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ON THE EXISTENCE OF STRONG TRAVELLING WAVE PROFILES TO 2×2 SYSTEMS OF VISCOUS CONSERVATION LAWS

By

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Abstract. We consider strong travelling wave profiles for a class of 2×2 viscous conservation laws. Our main assumption is that the product of nondiagonal elements within the Fŕechet derivative (Jacobian) of the flux is nonnegative. By using the regularization method improved by the author, we prove the existence of strong travelling wave profiles for those systems.

1. Introduction. In this paper, we consider a class of 2×2 systems of conservation laws with artificial viscosity

$$u_t + f(u,v)_x = \epsilon u_{xx}, \quad v_t + g(u,v)_x = \epsilon v_{xx}, \qquad t > 0, \ -\infty < x < \infty, \tag{1.1}$$

where $\epsilon > 0$ is a constant. Here u and v are functions of t and x, and $f(u,v)$ and $g(u,v)$ are smooth functions of two real variables u and v. Our main assumption is that

$$f_v g_u \geq 0 \quad \text{in } \mathbb{R}^2.$$

It should be noticed that the corresponding inviscid system (1.1) is not always strictly hyperbolic. For definiteness we shall assume that

$$f_v \leq 0, \quad g_u \leq 0 \quad \text{in } \mathbb{R}^2. \tag{1.2}$$

Of course, analogous results can be obtained for the case $f_v \geq 0$, $g_u \geq 0$.

Let (u_-, v_-) and (u_+, v_+) be constant states in $\mathbb{R}^2$. By strong travelling wave profiles of system (1.1) for a given constant s as the wave speed, we mean solutions $U = (u(\xi), v(\xi))$ with $\xi = x - st$, $s \neq 0$ that satisfy the boundary-value problem

$$\epsilon u''(\xi) = f(u,v)' - su'(\xi), \quad \epsilon v''(\xi) = g(u,v)' - sv'(\xi), \tag{1.3}$$

$$\big(u(-\infty), v(-\infty)\big) = (u_-, v_-), \quad \big(u(\infty), v(\infty)\big) = (u_+, v_+), \tag{1.4}$$

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where $'$ denotes differentiation with respect to ξ. It is well known that constant states (u_-, v_-) and (u_+, v_+) and constant s must satisfy the Rankine-Hugoniot condition

$$s(u_+ - u_-) = f(u_+, v_+) - f(u_-, v_-), \quad s(v_+ - v_-) = g(u_+, v_+) - g(u_-, v_-). \tag{1.5}$$

By using the regularization method introduced by Dafermos [1] (cf. [2], [3], [4] and [5]), Yang, Zhang and Zhu [7] prove the existence of strong travelling wave profiles to system (1.1) satisfying (1.2) and the following conditions: For

$$A_i = \Big\{(u, v) : \frac{\partial^i f(u, v)}{\partial v^i} = 0,\ (u, v) \in \mathbb{R}^2\Big\}, \quad i = 1, 2, \ldots, 2m,$$
$$B_j = \Big\{(u, v) : \frac{\partial^j g(u, v)}{\partial u^j} = 0,\ (u, v) \in \mathbb{R}^2\Big\}, \quad j = 1, 2, \ldots, 2n,$$

where m and n are any nonnegative integers,

$$A_1 \subset A_i,\ i = 2, 3, \cdots, 2m, \quad B_1 \subset B_j,\ j = 2, 3, \ldots, 2n, \tag{1.6}$$
$$\frac{\partial^{2m+1} f(u, v)}{\partial v^{2m+1}}\Big|_{A_1} < 0, \quad \frac{\partial^{2n+1} g(u, v)}{\partial u^{2n+1}}\Big|_{B_1} < 0, \tag{1.7}$$

and for every bounded subset U of $\mathbb{R}$,

$$|f(u, v)| \le C \quad \text{for all } u \in \mathbb{R},\ v \in U, \tag{1.8}$$
$$|g(u, v)| \le C \quad \text{for all } u \in U,\ v \in \mathbb{R}, \tag{1.9}$$

where C is a positive constant.

The purpose of this paper is to discuss the existence of strong travelling wave profiles to system (1.1) satisfying (1.2), (1.8) and (1.9). In [6], the author improves the regularization method introduced by Dafermos. By using the improved method, we prove the existence of strong travelling wave profiles to system (1.1) satisfying (1.2), (1.8) and (1.9).

2. Existence theorem for strong travelling wave profiles. In this section, we establish an existence theorem for strong travelling wave profiles to system (1.1). More precisely, we prove the following theorem:

THEOREM 2.1. Consider the two-parameter family of boundary-value problems

$$\epsilon u''(\xi) = \mu\Big(f(u, v) - \frac{v}{L}\Big)' - su'(\xi), \quad \epsilon v''(\xi) = \mu\Big(g(u, v) - \frac{u}{L}\Big)' - sv'(\xi), \tag{2.1}$$
$$\big(u(-L), v(-L)\big) = (\mu u_-, \mu v_-), \quad \big(u(L), v(L)\big) = (\mu u_+, \mu v_+), \tag{2.2}$$

with parameters $\mu \in [0, 1]$, $L \ge 1$. Assume that there exists a positive constant M depending at most on ϵ, f, g, (u_-, v_-), (u_+, v_+) (and thus independent of μ and L), such that every possible solution $(u(\xi), v(\xi))$ of (2.1), (2.2) satisfies

$$\sup_{-L<\xi<L} |u(\xi)| < M, \quad \sup_{-L<\xi<L} |v(\xi)| < M, \tag{2.3}$$
$$\sup_{-L<\xi<L} |u'(\xi)| < M, \quad \sup_{-L<\xi<L} |v'(\xi)| < M. \tag{2.4}$$

Then there exists a continuous solution of (1.3), (1.4).

Proof. Setting

$$U(\xi) = \begin{pmatrix} u(\xi) \\ v(\xi) \end{pmatrix}, \quad F_L(U) = \begin{pmatrix} f(U) - \dfrac{v}{L} \\ g(U) - \dfrac{u}{L} \end{pmatrix},$$

we rewrite (2.1), (2.2) as

$$\epsilon U''(\xi) = \mu F_L(U)' - sU'(\xi), \tag{2.5}$$

$$U(-L) = \mu U_-, \quad U(L) = \mu U_+, \tag{2.6}$$

where

$$U_- = \begin{pmatrix} u_- \\ v_- \end{pmatrix}, \quad U_+ = \begin{pmatrix} u_+ \\ v_+ \end{pmatrix}.$$

First, we establish the existence of solutions to (2.5), (2.6) for $\mu = 1$ by applying the Leray-Schauder fixed-point theorem. We define $T : C^0([-L, L]; \mathbb{R}^2) \to C^0([-L, L]; \mathbb{R}^2)$ which carries $V(\xi)$ into $U(\xi)$ with

$$\begin{aligned} U(\xi) = U_- + \omega \int_{-L}^{\xi} \exp\Big(-\frac{s}{\epsilon}\,\zeta\Big) d\zeta + \frac{1}{\epsilon}\int_{-L}^{\xi} F_L\big(V(\zeta)\big) d\zeta \\ - \frac{s}{\epsilon^2}\int_{-L}^{\xi}\int_0^{\zeta} F_L\big(V(\tau)\big)\exp\Big(\frac{s(\tau-\zeta)}{\epsilon}\Big) d\tau d\zeta, \end{aligned} \tag{2.7}$$

where ω is computed from

$$\begin{aligned} \omega \int_{-L}^{L} \exp\Big(-\frac{s}{\epsilon}\,\zeta\Big) d\zeta = U_+ - U_- - \frac{1}{\epsilon}\int_{-L}^{L} F_L\big(V(\zeta)\big) d\zeta \\ + \frac{s}{\epsilon^2}\int_{-L}^{L}\int_0^{\zeta} F_L\big(V(\tau)\big)\exp\Big(\frac{s(\tau-\zeta)}{\epsilon}\Big) d\tau d\zeta. \end{aligned} \tag{2.8}$$

It can be verified easily that every fixed point of μT is a solution of (2.5), (2.6).

It is clear that T is continuous. Moreover, we observe that the range of T is contained in $C^1([-L, L]; \mathbb{R}^2)$ and from (2.7) we obtain

$$U'(\xi) = \omega \exp\Big(-\frac{s}{\epsilon}\,\xi\Big) + \frac{1}{\epsilon} F_L\big(V(\xi)\big) - \frac{s}{\epsilon^2}\int_0^{\xi} F_L\big(V(\tau)\big)\exp\Big(\frac{s(\tau-\xi)}{\epsilon}\Big) d\tau, \tag{2.9}$$

which shows that T maps bounded sets of $C^0([-L, L]; \mathbb{R}^2)$ into uniformly equicontinuous sets of $C^0([-L, L]; \mathbb{R}^2)$. Therefore, T is compact. Finally, on account of (2.3), every fixed point of μT, $\mu \in [0, 1]$, is contained in the open ball of $C^0([-L, L]; \mathbb{R}^2)$ which is centered at 0 and has radius M. Thus, by the Leray-Schauder fixed-point theorem, T has a fixed point $U(\xi; L)$ which is a solution of (2.5), (2.6) for $\mu = 1$.

Next, we extend the domain of $U(\cdot; L)$ onto the entire real axis by setting $U(\xi; L) = U_-$ for $\xi < -L$, $U(\xi; L) = U_+$ for $\xi > L$ (cf. [1]). By virtue of (2.3) and (2.4), the set $\{U(\cdot; L) : L \geq 1\}$ is precompact in $C^0((-\infty, \infty); \mathbb{R}^2)$. Therefore, there exists a sequence $\{L_n\}$, $L_n \to \infty$ as $n \to \infty$, and a function $U(\xi) \in C^0((-\infty, \infty); \mathbb{R}^2)$ such that $U(\xi; L_n) \to U(\xi)$, $n \to \infty$, uniformly on $(-\infty, \infty)$.

We now show that $U(\xi)$ is a continuous solution of (1.3), (1.4). Since it is easy to check that $U(\xi)$ is a weak solution of (1.3), we only verify the initial condition (1.4). Suppose that $U(\xi)$ does not satisfy (1.4). Then we see that there exists $\delta \in \mathbb{R}$ with $\delta > 0$ such that for every $L \geq 1$,

$$|U(\xi_1) - U_+| \geq \delta \quad \text{or} \quad |U(\xi_1) - U_-| \geq \delta \quad \text{for some } |\xi_1| > L. \tag{2.10}$$

However, if L be sufficiently large, then we have

$$|U(\xi_1) - U_+| = |U(\xi_1) - U(\xi_1; L)| \leq \sup_\xi |U(\xi) - U(\xi; L)| < \delta, \tag{2.11}$$

$$|U(\xi_1) - U_-| = |U(\xi_1) - U(\xi_1; -L)| \leq \sup_\xi |U(\xi) - U(\xi; -L)| < \delta. \tag{2.12}$$

This is a contradiction. Thus the proof of Theorem 2.1 is complete. □

3. Existence of strong travelling wave profiles to system (1.1). In this section, we prove the existence of strong travelling wave profiles to system (1.1) satisfying (1.2), (1.8) and (1.9) by applying Theorems 2.1. More precisely, we prove the following theorem:

THEOREM 3.1. Assume that conditions (1.2), (1.8) and (1.9) are satisfied. Then, there exists a continuous solution of (1.3), (1.4).

In proving Theorem 3.1, one is helped by the following proposition, which asserts that one of the components of every solution to (2.1), (2.2) is monotone and the other component is monotone or bell-shaped:

PROPOSITION 3.2. Assume that condition (1.2) is satisfied and let $(u(\xi), v(\xi))$ be a solution of (2.1), (2.2). Then, one of the following holds:
(a) Both $u(\xi)$ and $v(\xi)$ are constant on $(-L, L)$.
(b) $u(\xi)$ is a strictly increasing (or decreasing) function with no critical point in $(-L, L)$, while $v(\xi)$ has, at most, one critical point, at which $v(\xi)$ necessarily must attain a maximum (or minimum), in $(-L, L)$.
(c) $u(\xi)$ has, at most, one critical point, at which $u(\xi)$ necessarily must attain a maximum (or minimum), in $(-L, L)$, while $v(\xi)$ is a strictly increasing (or decreasing) function with no critical point in $(-L, L)$.

Proof. Since Proposition 3.2 can be proved in a manner similar to Theorem 4.1 in [1], we omit the proof of Proposition 3.2. □

We now prove Theorem 3.1. Let $(u_L(\xi), v_L(\xi))$ be a solution of (2.1), (2.2). From Theorem 2.1 it is sufficient for the proof of Theorem 3.1 to prove

$$|u_L(\xi)|, \quad |v_L(\xi)| \leq M \quad \text{for all } \xi \in (-L, L), \tag{3.1}$$

$$|u_L'(\xi)|, \quad |v_L'(\xi)| \leq M \quad \text{for all } \xi \in (-L, L), \tag{3.2}$$

where M is a positive constant which is independent of μ and L. By Proposition 3.2, it is sufficient for the proof of inequalities (3.1) and (3.2) to deal with the following three cases:
Case 1: Both $u_L(\xi)$ and $v_L(\xi)$ are monotone.

Case 2: $u_L(\xi)$ is strictly increasing (or decreasing) on $(-L, L)$, while $v_L(\xi)$ is strictly increasing (or decreasing) on $(-L, \tau_L)$, attains a maximum (or minimum) at τ_L, and is strictly decreasing (or increasing) on (τ_L, L).

Case 3: $u_L(\xi)$ is strictly increasing (or decreasing) on $(-L, \tau_L)$, attains a maximum (or minimum) at τ_L, and is strictly decreasing (or increasing) on (τ_L, L), while $v_L(\xi)$ is strictly increasing (or decreasing) on $(-L, L)$.

We only prove inequalities (3.1) and (3.2) in Cases 1 and 3, because the proof of Case 2 is proved by arguments similar to the proof of Case 3.

First, we prove inequalities (3.1) and (3.2) in Case 1. Since inequality (3.1) clearly holds, we only prove inequality (3.2). For definiteness, we prove inequality (3.2) for $u_L(\xi)$ which is strictly increasing on $(-L, L)$, because all other cases are treated in an analogous fashion.

Note that for every $L \geq 1$, there exists $\zeta \in (0, 1)$ such that

$$0 \leq u_L'(\zeta) \leq u_+ - u_-. \tag{3.3}$$

Then, integrating the first equation in (2.1) over (ζ, ξ) and (ξ, ζ), and using (3.3), we obtain

$$\begin{aligned} |u_L'(\xi)| \leq & \Big(1 + \frac{|s|}{\epsilon}\Big)(u_+ - u_-) \\ & + \epsilon^{-1}\big(\big|f(u_L(\xi), v_L(\xi)) - f(u_L(\zeta), v_L(\zeta))\big| + |v_+ - v_-|\big). \end{aligned} \tag{3.4}$$

On account of (1.8), the right-hand side of (3.4) is bounded independently of μ and L. Thus inequality (3.2) in Case 1 holds.

Next, we prove inequalities (3.1) and (3.2) in Case 3. We only prove the case that $v_L(\xi)$ is strictly increasing on $(-L, L)$, because the case that $v_L(\xi)$ is strictly decreasing on $(-L, L)$ is proved by arguments similar to the proof of the case that $v_L(\xi)$ is strictly increasing on $(-L, L)$. Note that when $v_L(\xi)$ is strictly increasing on $(-L, L)$, $u_L(\xi)$ is strictly increasing on $(-L, \tau_L)$, attains a maximum at τ_L, and is strictly decreasing on (τ_L, L).

In proving inequality (3.1), it is sufficient to estimate $u_L(\tau_L)$ from above. If $s > 0$, then integrating the first equation in (2.1) over (τ_L, L), we obtain

$$u_L(\tau_L) \leq \mu u_+ - \frac{\mu}{s}\big(f(\mu u_+, \mu v_+) - f(u_L(\tau_L), v_L(\tau_L))\big) + \frac{\mu}{sL}(\mu v_+ - v_L(\tau_L)), \tag{3.5}$$

while if $s < 0$, then integrating the first equation in (2.1) over $(-L, \tau_L)$, we obtain

$$u_L(\tau_L) \leq \mu u_- + \frac{\mu}{s}\big(f(u_L(\tau_L), v_L(\tau_L)) - f(\mu u_-, \mu v_-)\big) - \frac{\mu}{sL}(v_L(\tau_L) - \mu v_-). \tag{3.6}$$

On account of (1.8), the right-hand sides of (3.5) and (3.6) are bounded independently of μ and L. Thus inequality (3.1) in Case 3 holds.

We proceed to prove inequality (3.2). Integrating the first equation in (2.1) over (τ_L, ξ) and (ξ, τ_L), we obtain

$$|u_L'(\xi)| \leq \epsilon^{-1}\big(\big|f(u_L(\xi), v_L(\xi)) - f(u_L(\tau_L), v_L(\tau_L))\big| + |v_+ - v_-| + 2M|s|\big), \tag{3.7}$$

where M is a positive constant which is independent of μ and L. On account of (1.8), the right-hand side of (3.7) is bounded independently of μ and L. Thus inequality (3.2) in Case 3 holds and the proof of Theorem 3.1 is complete.

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