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A GENERALIZED CONSERVED PHASE-FIELD SYSTEM BASED ON TYPE III HEAT CONDUCTION

By

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Abstract. In this paper, we are interested in the study of the asymptotic behavior, in terms of finite-dimensional attractors, of a generalization of the conserved phase-field system proposed by G. Caginalp. This model is based on a heat conduction law recently proposed in the context of thermoelasticity and known as type III law. In particular, we prove the existence of exponential attractors and, thus, of finite-dimensional global attractors.

1. Introduction. G. Caginalp introduced in [6] (see also [7]) the following phase-field system:

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta\theta, \tag{1.1}$$

$$\frac{\partial \theta}{\partial t} - \Delta\theta = -\frac{\partial u}{\partial t}, \tag{1.2}$$

where u is the order parameter and θ is the (relative) temperature. These equations model phase transition processes such as melting/solidification and have been studied, e.g., in [3], [4] and [22]; see also, e.g., [1], [11], [12], [18], [34] and [35] for a similar phase-field model with a memory term. Equations (1.1)-(1.2) consist of the coupling of the Cahn-Hilliard equation introduced in [8] and [9] with the heat equation.

These equations are known as the conserved phase-field model in the sense that, when endowed with Neumann boundary conditions, the spatial average of the order parameter is a conserved quantity. Indeed, in that case, integrating (1.1) over the spatial domain Ω (we assume, throughout this paper, that Ω is a bounded and regular domain of $\mathbb{R}^n$, $n = 2$ or 3), we have the conservation of mass,

$$\langle u(t)\rangle = \langle u(0)\rangle,\ t \geq 0, \tag{1.3}$$

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where

$$\langle . \rangle = \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega \cdot \, dx \tag{1.4}$$

denotes the spatial average. Furthermore, integrating (1.2) over Ω, we obtain

$$\langle H(t) \rangle = \langle H(0) \rangle, \ t \geq 0, \tag{1.5}$$

where

$$H = u + \theta \tag{1.6}$$

is the enthalpy, which also yields, owing to (1.3), the conservation of the temperature,

$$\langle \theta(t) \rangle = \langle \theta(0) \rangle, \ t \geq 0. \tag{1.7}$$

We note that the generalized heat equation (1.2) is based on the usual Fourier law for heat conduction. Indeed, we can rewrite this equation as

$$\frac{\partial H}{\partial t} = -\mathrm{div} q, \tag{1.8}$$

where q is the thermal flux vector and, assuming the Fourier law

$$q = -\nabla \theta, \tag{1.9}$$

we recover (1.2).

Now, one drawback of the Fourier law is that it predicts that thermal signals propagate with an infinite speed, which violates causality (see, e.g., [10]).

Therefore, several alternative laws have been proposed and studied in [25], [26], [27], [28], [29], [30] and [31]; one essential feature of these alternative models is that one ends up with a second-order (in time) equation for the temperature. We note that these references deal with the (nonconserved) Caginalp model introduced in [5],

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \theta, \tag{1.10}$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t}. \tag{1.11}$$

We consider in this paper the following generalization of the conserved Caginalp phase-field system:

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta \frac{\partial \alpha}{\partial t}, \tag{1.12}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t}, \tag{1.13}$$

associated with Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = \frac{\partial \alpha}{\partial \nu} = 0 \text{ on } \Gamma, \tag{1.14}$$

where Γ is the boundary of Ω and ν is the unit outer normal to Γ, with initial conditions

$$u|_{t=0} = u_0, \ \alpha|_{t=0} = \alpha_0, \ \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1. \tag{1.15}$$

In these equations, α is the thermal displacement variable, defined by

$$\alpha = \int_0^t \theta \, d\tau + \alpha_0 \ (\theta = \frac{\partial \alpha}{\partial t}) \tag{1.16}$$

(here, α_0 is a priori chosen arbitrarily).

Furthermore, they are obtained by considering the following heat conduction law:

$$q = -k\nabla\alpha - k^\star\nabla\theta,\ k,\ k^\star > 0. \tag{1.17}$$

This law was derived in the context of an alternative treatment for a thermomechanical theory of deformable media proposed by A. E. Green and P. M. Naghdi in [23] and [24]. In particular, (1.13) follows from (1.8) and (1.17), taking k and k' equal to one (recall that $\theta = \frac{\partial\alpha}{\partial t}$).

Remark 1.1. Actually, this system still has an infinite propagation speed, due to the parabolic nature of (1.12). A fully hyperbolic model, i.e., a hyperbolic relaxation of the equation for the order parameter, is considered in [26] in the nonconserved case for the Maxwell-Cattaneo law

$$\frac{\partial}{\partial t}(q + \frac{\partial q}{\partial t}) = -\nabla\theta. \tag{1.18}$$

However, the hyperbolic relaxation of the Cahn-Hilliard equation is a particularly difficult problem in space dimensions greater than one (see, e.g., [19], [20] and [21]).

Integrating (1.12) over Ω, we again have the conservation of mass,

$$\langle u(t)\rangle = \langle u_0\rangle,\ t \geq 0 \tag{1.19}$$

and, integrating (1.13) over Ω, we recover the conservation of the enthalpy,

$$\langle H(t)\rangle = \langle u(t) + \frac{\partial\alpha}{\partial t}(t)\rangle = \langle H(0)\rangle = \langle u_0 + \alpha_1\rangle,\ t \geq 0, \tag{1.20}$$

and we have the conservation of the thermal displacement variable,

$$\langle\frac{\partial\alpha}{\partial t}(t)\rangle = \langle\alpha_1\rangle,\ t \geq 0, \tag{1.21}$$

which also yields

$$\langle\alpha(t)\rangle = \langle\alpha_0\rangle + t\langle\alpha_1\rangle,\ t \geq 0. \tag{1.22}$$

In particular, it follows from (1.22) that the variable α is not dissipative in the sense that it is not bounded, independently of the initial conditions, for t large; we also note that, for every $c \neq 0$, $(u, \alpha) = (0, ct)$ is solution to (1.12)-(1.14).

Our aim in this paper is to study the asymptotic behavior in terms of finite-dimensional attractors of the dynamical system associated with (1.12)-(1.15). In particular, we prove the existence of exponential attractors and, thus, of finite-dimensional global attractors.

2. Setting of the problem. We rewrite, in view of (1.19) and (1.20), the problem in the following (at least formally) equivalent form:

$$\frac{\partial\overline{u}}{\partial t} + \Delta^2\overline{u} - \Delta(f(u) - \langle f(u)\rangle) = -\Delta\frac{\partial\overline{\alpha}}{\partial t}, \tag{2.1}$$

$$\frac{\partial^2\overline{\alpha}}{\partial t^2} - \Delta\frac{\partial\overline{\alpha}}{\partial t} - \Delta\overline{\alpha} = -\frac{\partial\overline{u}}{\partial t}, \tag{2.2}$$

$$\frac{\partial\overline{u}}{\partial\nu} = \frac{\partial\overline{\alpha}}{\partial\nu} = 0 \text{ on } \Gamma, \tag{2.3}$$

$$\overline{u}|_{t=0} = \overline{u}_0,\ \overline{\alpha}|_{t=0} = \overline{\alpha}_0,\ \frac{\partial\overline{\alpha}}{\partial t}|_{t=0} = \overline{\alpha}_1, \tag{2.4}$$

where

$$\overline{v} = v - \langle v \rangle,$$

and we have

$$u = \overline{u} + \langle u_0 \rangle, \tag{2.5}$$

$$\frac{\partial \alpha}{\partial t} = \frac{\partial \overline{\alpha}}{\partial t} + \langle \alpha_1 \rangle, \tag{2.6}$$

$$\alpha = \overline{\alpha} + \langle \alpha_0 \rangle + t \langle \alpha_1 \rangle. \tag{2.7}$$

As far as the nonlinear term is concerned, we make the following assumptions:

$$f \text{ is of class } \mathcal{C}^2, \ f(0) = 0, \tag{2.8}$$

$$f'(s) \geq -c_0, \ c_0 \geq 0, \ s \in \mathbb{R}, \tag{2.9}$$

$$f(s)s \geq c_1 F(s) - c_2, \ F(s) \geq -c_3, \ c_1 > 0, \ c_2, \ c_3 \geq 0, \ s \in \mathbb{R}, \tag{2.10}$$

$$|f(s)| \leq \epsilon F(s) + c_\epsilon, \ \forall \epsilon > 0, \ s \in \mathbb{R}, \tag{2.11}$$

where $F(s) = \int_0^s f(\tau)\, d\tau$.

Remark 2.1. In particular, these assumptions are satisfied by polynomials of degree $2p+1$ with a strictly positive leading coefficient, $p \geq 1$ (and, of course, by the usual cubic nonlinear term $f(s) = s^3 - s$).

Remark 2.2. We can also endow (1.12)-(1.13) with Dirichlet boundary conditions,

$$u = \Delta u = \alpha = 0 \text{ on } \Gamma.$$

In that case, we do not need assumption (2.11).

We denote by $\|\cdot\|$ the usual L^2-norm with associated scalar product $((\cdot,\cdot))$, and we note that

$$v \mapsto (\|\overline{v}\|_{-1}^2 + \langle v \rangle^2)^{\frac{1}{2}},$$

where $\|\cdot\|_{-1} = \|(-\Delta)^{-\frac{1}{2}} \cdot\|$, $-\Delta$ denoting the minus Laplace operator with Neumann boundary conditions and acting on functions with null average, is a norm in $H^{-1}(\Omega) = H^1(\Omega)'$ which is equivalent to the usual H^{-1}-one; it is understood here that

$$\langle \cdot \rangle = \frac{1}{\mathrm{Vol}(\Omega)} \langle \cdot, 1 \rangle_{H^{-1}(\Omega), H^1(\Omega)}.$$

We also note that

$$v \mapsto (\|\overline{v}\|^2 + \langle v \rangle^2)^{\frac{1}{2}},$$

$$v \mapsto (\|\nabla v\|^2 + \langle v \rangle^2)^{\frac{1}{2}}$$

and

$$v \mapsto (\|\Delta v\|^2 + \langle v \rangle^2)^{\frac{1}{2}}$$

are norms in $L^2(\Omega)$, $H^1(\Omega)$ and $H^2(\Omega)$, respectively, which are equivalent to the usual ones. We further have the generalized Poincaré inequality

$$\|\overline{v}\| \leq c \|\nabla v\|, \ \forall v \in H^1(\Omega). \tag{2.12}$$

Finally, we denote by $\|\cdot\|_X$ the norm in the Banach space X.

Throughout this paper, the same letter c (and, sometimes, c', c'' and c''') denotes constants which may vary from line to line, or even in the same line. Similarly, the same letter Q denotes monotone increasing (with respect to each argument) functions which may vary from line to line, or even in the same line.

3. A priori estimates. We assume that

$$|\langle u_0\rangle| \le M_1,\ |\langle \alpha_1\rangle| \le M_2 \tag{3.1}$$

for fixed positive constants M_1 and M_2, which yields, owing to (1.19), (1.21) and (1.22),

$$|\langle u(t)\rangle| \le M_1,\ |\langle \frac{\partial \alpha}{\partial t}\rangle| \le M_2,\ t \ge 0 \tag{3.2}$$

and

$$|\langle \alpha(t)\rangle| \le |\langle \alpha_0\rangle| + tM_1,\ t \ge 0. \tag{3.3}$$

We rewrite (2.1) in the following equivalent form:

$$(-\Delta)^{-1}\frac{\partial \overline{u}}{\partial t} - \Delta \overline{u} + f(u) - \langle f(u)\rangle = \frac{\partial \overline{\alpha}}{\partial t}. \tag{3.4}$$

We multiply (3.4) by $\frac{\partial \overline{u}}{\partial t}$ and have, integrating over Ω and by parts,

$$\frac{d}{dt}(\|\nabla u\|^2 + 2\int_\Omega F(u)\,dx) + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 = 2((\frac{\partial \overline{\alpha}}{\partial t}, \frac{\partial \overline{u}}{\partial t})). \tag{3.5}$$

We then multiply (2.2) by $\frac{\partial \overline{\alpha}}{\partial t}$ and obtain

$$\frac{d}{dt}(\|\nabla \alpha\|^2 + \|\frac{\partial \overline{\alpha}}{\partial t}\|^2) + 2\|\nabla \frac{\partial \alpha}{\partial t}\|^2 = -2((\frac{\partial \overline{\alpha}}{\partial t}, \frac{\partial \overline{u}}{\partial t})). \tag{3.6}$$

Summing (3.5) and (3.6), we find

$$\frac{dE_1}{dt} + 2(\|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\nabla \frac{\partial \alpha}{\partial t}\|^2) = 0, \tag{3.7}$$

where

$$E_1 = \|\nabla u\|^2 + 2\int_\Omega F(u)\,dx + \|\nabla \alpha\|^2 + \|\frac{\partial \overline{\alpha}}{\partial t}\|^2. \tag{3.8}$$

We now multiply (3.4) by $\overline{u}$ and have, owing to (2.10), (2.11) and (2.12),

$$\frac{d}{dt}\|\overline{u}\|_{-1}^2 + \|\nabla u\|^2 + c\int_\Omega F(u)\,dx \le c'\|\nabla \frac{\partial \alpha}{\partial t}\|^2 + c''_{M_1},\ c > 0. \tag{3.9}$$

Summing (3.7) and δ_1 times (3.9), where $\delta_1 > 0$ is small enough, we obtain

$$\frac{dE_2}{dt} + c(\|\nabla u\|^2 + 2\int_\Omega F(u)\,dx + \|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\nabla \frac{\partial \alpha}{\partial t}\|^2) \le c'_{M_1},\ c > 0, \tag{3.10}$$

where

$$E_2 = E_1 + \delta_1\|\overline{u}\|_{-1}^2. \tag{3.11}$$

We multiply (2.2) by $\overline{\alpha}$ and find, owing to (2.12),

$$\frac{d}{dt}(\|\nabla \alpha\|^2 + 2((\frac{\partial \overline{\alpha}}{\partial t}, \overline{\alpha}))) + \|\nabla \alpha\|^2 \le \|\frac{\partial u}{\partial t}\|_{-1}^2 + c\|\nabla \frac{\partial \alpha}{\partial t}\|^2. \tag{3.12}$$

We sum (3.10) and δ_2 times (3.12), where $\delta_2 > 0$ is small enough, and we get

$$\frac{dE_3}{dt} + c(E_3 + \|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\nabla \frac{\partial \alpha}{\partial t}\|^2) \le c'_{M_1},\ c > 0, \tag{3.13}$$

where

$$E_3 = E_2 + \delta_2(\|\nabla \alpha\|^2 + 2((\frac{\partial \overline{\alpha}}{\partial t}, \overline{\alpha}))). \tag{3.14}$$

We finally multiply (2.1) by $\overline{u}$ to obtain, owing to (2.8), (2.9) and (2.12),

$$\frac{d}{dt}\|\overline{u}\|^2 + \|\Delta u\|^2 \leq c(\|\nabla u\|^2 + \|\nabla \frac{\partial \alpha}{\partial t}\|^2). \tag{3.15}$$

We sum (3.13) and δ_3 times (3.15), where $\delta_3 > 0$ is small enough, and find, setting

$$E_4 = E_3 + \delta_3 \|\overline{u}\|^2 + \langle u \rangle^2 + \langle \frac{\partial \alpha}{\partial t} \rangle^2, \tag{3.16}$$

an inequality of the form

$$\frac{dE_4}{dt} + c(E_4 + \|u\|^2_{H^2(\Omega)} + \|\frac{\partial u}{\partial t}\|^2_{-1} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^1(\Omega)}) \leq c'_{M_1,M_2},\ c > 0, \tag{3.17}$$

where E_4 satisfies

$$c(\|u\|^2_{H^1(\Omega)} + \int_\Omega F(u)\,dx + \|\overline{\alpha}\|^2_{H^1(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2) + c'_{M_1,M_2} \leq E_4 \tag{3.18}$$

$$\leq c''(\|u\|^2_{H^1(\Omega)} + \int_\Omega F(u)\,dx + \|\overline{\alpha}\|^2_{H^1(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2) + c'''_{M_1,M_2},\ c,\ c'' > 0.$$

In particular, we deduce from (3.17) and (3.18) the dissipative inequality

$$E_4(t) \leq ce^{-c't}(\|u_0\|^2_{H^1(\Omega)} + \int_\Omega F(u_0)\,dx + \|\overline{\alpha}_0\|^2_{H^1(\Omega)} + \|\alpha_1\|^2) + c''_{M_1,M_2},\ c' > 0,\ t \geq 0. \tag{3.19}$$

Furthermore,

$$\int_t^{t+1} (\|u\|^2_{H^2(\Omega)} + \|\frac{\partial u}{\partial t}\|^2_{-1} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^1(\Omega)})\,d\tau \tag{3.20}$$

$$\leq ce^{-c't}(\|u_0\|^2_{H^1(\Omega)} + \int_\Omega F(u_0)\,dx + \|\overline{\alpha}_0\|^2_{H^1(\Omega)} + \|\alpha_1\|^2) + c''_{M_1,M_2},\ c' > 0,\ t \geq 0.$$

We now multiply (2.1) by $\frac{\partial \overline{u}}{\partial t}$ and have, owing to (2.8) and the continuous embedding $H^1(\Omega) \subset \mathcal{C}(\overline{\Omega})$,

$$\frac{d}{dt}\|\Delta u\|^2 + \|\frac{\partial u}{\partial t}\|^2 \leq Q(\|u\|_{H^2(\Omega)}) + 2((\nabla \frac{\partial \alpha}{\partial t}, \nabla \frac{\partial u}{\partial t})). \tag{3.21}$$

Multiplying then (2.2) by $-\Delta \frac{\partial \alpha}{\partial t}$, we obtain

$$\frac{d}{dt}(\|\Delta \alpha\|^2 + \|\nabla \frac{\partial \alpha}{\partial t}\|^2) + \|\Delta \frac{\partial \alpha}{\partial t}\|^2 = -2((\nabla \frac{\partial \alpha}{\partial t}, \nabla \frac{\partial u}{\partial t})). \tag{3.22}$$

Summing finally (3.21) and (3.22), we find

$$\frac{d}{dt}(\|\Delta u\|^2 + \|\Delta \alpha\|^2 + \|\nabla \frac{\partial \alpha}{\partial t}\|^2) + \|\frac{\partial u}{\partial t}\|^2 + \|\Delta \frac{\partial \alpha}{\partial t}\|^2 \leq Q(\|u\|_{H^2(\Omega)}). \tag{3.23}$$

In particular, setting

$$y = \langle u \rangle^2 + \|\Delta u\|^2 + \|\Delta \alpha\|^2 + \|\nabla \frac{\partial \alpha}{\partial t}\|^2 + \langle \frac{\partial \alpha}{\partial t} \rangle^2, \tag{3.24}$$

we deduce from (3.23) an inequality of the form

$$y' \leq Q_{M_1,M_2}(y). \tag{3.25}$$

Let z be the solution to the ordinary differential equation

$$z' = Q_{M_1,M_2}(z),\ z(0) = y(0). \tag{3.26}$$

It follows from the comparison principle that there exists $T_0 \in (0, \frac{1}{2})$ such that

$$y(t) \leq z(t),\ t \leq T_0, \tag{3.27}$$

which yields

$$y(t) \leq Q_{M_1,M_2}(\|u_0\|_{H^2(\Omega)}, \|\overline{\alpha}_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^1(\Omega)}),\ t \leq T_0, \tag{3.28}$$

and hence

$$\begin{aligned} &\|u(t)\|_{H^2(\Omega)} + \|\overline{\alpha}(t)\|_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}(t)\|_{H^1(\Omega)} \\ &\leq Q_{M_1,M_2}(\|u_0\|_{H^2(\Omega)}, \|\overline{\alpha}_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^1(\Omega)}),\ t \leq T_0. \end{aligned} \tag{3.29}$$

We differentiate (3.4) with respect to time and have, owing to (2.2),

$$(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t} - \Delta\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial t} - \langle f'(u)\frac{\partial u}{\partial t}\rangle = \Delta\frac{\partial \alpha}{\partial t} + \Delta\alpha - \frac{\partial u}{\partial t}. \tag{3.30}$$

We multiply (3.30) by $t\frac{\partial u}{\partial t}$ and obtain, owing to (2.9),

$$\frac{d}{dt}(t\|\frac{\partial u}{\partial t}\|_{-1}^2) + \frac{3}{2}t\|\nabla\frac{\partial u}{\partial t}\|^2 \leq ct(\|\frac{\partial u}{\partial t}\|^2 + \|\nabla\alpha\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2) + \|\frac{\partial u}{\partial t}\|_{-1}^2,$$

which yields, owing to the interpolation inequality

$$\|\frac{\partial u}{\partial t}\|^2 \leq c\|\frac{\partial u}{\partial t}\|_{-1}^2\|\nabla\frac{\partial u}{\partial t}\|, \tag{3.31}$$

the following inequality:

$$\frac{d}{dt}(t\|\frac{\partial u}{\partial t}\|_{-1}^2) + t\|\nabla\frac{\partial u}{\partial t}\|^2 \leq ct(\|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\nabla\alpha\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2) + \|\frac{\partial u}{\partial t}\|_{-1}^2. \tag{3.32}$$

We then deduce from (3.19), (3.20), (3.32) and Gronwall's lemma that

$$\|\frac{\partial u}{\partial t}(t)\|_{-1}^2 \leq \frac{1}{t}Q_{M_1,M_2}(\|u_0\|_{H^2(\Omega)}, \|\overline{\alpha}_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^1(\Omega)}),\quad 0 < t \leq T_0. \tag{3.33}$$

We now multiply (3.30) by $\frac{\partial u}{\partial t}$ and find, proceeding as above,

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\nabla\frac{\partial u}{\partial t}\|^2 \leq c(\|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\nabla\alpha\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2), \tag{3.34}$$

which yields, owing again to (3.19), (3.20) and Gronwall's lemma,

$$\|\frac{\partial u}{\partial t}(t)\|_{-1}^2 \leq e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2(\Omega)}, \|\overline{\alpha}_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^1(\Omega)})\|\frac{\partial u}{\partial t}(T_0)\|_{-1}^2,\ t \geq T_0. \tag{3.35}$$

Hence, in view of (3.33),

$$\|\frac{\partial u}{\partial t}(t)\|_{-1}^2 \leq e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2(\Omega)}, \|\overline{\alpha}_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^1(\Omega)}),\ t \geq T_0. \tag{3.36}$$

We rewrite, for $t \geq T_0$ fixed, (3.4) in the form

$$-\Delta\overline{u} + f(u) - \langle f(u)\rangle = h_u(t),\ \frac{\partial \overline{u}}{\partial \nu} = 0 \text{ on } \Gamma, \tag{3.37}$$

where

$$h_u(t) = -(-\Delta)^{-1}\frac{\partial \overline{u}}{\partial t} + \frac{\partial \overline{\alpha}}{\partial t} \tag{3.38}$$

satisfies, owing to (3.19) and (3.35),

$$\|h_u(t)\| \leq e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2(\Omega)}, \|\overline{\alpha}_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^1(\Omega)}),\ t \geq T_0. \tag{3.39}$$

We multiply (3.37) by $\overline{u}$ and have, owing to (2.10), (2.11) and (2.12),

$$\|\nabla u\|^2 + c\int_\Omega F(u)\,dx \le c'\|h_u(t)\|^2 + c''_{M_1},\ c>0. \tag{3.40}$$

We then multiply (3.37) by $-\Delta\overline{u}$ and obtain, owing to (2.9),

$$\|\Delta u\|^2 \le c(\|h_u(t)\|^2 + \|\nabla u\|^2). \tag{3.41}$$

It thus follows from (3.39), (3.40) and (3.41) that

$$\|\Delta u(t)\|^2 \le e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2(\Omega)}, \|\overline{\alpha}_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^1(\Omega)}),\ t\ge T_0, \tag{3.42}$$

and hence

$$\|u(t)\|_{H^2(\Omega)} \le e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2(\Omega)}, \|\overline{\alpha}_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^1(\Omega)}),\ t\ge T_0. \tag{3.43}$$

We come back to (3.22), from which it follows that

$$\frac{d}{dt}(\|\Delta\alpha\|^2 + \|\nabla\frac{\partial\alpha}{\partial t}\|^2) + \|\Delta\frac{\partial\alpha}{\partial t}\|^2 \le \|\frac{\partial u}{\partial t}\|^2. \tag{3.44}$$

Noting that it follows from (3.19), (3.20), (3.34) and (3.36) that

$$\int_{T_0}^t \|\nabla\frac{\partial u}{\partial t}\|^2\,d\tau \le e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2(\Omega)}, \|\overline{\alpha}_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^1(\Omega)}),\ t\ge T_0, \tag{3.45}$$

we deduce from (3.44) and (3.45) that

$$\|\Delta\alpha(t)\|^2 + \|\nabla\frac{\partial\alpha}{\partial t}(t)\|^2 \le e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2(\Omega)}, \|\overline{\alpha}_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^1(\Omega)})$$
$$+\|\Delta\alpha(T_0)\|^2 + \|\nabla\frac{\partial\alpha}{\partial t}(T_0)\|^2,\ t\ge T_0;$$

hence, owing to (3.29),

$$\|\overline{\alpha}(t)\|_{H^2(\Omega)} + \|\frac{\partial\alpha}{\partial t}(t)\|_{H^1(\Omega)} \le e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2(\Omega)}, \|\overline{\alpha}_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^1(\Omega)}),\ t\ge T_0. \tag{3.46}$$

We finally deduce from (3.29), (3.43) and (3.46) that

$$\|u(t)\|_{H^2(\Omega)} + \|\overline{\alpha}(t)\|_{H^2(\Omega)} + \|\frac{\partial\alpha}{\partial t}(t)\|_{H^1(\Omega)} \tag{3.47}$$
$$\le e^{ct}Q_{M_1,M_2}(\|u_0\|_{H^2(\Omega)}, \|\overline{\alpha}_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^1(\Omega)}),\ t\ge 0.$$

We then note that it follows from (3.20) that

$$\int_0^1 (\|u\|^2_{H^2(\Omega)} + \|\frac{\partial\alpha}{\partial t}\|^2_{H^1(\Omega)})\,d\tau \tag{3.48}$$
$$\le c(\|u_0\|^2_{H^1(\Omega)} + \int_\Omega F(u_0)\,dx + \|\overline{\alpha}_0\|^2_{H^1(\Omega)} + \|\alpha_1\|^2) + c'_{M_1,M_2}.$$

Therefore, there exists $T\in(0,1)$ such that

$$\|u(T)\|^2_{H^2(\Omega)} + \|\frac{\partial\alpha}{\partial t}(T)\|^2_{H^1(\Omega)} \tag{3.49}$$
$$\le c(\|u_0\|^2_{H^1(\Omega)} + \int_\Omega F(u_0)\,dx + \|\overline{\alpha}_0\|^2_{H^1(\Omega)} + \|\alpha_1\|^2) + c'_{M_1,M_2}.$$

Actually, proceeding similarly but starting from $t = T$ instead of $t = 0$, we can see that (3.49) holds for $t = 1$:

$$\|u(1)\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}(1)\|^2_{H^1(\Omega)} \tag{3.50}$$
$$\leq Q(\|u_0\|^2_{H^1(\Omega)} + \int_\Omega F(u_0)\,dx + \|\overline{\alpha}_0\|^2_{H^1(\Omega)} + \|\alpha_1\|^2) + c'_{M_1,M_2}.$$

We can now repeat the same calculations to find

$$\|u(t)\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}(t)\|^2_{H^1(\Omega)} \tag{3.51}$$
$$\leq Q(\|u(t-1)\|^2_{H^1(\Omega)} + \int_\Omega F(u(t-1))\,dx + \|\overline{\alpha}(t-1)\|^2_{H^1(\Omega)} + \|\frac{\partial \alpha}{\partial t}(t-1)\|^2)$$
$$+c'_{M_1,M_2},\ t \geq 1,$$

where the constants are independent of t; hence, owing to (3.19),

$$\|u(t)\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}(t)\|^2_{H^1(\Omega)} \tag{3.52}$$
$$\leq e^{-c't}Q_{M_1,M_2}(\|u_0\|^2_{H^1(\Omega)} + \int_\Omega F(u_0)\,dx + \|\overline{\alpha}_0\|^2_{H^1(\Omega)} + \|\alpha_1\|^2) + c''_{M_1,M_2},\ c' > 0,\ t \geq 1,$$

which, combined with (3.47) for $t = 1$, finally yields

$$\|u(t)\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}(t)\|^2_{H^1(\Omega)} \tag{3.53}$$
$$\leq e^{-ct}Q_{M_1,M_2}(\|u_0\|_{H^2(\Omega)}, \|\overline{\alpha}_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^1(\Omega)}) + c'_{M_1,M_2},\ t \geq 0.$$

We now note that it follows from (3.23) and (3.53) that

$$\frac{d}{dt}(\|\Delta u\|^2 + \|\Delta\alpha\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2) + \|\frac{\partial u}{\partial t}\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}\|^2 \tag{3.54}$$
$$\leq e^{-ct}Q_{M_1,M_2}(\|u_0\|_{H^2(\Omega)}, \|\overline{\alpha}_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^1(\Omega)}) + c'_{M_1,M_2},\ c > 0,\ t \geq 0.$$

Multiplying then (2.2) by $-\Delta\overline{\alpha}$, we have

$$\frac{d}{dt}(\|\Delta\alpha\|^2 + 2((\Delta\alpha, \frac{\partial\overline{\alpha}}{\partial t}))) + \|\Delta\alpha\|^2 \leq \|\frac{\partial u}{\partial t}\|^2 + 2\|\nabla\frac{\partial \alpha}{\partial t}\|^2. \tag{3.55}$$

Summing (3.18), (3.54) and δ_4 times (3.55), where $\delta_4 > 0$ is small enough, we finally obtain

$$\frac{dE_5}{dt} + c(E_5 + \|\frac{\partial u}{\partial t}\|^2_{-1} + \|\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}) \tag{3.56}$$
$$\leq e^{-c't}Q_{M_1,M_2}(\|u_0\|_{H^2(\Omega)}, \|\overline{\alpha}_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^1(\Omega)}) + c''_{M_1,M_2},\ c,\ c' > 0,\ t \geq 0,$$

where

$$E_5 = E_4 + \|\Delta u\|^2 + \|\Delta\alpha\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2 + \delta_4(\|\Delta\alpha\|^2 + 2((\Delta\alpha, \frac{\partial\overline{\alpha}}{\partial t}))) \tag{3.57}$$

satisfies

$$c(\|u\|^2_{H^2(\Omega)} + \int_\Omega F(u)\,dx + \|\overline{\alpha}\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^1(\Omega)}) + c'_{M_1,M_2} \leq E_5 \tag{3.58}$$
$$\leq c''(\|u\|^2_{H^2(\Omega)} + \int_\Omega F(u)\,dx + \|\overline{\alpha}\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^1(\Omega)}) + c'''_{M_1,M_2},\ c,\ c'' > 0.$$

Noting that, in (3.56), we can choose the constant c' such that $c' > c$, we finally deduce from (3.56) the dissipative estimate

$$\|u(t)\|^2_{H^2(\Omega)} + \|\overline{\alpha}(t)\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}(t)\|^2_{H^1(\Omega)} \tag{3.59}$$
$$\leq e^{-ct} Q_{M_1,M_2}(\|u_0\|_{H^2(\Omega)}, \|\overline{\alpha}_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^1(\Omega)}) + c''_{M_1,M_2},\ c > 0,\ t \geq 0.$$

4. The dissipative semigroup. We introduce the following spaces:

$$\Phi = H^1_0(\Omega) \times H^1_0(\Omega) \times L^2(\Omega),$$
$$\Psi = (H^2(\Omega) \cap H^1_0(\Omega)) \times (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega),$$
$$\Phi_{M_1,M_2} = \{(\varphi, \psi, \theta) \in \Phi,\ |\langle\varphi\rangle| \leq M_1,\ |\langle\psi\rangle| \leq M_2\},$$
$$\Psi_{M_1,M_2} = \Psi \cap \Phi_{M_1,M_2},$$
$$\overline{\Phi}_{M_1,M_2} = \{(\varphi, \psi, \theta) \in \Phi_{M_1,M_2},\ \langle\psi\rangle = 0\},$$
$$\overline{\Psi}_{M_1,M_2} = \Psi \cap \overline{\Phi}_{M_1,M_2},$$

where M_1 and M_2 are fixed positive constants. We have

THEOREM 4.1. We assume that $(u_0, \alpha_0, \alpha_1) \in \Psi_{M_1,M_2}$. Then, (1.12)-(1.15) possesses a unique solution such that $(u(t), \alpha(t), \frac{\partial \alpha}{\partial t}(t)) \in \Psi_{M_1,M_2}$, $\forall t \geq 0$.

Proof. The proof of existence is based on (2.7), (3.59) and, e.g., a standard Galerkin scheme.

Let now $(u^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t})$ and $(u^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t})$ be two solutions to (1.12)-(1.14) with initial data $(u_{0,1}, \alpha_{0,1}, \alpha_{1,1})$ and $(u_{0,2}, \alpha_{0,2}, \alpha_{1,2})$, respectively. We set

$$(u, \alpha, \frac{\partial \alpha}{\partial t}) = (u^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t}) - (u^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t})$$

and

$$(u_0, \alpha_0, \alpha_1) = (u_{0,1}, \alpha_{0,1}, \alpha_{1,1}) - (u_{0,2}, \alpha_{0,2}, \alpha_{1,2}).$$

We have

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f(u^{(1)}) - f(u^{(2)}) - \langle f(u^{(1)}) - f(u^{(2)})\rangle = \frac{\partial \overline{\alpha}}{\partial t}, \tag{4.1}$$

$$\frac{\partial^2 \overline{\alpha}}{\partial t^2} - \Delta \frac{\partial \overline{\alpha}}{\partial t} - \Delta \overline{\alpha} = -\frac{\partial \overline{u}}{\partial t}, \tag{4.2}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \alpha}{\partial \nu} = 0 \text{ on } \Gamma, \tag{4.3}$$

$$u|_{t=0} = u_0,\ \overline{\alpha}|_{t=0} = \overline{\alpha}_0,\ \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1. \tag{4.4}$$

We multiply (4.4) by $\frac{\partial u}{\partial t}$ and obtain

$$\frac{d}{dt}\|\nabla u\|^2 + \|\frac{\partial u}{\partial t}\|^2_{-1} + 2((f(u^{(1)}) - f(u^{(2)}), \frac{\partial u}{\partial t})) = ((\frac{\partial \overline{\alpha}}{\partial t}, \frac{\partial \overline{u}}{\partial t})). \tag{4.5}$$

We then multiply (4.2) by $\frac{\partial \overline{\alpha}}{\partial t}$ and find

$$\frac{d}{dt}(\|\nabla \alpha\|^2 + \|\frac{\partial \overline{\alpha}}{\partial t}\|^2) + \|\nabla \frac{\partial \alpha}{\partial t}\|^2 = -((\frac{\partial \overline{\alpha}}{\partial t}, \frac{\partial \overline{u}}{\partial t})). \tag{4.6}$$

Summing (4.5) and (4.6), we have

$$\frac{d}{dt}(\|\nabla u\|^2+\|\nabla\alpha\|^2+\|\frac{\partial\overline{\alpha}}{\partial t}\|^2)+\|\frac{\partial u}{\partial t}\|_{-1}^2+\|\nabla\frac{\partial\alpha}{\partial t}\|^2\leq\|\nabla(f(u^{(1)})-f(u^{(2)}))\|^2. \quad (4.7)$$

Noting that, owing to (3.59) and Hölder's inequality,

$$\|\nabla(f(u^{(1)})-f(u^{(2)}))\|=\|\nabla(\int_0^1 f'(u^{(1)}+s(u^{(2)}-u^{(1)}))\,ds\,u\| \quad (4.8)$$

$$\leq\|\int_0^1 f'(u^{(1)}+s(u^{(2)}-u^{(1)}))\,ds\,\nabla u\|$$

$$+\|u\int_0^1 f''(u^{(1)}+s(u^{(2)}-u^{(1)}))(\nabla u^{(1)}+s\nabla(u^{(2)}-u^{(1)}))\,ds\|$$

$$\leq Q_{M_1,M_2}(\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)},\|\overline{\alpha}_{0,1}\|_{H^2(\Omega)},\|\overline{\alpha}_{0,2}\|_{H^2(\Omega)},\|\alpha_{1,1}\|_{H^1(\Omega)},\|\alpha_{1,2}\|_{H^1(\Omega)})$$

$$\times(\|\nabla u\|+\||u||\nabla u^{(1)}|\|+\||u||\nabla u^{(2)}|\|)$$

$$\leq Q_{M_1,M_2}(\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)},\|\overline{\alpha}_{0,1}\|_{H^2(\Omega)},\|\overline{\alpha}_{0,2}\|_{H^2(\Omega)},\|\alpha_{1,1}\|_{H^1(\Omega)},\|\alpha_{1,2}\|_{H^1(\Omega)})$$

$$\times\|u\|_{H^1(\Omega)},$$

we finally obtain, setting

$$E_5=\langle u\rangle^2+\|\nabla\alpha\|^2+\|\frac{\partial\overline{\alpha}}{\partial t}\|^2+\langle\frac{\partial\alpha}{\partial t}\rangle^2, \quad (4.9)$$

an inequality of the form

$$\frac{dE_5}{dt}+c(\|\frac{\partial u}{\partial t}\|_{-1}^2+\|\nabla\frac{\partial\alpha}{\partial t}\|^2) \quad (4.10)$$

$$\leq Q_{M_1,M_2}(\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)},\|\overline{\alpha}_{0,1}\|_{H^2(\Omega)},\|\overline{\alpha}_{0,2}\|_{H^2(\Omega)},\|\alpha_{1,1}\|_{H^1(\Omega)},\|\alpha_{1,2}\|_{H^1(\Omega)})E_5$$

and hence

$$E_5(t)\leq e^{tQ_{M_1,M_2}}E_5(0), \quad (4.11)$$

where we have omitted the arguments of the function Q_{M_1,M_2}. Noting also that

$$|\langle\alpha(t)\rangle|\leq|\langle\alpha_0\rangle|+t|\langle\alpha_1\rangle|, \quad (4.12)$$

we deduce from (4.11) and (4.12) the uniqueness as well as the continuous dependence with respect to the initial data. □

Remark 4.2. We note that, even though it is hyperbolic, (1.13) exhibits some partial regularization effect. Indeed, it follows from (3.17) and (3.56) that the variable $\frac{\partial\alpha}{\partial t}$ is regularizing. Note that this is expected, since rewriting (1.13) in the form

$$\frac{\partial\beta}{\partial t}-\Delta\beta-\Delta\alpha=-\frac{\partial u}{\partial t},\ \beta=\frac{\partial\alpha}{\partial t},$$

we have a parabolic equation.

In particular, it follows from Theorem 4.1 that we can define the family of solving operators

$$S(t) : \Psi \to \Psi,\ (u_0, \alpha_0, \alpha_1) \mapsto (u(t), \alpha(t), \frac{\partial \alpha}{\partial t}(t)),\ t \geq 0,$$

which forms a continuous (for the topology of Φ) semigroup (i.e., $S(0) = \mathrm{I}$, $S(t+s) = S(t) \circ S(s)$, t, $s \geq 0$).

Now, as mentioned in the introduction, the variable α is not dissipative, so that $S(t)$ is also not dissipative. However, introducing the family of operators

$$\overline{S}(t) : (u_0, \overline{\alpha}_0, \alpha_1) \mapsto (u(t), \overline{\alpha}(t), \frac{\partial \alpha}{\partial t}(t)),\ t \geq 0,$$

we deduce from (3.59) the following

THEOREM 4.3. The family of operators $\overline{S}(t)$ forms a continuous and dissipative semigroup in $\overline{\Psi}_{M_1,M_2}$.

We also recall

DEFINITION 4.4. The semigroup $\overline{S}(t)$ is dissipative in the Banach space X if there exists a bounded set $\mathcal{B}_0 \subset X$ (called an absorbing set) such that, for every $B \subset X$ bounded, there exists $t_0 = t_0(B)$ such that $t \geq t_0$ implies $\overline{S}(t)B \subset \mathcal{B}_0$.

REMARK 4.5. It is easy to see that we can assume, without loss of generality, that $\mathcal{B}_0$ is positively invariant by $S(t)$; i.e., $S(t)\mathcal{B}_0 \subset \mathcal{B}_0$, $\forall t \geq 0$.

REMARK 4.6. It follows from (1.22) that the semigroup $S(t)$ is continuous and dissipative in

$$\tilde{\Psi}_{M_1,M_2} = \{(\varphi, \psi, \theta) \in \Psi,\ |\langle \varphi \rangle| \leq M_1,\ |\langle \psi \rangle| \leq M_2,\ \langle \theta \rangle = 0\}$$

for fixed constants M_1 and M_2.

5. Existence of exponential attractors. We first derive an asymptotic smoothing property on the difference of two solutions that is one of the key tools to construct exponential attractors (see [14], [15], [16], [17], [32] and [33]).

Let $(u^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t})$ and $(u^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t})$ be two solutions to (1.12)-(1.14) with initial data $(u_{0,1}, \alpha_{0,1}, \alpha_{1,1})$ and $(u_{0,2}, \alpha_{0,2}, \alpha_{1,2})$, respectively. We set

$$(u, \alpha, \frac{\partial \alpha}{\partial t}) = (u^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t}) - (u^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t})$$

and

$$(u_0, \alpha_0, \alpha_1) = (u_{0,1}, \alpha_{0,1}, \alpha_{1,1}) - (u_{0,2}, \alpha_{0,2}, \alpha_{1,2}).$$

We note that, owing to the transitivity of the exponential attraction (see [17]), it suffices to take $(u_0, \overline{\alpha}_0, \alpha_1)$ in the bounded absorbing set $\mathcal{B}_0$ constructed in the previous section. We have

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f(u^{(1)}) - f(u^{(2)}) - \langle f(u^{(1)}) - f(u^{(2)}) \rangle = \frac{\partial \overline{\alpha}}{\partial t}, \tag{5.1}$$

$$\frac{\partial^2 \overline{\alpha}}{\partial t^2} - \Delta \frac{\partial \overline{\alpha}}{\partial t} - \Delta \overline{\alpha} = -\frac{\partial \overline{u}}{\partial t}, \tag{5.2}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \alpha}{\partial \nu} = 0 \text{ on } \Gamma, \tag{5.3}$$

$$u|_{t=0} = u_0,\ \overline{\alpha}|_{t=0} = \overline{\alpha}_0,\ \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1. \tag{5.4}$$

We write $(u, \overline{\alpha}, \frac{\partial \alpha}{\partial t}) = (v, a, \frac{\partial a}{\partial t}) + (w, b, \frac{\partial w}{\partial t})$, where $(v, a, \frac{\partial a}{\partial t})$ and $(w, b, \frac{\partial w}{\partial t})$ are solutions to

$$(-\Delta)^{-1}\frac{\partial v}{\partial t} - \Delta v = \frac{\partial a}{\partial t}, \tag{5.5}$$

$$\frac{\partial^2 a}{\partial t^2} - \Delta\frac{\partial a}{\partial t} - \Delta a = -\frac{\partial v}{\partial t}, \tag{5.6}$$

$$\frac{\partial v}{\partial \nu} = \frac{\partial b}{\partial \nu} = 0 \text{ on } \Gamma, \tag{5.7}$$

$$v|_{t=0} = u_0,\ a|_{t=0} = \overline{\alpha}_0,\ \frac{\partial b}{\partial t}|_{t=0} = \alpha_1, \tag{5.8}$$

and

$$(-\Delta)^{-1}\frac{\partial w}{\partial t} - \Delta u + f(u^{(1)}) - f(u^{(2)}) - \langle f(u^{(1)}) - f(u^{(2)})\rangle = \frac{\partial b}{\partial t}, \tag{5.9}$$

$$\frac{\partial^2 b}{\partial t^2} - \Delta\frac{\partial b}{\partial t} - \Delta b = -\frac{\partial w}{\partial t}, \tag{5.10}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial b}{\partial \nu} = 0 \text{ on } \Gamma, \tag{5.11}$$

$$w|_{t=0} = 0,\ b|_{t=0} = 0,\ \frac{\partial b}{\partial t}|_{t=0} = 0. \tag{5.12}$$

First, repeating the estimates performed in Section 3 for $f \equiv 0$, we easily obtain an inequality of the form

$$\frac{dE_6}{dt} + cE_6 \le 0,\ c > 0, \tag{5.13}$$

where

$$c(\|v\|^2_{H^2(\Omega)} + \|a\|^2_{H^2(\Omega)} + \|\frac{\partial a}{\partial t}\|^2_{H^1(\Omega)}) \le E_6 \tag{5.14}$$

$$\le c'(\|v\|^2_{H^2(\Omega)} + \|a\|^2_{H^2(\Omega)} + \|\frac{\partial a}{\partial t}\|^2_{H^1(\Omega)}),\ c,\ c' > 0,$$

which yields

$$E_6(t) \le e^{-ct}E_6(0),\ c > 0,\ t \ge 0. \tag{5.15}$$

Next, we note that, proceeding exactly as in the previous section, we have

$$\|f(u^{(1)}) - f(u^{(2)}) - \langle f(u^{(1)}) - f(u^{(2)})\rangle\|_{H^1(\Omega)} \le c\|\nabla u\|, \tag{5.16}$$

where the constant c only depends on $\mathcal{B}_0$.

Similarly, rewriting (5.9) in the form

$$(-\Delta)^{-1}\frac{\partial w}{\partial t} - \Delta u = \frac{\partial b}{\partial t} + g, \tag{5.17}$$

where

$$g = -(f(u^{(1)}) - f(u^{(2)}) - \langle f(u^{(1)}) - f(u^{(2)})\rangle), \tag{5.18}$$

we have, performing again the same estimates as in Section 3 (with $f \equiv 0$ and g acting as a forcing term; note that $\langle g\rangle = 0$), we obtain an inequality of the form

$$\frac{dE_7}{dt} \le c\|g\|^2_{H^1(\Omega)}, \tag{5.19}$$

where

$$c(\|w\|^2_{H^2(\Omega)} + \|b\|^2_{H^2(\Omega)} + \|\frac{\partial b}{\partial t}\|^2_{H^1(\Omega)}) \leq E_7 \tag{5.20}$$

$$\leq c'(\|w\|^2_{H^2(\Omega)} + \|b\|^2_{H^2(\Omega)} + \|\frac{\partial b}{\partial t}\|^2_{H^1(\Omega)}),\ c,\ c' > 0,$$

which yields, noting that $E_7(0) = 0$ and employing (4.11),

$$E_7(t) \leq ce^{tc'_{M_1,M_2}}(\|u_0\|^2_{H^1(\Omega)} + \|\overline{\alpha}_0\|^2_{H^1(\Omega)} + \|\alpha_1\|^2),\ t \geq 0, \tag{5.21}$$

where the constants only depend on $\mathcal{B}_0$. We now derive a Hölder (both with respect to space and time) estimate. Actually, owing to (4.11), it suffices to prove the Hölder continuity with respect to time.

We have

$$\|u(t_1) - u(t_2)\|_{H^1(\Omega)} + \|\overline{\alpha}(t_1) - \overline{\alpha}(t_2)\|_{H^1(\Omega)} + \|\frac{\partial \alpha}{\partial t}(t_1) - \frac{\partial \alpha}{\partial t}(t_2)\| \tag{5.22}$$

$$\leq c|t_1 - t_2|^{\frac{1}{2}} \Big| \int_{t_1}^{t_2} (\|\frac{\partial u}{\partial t}\|^2_{H^1(\Omega)} + \|\frac{\partial \overline{\alpha}}{\partial t}\|^2_{H^1(\Omega)} + \|\frac{\partial^2 \alpha}{\partial t^2}\|)\, d\tau \Big|^{\frac{1}{2}}.$$

We note that it follows from (3.19), (3.20) and (3.34) that

$$|\int_{t_1}^{t_2} \|\frac{\partial u}{\partial t}\|^2_{H^1(\Omega)}\, d\tau| \leq c, \tag{5.23}$$

where the constant c only depends on $\mathcal{B}_0$ and $T \geq T_0$ such that $t_1,\, t_2 \in [T_0, T]$; actually, replacing $\mathcal{B}_0$ by $\overline{S}(1)\mathcal{B}_0$, we see that we can take $T_0 = 0$.

Then, it follows from (3.20) that

$$|\int_{t_1}^{t_2} \|\frac{\partial \overline{\alpha}}{\partial t}\|^2_{H^1(\Omega)}\, d\tau| \leq c, \tag{5.24}$$

where the constant c only depends on $\mathcal{B}_0$ and T such that $t_1,\, t_2 \in [0, T]$.

Finally, recalling that

$$\frac{\partial^2 \alpha}{\partial t^2} = \Delta \frac{\partial \alpha}{\partial t} + \Delta \alpha - \frac{\partial u}{\partial t},$$

it follows from (3.56) and (3.59) that

$$|\int_{t_1}^{t_2} \|\frac{\partial^2 \alpha}{\partial t^2}\|\, d\tau| \leq c, \tag{5.25}$$

where the constant c only depends on $\mathcal{B}_0$ and T such that $t_1,\, t_2 \in [0, T]$.

We thus deduce from (5.22), (5.23), (5.24) and (5.25) that

$$\|u(t_1) - u(t_2)\|_{H^1(\Omega)} + \|\overline{\alpha}(t_1) - \overline{\alpha}(t_2)\|_{H^1(\Omega)} + \|\frac{\partial \alpha}{\partial t}(t_1) - \frac{\partial \alpha}{\partial t}(t_2)\| \leq c|t_1 - t_2|^{\frac{1}{2}}, \tag{5.26}$$

where the constant c only depends on $\mathcal{B}_0$ and T such that $t_1,\, t_2 \in [0, T]$.

It now follows from (4.11), (5.15), (5.21), (5.26) and the transitivity of the exponential attraction that we have the following result (see, e.g., [13] and [17]).

THEOREM 5.1. The semigroup $\overline{S}(t)$ possesses an exponential attractor $\mathcal{M} \subset \mathcal{B}_0$; i.e.,

(i) $\mathcal{M}$ is compact in Φ;

(ii) $\mathcal{M}$ is positively invariant, $\overline{S}(t)\mathcal{M} \subset \mathcal{M}$, $\forall t \geq 0$;

(iii) $\mathcal{M}$ has finite fractal dimension in Φ;

(iv) $\mathcal{M}$ attracts the bounded subsets of Ψ_{M_1,M_2} exponentially fast,

$$\forall B \subset \Psi_{M_1,M_2} \text{ bounded, } \operatorname{dist}_\Phi(\overline{S}(t)B, \mathcal{M}) \leq Q(\|B\|_\Phi)e^{-ct}, \ c > 0, \ t \geq 0,$$

where the contant c is independent of B and dist_Φ denotes the Hausdorff semidistance between sets, defined by

$$\operatorname{dist}_\Phi(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_\Phi.$$

Since $\mathcal{M}$ is a compact attracting set, we deduce from Theorem 5.1 and standard results (see, e.g., [2], [33] and [36])

COROLLARY 5.2. The semigroup $\overline{S}(t)$ possesses the finite-dimensional global attractor $\mathcal{A} \subset \mathcal{B}_0$.

REMARK 5.3. We recall that the global attractor $\mathcal{A}$ is the smallest (for the inclusion) compact set of the phase space which is invariant by the flow (i.e., $\overline{S}(t)\mathcal{A} = \mathcal{A}$, $\forall t \geq 0$) and attracts all bounded sets of initial data as time goes to infinity; it thus appears to be a suitable object in view of the study of the asymptotic behavior of the system. Furthermore, the finite dimensionality means, roughly speaking, that even though the initial phase space is infinite dimensional, the reduced dynamics is, in some proper sense, finite dimensional and can be described by a finite number of parameters. We refer the reader to [2], [33] and [36] for more details and discussions on this.

REMARK 5.4. Compared to the global attractor, an exponential attractor is expected to be more robust under perturbations. Indeed, the rate of attraction of trajectories to the global attractor may be slow, and it is very difficult, if not impossible, to estimate this rate of attraction with respect to the physical parameters of the problem in general. As a consequence, global attractors may change drastically under small perturbations. We refer the reader to [13] and [33] for discussions on this subject.

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