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HARMONIC CIRCULAR INCLUSIONS FOR NON-UNIFORM FIELDS THROUGH THE USE OF MULTI-COATING

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Abstract. We propose a novel method for rendering a circular inclusion harmonic even in the presence of non-uniform loading. Most significantly, the condition that the inclusion be harmonic is shown to be *independent* of the specific form of the loading. In addition, we demonstrate that the harmonicity condition obtained actually leads to the stronger property of neutrality when the loading takes a particular form. Our method is based on the idea of 'multi-coating' (surrounding the inclusion with a specified number of coatings each with its own separate elastic properties) used in the design of cloaking structures for the conductivity problem. Consequently, the harmonic inclusions designed here can also be thought of as special kinds of 'near-cloaking' structures in plane elasticity (in the sense that they are invisible to any changes in mean stress in the structure).

1. Introduction. In the optimum design of composites, the concept of "harmonic shape" was first advocated by Bjorkman and Richards [2], [3]. A "harmonic shape" (hole or inclusion) is one which does not disturb the trace of the surrounding stress field when inserted into a uniformly or non-uniformly stressed solid. There are many important practical applications in which the design of harmonic inclusions is crucial. For example, in biomechanics, where implants are embedded in human bones, it is known that the mechanism responsible for loosening and failure of the implant/bone system is controlled by the disturbance in mean stress (Firoozbakhsh and Aleyaasin, [5]; Weinans et al., [10]).

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In Ru [8], the author extended the original concept of "harmonic holes" to a three-phase harmonic circular inclusion under uniform remote stresses and observed that the resulting harmonicity condition was, in fact, independent of the loading.

In anti-plane elasticity (or the mathematically equivalent two-dimensional conductivity), Ammari et al. [1] proposed a novel cancellation technique for the design of near-cloaking structures in which a circular elastic inclusion with a sufficiently large number of coatings will not cause any disturbance in the matrix under any kind of non-uniform remote anti-plane stresses. These near-cloaking N-phase structures have the amazing property that the generalized polarization tensors up to the $(N-2)$-th order vanish.

In this paper, we design the material parameters of a multi-coated circular elastic inclusion in such a way that the inclusion becomes harmonic under a general class of non-uniform loadings. Our idea, which originates from the analysis of the case of an inclusion with a single coating (Ru, [8]) and the near-cloaking multi-coated structure in anti-plane shear (Ammari et al., [1]), is based on a multi-coating which cancels the higher-order poles in the analytic function governing the mean stress in the surrounding matrix. It will be shown that the permissible set of non-uniform loadings is quite general (although not arbitrary). For comparison purposes, we examine the conditions under which a single-coated circular inclusion becomes harmonic when subjected to the same non-uniform loading considered by Bjorkman and Richards [3] for harmonic holes. We similarly establish results for a double-coated circular inclusion when subjected to the same non-uniform loading considered by Wang et al. [9]. In each of these aforementioned cases we are able to design harmonic inclusions independently of the remote non-uniform stresses. This is in sharp contrast to the results obtained in Bjorkman and Richards [3] and Wang et al. [9] where it is shown that the remote loading exerts a significant influence on the harmonic shape.

2. Formulation. For plane deformations of an isotropic elastic material, the in-plane displacements u and v, the two resultant forces f_x and f_y, and the in-plane stresses σ_{xx}, σ_{yy} and σ_{xy} can be expressed in terms of two analytic functions $\varphi(z)$ and $\psi(z)$ of the complex variable $z = x + iy$ as (Muskhelishvili, [6])

$$\begin{aligned} 2\mu(u+iv) &= \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \\ f_x + if_y &= -i\left[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}\right], \end{aligned} \tag{2.1}$$

$$\begin{aligned} \sigma_{xx} + \sigma_{yy} &= 2[\varphi'(z) + \overline{\varphi'(z)}], \\ \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} &= 2\left[\bar{z}\varphi''(z) + \psi'(z)\right], \end{aligned} \tag{2.2}$$

where $\kappa = 3 - 4\nu$ for plane strain and $\kappa = \frac{(3-\nu)}{1+\upsilon}$ for plane stress; μ and ν, where $\mu > 0$ and $0 \leqslant \nu \leqslant 0.5$, are the shear modulus and Poisson's ratio, respectively.

Consider an N-phase structure in which a circular elastic inclusion is bonded to an infinite matrix through $(N-2)$ coaxial coatings. Let $S_1,\ldots,S_k$ $(k = 2, 3,\ldots, N-1)$ and S_N denote the inclusion, the $(N-2)$ coatings, and the surrounding matrix, all of which are perfectly bonded across the $(N-1)$ concentric circles R_k, $(k = 1, 2,\ldots, N-1)$ $(R_1 < R_2 < \ldots < R_{N-1})$. The subscript j or the superscript (j) will be adopted to denote the quantities in S_j. In addition, the remote non-uniform loading is characterized by the functions

$$\varphi_N(z) = Az + Bz^2, \quad \psi_N(z) = \sum_{n=1}^{N-2} C_n z^n, \tag{2.3}$$

where A is a real constant, and B and C_n $(n = 1, 2, \ldots, N-2)$ are complex constants.

2.1. *Design of multi-coated harmonic circular inclusions.* The boundary value problem of the N-phase inclusion/matrix system takes the following form:

$$\begin{aligned}
&\tfrac{1}{\mu_j}\left[\kappa_j\varphi_j(z) - z\overline{\varphi_j'(z)} - \overline{\psi_j(z)}\right] = \tfrac{1}{\mu_{j+1}}\left[\kappa_{j+1}\varphi_{j+1}(z) - z\overline{\varphi_{j+1}'(z)} - \overline{\psi_{j+1}(z)}\right],\\
&\varphi_j(z) + z\overline{\varphi_j'(z)} + \overline{\psi_j(z)} = \varphi_{j+1}(z) + z\overline{\varphi_{j+1}'(z)} + \overline{\psi_{j+1}(z)}, \ |z| = R_j, \ (j = 1, 2, \cdots, N-1),
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
&\varphi_N(z) = Az + Bz^2, \quad (z \in S_N),\\
&\psi_N(z) = \sum_{n=1}^{N-2} C_n z^n + O(1), \quad |z| \to \infty.
\end{aligned} \tag{2.5}$$

Here, $\varphi_j(z)$ and $\psi_j(z)$, as defined in phase j, can be expanded into the following Laurent series

$$\begin{aligned}
\varphi_j(z) &= A_1^{(j)} z + A_2^{(j)} z^2 + \sum_{n=1}^{N-2}\left[A_{1n}^{(j)} z^{-n} + A_{2n}^{(j)} z^{n+2}\right],\\
\psi_j(z) &= B_1^{(j)} z^{-1} + B_2^{(j)} z^{-2} + \sum_{n=1}^{N-2}\left[B_{1n}^{(j)} z^{-(n+2)} + B_{2n}^{(j)} z^{n}\right],
\end{aligned} \qquad (z \in S_j) \tag{2.6}$$

in which the coefficients are to be determined.

By enforcing the continuity conditions (2.4) across the interface $|z| = R_j, j = 1, 2, \cdots, N-1$ and after some straightforward yet tedious algebraic operations, the coefficients in the Laurent series for the inner phase j can be expressed in terms of those for the outer phase $(j+1)$ as

$$\begin{aligned}
&\left[\begin{array}{cccc} A_{1n}^{(j)} & R_1^{-2}B_{1n}^{(j)} & R_1^{2n+2}\bar{A}_{2n}^{(j)} & R_1^{2n}\bar{B}_{2n}^{(j)} \end{array}\right]^T \\
&= \mathbf{P}_n^{(j)}\left[\begin{array}{cccc} A_{1n}^{(j+1)} & R_1^{-2}B_{1n}^{(j+1)} & R_1^{2n+2}\bar{A}_{2n}^{(j+1)} & R_1^{2n}\bar{B}_{2n}^{(j+1)} \end{array}\right]^T, n = 1, 2, \ldots, N-2
\end{aligned} \tag{2.7}$$

$$\left[\begin{array}{cc} R_1^{-2}B_2^{(j)} & R_1^2\bar{A}_2^{(j)} \end{array}\right]^T = \mathbf{Q}_j\left[\begin{array}{cc} R_1^{-2}B_2^{(j+1)} & R_1^2\bar{A}_2^{(j+1)} \end{array}\right]^T, \tag{2.8}$$

$$\left[\begin{array}{cc} R_1^{-2}B_1^{(j)} & A_1^{(j)} \end{array}\right]^T = \mathbf{R}_j\left[\begin{array}{cc} R_1^{-2}B_1^{(j+1)} & A_1^{(j+1)} \end{array}\right]^T, \tag{2.9}$$

where $\mathbf{P}_n^{(j)}$ is a 4×4 real matrix and $\mathbf{Q}_j$ and $\mathbf{R}_j$ are 2×2 real matrices. These matrices are specifically determined as

$$\mathbf{P}_n^{(j)} = \tau_j\begin{pmatrix}
1+\beta_j & 0 & \rho_j^{-n-1}(n+2)(\beta_j-\alpha_j) & \rho_j^{-n}(\beta_j-\alpha_j)\\
2n\rho_j^{-1}\beta_j & 1-\beta_j & \rho_j^{-n-2}\varpi_j & \rho_j^{-n-1}n(\beta_j-\alpha_j)\\
n\rho_j^{n+1}(\alpha_j-\beta_j) & \rho_j^{n+2}(\beta_j-\alpha_j) & 1+\beta_j & 0\\
\rho_j^{n}\varpi_j & \rho_j^{n+1}(n+2)(\alpha_j-\beta_j) & -2n\rho_j^{-1}(n+2)\beta_j & 1-\beta_j
\end{pmatrix}, \tag{2.10}$$

$$\mathbf{Q}_j = \tau_j \begin{bmatrix} 1-\beta_j & -\rho_j^{-2}(\beta_j+\alpha_j) \\ \rho_j^2(\beta_j-\alpha_j) & 1+\beta_j \end{bmatrix}, \tag{2.11}$$

$$\mathbf{R}_j = \tau_j \begin{bmatrix} 1+\alpha_j-2\beta_j & -4\rho_j^{-1}\beta_j \\ \rho_j(\beta_j-\alpha_j) & 1-\alpha_j+2\beta_j \end{bmatrix}, \tag{2.12}$$

with material parameters $\varpi_j = [n\,(n+2)\,(\beta_j-\alpha_j)-(\beta_j+\alpha_j)]$, $\tau_j = \frac{\frac{\mu_j}{\mu_{j+1}}(\kappa_{j+1}+1)+\kappa_j+1}{2(\kappa_j+1)}$ and geometric parameter $\rho_j = \frac{R_1^2}{R_j^2} \leqslant 1$. Here, α_j , β_j are two Dundurs parameters for the bi-material composed of phase j and phase $(j+1)$ (Dundurs, [4]):

$$\alpha_j = \frac{\frac{\mu_j}{\mu_{j+1}}(\kappa_{j+1}+1)-(\kappa_j+1)}{\frac{\mu_j}{\mu_{j+1}}(\kappa_{j+1}+1)+\kappa_j+1},\ \ \beta_j = \frac{\frac{\mu_j}{\mu_{j+1}}(\kappa_{j+1}-1)-(\kappa_j-1)}{\frac{\mu_j}{\mu_{j+1}}(\kappa_{j+1}+1)+\kappa_j+1}. \tag{2.13}$$

It follows from Eqs. (2.7), (2.8) and (2.9) that

$$\begin{bmatrix} 0 & 0 & R_1^{2n+2}\bar{A}_{2n}^{(1)} & R_1^{2n}\bar{B}_{2n}^{(1)} \end{bmatrix}^T = \mathbf{S}_n \begin{bmatrix} A_{1n}^{(N)} & R_1^{-2}B_{1n}^{(N)} & 0 & R_1^{2n}\bar{C}_n \end{bmatrix}^T, \quad (n=1,2,\ldots,N-2) \tag{2.14}$$

$$\begin{bmatrix} 0 & R_1^2\bar{A}_2^{(1)} \end{bmatrix}^T = \mathbf{K} \begin{bmatrix} R_1^{-2}B_2^{(N)} & R_1^2\bar{B} \end{bmatrix}^T, \tag{2.15}$$

$$\begin{bmatrix} 0 & A_1^{(1)} \end{bmatrix}^T = \mathbf{T} \begin{bmatrix} R_1^{-2}B_1^{(N)} & A \end{bmatrix}^T, \tag{2.16}$$

where

$$\mathbf{S}_n = \mathbf{P}_n^{(1)}\mathbf{P}_n^{(2)}\cdots\mathbf{P}_n^{(N-1)},\quad \mathbf{K} = \mathbf{Q}_1\mathbf{Q}_2\cdots\mathbf{Q}_{N-1},\quad \mathbf{T} = \mathbf{R}_1\mathbf{R}_2\cdots\mathbf{R}_{N-1}. \tag{2.17}$$

In writing Eqs. (2.14)-(2.16), we have imposed the regularity condition for the two analytic functions $\varphi_1(z)$ and $\psi_1(z)$ defined in the inclusion and in the far-field condition (2.3). The unknown coefficients in the inclusion and in the matrix can then be uniquely determined as

$$\begin{aligned} \begin{bmatrix} A_{1n}^{(N)} \\ R_1^{-2}B_{1n}^{(N)} \end{bmatrix} &= -R_1^{2n}\bar{C}_n(\mathbf{S}_n^{11})^{-1}\mathbf{S}_n^{12}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ \begin{bmatrix} R_1^{2n+2}A_{2n}^{(1)} \\ R_1^{2n}B_{2n}^{(1)} \end{bmatrix} &= R_1^{2n}C_n\left[\mathbf{S}_n^{22}-\mathbf{S}_n^{21}(\mathbf{S}_n^{11})^{-1}\mathbf{S}_n^{12}\right]\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ R_1^{-2}B_2^{(N)} = -\frac{K_{12}}{K_{11}}R_1^2\bar{B},&\quad R_1^2A_2^{(1)} = \left(K_{22}-\frac{K_{12}K_{21}}{K_{11}}\right)R_1^2B, \\ R_1^{-2}B_1^{(N)} = -\frac{T_{12}}{T_{11}}A,&\quad A_1^{(1)} = \left(T_{22}-\frac{T_{12}T_{21}}{T_{11}}\right)A, \end{aligned} \tag{2.18}$$

where the following partitioned forms of the matrices have been adopted:

$$\mathbf{S}_n = \begin{bmatrix} \mathbf{S}_n^{11} & \mathbf{S}_n^{12} \\ \mathbf{S}_n^{21} & \mathbf{S}_n^{22} \end{bmatrix},\quad \mathbf{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix},\quad \mathbf{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix},$$

with the dimension of $\mathbf{S}_n^{ij}$ $(i,j=1,2)$ being 2×2.

In order to make the multi-coated inclusion harmonic, the 2×2 matrix $(\mathbf{S}_n^{11})^{-1}\mathbf{S}_n^{12}$ should be a lower triangular matrix (or, equivalently, $[(\mathbf{S}_n^{11})^{-1}\mathbf{S}_n^{12}]_{12}=0$) for $n=1,2,\ldots,N-2$. In other words, the higher-order poles up to the order of $(N-2)$ in $\varphi_N(z)$ must be cancelled exactly. This condition is independent of the remote non-uniform stress field characterized by A, B and C_n $(n = 1, 2, \ldots, N-2)$, and takes the following general form:

$$f_n(\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_{N-1}, \beta_{N-1}, \rho_2, \rho_3, \cdots, \rho_{N-1}) = 0,\ (n = 1, 2, \cdots, N-2). \quad (2.19)$$

For example, when $N = 4$ and $\beta_1 = \beta_2 = \beta_3 = 0$, the explicit expression of Eq. (2.19) can be finally derived as

$$\begin{aligned}
&\alpha_1 + \rho_2^{-n}\alpha_2 + \rho_3^{-n}\alpha_3 + \rho_2^{n+2}\alpha_1^2\alpha_2 + \rho_3^{n+2}\alpha_1^2\alpha_3 + \rho_2^{-2n-2}\rho_3^{n+2}\alpha_2^2\alpha_3\\
&+ \left[(n+1)^2\rho_2^2 - 2n(n+2)\rho_2 + n(n+2)\right]\alpha_1\alpha_2^2\\
&+ \left[(n+1)^2\rho_3^2 - 2n(n+2)\rho_3 + n(n+2)\right]\alpha_1\alpha_3^2\\
&+ \left[\rho_2^{-n-2}\rho_3^2(n+1)^2 - 2n(n+2)\rho_2^{-n-1}\rho_3 + \rho_2^{-n}n(n+2)\right]\alpha_2\alpha_3^2\\
&+ \rho_2^{2n+2}\rho_3^{-n}\alpha_1^2\alpha_2^2\alpha_3 + \rho_2^{n}\left[(n+1)^2\rho_3^2 - 2n(n+2)\rho_2\rho_3 + n(n+2)\rho_2^2\right]\alpha_1^2\alpha_2\alpha_3^2 \qquad (2.20)\\
&+ \left[\left[\rho_2^{-1}\rho_3(n+1)^2 + n(n+2)(\rho_2 - \rho_3 - 1)\right]^2 + n(n+2)(\rho_2 - \rho_3)^2\right]\alpha_1\alpha_2^2\alpha_3^2\\
&+ \rho_2^{-n-2}\rho_3^{-n}(\rho_2^{2n+2} + \rho_3^{2n+2})\left[(1+\rho_2^2)(n+1)^2 - 2n(n+2)\rho_2\right]\alpha_1\alpha_2\alpha_3\\
&= 0, \quad (n = 1, 2).
\end{aligned}$$

It is observed that even for the above simple case of β_j =0, the specific expression is still rather tedious. In addition, it is easily found from Eq. (2.20) that if (α_1,α_2,α_3) is a solution, then ($-\alpha_1$,$-\alpha_2$,$-\alpha_3$) is also a solution.

For given geometric parameters $(\rho_2, \rho_3, \cdots, \rho_{N-1})$, there are in total $(N-2)$ non-linear equations for the $2(N-1)$ Dundurs parameters $(\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_{N-1}, \beta_{N-1})$ with the restriction that the admissible values of these α_j and β_j are within the parallelogram enclosed by $\alpha_j = \pm 1$ and $\alpha_j - 4\beta_j = \pm 1$ in the α_j,β_j-plane. These non-linear equations can be solved iteratively to arrive at these Dundurs parameters.

In the next section, several typical cases will be discussed in detail to illustrate the obtained general result.

3. Discussions.

3.1. *A single-coated harmonic inclusion under non-uniform loading.* We first consider a three-phase structure ($N = 3$) with asymptotic behavior characterized by

$$\psi_N(z) = C_M z^M + O(1), \quad |z| \to \infty. \quad (3.1)$$

It is of interest to note that the case $M = 2$ corresponds to the non-uniform loading considered by Bjorkman and Richards [3]. In order to make the single-coated inclusion harmonic under the remote loading (3.1), the following condition should be satisfied:

$$\begin{aligned}
&\rho_2^{2M+2}(\beta_1 - \alpha_1)(\beta_1 + \alpha_1)(\beta_2 - 1)(\beta_2 - \alpha_2)\\
&+\rho_2^{M+2}\left[M(M+2)(1-\beta_1)(\beta_1 - \alpha_1) - (1+\beta_1)(\beta_1+\alpha_1)\right](\beta_2 - \alpha_2)^2\\
&+\rho_2^{M+1}2M(M+2)(\beta_1 - 1)(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)^2 \qquad (3.2)\\
&+\rho_2^{M}(1-\beta_1)(\beta_1-\alpha_1)\left[(1-\beta_2)^2 + M(M+2)(\beta_2-\alpha_2)^2\right]\\
&+(1-\beta_1)(1+\beta_1)(1-\beta_2)(\beta_2-\alpha_2) = 0.
\end{aligned}$$

The above condition is again independent of the specific form of remote non-uniform stresses characterized by A, B and C_M. When $M = 1$, we recover the case of remote uniform loading considered in Ru [8]. In addition, if $(\mu_3-\mu_1)(\mu_3-\mu_2)<0$, Eq. (2.19) has at least one root ρ_2 $(0<\rho_2<1)$. Interestingly, this condition is independent of the specific value of M.

For example, if the inclusion is extremely rigid, we have

$$\alpha_1 = 1, \;\; \beta_1 = \frac{\kappa_2 - 1}{\kappa_2 + 1}. \tag{3.3}$$

Consequently Eq. (3.2) reduces to

$$\begin{gathered}\left[(M^2+2M+\kappa_2^2)\rho_2^{M+2} - 2M(M+2)\rho_2^{M+1} + M(M+2)\rho_2^M\right]\lambda^2 + \kappa_2(1+\rho_2^{2M+2})\lambda \\ + \rho_2^M = 0, \\ \lambda = \frac{\alpha_2-\beta_2}{1-\beta_2}, \quad \mu_3 > \mu_2, \quad -\frac{1}{\kappa_2} < \lambda < 0. \end{gathered} \tag{3.4}$$

Then λ can be simply determined as

$$\lambda = \frac{-\kappa_2(1+\rho_2^{2M+2}) + \sqrt{\kappa_2^2(1+\rho_2^{2M+2})^2 - 4\rho_2^M\left[(M^2+2M+\kappa_2^2)\rho_2^{M+2} - 2M(M+2)\rho_2^{M+1} + M(M+2)\rho_2^M\right]}}{2\left[(M^2+2M+\kappa_2^2)\rho_2^{M+2} - 2M(M+2)\rho_2^{M+1} + M(M+2)\rho_2^M\right]}. \tag{3.5}$$

At the other extreme, if the inclusion is extremely compliant, we have

$$\alpha_1 = -1. \tag{3.6}$$

Consequently Eq. (3.2) reduces to

$$\begin{gathered}\left[(M+1)^2\rho_2^{M+2} - 2M(M+2)\rho_2^{M+1} + M(M+2)\rho_2^M\right]\lambda^2 - (1+\rho_2^{2M+2})\lambda + \rho_2^M = 0, \\ \lambda = \frac{\alpha_2-\beta_2}{1-\beta_2}, \quad \mu_2 > \mu_3, \quad 0 < \lambda < 1, \end{gathered} \tag{3.7}$$

from which we arrive at

$$\lambda = \frac{(1+\rho_2^{2M+2}) - \sqrt{(1+\rho_2^{2M+2})^2 - 4\rho_2^M\left[(M+1)^2\rho_2^{M+2} - 2M(M+2)\rho_2^{M+1} + M(M+2)\rho_2^M\right]}}{2\left[(M+1)^2\rho_2^{M+2} - 2M(M+2)\rho_2^{M+1} + M(M+2)\rho_2^M\right]}. \tag{3.8}$$

The variations of λ as functions of ρ_2 and M determined by Eqs. (3.5) with $\kappa_2 = 2$ and (3.8) are plotted, respectively, in Figs. 1 and 2. Apparently when $M = 1$, our results reduce to those in Ru [8]. It is observed from Figs. 1 and 2 that the parameter λ is always a monotonic function of ρ_2 for a fixed M. This means that λ is uniquely determined by ρ_2 or vice versa. It is interesting to note that the resulting harmonic single-coated inclusion is independent of the loading parameters A, B and C_M in contrast to the shape of the harmonic hole designed by Bjorkman and Richards [3] which is, in fact, specifically dependent on these loading parameters.

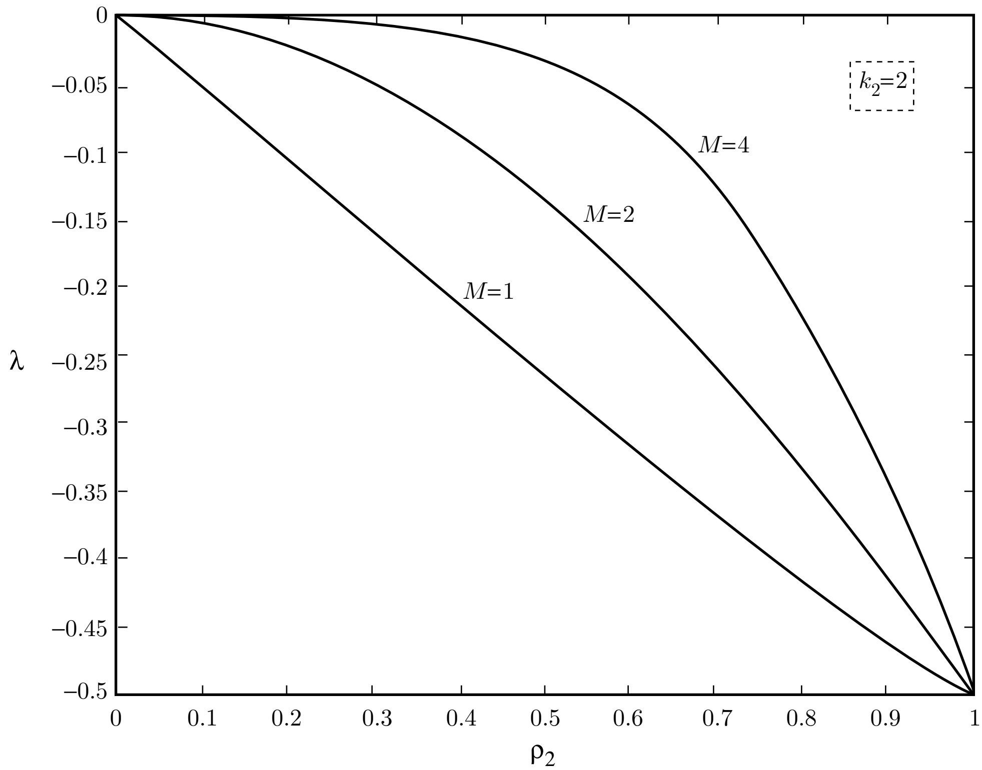

FIG. 1. The variations of λ as functions of ρ_2 and M for a single-coated harmonic rigid inclusion determined by Eq. (3.5) with $\kappa_2 = 2$.

3.2. *A double-coated harmonic inclusion.* Next, we consider a circular inclusion with a double coat ($N = 4$). The remote non-uniform loading characterized by Eq. (2.3) with $B = 0$ and $N = 4$ now reduces to the remote stress field considered by Wang et al. [9]. Table 1 lists the calculated Dundurs parameters α_j $(j = 1, 2, 3)$ with $\beta_1 = \beta_2 = \beta_3 = 0$ for a double-coated harmonic inclusion. During the calculation we directly adopt the explicit form (2.20): first we assign values of ρ_2, ρ_3 and α_1 with the restriction that $0 < \rho_3 < \rho_2 < 1$ and $0 < \alpha_1 \leqslant 1$, and then the other two parameters α_2 and α_3 $(-1 < \alpha_2, \alpha_3 < 1)$ can always be uniquely determined. It is observed from Table 1 that when the inclusion is rigid ($\alpha_1 = 1$), the inner coating is much softer than the outer coating such that $\alpha_2 \approx -1$. For fixed values of the geometric parameters ρ_2 and ρ_3, both $-\alpha_2$ and α_3 are increasing functions of α_1.

3.3. *A harmonic inclusion using three or more coatings.* Finally, we consider a circular inclusion with three or more coatings. Condition (2.19) is solved iteratively in the following manner: First we assign values of the geometric parameters $(\rho_2, \rho_3, \cdots, \rho_{N-1})$ and the N Dundurs parameters β_j $(j = 1, 2, \ldots, N-1)$, α_1. Next, the remaining $(N-2)$ Dundurs parameters α_j $(j = 2, 3, \ldots, N-1)$ are uniquely determined by solving Eq. (2.19). Our specific calculations indicate that as the number of coatings increases, the

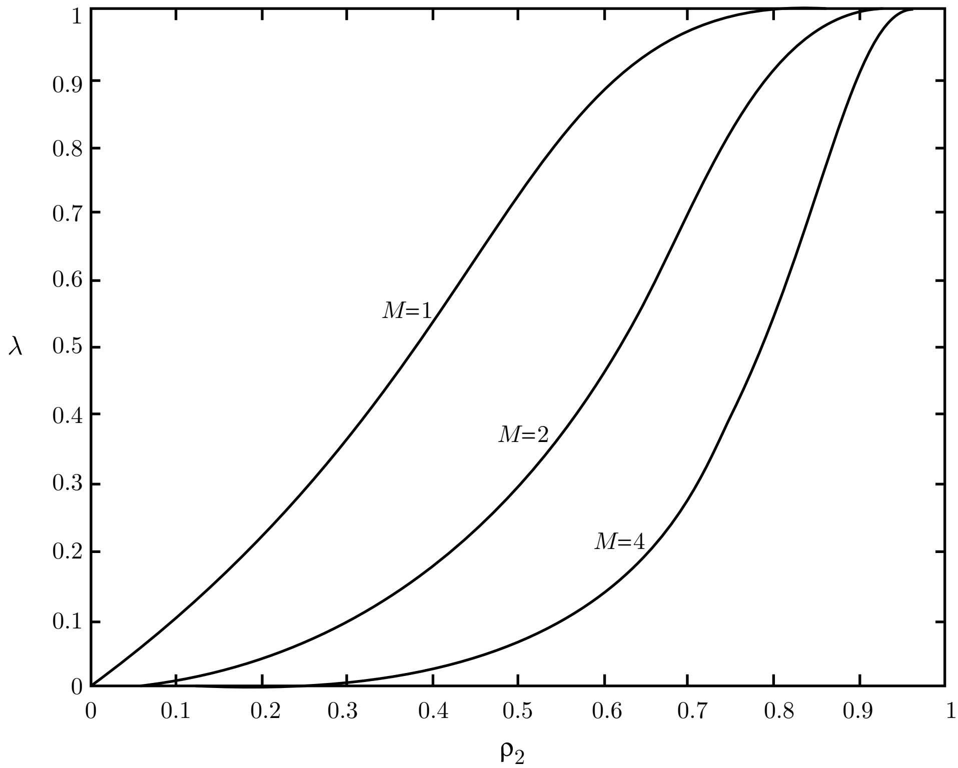

FIG. 2. The variations of λ as functions of ρ_2 and M for a single-coated harmonic inclusion determined by Eq. (3.8).

	α_1	α_2	α_3
ρ_2 =0.8, ρ_3 =0.6	0.2	−0.2522	0.0715
	0.4	−0.4839	0.1400
	0.6	−0.6833	0.2040
	0.8	−0.8508	0.2654
	1.0	−0.9997	0.7214
ρ_2 =0.9, ρ_3 =0.3	0.2	−0.1890	0.0031
	0.4	−0.3783	0.0064
	0.6	−0.5687	0.0106
	0.8	−0.7635	0.0165
	1.0	−0.9997	0.0840

TABLE 1. The calculated Dundurs parameters α_j (j =1,2,3) with $\beta_1 = \beta_2 = \beta_3 = 0$ for a harmonic doubly coated inclusion.

permissible maximum absolute value of α_1 always decreases. It is not clear if this observation is a consequence of the physical problem under consideration or of our iteration algorithm. Listed in Tables 2 and 3 are the calculated Dundurs parameters α_j with

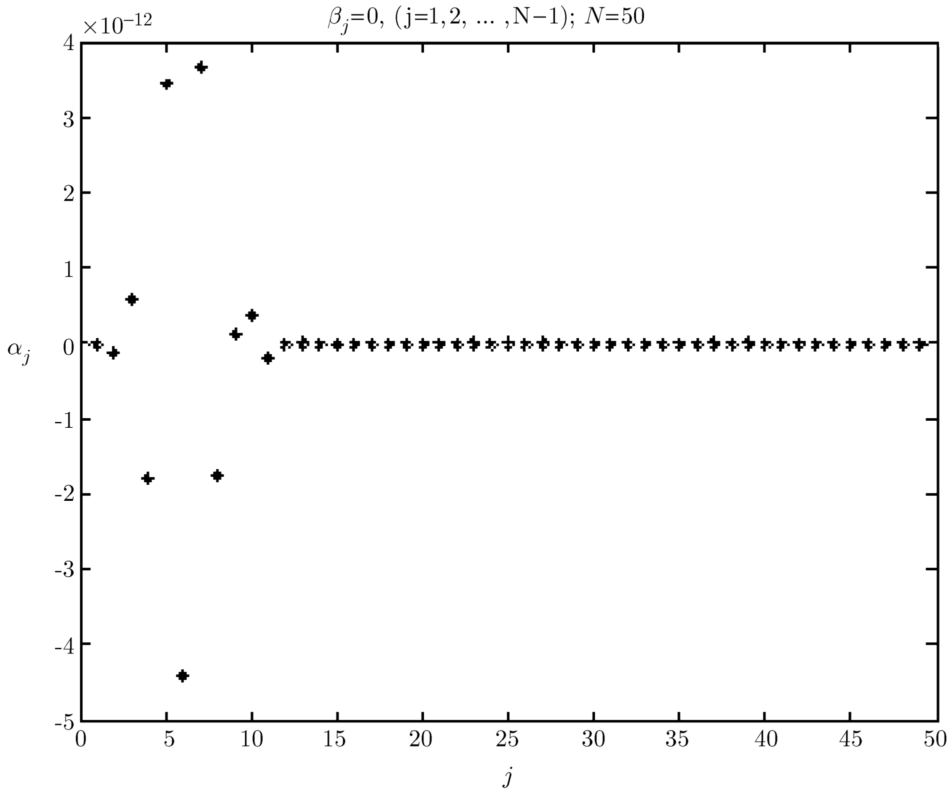

FIG. 3. The calculated α_j ($j = 1, 2, \ldots, N-1$) with $\beta_j = 0$, $j = 1, 2, \ldots, N-1$) and $\rho_j = \dfrac{1 - 0.75(j-1)}{(N-2)}$, ($j = 2, 3, \cdots, N-1$) for $N = 50$.

$\beta_j = 0$ ($j = 2, 3, \ldots, N-1$) for $N = 6$ and $N = 9$. It is observed from Tables 2 and 3 that α_j and α_{j+1} have opposite signs and that $(-1)^{j-1}\alpha_j$ ($j = 2, 3, \ldots, N-1$) are increasing functions of α_1.

α_1	α_2	α_3	α_4	α_5
0.1	−0.1723	0.0906	−0.0145	0.0004
0.2	−0.3332	0.1756	−0.0282	0.0008
0.4	−0.5964	0.3179	−0.0515	0.0014
0.5	−0.6971	0.3745	−0.0609	0.0017
0.526	−0.7203	0.3877	−0.0631	0.0017

TABLE 2. The calculated α_j ($j = 1, 2, \ldots, N-1$) with $\beta_j = 0$ ($j = 1, 2, \ldots, N-1$) for $N = 6$ using $\rho_2 = 0.8125$, $\rho_3 = 0.6250$, $\rho_4 = 0.4375$, $\rho_5 = 0.2500$.

We illustrate in Figs. 3 and 4 the calculated Dundurs parameters α_j ($j = 2, 3, \ldots, N-1$) with $\beta_j = 0$ ($j = 2, 3, \ldots, N-1$) for the larger values of $N = 50$ and $N = 495$,

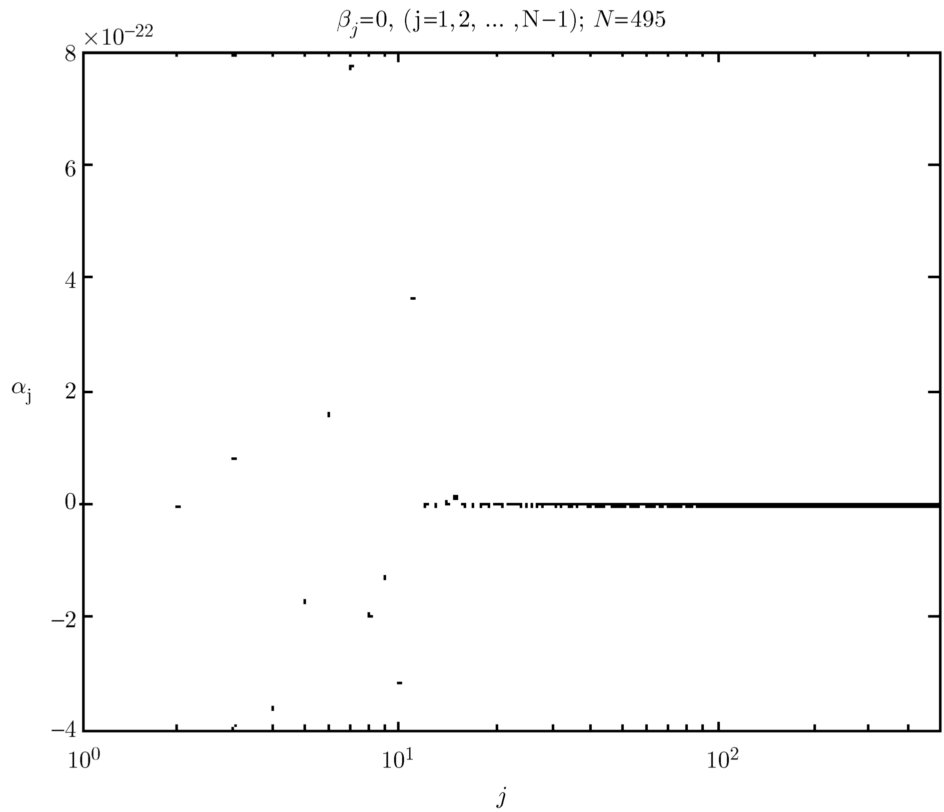

FIG. 4. The calculated α_j $(j = 1, 2, \ldots, N-1)$ with $\beta_j = 0$, $(j = 1, 2, \ldots, N-1)$ and $\rho_j = \dfrac{1-0.75(j-1)}{(N-2)}$, $(j = 2, 3, \cdots, N-1)$ for $N = 495$.

α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
0.05	−0.1539	0.1848	−0.1103	0.0334	−0.0047	0.0003	-3×10^{-6}
0.1	−0.2887	0.3322	−0.1986	0.0613	−0.0089	0.0005	-5×10^{-6}

TABLE 3. The calculated α_j $(j = 1,2,\ldots,N-1)$ with $\beta_j = 0$ $(j = 1,2,\ldots,N-1)$ for $N = 9$. The geometric parameters are: $\rho_2 = 0.8929$, $\rho_3 = 0.7857$, $\rho_4 = 0.6786$, $\rho_5 = 0.5714$, $\rho_6 = 0.4643$, $\rho_7 = 0.3571$, $\rho_8 = 0.25$.

respectively. It is observed from Fig. 3 that the absolute values of α_j $(20 \leqslant j \leqslant 49)$ are negligible as compared with those of α_j $(1 \leqslant j \leqslant 11)$. Also, from Fig. 4 those α_j having relatively large absolute values are mainly those for which $j = 3, 4, \ldots, 11$.

3.4. *A neutral coated circular inclusion under non-uniform loading.* It is well established that a coated circular elastic inclusion can be made neutral in the presence of a remote uniform hydrostatic stress field (see for example Ru, [7]). In what follows we discuss an interesting consequence of the present research, namely the identification of single or multi-coated circular elastic inclusions which can be made neutral under a general class of non-uniform loadings. In fact, from Eqs. (2.6) and (2.18), it follows that when $K_{12} = T_{12} = 0$, a multi-coated inclusion will be neutral when subjected to the following remote non-uniform loading

$$\varphi_N(z) = Az + Bz^2, \quad \psi_N(z) = 0. \tag{3.9}$$

Below we will design neutral single- and double-coated inclusions under the non-uniform loading given by (3.9).

3.5. *A neutral single-coated inclusion ($N = 3$).* When N =3, the condition $K_{12} = T_{12}$ =0 leads to the two linear algebraic equations

$$\begin{aligned} &\rho_2^2(\beta_1+\alpha_1)(1+\beta_2)+(1-\beta_1)(\beta_2+\alpha_2)=0,\\ &\rho_2\beta_1(1-\alpha_2+2\beta_2)+\beta_2(1+\alpha_1-2\beta_1)=0. \end{aligned} \tag{3.10}$$

We can first assign suitable values of ρ_2, α_1 and β_1, and then α_2 and β_2 can be uniquely determined by solving (3.10).

For example, if the inclusion is extremely rigid, we can obtain from Eq. (3.10) that

$$\alpha_2 = -\frac{\rho_2\left[\kappa_2(1-\rho_2)^2+(1+\rho_2\kappa_2)^2\right]}{4+\rho_2(\kappa_2-1)(3+\rho_2^2\kappa_2)},\ \beta_2 = -\frac{\rho_2(\kappa_2-1)(1+\rho_2^2\kappa_2)}{4+\rho_2(\kappa_2-1)(3+\rho_2^2\kappa_2)}. \tag{3.11}$$

The above obtained α_2 and β_2 for different values of ρ_2 ($0\leqslant \rho_2 \leqslant 1$) and κ_2 ($1\leqslant \kappa_2 \leqslant 3$) always lie within the parallelogram enclosed by $\alpha_2 = \pm 1$ and $\alpha_2 - 4\beta_2 = \pm 1$ in the α_2, β_2-plane. Physically we can always design a single-coated neutral rigid inclusion by choosing the appropriate shear modulus of the coating and Poisson's ratio of the surrounding matrix.

On the other hand, if the inclusion is extremely compliant ($\alpha_1 = -1$), we can obtain, from Eq. (3.10), that

$$\alpha_2 = -\frac{\rho_2}{2+\rho_2},\ \beta_2 = \frac{\rho_2(1+\rho_2)}{(2+\rho_2)(1-\rho_2)}. \tag{3.12}$$

In this case, α_2 and β_2 lie within the aforementioned parallelogram only when $0 \leqslant \rho_2 \leqslant \frac{(\sqrt{17}-3)}{4} \approx 0.2808$. Physically, in order to make a single-coated inclusion neutral, the coating must be sufficiently thick.

If $\beta_1 = \beta_2 = 0$, it then follows from Eq. (3.10) that $\rho_2 = \sqrt{\frac{-\alpha_2}{\alpha_1}}$, which is valid when $\alpha_1 \cdot \alpha_2 < 0$ and $|\alpha_1| \geqslant |\alpha_2|$.

In the case when a single-coated inclusion cannot be made neutral when subjected to the non-uniform loading (3.9), it is necessary to resort to the double-coated inclusion to achieve neutrality.

3.6. *A double-coated neutral inclusion ($N = 4$).* When $N = 4$, the condition $K_{12} = T_{12} = 0$ can be explicitly written as

$$\begin{aligned} &(\beta_1+\alpha_1)(1+\beta_2)(1+\beta_3)+\rho_2^{-2}(1-\beta_1)(\beta_2+\alpha_2)(1+\beta_3)+\rho_3^{-2}(1-\beta_1)(1-\beta_2)\\ &\cdot(\beta_3+\alpha_3)-\rho_2^2\rho_3^{-2}(\beta_1+\alpha_1)(\beta_2-\alpha_2)(\beta_3+\alpha_3)=0, \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\beta_1(1-\alpha_2+2\beta_2)(1-\alpha_3+2\beta_3)+\rho_2^{-1}\beta_2(1+\alpha_1-2\beta_1)(1-\alpha_3+2\beta_3)\\ &+\rho_3^{-1}\beta_3(1+\alpha_1-2\beta_1)(1+\alpha_2-2\beta_2)-4\rho_2\rho_3^{-1}\beta_1\beta_3(\beta_2-\alpha_2)=0. \end{aligned} \tag{3.14}$$

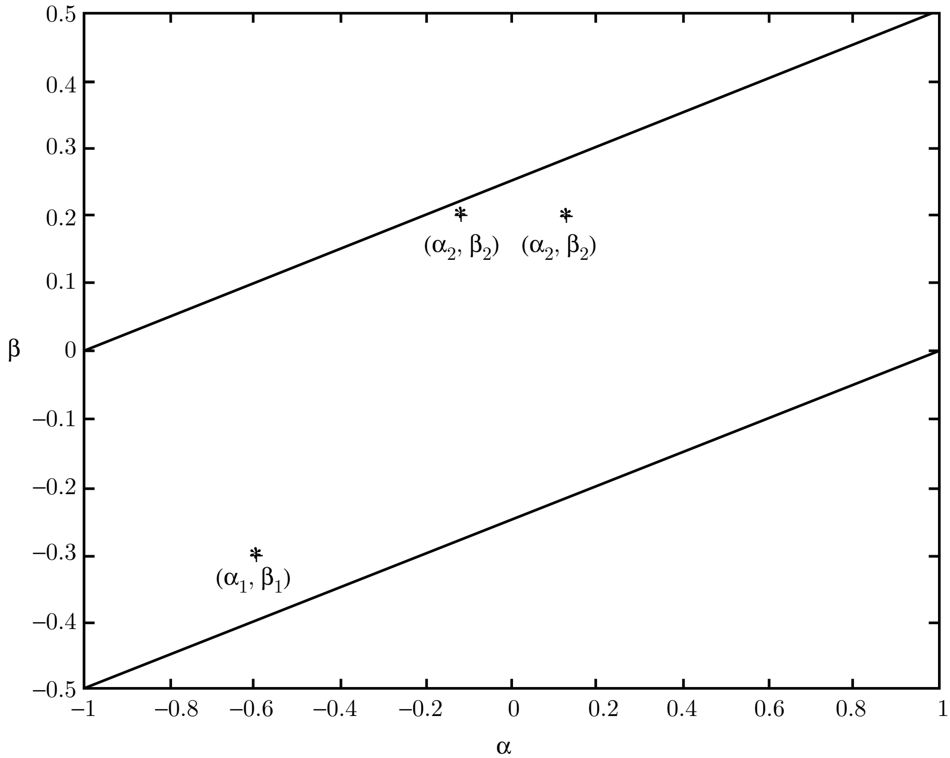

FIG. 5. The distributions of the obtained Dundurs parameters α_j, β_j, $(j = 1, 2{,}3)$ in the α, β-plane for a neutral double-coated circular inclusion with $\rho_2 = 0.8$and $\rho_3 = 0.6$.

For given geometric parameters ρ_2 and ρ_3, Eqs. (3.13) and (3.14) are two nonlinear equations for the Dundurs parameters α_j, β_j $(j = 1, 2, 3)$. We can first assign values of β_1, β_2, β_3, and α_1, following which, α_2 and α_3 can be determined from Eqs. (3.13) and (3.14) as follows:

$$\alpha_3 = \frac{-c_1 + \sqrt{c_1^2 + 4c_0 c_2}}{2c_2},$$
$$\alpha_2 = \frac{\left[\begin{array}{l} (\beta_1+\alpha_1)\left[(1+\beta_2)(1+\beta_3) - \rho_2^2\rho_3^{-2}\beta_2\beta_3\right] + (1-\beta_1)\left[\rho_2^{-2}\beta_2(1+\beta_3) + \rho_3^{-2}\beta_3(1-\beta_2)\right] \\ +\rho_3^{-2}\left[(1-\beta_1)(1-\beta_2) - \rho_2^2\beta_2(\beta_1+\alpha_1)\right]\alpha_3 \end{array}\right]}{\left[\rho_2^{-2}(\beta_1-1)(1+\beta_3) - \rho_2^2\rho_3^{-2}\beta_3(\beta_1+\alpha_1) - \rho_2^2\rho_3^{-2}(\beta_1+\alpha_1)\alpha_3\right]} \tag{3.15}$$

where

$$\begin{aligned}
c_2 &= \rho_3^{-2}\beta_1(1-\beta_1)(1-\beta_2) + \rho_2^2\rho_3^{-2}\beta_1(\beta_1+\alpha_1)(1+\beta_2) + \rho_2\rho_3^{-2}\beta_2(\beta_1+\alpha_1) \\
&\qquad \cdot (1+\alpha_1-2\beta_1), \\
c_1 &= \beta_1(\beta_1+\alpha_1)(1+\beta_2)(1+\beta_3)(1-\rho_2^2\rho_3^{-2}) + \rho_2^{-2}\beta_1(1-\beta_1)(1+3\beta_2)(1+\beta_3) \\
&\quad + (1-\beta_1)(1+\alpha_1-2\beta_1)\left[\rho_2^{-3}\beta_2(1+\beta_3) + \rho_3^{-3}\beta_3(1-\beta_2)\right] + \rho_3^{-3}\beta_1(1-\beta_1)(1-\beta_2) \\
&\qquad \cdot [4\rho_2\beta_3 - \rho_3(1+\beta_3)] \\
&\quad - \rho_2\rho_3^{-3}(\beta_1+\alpha_1)(1+\alpha_1-2\beta_1)\left[\rho_2\beta_3(1-\beta_2) + \rho_3\beta_2(1+\beta_3)\right], \\
c_0 &= \left[\beta_1(1+2\beta_3) - 4\rho_2\rho_3^{-1}\beta_1\beta_3 - \rho_3^{-1}\beta_3(1+\alpha_1-2\beta_1)\right] \\
&\quad \times \left\{(\beta_1+\alpha_1)\left[(1+\beta_2)(1+\beta_3) - \rho_2^2\rho_3^{-2}\beta_2\beta_3\right] + (1-\beta_1)\right. \\
&\qquad \left.\cdot \left[\rho_2^{-2}\beta_2(1+\beta_3) + \rho_3^{-2}\beta_3(1-\beta_2)\right]\right\} \\
&\quad + \left[\rho_2^{-2}(1-\beta_1)(1+\beta_3) + \rho_2^2\rho_3^{-2}\beta_3(\beta_1+\alpha_1)\right] \\
&\quad \times \left\{\beta_1(1+2\beta_2)(1+2\beta_3) - 4\rho_2\rho_3^{-1}\beta_1\beta_2\beta_3 + (1+\alpha_1-2\beta_1)\right. \\
&\qquad \left.\cdot \left[\rho_2^{-1}\beta_2(1+2\beta_3) + \rho_3^{-1}\beta_3(1-2\beta_2)\right]\right\}.
\end{aligned} \tag{3.16}$$

In addition, the obtained α_j and β_j should be within the parallelogram enclosed by $\alpha_j = \pm 1$ and $\alpha_j - 4\beta_j = \pm 1$ in the α_j,β_j-plane. For example, if we take $\rho_2 = 0.8$ and $\rho_3 = 0.6$, then the Dundurs parameters are determined as: $\alpha_1 = -0.6, \beta_1 = -0.3$, $\alpha_2 = -0.1182, \beta_2 = 0.2$, $\alpha_3 = 0.1227, \beta_3 = 0.2$, all of which are within the parallelogram as shown in Fig. 5 and will make the double-coated inclusion neutral to the non-uniform loading characterized by Eq. (3.9).

Conclusions. In this study we are concerned with the inverse problem associated with the design of multi-coated harmonic circular elastic inclusions subjected to a general class of remote non-uniform loading characterized by Eq. (2.3). In stark contrast to the design of harmonic holes where the remote (non-uniform) loading exerts a significant influence on the shape of the hole (Bjorkman and Richards, [2]; Wang et al., [9]), the multi-coated harmonic inclusion designed here is independent of the remote non-uniform stresses. Generally, we have found that in order to achieve harmonicity of these inclusions, the number of intermediate coatings $(N-2)$ should be equal to the number of terms in the asymptotic behavior of $\psi_N(z)$. As a consequence of our analysis, we have also found conditions under which our method will lead to neutral single- and double-coated circular inclusions under the non-uniform loading (3.9).

References

[1] Ammari, H., Kang, H., Lee, H., and Lim, M., 2011. Enhancement of near cloaking using generalized polarization tensors vanishing structure. Part I: the conductivity problem. arXiv:1104.3936v1.

[2] Bjorkman, G.S., Richards, R., 1976. Harmonic holes—an inverse problem in elasticity. ASME J. Appl. Mech. 43, 414–418.

[3] Bjorkman, G.S., Richards, R., 1979. Harmonic holes for nonconstant fields. ASME J. Appl. Mech. 46, 573–576.

[4] Dundurs, J., 1969. Elastic interaction of dislocations with inhomogeneities, in Mathematical Theory of Dislocations, T. Mura Ed., ASME, New York, pp. 70–115.

[5] Firoozbakhsh, K., Aleyaasin, M., 1989. The effect of stress concentration on the bone remodeling: Theoretical predictions. J. Biomech. Engrg. 111, 355–360.

[6] N. I. Muskhelishvili, *Some basic problems of the mathematical theory of elasticity. Fundamental equations, plane theory of elasticity, torsion and bending*, P. Noordhoff Ltd., Groningen, 1953. Translated by J. R. M. Radok. MR0058417 (15,370d)

[7] Ru, C.Q., 1998. Interface design of neutral elastic inclusions. Int. J. Solids Struct. 35, 559–572.

[8] Ru, C.Q., 1999. A new method for an inhomogeneity with stepwise graded interphase layer under thermomechanical loadings. J. Elasticity 56, 107−127.

[9] G. F. Wang, P. Schiavone, and C.-Q. Ru, *Harmonic shapes in finite elasticity under nonuniform loading*, Trans. ASME J. Appl. Mech. **72** (2005), no. 5, 691–694, DOI 10.1115/1.1979514. MR2168196 (2006c:74039)

[10] Weinans, H., Huiskes, R., Grootenboer, H.J., 1994. Effect of fit and bonding characteristics of femoral stems on adaptive bone remodeling. J. Biomech. Engrg. 116, 393−400.