We have (see [19, 31])

$$\|\eta(t)e_j\|_{X^{s,b}}(\mathbb{R}\times\mathbb{R})\lesssim\|e_j(0)\|_{H^s}+\|\eta(t)E_j\|_{X^{s,b-1}}(\mathbb{R}\times\mathbb{R}).$$

Thus for the terms E_j, $j=1,2,3$ which have enough negative powers of N we use

$$\|\eta(t)e_j\|_{X^{1/4,b}}\lesssim\sup_{0\le t\le 1}\|E_j\|_{H^{1/4}},$$

and for the term E_4 for which there is not enough of a negative power on N, we compute

$$\widetilde{\eta(t)e_4}(\tau,\xi)=\frac{6}{\sqrt{N}}\widetilde{\eta u^3}(\tau+6N^3-3N^2\xi,\sqrt{3N}(\xi-3N)),$$

and using the definition of Bourgain space,

$$\begin{aligned}&\|\eta(t)e_4\|^2_{X^{1/4,b}}\lesssim\frac{1}{N}\int_{\mathbb{R}^2}\left\langle\xi\right\rangle^{1/2}\left\langle\tau-\xi^3\right\rangle^{2(b-1)}|\widetilde{\eta u^3}(\tau+6N^3-3N^2\xi,\sqrt{3N}(\xi-3N))|^2d\tau d\xi\\ =&\frac{1}{N}\int_{|\xi-3N|>1}\int_{|\tau-a|>1}+\int_{|\xi-3N|\le1}\int_{|\tau-a|>1}+\int_{|\xi-3N|\le1}\int_{|\tau-a|\le1}+\int_{|\xi-3N|>1}\int_{|\tau-a|\le1}(\ )d\tau d\xi\\ =&I_1+I_2+I_3+I_4,\end{aligned}$$

where $a = 3N^2\xi - 6N^3$. We estimate I_1, considering $\xi - 3N = x/\sqrt{3N}, \tau - a = t$,

$$
\begin{aligned}
I_1 =& \frac{1}{N^{3/2}} \int_{|x| \gtrsim N^{1/2}} \int_{|t|>1} \langle 3N + x/\sqrt{3N} \rangle^{1/2} |\widetilde{\eta u^3}(t,x)|^2 dt dx \\
\lesssim& \frac{1}{N} \int_{|x| \gtrsim N^{1/2}} \int_{|t|>1} \frac{\langle x \rangle^{2k+1/2} |\widetilde{\eta u^3}(t,x)|^2 dt dx}{\langle x \rangle^{2k}} \\
\lesssim& \frac{1}{N^{k+1}} \int_{\mathbb{R}} \langle x \rangle^{2k+1/2} \int_{\mathbb{R}} |\widetilde{\eta u^3}(t,x)|^2 dt dx \\
\lesssim& \frac{1}{N^{k+1}} \|\eta\|_{L^2}^2 \sup_t \|u^3(t)\|_{H^{k+1/4}}^2 \\
\lesssim& \frac{1}{N^{k+1}} \|\eta\|_{L^2}^2 \sup_t \|u(t)\|_{H^{k+1/4}}^6,
\end{aligned}
$$

where we used Plancherel in the temporal variable and the fact that H^k is closed under multiplication. The estimate of I_4 is the same as that of I_1. Next, we estimate I_3. Similarly as above we have

$$
\begin{aligned}
I_3 \lesssim& \frac{1}{N} \int_{|\xi| \lesssim N^{1/2}} \int_{|t| \le 1} \frac{\langle t \rangle^{2k} |\widetilde{\eta u^3}(t,\xi)|^2}{\langle t - A(\xi) \rangle^{2(1-b)} \langle t \rangle^{2k}} dt d\xi \\
\lesssim& \frac{1}{N} \int_{|\xi| \lesssim N^{1/2}} \int_{|t| \le 1} \frac{|\widetilde{\eta u^3}(t,\xi)|^2}{\langle A(\xi) \rangle^{2(1-b)}} dt d\xi \\
\lesssim& \frac{1}{N^{6(1-b)+1}} \int_{\mathbb{R}^2} |\widetilde{\eta u^3}(t,\xi)|^2 dt d\xi,
\end{aligned}
$$

where $A(\xi) = (\xi N^{-1/2} + 3N)^3 - 3\xi N^{3/2} - 3N^3$ satisfies $|A(\xi)| \ge 2N^3$ if $|\xi| \lesssim N^{1/2}$ and $N \gg 1$.

Finally we estimate I_2. Letting $\varphi(\xi) \in C_0^\infty(\mathbb{R})$ with $\varphi(\xi) = 1$ if $|\xi| \le \sqrt{3N}$ and $\varphi(\xi) = 0$ if $|\xi| > \sqrt{6N}$, we have

$$
\begin{aligned}
I_2 \lesssim& \frac{1}{N} \int_{|\xi| \le (3N)^{1/2}} \int_{|t|>1} \frac{|\widetilde{\eta u^3}(t,\xi)|^2}{\langle t - A(\xi) \rangle^{2(1-b)}} dt d\xi \\
\lesssim& \frac{1}{N} \int_{\mathbb{R}} \varphi(\xi) \int_{\mathbb{R}} \left| \widehat{\frac{1}{|\tau|^b}} (t - A(\xi)) \right|^2 |\widetilde{\eta u^3}(t,\xi)|^2 dt d\xi \\
\lesssim& \frac{1}{N} \int_{\mathbb{R}} \varphi(\xi) \int_{\mathbb{R}} \left| \widehat{\frac{e^{iA(\xi)\tau}}{|\tau|^b}} (t) \right|^2 |\widetilde{\eta u^3}(t,\xi)|^2 dt d\xi \\
\lesssim& \frac{1}{N} \int_{\mathbb{R}} \varphi(\xi) \int_{\mathbb{R}} \left| \frac{e^{iA(\xi)\tau}}{|\tau|^b} * \widehat{\eta u^3} \right|^2 |(t,\xi)|^2 dt d\xi.
\end{aligned}
$$

Using the Fubini Theorem we obtain

$$
I_2 \lesssim \frac{1}{N} \left\| \varphi(\xi) \frac{e^{iA(\xi)\tau}}{|\tau|^b} * (\eta \widehat{u^3}) \right\|_{L^2(\mathbb{R}^2)} = \left\| \int_{\mathbb{R}} \varphi(\xi) \eta(\tau) \frac{e^{-iA(\xi)\tau} \widehat{u^3}(\tau,\xi)}{|t-\tau|^b} d\tau \right\|_{L^2(\mathbb{R}^2)}.
$$

$\square$

LEMMA 2.6. Let u be a Schwartz solution to the complex mKdV equation (2.22) and v be a Schwartz solution to the approximated complex mKdV equation

$$\partial_t v + \partial_x^3 v + v^2 \partial_x \overline{v} = E$$

for some error function E. Let e be the solution to the inhomogeneous problem

$$\partial_t e + \partial_x^3 e = E, \quad e(0) = 0.$$

Suppose that

$$\|u(0)\|_{H^{3/4}}, \|v(0)\|_{H^{3/4}}, \quad \|\eta(t)e\|_{X^{3/4,b}} \lesssim \epsilon,$$

and let e be the solution for some sufficiently small absolute constant $0 < \epsilon \ll 1$. Then we have

$$\|\eta(t)(u - v)\|_{X^{3/4,b}} \lesssim \|u(0) - v(0)\|_{H^{3/4}} + \|\eta(t)e\|_{X^{3/4,b}}.$$

In particular, we have

$$\sup_{0 \le t \le 1} \|u(t) - v(t)\|_{X^{3/4,b}} \lesssim \|u(0) - v(0)\|_{H^{3/4}} + \|\eta(t)e\|_{X^{3/4,b}}.$$

Proof. The proof is very similar to that of [10], Lemma 5.1 (see also [27], Lemma 4.5). □

2.2.1. *Proof of Theorem* 1.2.

Proof. Letting $0 < \delta \ll \epsilon \ll 1$ and $T > 0$, by Theorem 2.3 we find two global solutions u_1, u_2 satisfying

$$\|u_j(0)\|_{H^s} \lesssim \epsilon, \tag{2.30}$$

$$\|u_1(0) - u_2(0)\|_{H^s} \lesssim \delta, \tag{2.31}$$

$$\sup_{0 \le t \le T} \|u_1(t) - u_2(t)\|_{H^s} \gtrsim \epsilon, \tag{2.32}$$

$$\sup_{0 \le t \le \infty} \|u_j(t)\|_{H^k} \lesssim \epsilon, \tag{2.33}$$

for $s < 0$ and $k > 6$. As in (2.24) we define

$$U_{ap,j} = \frac{2}{\sqrt{N}} \operatorname{Re} e^{itN^3 + iNx} u_j(t, \frac{x}{\sqrt{3N}} + \sqrt{3N} Nt), \quad j = 1, 2,$$

and let U_1, U_2 be smooth global solutions of (2.22) with initial data $U_j(0, x) = U_{ap,j}(0, x)$. Set

$$U_j^\lambda(t, x) := \lambda U_j(\lambda^3 t, \lambda x), \quad j = 1, 2,$$

the rescale solutions of (2.22). We need to make them satisfy the conditions (1.10), (1.11) and (1.12).

Applying Lemma 2.4 (i) when $s \ge 0$ we have

$$\|U_j^\lambda(0)\|_{H^s} \lesssim \lambda^{s+1/2} N^{s-1/4} \|u_j(0)\|_{H^s},$$

and when $s < 0$, we use Lemma 2.4 (ii) with k large to obtain

$$\|U_j^\lambda(0)\|_{H^s} \lesssim \lambda^{s+1/2} N^{s-1/4} \|u_j(0)\|_{H^k}.$$

If $\lambda = N^{(1/4-s)/(1/2+s)}$, then we have (1.10) for $U_j^\lambda(0)$ by (2.30). Similarly we have (1.11) for $U_1^\lambda(0) - U_2^\lambda(0)$ from (2.31). To complete the proof we will show (1.12). Using (2.32) we find $t_0 > 0$ (see [10]) such that

$$\|u_1(t_0) - u_2(t_0)\|_{L^2} \gtrsim \epsilon.$$

By Lemma 2.4 (iii) we thus have

$$\|U^\lambda_{ap,1}(t_0/\lambda^3) - U^\lambda_{ap,2}(t_0/\lambda^3)\|_{H^s} \gtrsim \lambda^{s+1/2} N^{s-1/4}\epsilon \sim \epsilon. \tag{2.34}$$

On the other hand, using Lemma 2.5 and the hypothesis $s > -5/16$,

$$\|U^\lambda_{ap,j}(t) - U^\lambda_j(t)\|_{H^s} \lesssim \lambda^{\max\{s,0\}+1/2} N^{-3/2}\epsilon \ll \epsilon, \quad j = 1, 2. \tag{2.35}$$

Using the triangle inequality we show that

$$\begin{aligned}\|U_1^\lambda(t_0/\lambda^3) - U_2^\lambda(t_0/\lambda^3)\|_{H^s} \geq \|U^\lambda_{ap,1}(t_0/\lambda^3) - U^\lambda_{ap,2}(t_0/\lambda^3)\|_{H^s}\\ - \sum_j \|U^\lambda_{ap,j}(t_0/\lambda^3) - U^\lambda_j(t_0/\lambda^3)\|_{H^s},\end{aligned}$$

and by (2.34), (2.35) we obtain (1.12) and this concludes the proof of the theorem. □

Next we prove Theorem 1.3.

3. Proof of Theorem 1.3 (Bourgain Method). In this section we shall consider all the parameters in the equation (1.1), i.e.

$$\begin{cases}\partial_t u + ia\,\partial_x^2 u + b\,\partial_x^3 u + ic|u|^2 u + d\,|u|^2\partial_x u + eu^2\partial_x\overline{u} = 0, \quad x \in \mathbb{R},\ t > 0,\\ u(x,0) = \eta\,\varphi(x).\end{cases} \tag{3.36}$$

Differentiating the solution $u = u(\eta, x, t)$ of the IVP (3.36) we obtain the relations:

$$\begin{aligned}\frac{\partial u}{\partial \eta}\Big|_{\eta=0} &= V(t)\,\varphi,\\ \frac{\partial^2 u}{\partial \eta^2}\Big|_{\eta=0} &= 0,\\ \frac{\partial^3 u}{\partial \eta^3}\Big|_{\eta=0} &= C\int_0^t V(t-t')\big(F(V(t')\varphi)\big)\,dt',\end{aligned}$$

where as before $V(t)$ denotes the unitary group associated to the linear IVP and F denotes the nonlinearity:

$$F(u) = ic|u|^2u + d\,|u|^2\partial_x u + eu^2\partial_x\overline{u}.$$

In what follows, we will consider $N \gg 1$ and $0 < \delta \ll 1$. Let

$$\widehat{\varphi}(\xi) = \delta^{-1/2}N^{-s}\chi_I(\xi), \tag{3.37}$$

where $I = [N, N+\delta]$.

From the definition of the unitary group $V(t)$ we can write

$$\int_0^t V(t-t')\,F(V(t')\varphi)(x)\,dt' =: I_1 + I_2 + I_3,$$

where

$$I_1 = i\,c\int_0^t\int_{\mathbb{R}} \exp\{ix\xi + i(t-t')\phi(\xi)\}\mathcal{F}\{\,|V(t')\varphi|^2\,V(t')\varphi\,\}\,d\xi dt',$$

$$I_2 = i\,d\int_0^t\int_{\mathbb{R}} \exp\{ix\xi + i(t-t')\phi(\xi)\}\,\mathcal{F}\{\,|V(t')\varphi|^2\,\partial_{\xi'}\{V(t')\varphi\}\,\}\,d\xi dt',$$

and

$$I_3 = i\,e\int_0^t\int_{\mathbb{R}} \exp\{ix\xi + i(t-t')\phi(\xi)\}\,\mathcal{F}\{\,(V(t')\varphi)^2\,\overline{\partial_{\xi'}\{V(t')\varphi\}}\,\}d\xi dt'.$$

To simplify the notation we set $\Psi(x,t,\xi) = \exp\{i(x\xi + t\phi(\xi))\}$. Using the definition of $V(t)$ and (3.37) we obtain

$$\begin{aligned} I_1 &= i\,c\int_0^t\int_{\mathbb{R}}\Psi(x,t,\xi)\int_{\mathbb{R}^2}\exp\{it'\phi_1(\xi_1,\xi_2,\xi)\}\widehat{\varphi}(\xi-\xi_1-\xi_2)\widehat{\varphi}(\xi_1)\,\widehat{\overline{\varphi}}(\xi_2)\,d\xi_1 d\xi_2 d\xi dt' \\ &= i\,c\int_{\mathbb{R}^3}\Psi(x,t,\xi)\widehat{\varphi}(\xi-\xi_1-\xi_2)\,\widehat{\varphi}(\xi_1)\,\widehat{\overline{\varphi}}(\xi_2)\int_0^t \exp\{it'\phi_1(\xi_1,\xi_2,\xi)\}\,dt' d\xi_1 d\xi_2 d\xi. \end{aligned}$$

Thus

$$\widehat{I_1} = \frac{i\,c\,\exp\{it\phi(\xi)\}}{\delta^{3/2}N^{3s}}\int_{A(\xi)}\frac{\exp\{it\phi_1(\xi_1,\xi_2,\xi)\}-1}{\phi_1(\xi_1,\xi_2,\xi)}\,d\xi_1 d\xi_2 d\xi,$$

where

$$\phi_1(\xi_1,\xi_2,\xi) = \phi(\xi-\xi_1-\xi_2)+\phi(\xi_1)-\phi(-\xi_2)-\phi(\xi) = -(\xi_1+\xi_2)(\xi-\xi_1)(2a+3b(\xi-\xi_2))$$

and

$$A(\xi) := \{(\xi_1,\xi_2)\in\mathbb{R}^2 :\ \xi_1\in I,\ -\xi_2\in I,\ \xi-\xi_1-\xi_2\in I\}.$$

Similarly, we have

$$\widehat{I_2} = \frac{i\,d\,\exp\{it\phi(\xi)\}}{\delta^{3/2}N^{3s}}\int_{A(\xi)}\frac{\exp\{it\phi_1(\xi_1,\xi_2,\xi)\}-1}{\phi_1(\xi_1,\xi_2,\xi)}\,\xi_1\ d\xi_1 d\xi_2 d\xi,$$

$$\widehat{I_3} = \frac{i\,e\,\exp\{it\phi(\xi)\}}{\delta^{3/2}N^{3s}}\int_{\widetilde{A}(\xi)}\frac{\exp\{it\widetilde{\phi}_1\,(\xi_1,\xi_2,\xi)\}-1}{\widetilde{\phi}_1(\xi_1,\xi_2,\xi)}\,\xi_1\ d\xi_1 d\xi_2 d\xi,$$

where $\widetilde{A}(\xi) := \{(\xi_1,\xi_2)\in\mathbb{R}^3 :\ (\xi_2,\xi_1)\in A(\xi)\}$ and $\widetilde{\phi}_1(\xi_1,\xi_2,\xi) = \phi_1(\xi_2,\xi_1,\xi)$.

Let $\widehat{G}(x,t) := \widehat{I_1}+\widehat{I_2}+\widehat{I_3}$. It is easy to check that the next statements are satisfied:

1) For $(\xi_1,\,\xi_2)\in A(\xi)$ we have

$$|\phi_1(\xi_1,\xi_2,\xi)| \lesssim C_{\phi_1,\,b} = \begin{cases}\delta^2\,N & \text{if}\quad b\neq 0,\\ \delta^2 & \text{if}\quad b=0,\ a\neq 0.\end{cases} \tag{3.38}$$

2) If $N<\xi<N+\delta$, then

$$|A(\xi)| \gtrsim \delta^2, \tag{3.39}$$

where $|\cdot|$ denotes the Lebesgue measure of the set A.

3)

$$\operatorname{supp}\,\widehat{G}(\cdot,t)(\xi)\subseteq[N-\delta,\,N+2\delta].$$

Similar statements are satisfied when we replace $\widetilde{\phi}_1$ and $\widetilde{A}(\xi)$ with ϕ_1 and $A(\xi)$.

Let

$$\delta_b := \delta = \begin{cases} N^{(-\theta-1)/2} & \text{if } b \neq 0, \\ N^{-\theta/2} & \text{if } b = 0,\ a \neq 0, \end{cases} \tag{3.40}$$

where $\theta > 0$.

If $d \neq 0$ or $e \neq 0$, then

$$|\widehat{G}(\xi)| \gtrsim \frac{1}{N^{3s}\,\delta^{3/2}}\,|A(\xi) \cup \widetilde{A}(\xi)|\,(N \min\{|d|, |e|\} - |c|).$$

If $d = e = 0$, then

$$|\widehat{G}(\xi)| \gtrsim \frac{1}{N^{3s}\delta^{3/2}}\,|A(\xi) \cup \widetilde{A}(\xi)|\,|c|.$$

We are ready to finish the proof of the theorem. Let us suppose that the map data–solution is C^3. Then we would have

$$\|\varphi\|^3_{H^s} \gtrsim \left\| \int_0^t U(t-t')\,F(U(t')\varphi)(\cdot)\,dt' \right\|_{H^s} = \|G\|_{H^s}$$

$$\gtrsim \begin{cases} \dfrac{\delta_b^2\,N|t|}{\delta_b^{3/2}\,N^{3s}}\,N^s\delta_b^{1/2} = N^{-2s}N\delta_b\,|t|, & d \neq 0 \text{ or } e \neq 0, \\[2ex] \dfrac{\delta_b^2\,c\,|t|}{\delta_b^{3/2}\,N^{3s}}\,N^s\delta_b^{1/2} = N^{-2s}c\,\delta_b\,|t|, & d = e = 0. \end{cases}$$

Taking δ_b as in (3.40) we obtain a contradiction for N large and θ suitably chosen in each case i), ii), iii) and iv) given in the statement of the theorem. Hence the map data–solution is not C^3 in these cases. This completes the proof.

Remark 3.1.

1) For $b \neq 0$, $d = e$ and $c = 0$ we can use the transformation (1.4) and work with (1.5).
2) Note that the conditions (2.16), on the parameters of the explicit solution (2.17), are a particular case of (i), i.e. $b \neq 0$, $e \neq 0$ or $d \neq 0$, but in Theorem 1.1 we have a stronger result.

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