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BOUNDARY LAYERS FOR SELF-SIMILAR VISCOUS APPROXIMATIONS OF NONLINEAR HYPERBOLIC SYSTEMS

By

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Dedicated to Professor Constantine M. Dafermos on the occasion of his 70th birthday

Abstract. We provide a precise description of the set of residual boundary conditions generated by the self-similar viscous approximation introduced by Dafermos et al. We then apply our results, valid for both conservative and nonconservative systems, to the analysis of the boundary Riemann problem and we show that, under appropriate assumptions, the limits of the self-similar and the classical vanishing viscosity approximation coincide. We require neither genuine nonlinearity nor linear degeneracy of the characteristic fields.

1. Introduction and main results. We are interested in systems of conservation laws in one space dimension

$$\partial_t U + \partial_x \big[F(U) \big] = 0, \tag{1.1}$$

where $U = U(t, x)$ attains values in $\mathbb{R}^n$, $t \geq 0$ and $x \in \mathbb{R}$ are the scalar independent variables and the flux function $F : \mathbb{R}^n \to \mathbb{R}^n$ is regular. System (1.1) is strictly hyperbolic if the Jacobian matrix $DF(U)$ has n real, distinct eigenvalues

$$\lambda_1(U) < \lambda_2(U) < \cdots < \lambda_n(U) \tag{1.2}$$

for every $U \in \mathbb{R}^n$. We refer to the books by Dafermos [10] and Serre [25] for an exposition of the current state of the theory of hyperbolic conservation laws.

In the present paper, we characterize the set of residual boundary conditions generated by the self-similar viscous approximation introduced by Dafermos [9] et al. and we then

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apply our results, valid for both conservative and nonconservative systems, to the analysis of the so-called boundary Riemann problem.

The theory of systems of conservation laws poses several challenges: in general, classical solutions of Cauchy problems starting out from smooth initial data develop discontinuities and break down in finite time. Thus, one needs to look for distributional solutions. However, distributional solutions are not unique and therefore, various admissibility criteria, often motivated by physical considerations, have been introduced in the attempt at singling out a unique solution. We refer the reader to Dafermos [10] for an extended discussion on admissibility criteria. The Cauchy problem obtained by coupling (1.1) with the initial data $U(0,x) = U_0(x)$ has been studied extensively. Existence and uniqueness results for global in time, admissible distributional solutions have been established via the random choice method of Glimm [14] and the front tracking algorithm; see Bressan et al. [6] and Holden and Risebro [15]. These results hold true under the assumption that the total variation $\mathrm{TotVar}\, U_0$ of the initial data is sufficiently small. If this condition is violated, then the oscillation or total variation of the solution may blow up in finite time. Existence and uniqueness results for data with large total variation have been achieved by imposing additional conditions on the structure of the flux function F. In the present paper, we only require that the flux F satisfies the strict hyperbolicity condition (1.2) and hence, we focus on data of sufficiently small total variation.

In view of applications coming from physics, it would be natural to construct solutions of (1.1) by taking the limit $\varepsilon \to 0^+$ of the viscous approximation

$$\partial_t Z^\varepsilon + \partial_x \big[F(Z^\varepsilon)\big] = \varepsilon \partial_x \big[B(Z^\varepsilon) \partial_x Z^\varepsilon\big]. \tag{1.3}$$

Here, $B(Z)$ is an $n \times n$ matrix which depends on the physical model under consideration. Convergence results have been established in specific cases; see Dafermos [10] and the references therein. In particular, Bianchini and Bressan [4] proved convergence in the case when $B \equiv I$, while the proof of the convergence in the general physical case is still a challenging open problem.

A key tool in the analysis of the system of conservation laws (1.1) is the study of self-similar solutions in the form $U(t,x) = Q(x/t)$. Indeed, the aforementioned random choice and front-tracking schemes are implemented by constructing approximate solutions that locally have the structure of a self-similar function. Also, the analysis of self-similar solutions provides information on both the local (in space-time) and the long-time behavior of the solution of a general Cauchy problem. Self-similar solutions and the so-called Riemann problem have been extensively studied; see in particular the works by Lax [20], Liu [22, 23], Tzavaras [28] and Bianchini [3] in the context of data of small total variation.

To investigate the structure of self-similar solutions, Dafermos [9], Kalašnikov [18] and Tupčiev [26] independently introduced the self-similar viscous approximation

$$\partial_t U^\varepsilon + \partial_x \big[F(U^\varepsilon)\big] = \varepsilon\, t\, \partial_x \big[B(U^\varepsilon) \partial_x U^\varepsilon\big]. \tag{1.4}$$

Indeed, because of the "t" factor in front of the second-order term, the viscous approximation (1.4) admits self-similar solutions $U^\varepsilon(t,x) = Q^\varepsilon(x/t)$ which satisfy the ordinary differential equation

$$\varepsilon \frac{d}{d\xi}\Big[B(Q^\varepsilon)\frac{dQ^\varepsilon}{d\xi}\Big] = \frac{d}{d\xi}\big[F(Q^\varepsilon)\big] - \xi\frac{dQ^\varepsilon}{d\xi} \ . \tag{1.5}$$

Convergence results for (1.5) have been established, under suitable conditions imposed on the matrix B, by various authors; see in particular Tzavaras [27, 28], Andreianov [2], Joseph and LeFloch [17] and the analysis and the references in Dafermos [10, Section 9.8].

In this paper, we consider the initial-boundary value problem obtained by focusing on the domain $(t, x) \in [0, +\infty[\times[0, +\infty[$. The initial-boundary value problem poses all the challenges of the Cauchy problem that have been mentioned (breakdown of classical solutions, nonuniqueness of distributional solutions), as well as additional difficulties due to the presence of the boundary. For instance, consider the problem obtained by coupling the viscous approximation (1.3) with the Cauchy and Dirichlet data

$$Z^\varepsilon(0, x) = U_0, \qquad Z^\varepsilon(t, 0) = U_b, \tag{1.6}$$

where U_0 and U_b are given constant states in $\mathbb{R}^n$. Since we are interested in small total variation solutions, we focus on the case when $|U_0 - U_b|$ is sufficiently small. If the matrix B is singular (which is the case for most of the physically relevant systems), then problem (1.3), (1.6) may be ill-posed, namely admit no solutions. However, one can obtain a well-posed problem, at the price of higher technicalities, by relying on a more complicated formulation of the boundary condition (see Bianchini and Spinolo [5]). To simplify the exposition, in the following we assume that problem (1.3), (1.6) is well-posed, but our considerations can be extended to the more general case of a suitable class of singular matrices considered in [5].

We consider the family of initial-boundary value problems obtained by coupling (1.3) and (1.6) and we assume that Z^ε converge as $\varepsilon \to 0^+$ to a limit U in a suitable topology. Having the data (1.6), one expects that the solution U admits the self-similar form $U(t, x) = Q(x/t)$ for some function Q and, hence, one recovers in the limit a solution to the so-called *boundary Riemann problem.* Moreover, because of the assumption that $|U_0 - U_b|$ is sufficiently small, one expects that the trace $\bar{U} \doteq \lim_{\xi\to 0^+} Q(\xi)$ is well-defined. It should be mentioned that such results have been established under additional assumptions; see for instance the analysis in Gisclon [12] and in Ancona and Bianchini [1]. We emphasize that, in general, one has

$$\bar{U} \doteq \lim_{x\to 0^+} U(t, x) = \lim_{\xi\to 0^+} Q(\xi) \neq U_b$$

and, also, the limit U of (1.3), (1.6) varies if the matrix B varies, and we refer to the articles of Gisclon and Serre [12, 13] for these observations.

To investigate the relation between the boundary data U_b and the so-called *residual boundary condition* $\bar{U} \doteq \lim_{\xi\to 0^+} Q(\xi)$, we focus on the *noncharacteristic case*; namely we assume that, for every $U \in \mathbb{R}^n$, all the eigenvalues of the Jacobian matrix $DF(U)$ are bounded away from zero:

$$\lambda_1(U) < \cdots < \lambda_k(U) < -c < 0 < c < \lambda_{k+1}(U) < \cdots < \lambda_n(U) \tag{1.7}$$

for some integer $1 \le k \le n-1$ and for a suitable constant $c > 0$. Then, one expects (see again [1, 12] for rigorous results in specific cases) that there exists a function V, the so-called boundary layer, which solves the following system:

$$\begin{cases} B(V)V' = F(V) - F(\bar{U}), \\ V(0) = U_b, \qquad \lim_{y\to+\infty} V(y) = \bar{U} \ , \end{cases} \tag{1.8}$$

where V' denotes the first derivative of the function V.

Regarding the self-similar viscous approximation of a boundary Riemann problem obtained by coupling equation (1.4) with the data

$$U^\varepsilon(0,x) = U_0, \qquad U^\varepsilon(t,0) = U_b, \tag{1.9}$$

Joseph and LeFloch established compactness and convergence results in the case when B is the identity matrix (see [16]) or close to the identity matrix (see [17]). Moreover, they described the self-similar limit $U(t,x) = Q(x/t)$ and showed that Q has bounded total variation. Hence, in particular, the trace $\lim_{\xi\to0^+} Q(\xi)$, which we denote by $\bar{U}$ again, is well-defined. Furthermore, they investigated the boundary layer and proved in Theorem 4.2 of [16] that there exists a boundary layer V satisfying (1.8). Their analysis of the boundary layer involves delicate manipulations of the equations that rely on the conservative form of (1.4).

In this paper, we provide a different approach to the analysis of the boundary layers of the self-similar viscous approximation (1.4), (1.9). For simplicity, we focus on the case when $B \equiv I$, but our analysis can be extended to the more general case considered in [5]. The main differences of our analysis from the analysis in [16] are the following: first, we rely on completely different techniques since here we use center-stable manifold tools coming from the area of dynamical systems in the spirit of Bianchini and Bressan [4]. Our approach was inspired by the work of Dafermos [10, Section 9.8] concerning the limit of the self-similar viscous approximation for the Riemann problem on the whole real line. However, the presence of the boundary amounts for additional challenges which are tackled by employing the Center-Stable Manifold Theorem. As a by-product, our results apply directly to the analysis of nonconservative systems; namely we handle the limit of the self-similar viscous approximation

$$\partial_t U^\varepsilon + A(U^\varepsilon)\partial_x U^\varepsilon = \varepsilon\, t\, \partial_{xx} U^\varepsilon \tag{1.10}$$

in the case when the $n \times n$ matrix $A(U)$ does not necessarily coincide with the Jacobian matrix $DF(U)$ of some flux function $F : \mathbb{R}^n \to \mathbb{R}^n$. On the other side, the limit analysis in [16] applies to both the noncharacteristic and (under genuine nonlinearity assumptions) to the boundary characteristic case, while here we restrict to the noncharacteristic case.

It should be noted that in the nonconservative case the distributional solutions of the quasilinear hyperbolic system

$$\partial_t U + A(U)\partial_x U = 0$$

are not defined, and we refer to Dal Maso, LeFloch and Murat [11] for possible definitions of weak solutions. We also quote Le Floch and Tzavaras [21] and the numerous references therein for the analysis of the Cauchy problems associated to the nonconservative systems obtained by taking the limit $\varepsilon \to 0^+$ of (1.10).

Before stating our main results, we introduce some notation. We consider system

$$\begin{cases} V'' = A(V)V', \\ V(0) = U_b, \qquad \lim_{y\to+\infty} V(y) = \bar{U} \end{cases} \tag{1.11}$$

and note that systems (1.8) and (1.11) coincide in the case when $B \equiv I$ and $A(V) = DF(V)$. Next, we write the ordinary differential equation at the first line of (1.11) as a first-order system by setting

$$\begin{cases} V' = W, \\ W' = A(V)W \end{cases} \tag{1.12}$$

and, by relying on the noncharacteristic condition (1.7) and the Stable Manifold Theorem (see e.g. Perko [24, Section 2.7]), we conclude that there is a k-dimensional stable manifold $\mathcal{M}^s(\bar{U}) \subseteq \mathbb{R}^n \times \mathbb{R}^n$ which is invariant under (1.12) and consists of the data (V_0, W_0) such that the solution of (1.12) satisfying $V(0) = V_0$ and $W(0) = W_0$ has the following asymptotic behavior:

$$\lim_{y\to+\infty} |V(y) - \bar{U}| = 0, \qquad \lim_{y\to+\infty} |W(y)|e^{cy/4} = 0,$$

where c is the same constant as in (1.7). We then define the projection $\mathcal{M}^s_V(\bar{U}) \subseteq \mathbb{R}^n$ of $\mathcal{M}^s(\bar{U})$ onto the first component as follows:

$$\mathcal{M}^s_V(\bar{U}) = \left\{ V \text{ such that } (V, W) \in \mathcal{M}^s(\bar{U}) \text{ for some } W \right\}. \tag{1.13}$$

Now, we state the main result of the paper.

THEOREM 1.1. Let condition (1.7) hold for some constant $c > 0$ and a natural number k, $1 \le k \le n-1$. Given a state $U_0 \in \mathbb{R}^n$, there is a sufficiently small constant $\delta > 0$ and a constant $\beta > 0$ such that, for any U_b and $\bar{U}$ satisfying $|U_b - U_0| \le \beta\delta$ and $|U_b - \bar{U}| \le \beta\delta$, we have that $U_b \in \mathcal{M}^s_V(\bar{U})$ if and only if the following two properties hold.

(1) For every sequence $\{\bar{U}^\varepsilon\} \subseteq \mathbb{R}^n$ such that $\bar{U}^\varepsilon \to \bar{U}$ as $\varepsilon \to 0^+$, there exists a sequence of solutions $Q^\varepsilon : [0, \delta] \to \mathbb{R}^n$ to the problem

$$\begin{cases} \varepsilon \dfrac{d^2 Q^\varepsilon}{d\xi^2} = \left[A(Q^\varepsilon) - \xi I\right] \dfrac{dQ^\varepsilon}{d\xi}, \\ Q^\varepsilon(0) = U_b, \\ Q^\varepsilon(\delta) = \bar{U}^\varepsilon, \end{cases} \tag{1.14}$$

where I denotes the $n \times n$ identity matrix.

(2) By setting $V^\varepsilon(\zeta) := Q^\varepsilon(\varepsilon\zeta)$, the family $\{V^\varepsilon\}$ converges to a limit function $V^0 : [0, +\infty[\to \mathbb{R}^n$, as $\varepsilon \to 0^+$, uniformly on compact sets. The function V^0 solves

the problem

$$\begin{cases} \dfrac{d^2V^0}{d\zeta^2} = A(V^0)\dfrac{dV^0}{d\zeta}, \\ V^0(0) = U_b, \\ \lim\limits_{\zeta\to+\infty} V^0(\zeta) = \bar{U}, \\ \lim\limits_{\zeta\to+\infty} \left|\dfrac{dV^0}{d\zeta}\right| e^{c\zeta/4} = 0. \end{cases} \tag{1.15}$$

Before we proceed, we make some remarks. First, it is obvious that the solutions $Q^\varepsilon(\xi)$ to the ordinary differential equations (1.14) yield the self-similar solutions $U^\varepsilon(t,x)$ of (1.10) by writing $U^\varepsilon(t,x) = Q^\varepsilon(x/t)$. Second, by relying on the Stable Manifold Theorem, it follows immediately that if system (1.15) admits a solution, then $U_b \in \mathcal{M}^s_V(\bar{U})$. However, the opposite direction of the implication in the statement of the theorem is not trivial. In other words, the proof is mainly devoted to showing that from $U_b \in \mathcal{M}^s_V(\bar{U})$, then properties (1) and (2) of the theorem follow. Third, Theorem 1.1 can be reformulated by stating that the boundary layers generated by the self-similar viscous approximation are the same as the boundary layers (1.11) generated by the classical vanishing viscosity approximation. Below, we discuss an application of this fact concerning the boundary Riemann problem; the corresponding precise result is stated in Section 2 as Proposition 2.1.

Consider the self-similar vanishing viscosity approximation

$$\begin{cases} \partial_t U^\varepsilon + A(U^\varepsilon)\partial_x U^\varepsilon = \varepsilon t \partial^2_{xx} U^\varepsilon, \\ U^\varepsilon(t,0) = U_b, \\ U^\varepsilon(0,x) = U_0 \end{cases} \tag{1.16}$$

and the classical vanishing viscosity approximation

$$\begin{cases} \partial_t Z^\varepsilon + A(Z^\varepsilon)\partial_x Z^\varepsilon = \varepsilon \partial^2_{xx} Z^\varepsilon, \\ Z^\varepsilon(t,0) = U_b, \\ Z^\varepsilon(0,x) = U_0 \end{cases} \tag{1.17}$$

of a Riemann problem. Under the assumption that $|U_b - U_0|$ is sufficiently small, convergence results for (1.16) and (1.17) have been established by Joseph and LeFloch [16, 17] and by Ancona and Bianchini [1], respectively.

By combining the analysis in Dafermos [10, Section 9.8] and Theorem 1.1, we infer that, under appropriate assumptions (for completeness we give the precise result in Proposition 2.1 of Section 2.2), the limits of (1.16) and (1.17) coincide. An analogous result has been proved for conservative systems in Christoforou and Spinolo [8] in both the characteristic and the noncharacteristic boundary case. Here we extend this analysis to nonconservative systems, but we focus on the noncharacteristic case. The reason why these results are not *a priori* obvious is that, in general, for initial-boudary value problems, the limit of a viscous approximation (1.3), (1.6) depends on the choice of the viscosity matrix B and hence, on the type of viscous approximation.

The exposition of the paper is organized as follows: in Section 2 we compare the limits of the classical vanishing viscosity and self-similar viscous approximation of a boundary Riemann problem. More precisely, in Subsection 2.1, we present existing results concerning the Riemann problem established by Dafermos [10] for the self-similar viscous approximation and by Bianchini and Bressan [4] and Ancona and Bianchini [1] for the vanishing viscosity approximation. Using these results and applying Theorem 1.1, in Subsection 2.2 we show that, under reasonable assumptions, the limits of (1.16) and (1.17) coincide. The proof of Theorem 1.1 is provided in Section 3.

2. On the classical and the self-similar viscous approximation of a boundary Riemann problem. The aim of this section is to compare the limits of the self-similar (1.16) and the classical vanishing viscosity (1.17) approximation of a boundary Riemann problem. We first quote existing results related to the wave fan curves corresponding to these two different approximations and then we show in Proposition 2.1 that the two limits are the same.

2.1. *Preliminary results: construction of the wave fan curves.* In [20], Lax studied self-similar, distributional solutions of the system of conservation laws (1.1) by imposing some technical assumptions on the structure of the flux function F. In particular, for every $i = 1, \dots, n$, he constructed the *i-wave fan curve* $T^i(s_i, U^+)$ passing through a given state $U^+ \in \mathbb{R}^n$ with $s_i \in \mathbb{R}$ being the variable parameterizing the curve. The curve $T^i(s_i, U^+)$ takes values in $\mathbb{R}^n$ as s_i varies in a neighborhood of 0, and for fixed s_i small enough, the Riemann problem

$$\begin{cases} \partial_t U + \partial_x \big[F(U)\big] = 0, \\ U(0,x) = \begin{cases} U^+ & x > 0, \\ T^i(s_i, U^+) & x < 0 \end{cases} \end{cases} \tag{2.1}$$

admits a self-similar solution which is either a rarefaction, or a single contact discontinuity or a single shock satisfying the Lax admissibility condition. This condition was introduced in the same paper [20]. Lax's construction was later extended to very general cases in the works by Liu [22, 23], Tzavaras [28] and Bianchini [3].

Also, Bianchini and Bressan [4] and Dafermos [10, Section 9.8] described the *i-wave fan curve* obtained by taking the limit $\varepsilon \to 0^+$ in the classical vanishing viscosity

$$\partial_t Z^\varepsilon + A(Z^\varepsilon)\partial_x Z^\varepsilon = \varepsilon \partial^2_{xx} Z^\varepsilon \tag{2.2}$$

and in the self-similar viscous approximation (1.10), respectively. We go over the constructions in [4] and [10, Section 9.8] in Subsections 2.1.1 and 2.1.3, respectively, while in Subsection 2.1.2 we quote the description of the limit of (1.17) in the boundary case provided by Ancona and Bianchini [1]. We note that the constructions in [1, 4, 10] include the case of nonconservative systems.

2.1.1. *Wave fan curves induced by the classical vanishing viscosity approximation.* Here, we describe the *i-wave fan curve* $T^i(s_i, U^+)$ as constructed in [4, Section 14].

Given a matrix $A(U)$ satisfying strict hyperbolicity (1.2), we denote by $R_1(U), \dots, R_n(U)$ a basis of right eigenvectors.

We first assume $s_i > 0$ and then, consider the following fixed problem:

$$\begin{cases} V_i(\tau) & = U^+ + \displaystyle\int_0^\tau \hat{R}_i\big(V_i(s),\, \omega_i(s), \xi_i(s)\big)\, ds, \\ \omega_i(\tau) & = \hat{f}(\tau) - \hat{g}(\tau), \\ \xi_i(\tau) & = \dfrac{d\hat{g}}{d\tau}(\tau), \end{cases} \tag{2.3}$$

where we used the notation

$$\hat{f}(\tau) \doteq \int_0^\tau \hat{\lambda}_i\big(V_i(s),\, \omega_i(s), \xi_i(s)\big)\, ds,$$

while $\hat{g}$ denotes the concave envelope of $\hat{f}$ on the interval $[0,\, s_i]$. Also, the so-called generalized eigenvectors $\hat{R}_i$ and *generalized eigenvalues* $\hat{\lambda}_i$ are defined in [4, Section 14] by considering the travelling waves V to (2.2), namely solutions of the system

$$\begin{cases} V' & = W, \\ W' & = \Big[A(V) - \xi I\Big] W, \\ \xi' & = 0. \end{cases} \tag{2.4}$$

By restricting system (2.4) on a center manifold about the equilibrium point $V = U^+$, $W = \vec{0}$, $\xi = \lambda_i(U^+)$, one eventually determines the system

$$\begin{cases} V' & = \hat{R}_i(V,\, \omega_i, \xi)\omega_i, \\ \omega_i' & = \big(\hat{\lambda}_i(V,\, \omega_i, \xi) - \xi\big)\omega_i, \\ \xi' & = 0, \end{cases} \tag{2.5}$$

where ω_i is a real-valued unknown function and the generalized eigenvector $\hat{R}_i$ is a suitable function taking values in $\mathbb{R}^n$ and satisfying $\hat{R}_i(U^+,\, 0, \lambda_i(U^+)) = R_i(U^+)$.

By relying on the Contraction Map Theorem, one can show that problem (2.3) has a unique solution (V_i, ω_i, ξ_i) belonging to a suitable metric space and then set $T^i(s_i, U^+) = V_i(s_i)$. The construction in the case when $s_i < 0$ follows similarly by taking $\hat{g}$ in (2.3) to be the convex envelope of $\hat{f}$ instead of the concave one.

2.1.2. *Vanishing viscosity limit of a boundary Riemann problem.* In [1], Ancona and Bianchini showed that, if $|U_0 - U_b|$ is sufficiently small and the noncharacteristic condition (1.7) holds, then the limit $\varepsilon \to 0^+$ of the vanishing viscosity approximation (1.17) is determined by imposing the condition

$$U_b = \mathcal{G}_{U_0}(S, s_{k+1}, \dots, s_n) \doteq \phi_s\Big(S, T^{k+1}\big(s_{k+1}, \dots, T^n(s_n, U_0)\dots\big)\Big)\,. \tag{2.6}$$

Here, $S \in \mathbb{R}^k$ and ϕ_s is a map parameterizing the stable manifold of (1.12) about the equilibrium point $(\bar{U}, \vec{0})$; i.e. the manifold $\mathcal{M}^s_V(\bar{U})$ in (1.13) can be expressed as follows:

$$\mathcal{M}^s_V(\bar{U}) = \{\phi_s(S, \bar{U}) \text{ for some vector } S \in \mathbb{R}^k\}.$$

Also, $T^i(s_i, \cdot)$ is the *curve of admissible states* defined by Bianchini and Bressan in [4] whose construction is sketched in Subsection 2.1.1.

It can be shown that (2.6) uniquely determines the values of S, $s_{k+1}, \dots, s_n$ and that the solution $Z(t,x)$ obtained as the $\varepsilon \to 0^+$ limit of the classical vanishing viscosity approximation (1.16) admits the following representation:

$$Z(t,x) = \begin{cases} \bar{U} & \text{if} \quad x/t < \xi_{k+1}(s_{k+1}), \\ V_j(\tau) & \text{if} \quad x/t = \xi_j(\tau), \quad \text{for } \tau \in]0, s_j[\text{ or }]s_j, 0[, \ j = k+1, \dots, n, \\ V_j(s_j) & \text{if} \quad \xi_j(0) < x/t < \xi_{j+1}(s_{j+1}) \quad \text{for } j = k+1, \dots, n-1, \\ U_0 & \text{if} \quad x/t > \xi_n(0), \end{cases} \tag{2.7}$$

where $(s_{k+1}, \dots, s_n)$ are determined by using (2.6), V_j and ξ_j are the solutions of the fixed point problem (2.3) and the trace $\bar{U}$ is given by

$$\bar{U} = T^{k+1}\big(s_{k+1}, \dots, T^n(s_n, U_0) \dots\big) \, .$$

2.1.3. *Wave fan curves induced by the self-similar viscous approximation.* Now, we present the construction of the wave fan curves induced by the self-similar viscous approximation as was constructed by Dafermos in [10, Section 9.8].

To construct the i-wave fan curve $\phi_i(s_i, U^+)$ that emanates from some state U^+, we consider the system

$$\varepsilon \frac{d^2 Q^\varepsilon}{d\xi^2} = \big[A(Q^\varepsilon) - \xi I\big] \frac{dQ^\varepsilon}{d\xi} \tag{2.8}$$

satisfied by the self-similar solutions and we decompose the first derivative along the right eigenvectors of $A(U)$:

$$\frac{dQ^\varepsilon}{d\xi} = \sum_{j=1}^{n} a_j(\xi) R_j(Q^\varepsilon(\xi)). \tag{2.9}$$

By taking a dual basis of left eigenvectors $L_1(U), \dots, L_n(U)$, the component a_j is given by $a_j = \langle L_j, Q^\varepsilon \rangle$, $j = 1, \dots, n$, where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product.

Substituting (2.9) into (1.5), we arrive at

$$\varepsilon \frac{da_j}{d\xi} = [\lambda_j(V) - \xi] a_j + \varepsilon \sum_{h,l=1}^{n} \beta_{jhl} a_h \, a_l,$$

where the coefficients β_{jhl} are given by

$$\beta_{jhl}(Q^\varepsilon) = -\langle DL_j(Q^\varepsilon) R_h(Q^\varepsilon), R_l(Q^\varepsilon) \rangle \tag{2.10}$$

and DL_j denotes the Jacobian matrix of the field L_j. We now perform a change of variables by setting $\xi = \varepsilon \zeta$ and rescale the components a_j by setting $\omega_j := \varepsilon a_j$ for every $j = 1, \dots, n$. The self-similar viscous approximation Q^ε to (2.8) is denoted by V^ε in the new variable ζ, i.e. $V^\varepsilon(\zeta) \doteq Q^\varepsilon(\varepsilon\zeta)$. However, for the convenience of the reader, we abuse the notation, for now, and we drop the index ε from V^ε since the ε-dependence is clear in what follows. In Subsection 3.2, we will return to the original notation V^ε.

Using decomposition (2.9), we rewrite system (2.8) as an autonomous first-order system

$$\begin{cases} V' &= \sum_{j=1}^{n} \omega_j R_j(V), \\ \omega_j' &= [\lambda_j(V) - \xi]\omega_j + \sum_{h,l=1}^{n} \beta_{jhl}\omega_h \,\omega_l \quad \text{for every } j = 1, \dots, n, \\ \xi' &= \varepsilon, \\ \varepsilon' &= 0\,, \end{cases} \tag{2.11}$$

where $'$ denotes differentiation with respect to ζ. By linearizing (2.11) about the equilibrium point,

$$V = U^+,\ \ \omega_1 = 0, \dots, \omega_n = 0,\ \ \xi = \lambda_i(U^+),\ \ \varepsilon = 0,$$

we get the linear system

$$\begin{cases} V' &= \sum_{j=1}^{n} \omega_j R_j(U^+), \\ \omega_j' &= [\lambda_j(U^+) - \lambda_i(U^+)]\omega_j \quad \text{for every } j = 1, \dots, n, \\ \xi' &= \varepsilon, \\ \varepsilon' &= 0\,. \end{cases} \tag{2.12}$$

The center subspace $\mathcal{N}_i$ of system (2.12) consists of the vectors $(V, \omega, \xi, \varepsilon) \in \mathbb{R}^{2n+2}$, with $\omega_j = 0$ for $j \neq i$; thus, $\mathcal{N}_i$ has dimension $n+3$. By the Center Manifold Theorem, there is an $(n+3)$-dimensional manifold containing all the solutions of (2.11) that sojourn in a small enough neighbourhood of the equilibrium point $(U^+, \vec{0}, \lambda_i(U^+), 0)$. Such a center manifold is parameterized by $\mathcal{N}_i$ and it is tangent to the center subspace at the equilibrium. In general, the center manifold of (2.11) about $(U^+, \vec{0}, \lambda_i(U^+), 0)$ is not unique and therefore, in the following, we fix one and we denote it by $\mathcal{M}_i$. We refer the reader to the notes by Bressan in [7] for an extended discussion about the Center Manifold Theorem.

As is shown in [10, Section 9.8], system (2.11) on $\mathcal{M}_i$ is equivalent to

$$\begin{cases} V' &= R_i^\sharp(V, \omega_i, \xi, \varepsilon)\omega_i, \\ \omega_i' &= \big(\lambda_i^\sharp(V, \omega_i, \xi, \varepsilon) - \xi\big)\omega_i, \\ \xi' &= \varepsilon, \\ \varepsilon' &= 0\,, \end{cases} \tag{2.13}$$

where the functions $R_i^\sharp$ and $\lambda_i^\sharp$ take values in $\mathbb{R}^n$ and $\mathbb{R}$, respectively, and satisfy

$$R_i^\sharp(U^+, 0, \lambda_i(U^+), 0) = R_i(U^+) \quad \text{and} \quad \lambda_i^\sharp(U^+, 0, \lambda_i(U^+), 0) = \lambda_i(U^+). \tag{2.14}$$

It should be noted that system (2.13) consists of $n+3$ equations.

To define the value $\phi_i(s_i, U^+)$ attained by the i-wave fan curve of admissible states emanating from U^+ at $s = s_i > 0$, we consider the fixed point problem

$$\begin{cases} V_i(\tau) &= U^+ + \displaystyle\int_0^\tau R_i^\sharp\big(V_i(s),\, \omega_i(s), \xi_i(s), 0\big)\, ds, \\ \omega_i(\tau) &= f^\sharp(\tau) - g^\sharp(\tau), \\ \xi_i(\tau) &= \dfrac{dg^\sharp}{d\tau}(\tau) \end{cases} \tag{2.15}$$

over the interval $[0, s_i]$. Here,

$$f^\sharp(\tau) \doteq \int_0^\tau \lambda_i^\sharp\big(V_i(s),\, \omega_i(s), \xi_i(s), 0\big)\, ds,$$

while $g^\sharp$ denotes the concave envelope of $f^\sharp$ on the interval $[0,\, s_i]$. By relying on a fixed point argument, it can be shown that (2.15) admits a unique solution $\big(V_i(\tau), \omega_i(\tau), \xi_i(\tau)\big)$ over $[0, s_i]$. Then, the i–wave fan curve of admissible states is defined by setting $\phi_i(s_i, U^+)$ $:= V_i(s_i)$.

If $s_i < 0$, the construction is similar, except that the function g in (2.15) is the *convex* envelope of f over the interval $[s_i, 0]$. Moreover, it can be shown that, by construction, the curve $\phi_i(s_i, U^+)$ is tangent to the eigenvector $R_i(U^+)$ at $s_i = 0$.

2.2. *Comparison between the self-similar and the classical vanishing viscosity approximation.* Here, we prove that under appropriate conditions on the limit obtained by the self-similar viscous approximation (1.16), the solution to the boundary Riemann problem obtained via the classical vanishing viscosity approximation (1.17) coincides with the one obtained via the self-similar viscous approximation (1.16). This is not a priori obvious since it is known that, in general, for initial-boundary value problems the limit depends on the type of viscous approximation.

The comparison between the two limits is established using Theorem 1.1 and the construction of the wave fan curves described in Subsection 2.1. The following proposition states the result:

PROPOSITION 2.1. Let $U_0 \in \mathbb{R}^n$ and assume that condition (1.7) holds for an appropriate constant $c > 0$ and for some natural number k, $1 \le k \le n-1$. Then, there exists a sufficiently small constant $\delta > 0$ such that given U_b with $|U_0 - U_b| \le \delta$, the family of self-similar viscous approximations U^ε satisfying (1.16) converges, up to subsequences, to a self-similar limit function $U(t,x) = Q(x/t)$. Assume that the following two conditions are both satisfied:

(a) let $\bar{U}$ denote the trace

$$\bar{U} \doteq \lim_{\xi \to 0^+} Q(\xi).$$

Then $\bar{U}$ satisfies conditions (1) and (2) in the statement of Theorem 1.1.

(b) The function U admits the following representation:

$$U(t,x) = \begin{cases} \bar{U} & \text{if } x/t < \xi_{k+1}(s_{k+1}), \\ V_j(\tau) & \text{if } x/t = \xi_j(\tau) \quad \text{for } \tau \in \,]0, s_j[\;\text{ or }]s_j, 0[,\; j = k+1, \dots, n, \\ V_j(s_j) & \text{if } \xi_j(0) < x/t < \xi_{j+1}(s_{j+1}) \quad \text{for } j = k+1, \dots, n-1, \\ U_0 & \text{if } x/t > \xi_n(0), \end{cases} \tag{2.16}$$

for suitable small values $s_{k+1}, \dots, s_n \in \mathbb{R}$, where the functions V_j and ξ_j are solutions of the fixed point problem (2.15).

Then the limit $U(t,x)$ of the self-similar viscous approximation U^ε defined by (1.16) coincides with the unique limit $Z(t,x)$ of the classical vanishing viscosity approximation Z^ε defined by (1.17).

Proof. The convergence result follows from the analysis in Joseph and LeFloch [16]. Indeed, the proof of the compactness result in the first part of [16] can be straightforwardly extended to the nonconservative case. We now focus on the characterization of the limit.

Theorem 1.1 states that condition (a) in the statement of Proposition 2.1 is equivalent to assuming that $U_b \in \mathcal{M}^s_V(\bar{U})$ and hence by relying on conditions (a) and (b), we deduce the relation

$$U_b = \mathcal{F}_{U_0}(S, s_{k+1}, \dots, s_n) = \phi_s\Big(S, \phi_{k+1}\big(s_{k+1}, \dots \phi_n(s_n, U_0) \dots\big)\Big), \tag{2.17}$$

where ϕ_s is as in (2.6) a function parameterizing the stable manifold and the function ϕ_j, $j = k+1, \dots, n$ is the j–wave fan curve induced by the self-similar viscous approximation and is defined by (2.15). We note that $\mathcal{F}_{U_0}$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ and we claim that it is locally invertible. Indeed, we recall that the stable manifold $\mathcal{M}^s(\bar{U})$ is tangent at the origin to the stable space $V^s(\bar{U})$ and the curve $\phi_j(s_j, U^+)$ is tangent at $s_j = 0$ to the eigenvector $R_j(U^+)$ and therefore, the Jacobian matrix of $\mathcal{F}_{U_0}$ evaluated at the point $S = \vec{0}$, $s_{k+1} = \dots = s_n = 0$ is the $n \times n$ matrix whose columns are the eigenvectors $R_1(U_0), \dots, R_n(U_0)$. By relying on the Local Invertibility Theorem, we conclude that, if $|U_0 - U_b|$ is sufficiently small, then relation (2.17) uniquely determines the values of $S, s_{k+1}, \dots, s_n$.

We can now conclude the proof by combining the following two remarks. First, it should be noted that, in general, the center manifold about an equilibrium point is not unique and, hence, the functions $\hat{R}_i$ and $\hat{\lambda}_i$ in (2.3), (2.5) *depend* on the choice of the center manifold. However, as proven in [4, Remark 14.2, page 305], the limit $Z(t,x)$ that admits representation (2.7) *does not* depend on this choice. Second, by construction, the solutions of system (2.13) satisfying the additional condition $\varepsilon = 0$ are solutions of (2.4) lying on a center manifold about the equilibrium point $\big(V_i = U^+, W = \vec{0}, \xi = \lambda_i(U^+)\big)$. By comparing (2.15) and (2.3) and using the fact that the fixed point of (2.3) does not depend on the choice of the center manifold, we conclude that the function U in (2.16) coincides with the unique function Z in (2.7). The proof is complete. □

Remark 2.2. It should be noted that Joseph and LeFloch in [16, Theorem 4.2] showed that the limit U of the self-similar approximation (1.16) satisfies condition (a) of Proposition 2.1 for the conservative case, $A(U) = DF(U)$.

3. Proof of Theorem 1.1.

In this section, we provide the proof of Theorem 1.1.

By relying on the proof of the Stable Manifold Theorem (see Perko [24, Section 2.7]) we have that, if there exists a function V^0 satisfying (1.15), then $U_b \in \mathcal{M}^s_V(\bar{U})$. Establishing the opposite implication amounts to showing that, if $U_b \in \mathcal{M}^s_V(\bar{U})$, then properties (1) and (2) of the theorem hold. To accomplish this, we proceed in four steps:

- in Section 3.1, we restrict system (2.11) to a center-stable manifold;
- in Section 3.2, we construct the approximating sequence;
- in Section 3.3, we establish the convergence of the sequence and study the limit;
- in Section 3.4, we eventually conclude the proof of Theorem 1.1.

Note that in the following we fix a state $\bar{U} \in \mathbb{R}^n$ and we exhibit constants c_V and δ such that, for every U_b satisfying $|\bar{U} - U_b| \leq c_V \delta$ and $U_b \in \mathcal{M}^s_V(\bar{U})$, properties (1) and (2) in the statement of Theorem 1.1 are satisfied. The values of c_V and δ *a priori* depend on the state $\bar{U}$; however, as a matter of fact, the values of these constants only depend on some regularity coefficients of suitable functions constructed in Section 3.1. It turns out that the values of these functions are uniformly bounded on sufficiently small compact sets and hence, the statement of Theorem 1.1 is valid for every $\bar{U}$ in a small enough neighborhood of a given state U_0.

3.1. *Solutions lying on a center-stable manifold.* In this subsection, we analyze the solutions lying on a center-stable manifold for the boundary layers.

By linearizing system (2.11) about the equilibrium point $(V, \omega, \xi, \varepsilon) = (\bar{U}, \vec{0}, 0, 0)$, we obtain

$$\begin{cases} V' &= \displaystyle\sum_{j=1}^{n} \omega_j R_j(\bar{U}), \\ \omega_j' &= \lambda_j(\bar{U})\omega_j \quad \text{for every } j = 1, \dots, n, \\ \xi' &= \varepsilon, \\ \varepsilon' &= 0. \end{cases} \tag{3.1}$$

By the noncharacteristic condition (1.7), we have that $\lambda_j(\bar{U}) \neq 0$ for every $j = 1, \dots, n$ and therefore, the center-stable space to (3.1) is described by the vectors $(V, \omega_1, \dots, \omega_n, \xi, \varepsilon)$ satisfying $V \in \mathbb{R}^n$ and $\omega_{k+1} = 0, \dots, \omega_n = 0$. This implies that its dimension is $n + k + 2$. In the following, to simplify the exposition we use the notation $W = (\omega_1, \dots, \omega_n)^t$.

Next, we apply the Center-Stable Manifold Theorem and we refer the reader to the book by Katok and Hasselblatt [19] for an extended discussion on this subject. Here, we only recall the properties needed in this article. By the Center-Stable Manifold Theorem, it follows that in a sufficiently small neighbourhood of the equilibrium point $(\bar{U}, \vec{0}, 0, 0)$ there is a so-called center-stable manifold, which is locally invariant for system (2.11) and it has dimension $n + k + 2$ as the center-stable space. In general, the center-stable manifold of a given equilibrium point is not unique. However, any center-stable manifold of the equilibrium $(\bar{U}, \vec{0}, 0, 0)$ for (2.11) is always tangent to the center-stable space at $(\bar{U}, \vec{0}, 0, 0)$. Moreover, let c be the same constant as in (1.7). Then any center-stable manifold contains all the orbits $\big(V(\zeta), W(\zeta), \xi(\zeta), \varepsilon(\zeta)\big)$ that are confined in a sufficiently small neighborhood of $(\bar{U}, \vec{0}, 0, 0)$ and satisfy

$$\lim_{\zeta \to +\infty} \Big[|V(\zeta) - \bar{U}| + |W(\zeta)| + |\xi(\zeta)| + |\varepsilon(\zeta)|\Big] \exp(-c\zeta/2) = 0 \,.$$

By taking the value of δ in the statement of Theorem 1.1 sufficiently small, we now fix a center-stable manifold $\mathcal{M}^{cs}$ defined in the neighborhood of $(\bar{U}, \vec{0}, 0, 0)$ given by

$$|V - \bar{U}| \leq \delta, \quad |W| \leq \delta, \quad |\xi| \leq \delta, \quad |\varepsilon| \leq \delta. \tag{3.2}$$

Note that actually $\mathcal{M}^{cs}$ depends on $\bar{U}$, but to simplify notation we do not explicitly indicate this dependence. Also, using the notation $W_{cs} = (\omega_1, \dots, \omega_k)^t$, the parameterization of $\mathcal{M}^{cs}$ can be chosen in such a way that

$$\mathcal{M}^{cs} = \Big\{ (V, W, \xi, \varepsilon) \text{ such that } W = \Phi(V, W_{cs}, \xi, \varepsilon) \Big\} \tag{3.3}$$

for a suitable function Φ. In particular, if we restrict system (2.11) to $\mathcal{M}^{cs}$, then our unknowns are the functions $V(\zeta)$, $W_{cs}(\zeta)$, $\xi(\zeta)$, $\varepsilon(\zeta)$. In the following lemma, we derive the ODE satisfied by $(V, W_{cs}, \xi, \varepsilon)$.

Lemma 3.1. Let $\mathcal{M}^{cs}$ be the aforementioned center-stable manifold of (2.11). If the constant δ is sufficiently small, then any solution of (2.11) lying on $\mathcal{M}^{cs}$ and satisfying bounds (3.2) is also a solution of the system

$$\begin{cases} V' &= \Psi \, W_{cs}, \\ W'_{cs} &= \Big(\Lambda - \xi \Upsilon \Big) W_{cs}, \\ \xi' &= \varepsilon, \\ \varepsilon' &= 0. \end{cases} \tag{3.4}$$

Here, the function $\Psi = \Psi(V, W_{cs}, \xi, \varepsilon)$ takes values in the space of $n \times k$ matrices and both the functions $\Lambda = \Lambda(V, W_{cs}, \xi, \varepsilon)$ and $\Upsilon = \Upsilon(V, W_{cs}, \xi, \varepsilon)$ take values in the space of $k \times k$ matrices. Moreover, $\Lambda(\bar{U}, \vec{0}, 0, 0)$ is the diagonal matrix with eigenvalues $\lambda_1(\bar{U}), \dots, \lambda_k(\bar{U})$, while $\Upsilon(\bar{U}, \vec{0}, 0, 0)$ is the identity matrix.

Proof. First, we observe that the vector $(V, W = \vec{0}, \xi, \varepsilon)$ belongs to $\mathcal{M}^{cs}$ for any V, ξ and ε. Hence,

$$\Phi(V, W_{cs} = \vec{0}, \xi, \varepsilon) = \vec{0} \quad \text{for all } V, \ \xi, \ \varepsilon,$$

where Φ is the function given in (3.3). By relying on the regularity of the map Φ and using (3.3), we get the expression

$$W = \Phi(V, W_{cs}, \xi, \varepsilon) = \Phi_{cs}(V, W_{cs}, \xi, \varepsilon) W_{cs} \tag{3.5}$$

for a suitable regular map Φ_{cs} taking values in the space of the $n \times k$ matrices. Since the manifold $\mathcal{M}^{cs}$ is tangent to the center-stable space at the equilibrium $(V, W_{cs}, \xi, \varepsilon) = (\bar{U}, \vec{0}, 0, 0)$, then the columns of the matrix $\Phi_{cs}(\bar{U}, \vec{0}, 0, 0)$ are the eigenvectors $R_1(\bar{U}), \dots R_k(\bar{U})$:

$$\Phi_{cs}(\bar{U}, \vec{0}, 0, 0) = \Big(R_1(\bar{U}) | \dots | R_k(\bar{U}) \Big). \tag{3.6}$$

It should be noted that the first equation of system (2.11) can be written as

$$V' = \Big(R_1(V) | \dots | R_n(V) \Big) W$$

and thus, by substituting (3.5) above, we get that

$$V' = \Psi(V, W_{cs}, \xi, \varepsilon) W_{cs}, \tag{3.7}$$

for a suitable matrix Ψ taking values in the space of the $n \times k$ matrices.

Next, we differentiate (3.5) to get

$$W' = \Phi'_{cs} W_{cs} + \Phi_{cs} W'_{cs}.$$

We also have

$$\begin{aligned} \Phi'_{cs}(V, W_{cs}, \xi, \varepsilon) W_{cs} &= BV' + CW'_{cs} + \xi' \partial_\xi \Phi_{cs} W_{cs} \\ &= B \Psi W_{cs} + CW'_{cs} + \varepsilon (\partial_\xi \Phi_{cs}) W_{cs} \ , \end{aligned} \tag{3.8}$$

where the functions $B = B(V, W_{cs}, \xi, \varepsilon)$ and $C = C(V, W_{cs}, \xi, \varepsilon)$ take values in the spaces of $n \times n$ and $n \times k$ matrices, respectively, and satisfy

$$B(V, \vec{0}, \xi, \varepsilon) = \mathbf{0}_{n \times n}, \quad C(V, \vec{0}, \xi, \varepsilon) = \mathbf{0}_{n \times k} \quad \text{for every } V, \ \xi \text{ and } \varepsilon. \tag{3.9}$$

The exact expressions of B and C are not relevant here.

By substituting (3.8) into the second equation of (2.11), we obtain

$$\begin{aligned} W' &= \Phi_{cs} W'_{cs} + B \Psi W_{cs} + CW'_{cs} + \varepsilon (\partial_\xi \Phi_{cs}) W_{cs} \\ &= \Big(\mathrm{Diag}\big(\lambda_1, \dots, \lambda_n\big) - \xi I \Big) \Phi_{cs} W_{cs} + F W_{cs}, \end{aligned} \tag{3.10}$$

where $\mathrm{Diag}\big(\lambda_1, \dots, \lambda_n\big)$ denotes the $n \times n$ diagonal matrix with eigenvalues $\lambda_1(V), \dots, \lambda_n(V)$ and I stands for the $n \times n$ identity matrix. The function $F(V, W_{cs}, \xi, \varepsilon) W_{cs}$ takes values in the space of the $n \times n$ matrices and comes from the quadratic terms in the second equation of (2.11). Hence, by choosing a sufficiently small constant δ in (3.2), we have the bound

$$|F(V, W_{cs}, \xi, \varepsilon)| \le K |W_{cs}| \tag{3.11}$$

for some suitable constant $K > 0$. By rewriting (3.10), we infer that

$$\Big(C + \Phi_{cs} \Big) W'_{cs} = \Big(\mathrm{Diag}\big(\lambda_1, \dots, \lambda_n\big) - \xi I_n \Big) \Phi_{cs} W_{cs} + F W_{cs} - B \Psi W_{cs} - \varepsilon (\partial_\xi \Phi_{cs}) W_{cs} \tag{3.12}$$

and by relying on (3.6) and (3.9), we deduce that

$$\Big(C + \Phi_{cs} \Big) (\bar{U}, \vec{0}, 0, 0) = \Big(R_1(\bar{U}) | \dots | R_k(\bar{U}) \Big);$$

hence all the columns of this matrix are linearly independent at $(\bar{U}, \vec{0}, 0, 0)$. By continuity, the columns of the matrix $C + \Phi_{cs}$ computed at $(V, W_{cs}, \xi, \varepsilon)$ are linearly independent, provided that $(V, W_{cs}, \xi, \varepsilon)$ satisfies (3.2) for a sufficiently small constant δ. Hence, there exists a function $L(V, W_{cs}, \xi, \varepsilon)$ taking values in the space of the $k \times n$ matrices and satisfying

$$L(V, W_{cs}, \xi, \varepsilon) \Big(C + \Phi_{cs} \Big) (V, W_{cs}, \xi, \varepsilon) \equiv I_k \ .$$

By multiplying (3.12) on the left by L, we arrive at

$$W'_{cs} = \Big(\Lambda - \xi \Upsilon \Big) W_{cs},$$

where

$$\Lambda(V, W_{cs}, \xi, \varepsilon) = L \mathrm{Diag}\big(\lambda_1, \dots, \lambda_n\big) \Phi_{cs} + LF - LB\Psi - \varepsilon L (\partial_\xi \Phi_{cs})$$

is a function taking values in the space of $k \times k$ matrices and satisfying

$$\Lambda(\bar{U}, \vec{0}, 0, 0) = \mathrm{Diag}\big(\lambda_1(\bar{U}), \dots, \lambda_k(\bar{U})\big).$$

To get the above equality, we use properties (3.6), (3.9) and (3.11). Finally, $\Upsilon(V, W_{cs}, \xi, \varepsilon) = L \Phi_{cs}$ takes values in the space of $k \times k$ matrices and by relying on (3.6) and (3.9), we deduce that $\Upsilon(\bar{U}, \vec{0}, 0, 0)$ is the $k \times k$ identity matrix. This completes the proof of Lemma 3.1. □

3.2. *Construction of the approximating sequence.* Here, we construct a solution to the ODE (3.4) derived in the previous subsection. Let $\{\bar{U}^\varepsilon\} \subseteq \mathbb{R}^n$ be a family of data as in the statement of Theorem 1.1, namely $\bar{U}^\varepsilon \to \bar{U}$ as $\varepsilon \to 0^+$.

To begin with, we fix a positive constant $\bar{\varepsilon} > 0$ and a vector $\bar{W}_{cs} \in \mathbb{R}^k$. Roughly speaking, the goal is to construct a solution to (3.4), hence lying on $\mathcal{M}^{cs}$, such that $W_{cs}(0) = \bar{W}_{cs}$, $\varepsilon(\zeta) = \bar{\varepsilon}$, $\lim_{\zeta \to +\infty} V(\zeta) = \bar{U}^\varepsilon$ and $\xi(0) = 0$. However, in this problem a technical difficulty arises since by imposing $\xi(0) = 0$, system (3.4) yields $\xi(\zeta) = \bar{\varepsilon}\zeta$, whereas the manifold $\mathcal{M}^{cs}$ is defined for $|\xi| \leq \delta$ (see formula (3.2)). To overcome this difficulty, we preserve this bound $|\xi| \leq \delta$ by restricting to the interval $\zeta \in [0, \delta/\bar{\varepsilon}]$ and replacing the condition $\lim_{\zeta \to +\infty} V(\zeta) = \bar{U}^\varepsilon$ with $V(\delta/\bar{\varepsilon}) = \bar{U}^\varepsilon$. Also, to simplify the notation, we use ε instead of $\bar{\varepsilon}$ and we write V^ε and W^ε_{cs} to emphasize the dependence of the first two components of the solution to (3.4) on the parameter ε. After applying the variation of constants formula to (3.4), we obtain the fixed point problem

$$
\begin{cases}
V^\varepsilon(\zeta) = \bar{U}^\varepsilon + \displaystyle\int_{\delta/\varepsilon}^{\zeta} \Psi\Big(V^\varepsilon(y), W^\varepsilon_{cs}(y), \varepsilon y, \varepsilon\Big) W^\varepsilon_{cs}(y)\, dy \,, \\[2ex]
W^\varepsilon_{cs}(\zeta) = \exp\Big(\bar{\Lambda}\zeta - \varepsilon\zeta^2/2\, I_k\Big)\bar{W}_{cs} \\
\qquad\qquad + \displaystyle\int_0^{\zeta} \exp\Big(\bar{\Lambda}(\zeta - y) - I_k\varepsilon\zeta^2/2 + I_k\varepsilon y^2/2\Big)\bigg\{\bigg[\Lambda\Big(V^\varepsilon(y), W^\varepsilon_{cs}(y), \varepsilon y, \varepsilon\Big) - \bar{\Lambda}\bigg] \\
\qquad\qquad - \varepsilon y\bigg[\Upsilon\Big(V^\varepsilon(y), W^\varepsilon_{cs}(y), \varepsilon y, \varepsilon\Big) - I_k\bigg]\bigg\} W^\varepsilon_{cs}(y)\, dy,
\end{cases}
\tag{3.13}
$$

where $\bar{\Lambda} = \Lambda(\bar{U}, \vec{0}, 0, 0)$. We recall that $\bar{\Lambda}$ is a diagonal matrix having only negative eigenvalues and $\Upsilon(\bar{U}, \vec{0}, 0, 0) = I_k$. The following lemma provides existence and uniqueness results for the solution of (3.13).

Lemma 3.2. Let V^ε and W^ε_{cs} be as in the statement of Lemma 3.1. If the constant δ in the statement of Lemma 3.1 is sufficiently small, then we can find constants c_V, θ and c_W which do not depend on ε and satisfy the following properties: $0 < c_V, \theta, c_W < 1$ and for every sufficiently small $\varepsilon > 0$ and for $\bar{W}_{cs} \in \mathbb{R}^k$ with

$$|\bar{W}_{cs}| \leq \theta\delta, \tag{3.14}$$

we have that system (3.13) admits a unique solution satisfying

$$|V^\varepsilon(\zeta) - \bar{U}| \leq c_V\delta \quad \text{and} \quad |W^\varepsilon_{cs}(\zeta)| \leq c_W \exp\big(-c\,\zeta/2\big)\delta\,, \tag{3.15}$$

for every $\zeta \in [0, \delta/\varepsilon]$.

Proof. The proof relies on a standard fixed point argument based on the Contraction Map Theorem. So here, we provide only a sketch of the proof.

Fix ε and $\bar{W}_{cs}$ as above and define the spaces

$$X^\varepsilon_V := \big\{V^\varepsilon \in \mathcal{C}^0\big([0, \delta/\varepsilon]\,; \mathbb{R}^n\big) \text{ such that } |V^\varepsilon(\zeta) - \bar{U}| \leq c_V\delta \text{ for every } \zeta \in [0, \delta/\varepsilon]\big\}$$

and

$$X^\varepsilon_W := \big\{W^\varepsilon_{cs} \in \mathcal{C}^0\big([0, \delta/\varepsilon]\,; \mathbb{R}^k\big) : \ |W^\varepsilon_{cs}(\zeta)| \leq c_W\delta \exp\big(-c\,\zeta/2\big) \text{for every } \zeta \in [0, \delta/\varepsilon]\big\}\,.$$

The constants c_V and c_W are to be determined. By equipping X_V and X_W with the norms

$$\|V^\varepsilon\|_V := \eta\|V^\varepsilon\|_{\mathcal{C}^0([0,\delta/\varepsilon])} \quad \text{and} \quad \|W^\varepsilon_{cs}\|_W := \sup_{\zeta\in[0,\delta/\varepsilon]} \exp\big(c\,\zeta/2\big)|W^\varepsilon_{cs}(\zeta)|,$$

we obtain two closed metric spaces. In the previous expression, η denotes a real constant to be determined and c is given by (1.7).

Now, we define the map

$$T^\varepsilon : X^\varepsilon_V \times X^\varepsilon_W \to \mathcal{C}^0\big([0,\delta/\varepsilon]\,;\mathbb{R}^n\big) \times \mathcal{C}^0\big([0,\delta/\varepsilon]\,;\mathbb{R}^k\big)$$

by setting its components $T^\varepsilon_1(V^\varepsilon, W^\varepsilon_{cs})$ and $T^\varepsilon_2(V^\varepsilon, W^\varepsilon_{cs})$ equal to the first and the second component of the right-hand side of (3.13), respectively.

By direct computations, one can show that it is possible to choose the constants c_W, c_V, θ and η in such a way that T^ε takes values in the set $X^\varepsilon_V \times X^\varepsilon_W$ and is, moreover, a strict contraction with respect to both the variables V^ε and W^ε_{cs}. We recall that θ is the same constant as in (3.14). The values of c_W, c_V, θ and η depend on δ, but are independent of ε. By applying the Contraction Map Theorem, we conclude that (3.13) admits a unique solution belonging to $X^\varepsilon_V \times X^\varepsilon_W$.

To give a flavour of the construction, we choose to show that

$$\|T^\varepsilon_1(V^\varepsilon_1, W^\varepsilon_{cs1}) - T^\varepsilon_1(V^\varepsilon_2, W^\varepsilon_{cs2})\|_V \le \frac{1}{2}\big(\|V^\varepsilon_1 - V^\varepsilon_2\|_V + \|W^\varepsilon_{cs1} - W^\varepsilon_{cs2}\|_W\big) \tag{3.16}$$

for $(V^\varepsilon_i, W^e_{cs\,i}) \in X^\varepsilon_V \times X^\varepsilon_W$, $i = 1, 2$. Indeed, we estimate

$$\begin{aligned}
&|T^\varepsilon_1(V^\varepsilon_1,W^\varepsilon_{cs1}) - T^\varepsilon_1(V^\varepsilon_2, W^\varepsilon_{cs2})|(\zeta)\\
&\qquad\le \int_0^{\delta/\varepsilon} \Big|\Big[\Psi\big(V^\varepsilon_1(y), W^\varepsilon_{cs1}(y), \varepsilon y, \varepsilon\big) - \Psi\big(V^\varepsilon_2(y), W^\varepsilon_{cs2}(y), \varepsilon y, \varepsilon\big)\Big] W^\varepsilon_{cs1}(y)\Big|\, dy\\
&\qquad\quad + \int_0^{\delta/\varepsilon} \Big|\Psi\big(V^\varepsilon_2(y), W^\varepsilon_{cs2}(y), \varepsilon y, \varepsilon\big)\Big[W^\varepsilon_{cs1}(y) - W^\varepsilon_{cs2}(y)\Big]\Big| dy\\
&\qquad\le \int_0^{\delta/\varepsilon} \tilde{L}\Big(|V^\varepsilon_1 - V^\varepsilon_2|(y) + |W^\varepsilon_{cs1} - W^\varepsilon_{cs2}|(y)\Big) c_W \delta \exp\big(-c\,y/2\big)\, dy\\
&\qquad\quad + \int_0^{\delta/\varepsilon} M|W^\varepsilon_{cs1} - W^\varepsilon_{cs2}|(y)\, dy\ ,
\end{aligned}$$

where $\tilde{L}$ denotes a Lipschitz constant of the function Ψ with respect to both the variables V^ε and W^ε_{cs}, while M is a uniform bound for Ψ in the neighborhood of $(\bar{U}, \vec{0}, 0, 0)$ given by (3.2), namely

$$|\Psi\big(V^\varepsilon, W^\varepsilon_{cs}, \xi, \varepsilon\big)| \le M\ , \tag{3.17}$$

for every $(V^\varepsilon, W^\varepsilon_{cs}, \xi, \varepsilon)$ such that $|V^\varepsilon - \bar{U}| \le \delta$, $|W^\varepsilon_{cs}| \le \delta$, $|\xi| \le \delta$ and $|\varepsilon| \le \delta$. By using the estimate

$$|W^\varepsilon_{cs1} - W^\varepsilon_{cs2}|(y) \le \exp\big(-c\,y/2\big)\|W^\varepsilon_{cs1} - W^\varepsilon_{cs2}\|_W \quad \text{for } 0 \le y \le \delta/\varepsilon,$$

we arrive at

$$\begin{aligned}
\|T^\varepsilon_1(V^\varepsilon_1, W^\varepsilon_{cs1}) - T^\varepsilon_1(V^\varepsilon_2, W^\varepsilon_{cs2})\|_V \le \frac{2}{c}\Big[&\tilde{L}c_W\delta\|V^\varepsilon_1 - V^\varepsilon_2\|_V\\
&+ (\tilde{L}c_W\delta\eta + M\eta)\|W^\varepsilon_{cs1} - W^\varepsilon_{cs2}\|_W\Big],
\end{aligned}$$

which implies (3.16) provided that δ and η are sufficiently small. □

3.3. *The structure of the limit.* We now establish the convergence result and analyze the limit.

LEMMA 3.3. Let the hypotheses of Lemma 3.2 be satisfied, let δ, c_V and c_W be the same constants as in the statement of Lemma 3.2 and denote by $(V^\varepsilon, W^\varepsilon_{cs})$ the solution of system (3.13) satisfying bounds (3.15). Then, as $\varepsilon \to 0^+$, the family $\{(V^\varepsilon, W^\varepsilon_{cs})\}$ converges uniformly on compact subsets of $[0, +\infty[$. The limit V^0 is the unique solution of the ordinary differential equation

$$(V^0)'' = A(V^0)(V^0)' \tag{3.18}$$

satisfying the conditions

$$(V^0)'(0) = \bar{W}_{cs} \quad \text{and} \quad \lim_{\zeta \to +\infty} V^0(\zeta) = \bar{U} \tag{3.19}$$

and the bounds

$$|(V^0)'(\zeta)| \le c_W \exp\big(-c\,\zeta/2\big)\delta, \qquad |V^0(\zeta) - \bar{U}| \le c_V\delta \quad \text{for every } \zeta \in [0, +\infty[. \tag{3.20}$$

Moreover, V^0 satisfies

$$|V^0(\zeta) - \bar{U}| \le \tilde{c}_V\delta \exp\big(-c\,\zeta/2\big) \quad \text{for every } \zeta \in [0, +\infty[, \tag{3.21}$$

for a suitable constant $\tilde{c}_V > 0$ depending only on $\bar{U}$, c_W, c and on the upper bound M given by (3.17).

Proof. We proceed in three steps.

STEP 1. First, we establish compactness of the sequence $\{(V^\varepsilon, W^\varepsilon_{cs})\}$. Let M denote a constant such that

$$|\Psi\big(V^\varepsilon, W^\varepsilon_{cs}, \xi, \varepsilon\big)|,\ |\Lambda\big(V^\varepsilon, W^\varepsilon_{cs}, \xi, \varepsilon\big)|,\ |\Upsilon\big(V^\varepsilon, W^\varepsilon_{cs}, \xi, \varepsilon\big)| \le M \tag{3.22}$$

for every $|V^\varepsilon - \bar{U}| \le \delta$, $|W^\varepsilon_{cs}| \le \delta$, $|\xi| \le \delta$ and $|\varepsilon| \le \delta$ and fix a compact subset $[0, b]$ of $[0, +\infty[$. By relying on (3.15), we get that V^ε satisfies the bounds

$$\|V^\varepsilon - \bar{U}\|_{\mathcal{C}^0([0,b])} \le c_V\delta \quad \text{and} \quad \|(V^\varepsilon)'\|_{\mathcal{C}^0([0,b])} = \|\Psi W^\varepsilon_{cs}\|_{\mathcal{C}^0([0,b])} \le M c_W\delta \tag{3.23}$$

for every $0 < \varepsilon \le \delta/b$. Also, W^ε_{cs} satisfies

$$\|W^\varepsilon_{cs}\|_{\mathcal{C}^0([0,b])} \le c_W\delta \tag{3.24}$$

and

$$\big\|(W^\varepsilon_{cs})'\big\|_{\mathcal{C}^0([0,b])} = \big\|\big(\Lambda - \varepsilon\zeta\Upsilon\big)W^\varepsilon_{cs}\big\|_{\mathcal{C}^0([0,b])} \le M(1+\delta)c_W\delta\,. \tag{3.25}$$

From (3.23)–(3.25), we get that the family $\{(V^\varepsilon, W^\varepsilon_{cs})\}$ satisfies all the hypotheses of Ascoli-Arzelà's Theorem. Hence, for any given sequence $\varepsilon_n \to 0^+$, there exists a subsequence $\{\varepsilon_{n_k}\}$ and a pair (V^0, W^0_{cs}) such that

$$V^{\varepsilon_{n_k}} \to V^0 \quad \text{and} \quad W^{\varepsilon_{n_k}}_{cs} \to W^0_{cs},$$

uniformly on $[0, b]$ as $\varepsilon_{n_k} \to 0^+$. We then consider a sequence $\{b_j\}$ of points such that $b_j \to +\infty$ and apply the previous argument on any interval $[0, b_j]$. By relying on a standard diagonalization procedure, we conclude that for any sequence $\varepsilon_n \to 0^+$ there exists a subsequence ε_{n_k} and a pair (V^0, W^0_{cs}) such that

$$V^{\varepsilon_{n_k}} \to V^0 \quad \text{and} \quad W^{\varepsilon_{n_k}}_{cs} \to W^0_{cs}$$

uniformly on the compact subsets of $[0, +\infty[$ as $\varepsilon_{n_k} \to 0^+$. In particular, $\{(V^{\varepsilon_{n_k}}, W^{\varepsilon_{n_k}}_{cs})\}$ converges pointwise on $[0, +\infty[$ and hence from bounds (3.15) we deduce

$$|V^0(\zeta) - \bar{U}| \le c_V \delta \quad \text{and} \quad |W^0_{cs}(\zeta)| \le c_W \exp\big(-c\,\zeta/2\big)\delta \quad \text{for every } \zeta \in [0, +\infty[. \tag{3.26}$$

STEP 2. Next, we prove that the limit (V^0, W^0_{cs}) is independent of the subsequence $\{\varepsilon_{n_k}\}$. By applying Lebesgue's Dominated Convergence Theorem to (3.13), we get the identities

$$\begin{cases} V^0(\zeta) = \bar{U} + \displaystyle\int_{+\infty}^{\zeta} \Psi\Big(V^0(y), W^0_{cs}(y), 0, 0\Big) W^0_{cs}(y)\, dy, \\ \\ W^0_{cs}(\zeta) = \exp\big(\bar{\Lambda}\zeta\big)\bar{W}_{cs} + \displaystyle\int_0^{\zeta} \exp\big(\bar{\Lambda}(\zeta - y)\big)\Big[\Lambda\Big(V^0(y), W^0_{cs}(y), 0, 0\Big) - \bar{\Lambda}\Big] W^0_{cs}(y) dy\,. \end{cases} \tag{3.27}$$

By relying on a fixed point argument similar to the one in the proof of Lemma 3.2, we get that the solution (V^0, W^0_{cs}) of (3.27) satisfying bounds (3.26) is unique.

STEP 3. Last, we establish bound (3.21). By relying on the representation formula (3.27) and on estimates (3.22) and (3.26), we deduce that, for every $\zeta \in [0, +\infty[$,

$$|V^0(\zeta) - \bar{U}| \le \frac{2Mc_W}{c} \exp\big(-c\,\zeta/2\big)\delta := \tilde{c}_V \exp\big(-c\,\zeta/2\big)\delta.$$

Finally, by comparing (3.27) with (2.11) and (3.4), we obtain that V^0 is a solution of the ODE (3.18) satisfying conditions (3.19). $\square$

3.4. *Conclusion of the proof of Theorem* 1.1. Here, we complete the proof of Theorem 1.1.

We first denote by $V^s(\bar{U})$ and $\mathcal{M}^s(\bar{U})$ the stable space and stable manifold, respectively, for system (1.12) corresponding to the equilibrium point $(\bar{U}, \vec{0})$. Then, we change coordinates using matrix (3.6). More precisely, every point $(V, W) \in V^s(\bar{U})$ can be written in the form $W = \Phi_{cs}(\bar{U}, 0, 0, 0)\bar{W}_{cs}$ for some $\bar{W}_{cs} \in \mathbb{R}^k$. Next, we let θ and δ be as in Lemma 3.2 and for given $\bar{W}_{cs} \in \mathbb{R}^k$ satisfying $|\bar{W}_{cs}| \le \theta\delta$, we consider the unique solution (V^0, W^0_{cs}) of (3.27) satisfying bounds (3.26). Moreover, by varying $\bar{W}_{cs}$, we define the set

$$\mathcal{M} \doteq \Big\{\big(V^0(0), \Phi_{cs}(\bar{U}, 0, 0, 0)\bar{W}_{cs}\big) :\, \bar{W}_{cs} \in \mathbb{R}^k, \quad |\bar{W}_{cs}| \le \theta\delta\Big\} \subseteq \mathbb{R}^n \times \mathbb{R}^n.$$

Now, we claim that the set $\mathcal{M}$ coincides with the stable manifold $\mathcal{M}^s(\bar{U})$. To prove this claim, we first observe that system (1.12) is equivalent to system (2.11) provided that (2.11) is restricted to the invariant space $\{(V, W, \xi, \varepsilon) :\ \xi = \varepsilon = 0\} \subseteq \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$. Hence, $\{(V, W, \xi, \varepsilon) :\ (V, W) \in \mathcal{M}^s(\bar{U}),\ \xi = \varepsilon = 0\} \subseteq \mathcal{M}^{cs}$ for any given center-stable manifold $\mathcal{M}^{cs}$ of (2.11) corresponding to the equilibrium point $(\bar{U}, 0, 0, 0)$, and we recall that, by Lemma 3.1, solutions lying on $\mathcal{M}^{cs}$ satisfy (3.4). Thus, to study the stable manifold $\mathcal{M}^s(\bar{U})$ of (1.12), it suffices to study the stable manifold to system (3.4). By recalling the proof of the Stable Manifold Theorem (see for example Perko [24, pp. 104-108]) and the fact that (V^0, W^0_{cs}) satisfies (3.26) and (3.27) we finally establish that $\mathcal{M} = \mathcal{M}^s(\bar{U})$. By taking $U_b = V^0(0)$ and by relying on Lemma 3.3, the proof of Part (2) of Theorem 1.1 is complete. Part (1) follows easily from Part (2) by defining $Q^\varepsilon(\xi) = V^\varepsilon(\xi/\varepsilon)$ for $\xi \in [0, \delta]$.

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