QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXXII, NUMBER 2
JUNE 2014, PAGES 347–361
S 0033-569X(2014)01344-1
Article electronically published on March 14, 2014

GLOBAL SOLUTION TO A NON-CLASSICAL HEAT PROBLEM IN THE SEMI-SPACE $\mathbb{R}^+ \times \mathbb{R}^{n-1}$

By

MAHDI BOUKROUCHE (*Lyon University, F-42023 Saint-Etienne, Institut Camille Jordan CNRS UMR 5208, 23 rue Paul Michelon 42023 Saint-Etienne Cedex 2, France*)

AND

DOMINGO A. TARZIA (*Departamento de Matemática-CONICET, FCE, Univ. Austral, Paraguay 1950, S2000FZF Rosario, Argentina*)

Abstract. We consider the non-classical heat equation in the n-dimensional domain $D = \mathbb{R}^+ \times \mathbb{R}^{n-1}$ for which the internal energy supply depends on the heat flux on the boundary $S = \partial D$. The problem is motivated by the modeling of temperature regulation in the medium. Using the Green function for the domain D, the solution is found for an integral representation depending on the heat flux V on S which is an additional unknown of the problem. We obtain that V must satisfy a Volterra integral equation of second kind at time t with a parameter in $\mathbb{R}^{n-1}$. Under some conditions on data, we show that there exists a unique local solution which can be extended globally in time. This work generalizes the results obtained in the one-dimensional case.

1. Introduction. The aim of this paper is to study a problem on the non-classical heat equation, in the semi-n-dimensional space domain D for which the internal energy supply depends on the heat flux on the boundary S. In order to facilitate the notation we denote a point in $\mathbb{R}^n$ as follows: $(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$. The domain D and its boundary S are defined by

$$D = \mathbb{R}^+ \times \mathbb{R}^{n-1} = \{(x, y) \in \mathbb{R}^n : \quad x = x_1 > 0, \quad y = (x_2, \cdots, x_n) \in \mathbb{R}^{n-1}\}, \tag{1.1}$$

$$S = \partial D = \{0\} \times \mathbb{R}^{n-1} = \{(x, y) \in \mathbb{R}^n : \quad x = 0, \quad y \in \mathbb{R}^{n-1}\}. \tag{1.2}$$

Received July 19, 2012.
2010 *Mathematics Subject Classification.* Primary 35C15, 35K05, 35K20, 35K60, 80A20.
Key words and phrases. Non-classical n-dimensional heat equation, Volterra integral equation, existence and uniqueness of solution, integral representation of the solution.
E-mail address: Mahdi.Boukrouche@univ-st-etienne.fr
E-mail address: DTarzia@austral.edu.ar

PROBLEM 1.1. Find the temperature u at (x, y, t) such that it satisfies the following conditions:

$$\begin{array}{rcl} u_t - \Delta u &=& -F(u_x(0,y,t)), \qquad x>0, \quad y \in \mathbb{R}^{n-1}, \quad t>0 \\ u(0,y,t) &=& 0, \quad y \in \mathbb{R}^{n-1}, \quad t>0, \\ u(x,y,0) &=& h(x,y), \qquad x>0, \quad y \in \mathbb{R}^{n-1}, \end{array}$$

where Δ is the Laplacian in $\mathbb{R}^n$.

This problem is motivated by modeling the temperature in an isotropic medium with a non-uniform source that provides a cooling or heating system, according to the properties of F with respect to development of heat flow at the boundary S. For example, assuming that

$$V\,F(V,t) > 0, \quad \forall V \neq 0, \quad F(0) = 0, \tag{1.3}$$

then it is a cooling source when $u_x(0,y,t) > 0$ and a heating source when $u_x(0,y,t) < 0$ [6, 8].

Some references on the subject include [1], [18], [19], where the following semi-one-dimension of this nonlinear problem has been considered. The non-classical one-dimensional heat equation in a slab with fixed or moving boundaries was studied in [16]. See also other references on the subject, including [7], [10]-[13]. To our knowledge, this is the first time that the solution to a non-classical heat conduction of the type of Problem 1.1 is given. Other non-classical problems can be found in [2].

Section 2 provides the basic solution to the n-dimensional heat equation, which will be used in Section 3 to show that, under certain conditions on data F and h of Problem 1.1, there exists a unique local solution, which can be globally extended in time.

We also give in Lemma 3.4 several observations concerning the forcing function V_0, describing the flux on the boundary S. In Section 4 we study some particular cases of this problem. In Section 5 we also give a general conclusion.

2. Basic solutions for the n-dimensional heat equation. In this section we recall results on the integral representation of solutions of some classical problems of heat distribution in n-dimensional cases. For the convenience of the reader we will provide a proof for the generalization of classical one-dimensional results (see Lemma 2.1 and Lemma 2.2). We end this section with a technical Lemma 2.4, which contains some mathematical formulas useful for the study of our Problem 1.1.

It is classical that, by using the partial Fourier's transform, the solution of the following Cauchy problem for the n-dimensional heat equation

$$u_t - \Delta u = 0, \qquad (x,y) \in \mathbb{R}^n, \quad t>0 \tag{2.1}$$

$$u(x,y,0) = h(x,y), \qquad (x,y) \in \mathbb{R}^n \tag{2.2}$$

is known as Poisson's formula, given by the expression [9, 14]

$$u(x,y,t) = \int_{\mathbb{R}^n} K(x,y,t;\xi,\eta,0)h(\xi,\eta)d\xi d\eta \tag{2.3}$$

where K is the fundamental solution of an n-dimensional heat equation defined by

$$K(x,y,t;\xi,\eta,\tau)=\frac{\exp\left[-\frac{(x-\xi)^2+\|y-\eta\|^2}{4(t-\tau)}\right]}{\left(2\sqrt{\pi(t-\tau)}\right)^n},\quad (x,y)\in\mathbb{R}^n,\quad (\xi,\eta)\in\mathbb{R}^n,\quad t>\tau,\quad (2.4)$$

with $\xi=\xi_1\in\mathbb{R}$, $\eta=(\xi_2,\cdots,\xi_n)\in\mathbb{R}^{n-1}$ and $\|y-\eta\|=\sqrt{\sum_{i=2}^n(x_i-\xi_i)^2}$.

LEMMA 2.1. The solution of the problem

$$\begin{aligned} u_t-\Delta u &= 0,\qquad x>0,\quad y\in\mathbb{R}^{n-1},\quad t>0 & (2.5)\\ u(0,y,t) &= 0,\quad y\in\mathbb{R}^{n-1},\quad t>0 & (2.6)\\ u(x,y,0) &= h(x,y),\qquad x>0,\quad y\in\mathbb{R}^{n-1}, & (2.7)\end{aligned}$$

is given by the following formula:

$$u(x,y,t)=\int_D G_1(x,y,t;\xi,\eta,0)h(\xi,\eta)d\xi d\eta \quad (2.8)$$

where G_1 is the Green's function for the n-dimensional heat equation with homogeneous Dirichlet's boundary conditions, given the following expression:

$$\begin{aligned} G_1(x,y,t;\xi,\eta,\tau) &= K(x,y,t;\xi,\eta,\tau)-K(-x,y,t;\xi,\eta,\tau)\\ &= \frac{\exp\left[-\frac{\|y-\eta\|^2}{4(t-\tau)}\right]}{\left(2\sqrt{\pi(t-\tau)}\right)^{n-1}}G(x,t,\xi,\tau), \qquad (2.9)\end{aligned}$$

where K is given by (2.4) and G is the Green's function for the one-dimensional case given by

$$G(x,t,\xi,\tau)=\frac{e^{-\frac{(x-\xi)^2}{4(t-\tau)}}-e^{-\frac{(x+\xi)^2}{4(t-\tau)}}}{2\sqrt{\pi(t-\tau)}}\qquad t>\tau.$$

Proof. Define $\tilde{h}$ on $\mathbb{R}^n$ by

$$\tilde{h}(\xi,\eta)=\begin{cases} h(\xi,\eta) & \text{if}\quad (\xi,\eta)\in\mathbb{R}^+\times\mathbb{R}^{n-1}\\ -h(-\xi,\eta) & \text{if}\quad (\xi,\eta)\in\mathbb{R}^-\times\mathbb{R}^{n-1},\end{cases}$$

so the solution of the Cauchy problem

$$\begin{aligned} u_t-\Delta u &= 0\quad\text{in}\quad\mathbb{R}^n,\quad t>0,\\ u(x,y,0) &= \tilde{h}(x,y),\quad\text{in}\quad\mathbb{R}^n,\end{aligned}$$

is given by

$$\begin{aligned} u(x,t)=\int_{\mathbb{R}^n}K(x,y,t;\xi,\eta,0)\tilde{h}(\xi,\eta)d\xi d\eta &= \int_{\mathbb{R}^+\times\mathbb{R}^{n-1}}K(x,y,t;\xi,\eta,0)h(\xi,\eta)d\xi d\eta\\ &\quad -\int_{\mathbb{R}^-\times\mathbb{R}^{n-1}}K(x,y,t;\xi,\eta,0)h(-\xi,\eta)d\xi d\eta.\end{aligned}$$

With the change of the variables $\xi_1 = -\xi$ in the integral on $\mathbb{R}^- \times \mathbb{R}^{n-1}$ and $\xi_1 = \xi$ in the integral on $\mathbb{R}^+ \times \mathbb{R}^{n-1}$, we get

$$u(x,t) = \int_{\mathbb{R}^+ \times \mathbb{R}^{n-1}} [K(x,y,t;\xi_1,\eta,0) - K(x,y,t;-\xi_1,\eta,0)]h(\xi_1,\eta)d\xi_1 d\eta.$$

As

$$K(x,y,t;-\xi_1,\eta,0) = K(-x,y,t;\xi_1,\eta,0),$$

we deduce the formula (2.8) taking into account (2.9) and $G_1(0,y,t;\xi,\eta,\tau) = 0$; thus

$$u(0,y,t) = 0.$$

So u is the solution of the problem (2.5)-(2.7). $\square$

LEMMA 2.2. The solution of the problem

$$u_t - \Delta u = 0, \quad x > 0, \quad y \in \mathbb{R}^{n-1}, \quad t > 0 \tag{2.10}$$
$$u_x(0,y,t) = 0, \quad y \in \mathbb{R}^{n-1}, \quad t > 0 \tag{2.11}$$
$$u(x,y,0) = h(x,y), \quad x > 0, \quad y \in \mathbb{R}^{n-1}, \tag{2.12}$$

is given by the following formula:

$$u(x,y,t) = \int_D N_1(x,y,t;\xi,\eta,0)h(\xi,\eta)d\xi d\eta \tag{2.13}$$

where N_1 is the Green's function for the n-dimensional heat equation with homogeneous Neumann's boundary conditions, given by the following expression:

$$\begin{aligned} N_1(x,y,t;\xi,\eta,\tau) &= K(x,y,t;\xi,\eta,\tau) + K(-x,y,t;\xi,\eta,\tau) \\ &= \frac{\exp\left[-\frac{\|y-\eta\|^2}{4(t-\tau)}\right]}{\left(2\sqrt{\pi(t-\tau)}\right)^{n-1}} N(x,t,\xi,\tau), \end{aligned} \tag{2.14}$$

where K is given by (2.4) and N is the Neumann's function for the one-dimensional case defined by

$$N(x,t,\xi,\tau) = \frac{e^{-\frac{(x-\xi)^2}{4(t-\tau)}} + e^{-\frac{(x+\xi)^2}{4(t-\tau)}}}{2\sqrt{\pi(t-\tau)}} \quad t > \tau.$$

Proof. Define $\tilde{h}$ on $\mathbb{R}^n$ by

$$\tilde{h}(\xi,\eta) = \begin{cases} h(\xi,\eta) & \text{if} \quad (\xi,\eta) \in \mathbb{R}^+ \times \mathbb{R}^{n-1} \\ h(-\xi,\eta) & \text{if} \quad (\xi,\eta) \in \mathbb{R}^- \times \mathbb{R}^{n-1} \end{cases}$$

so the solution of the Cauchy problem

$$\begin{aligned} u_t - \Delta u &= 0, \quad \text{in} \quad \mathbb{R}^n, \quad t > 0, \\ u(x,y,0) &= \tilde{h}(x,y), \quad \text{in} \quad \mathbb{R}^n, \end{aligned}$$

is given by

$$u(x,t) = \int_{\mathbb{R}^n} K(x,y,t;\xi,\eta,0)\tilde{h}(\xi,\eta)d\xi d\eta = \int_{\mathbb{R}^+\times\mathbb{R}^{n-1}} K(x,y,t;\xi,\eta,0)h(\xi,\eta)d\xi d\eta$$
$$+ \int_{\mathbb{R}^-\times\mathbb{R}^{n-1}} K(x,y,t;\xi,\eta,0)h(-\xi,\eta)d\xi d\eta.$$

With the change of the variables $\xi_1 = -\xi$ in the integral on $\mathbb{R}^- \times \mathbb{R}^{n-1}$ and $\xi_1 = \xi$ in the integral on $\mathbb{R}^+ \times \mathbb{R}^{n-1}$, we get

$$u(x,t) = \int_{\mathbb{R}^+\times\mathbb{R}^{n-1}} [K(x,y,t;\xi_1,\eta,0) + K(x,y,t;-\xi_1,\eta,0)]h(\xi_1,\eta)d\xi_1 d\eta.$$

As $K(x,y,t;-\xi_1,\eta,0)] = K(-x,y,t;\xi_1,\eta,0)]$, we deduce the formula (2.13) taking into account (2.14). We have

$$N_{1,x}(x,y,t;\xi,\eta,\tau) = -\frac{(x-\xi)}{2(t-\tau)}K(x,y,t;\xi,\eta,\tau) - \frac{(x+\xi)}{2(t-\tau)}K(-x,y,t;\xi,\eta,\tau),$$

and then $N_{1,x}(0,y,t;\xi,\eta,\tau) = 0$ thus $u_x(0,y,t) = 0$. So u is the solution of the problem (2.10)-(2.12). □

Remark 2.3. In the proof of Lemma 2.1, we chose a function $\tilde{h}$ odd, whereas in the proof of Lemma 2.2, we choose a function $\tilde{h}$ even. This is in order to obtain $G_1(0,y,t;\xi,\eta,\tau) = 0$ and to satisfy the boundary condition (2.6). If we keep, in the proof of Lemma 2.2, the function $\tilde{h}$ odd, the boundary condition (2.11) cannot be satisfied without adding another term to (2.13).

The same problem occurs if, in the proof of Lemma 2.1, we take the function $\tilde{h}$ even, instead of odd. Thus the boundary condition (2.6) cannot be satisfied without adding another term to (2.8).

We now present the following technical Lemma 2.4, which contains mathematical formulas useful for the study of our Problem 1.1.

Lemma 2.4. The functions G_1 and N_1 have the following fundamental properties:

$$\int_{\mathbb{R}^{n-1}} \exp\left[-\frac{\|y-\eta\|^2}{4(t-\tau)}\right] d\eta = \left(2\sqrt{\pi(t-\tau)}\right)^{n-1} \tag{2.15}$$

$$\int_{\mathbb{R}^{n-1}} \|y-\eta\|^2 \exp\left[-\frac{\|y-\eta\|^2}{4(t-\tau)}\right] d\eta = \frac{(n-1)(t-\tau)}{\sqrt{2}} \left(2\sqrt{\pi(t-\tau)}\right)^{n-1} \tag{2.16}$$

$$\int_{\mathbb{R}^{n-1}} G_1(x,y,t,\xi,\eta,\tau)d\eta = G(x,t,\xi,\tau) \tag{2.17}$$

$$\int_{\mathbb{R}^{n-1}} N_1(x,y,t,\xi,\eta,\tau)d\eta = N(x,t,\xi,\tau) \tag{2.18}$$

$$\int_{\mathbb{R}^{n-1}} G_{1,x}(0,y,t,\xi,\eta,\tau)d\eta = G_x(0,t,\xi,\tau) = \frac{\xi}{2\sqrt{\pi}(t-\tau)^{3/2}} \exp\left[-\frac{\xi^2}{4(t-\tau)}\right] \tag{2.19}$$

$$\int_0^\infty N_1(x,y,t,\xi,\eta,\tau)d\xi = \frac{1}{(2\sqrt{\pi(t-\tau)})^{n-1}} \exp\left[-\frac{\|y-\eta\|^2}{4(t-\tau)}\right] \tag{2.20}$$

$$\int_0^\infty G_{1,x}(0,y,t,\xi,\eta,\tau)d\xi = \frac{2}{(2\sqrt{\pi}(t-\tau))^n}\exp\left[-\frac{\|y-\eta\|^2}{4(t-\tau)}\right]. \tag{2.21}$$

$$\int_0^\infty G_1(x,y,t,\xi,\eta,\tau)d\xi = \frac{1}{(2\sqrt{\pi}(t-\tau))^{n-1}}\exp\left[-\frac{\|y-\eta\|^2}{4(t-\tau)}\right]\operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right), \tag{2.22}$$

where

$$\operatorname{erf}(\zeta) = \left(\frac{2}{\sqrt{\pi}}\int_0^\zeta e^{-X^2}dX\right)$$

is the error function.

Proof. Taking into account that

$$\int_{\mathbb{R}} \exp\left[-\frac{(y_j-\eta_j)^2}{4(t-\tau)}\right]d\eta_j = 2\sqrt{\pi(t-\tau)} \quad \forall j\in\mathbb{N}, \tag{2.23}$$

we have

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \exp\left[-\frac{\|y-\eta\|^2}{4(t-\tau)}\right]d\eta &= \Pi_{i=2}^n \int_{-\infty}^\infty \exp\left[-\frac{(y_i-\eta_i)^2}{4(t-\tau)}\right]d\eta_i \\ &= \left(2\sqrt{\pi(t-\tau)}\right)^{n-1}, \end{aligned}\tag{2.24}$$

and thus (2.15) holds.

Taking into account (2.23) and

$$\begin{aligned}&\int_{\mathbb{R}}(y_i-\eta_i)^2\exp\left[-\frac{(y_i-\eta_i)^2}{4(t-\tau)}\right]d\eta_i \\ &\quad= \int_{\mathbb{R}} 2(t-\tau)z^2\exp(-z^2)\sqrt{2(t-\tau)}dz = (t-\tau)\sqrt{2\pi(t-\tau)},\end{aligned}$$

we have

$$\begin{aligned}\int_{\mathbb{R}^{n-1}}\|y-\eta\|^2\exp\left[-\frac{\|y-\eta\|^2}{4(t-\tau)}\right]d\eta &= \int_{\mathbb{R}^{n-1}}\left(\sum_{i=2}^n(y_i-\eta_i)^2\right)\Pi_{j=2}^n\exp\left[-\frac{(y_j-\eta_j)^2}{4(t-\tau)}\right]d\eta_j \\ &= \sum_{i=2}^n\int_{\mathbb{R}^{n-1}}(y_i-\eta_i)^2\Pi_{j=2}^n\exp\left[-\frac{(y_j-\eta_j)^2}{4(t-\tau)}\right]d\eta_j \\ = \Pi_{j=2,j\neq i}^n\left(\int_{\mathbb{R}}\exp\left[-\frac{(y_j-\eta_j)^2}{4(t-\tau)}\right]d\eta_j\right)&\sum_{i=2}^n\left(\int_{\mathbb{R}}(y_i-\eta_i)^2\exp\left[-\frac{(y_i-\eta_i)^2}{4(t-\tau)}\right]d\eta_i\right),\end{aligned}$$

and so (2.16) follows.

We also have

$$\int_{\mathbb{R}^{n-1}} G_1(x,y,t,\xi,\eta,\tau)d\eta = \frac{G(x,t,\xi,\tau)}{\left(2\sqrt{\pi(t-\tau)}\right)^{n-1}}\int_{\mathbb{R}^{n-1}}\exp\left[-\frac{\|y-\eta\|^2}{4(t-\tau)}\right]d\eta; \tag{2.25}$$

then (2.17) follows using (2.15), and similarly for (2.18). We obtain (2.19) using (2.17).

We have

$$\begin{aligned}\int_0^{+\infty} e^{\frac{-(x-\xi)^2}{4(t-\tau)}} d\xi &= 2\sqrt{t-\tau}\left(\int_{-\infty}^0 e^{-X^2} dX + \int_0^{\frac{x}{2\sqrt{t-\tau}}} e^{-X^2} dX\right) \\ &= \sqrt{\pi(t-\tau)}\left(1+\operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right)\right) \qquad (2.26)\end{aligned}$$

and

$$\begin{aligned}\int_0^{+\infty} e^{\frac{-(x+\xi)^2}{4(t-\tau)}} d\xi &= 2\sqrt{t-\tau}\left(\int_0^{+\infty} e^{-X^2} dX - \int_0^{\frac{x}{2\sqrt{t-\tau}}} e^{-X^2} dX\right) \\ &= \sqrt{\pi(t-\tau)}\left(1-\operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right)\right). \qquad (2.27)\end{aligned}$$

By using (2.26) and (2.27), we obtain

$$\int_0^{+\infty} N(x,t,\xi,\tau) d\xi = 1,$$

and, moreover, using the definition of (2.14), we deduce (2.20).

From (2.4) and (2.9), by derivation with respect to x, we obtain

$$K_{,x}(x,y,t;\xi,\eta,\tau) = \frac{-2(x-\xi)e^{-(x-\xi)^2+\|y-\eta\|^2}}{4(t-\tau)(2\sqrt{\pi(t-\tau)})^n} = \frac{-(x-\xi)}{2(t-\tau)}K(x,y,t;\xi,\eta,\tau)$$

$$G_{1,x}(x,y,t;\xi,\eta,\tau) = \frac{-(x-\xi)}{2(t-\tau)}K(x,y,t;\xi,\eta,\tau) + \frac{(x+\xi)}{2(t-\tau)}K(-x,y,t;\xi,\eta,\tau), \qquad (2.28)$$

then

$$G_{1,x}(0,y,t;\xi,\eta,\tau) = \frac{\xi}{t-\tau}K(0,y,t;\xi,\eta,\tau) = \frac{\xi}{(t-\tau)^{\frac{n+2}{2}}(2\sqrt{\pi})^n} e^{-\frac{\xi^2+\|y-\eta\|^2}{4(t-\tau)}}. \qquad (2.29)$$

Thus

$$\int_0^{+\infty} G_{1,x}(0,y,t;\xi,\eta,\tau) d\xi = \frac{e^{-\frac{\|y-\eta\|^2}{4(t-\tau)}}}{(t-\tau)^{\frac{n+2}{2}}(2\sqrt{\pi})^n}\int_0^{+\infty} \xi e^{-\frac{\xi^2}{4(t-\tau)}} d\xi = \frac{2e^{-\frac{\|y-\eta\|^2}{4(t-\tau)}}}{(2\sqrt{\pi(t-\tau)})^n},$$

as

$$\int_0^{+\infty} \xi e^{-\frac{\xi^2}{4(t-\tau)}} d\xi = 2(t-\tau), \qquad (2.30)$$

and (2.21) holds.

By using (2.26) and (2.27), we obtain

$$\int_0^{+\infty} G(x,t,\xi,\tau) d\xi = \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right), \qquad (2.31)$$

so by the definition of (2.9) we obtain (2.22), and close the proof of this lemma. □

3. Existence and uniqueness of the solution to Problem 1.1. In this section, we give first in Theorem 3.1 the integral representation (3.1) of the solution of our Problem 1.1, but it depends on the heat flow on the boundary S, which satisfies the Volterra integral equation (3.3) with initial condition (3.2). Then we prove, in Theorem 3.3, under some assumption on the data, that there exists a unique solution of the Problem 1.1 locally in time which can be extended globally in time.

THEOREM 3.1. The integral representation of the solution of the problem 1.1 is given by the following expression:

$$u(x,y,t) = u_0(x,y,t) - \int_0^t \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right)}{(2\sqrt{\pi(t-\tau)})^{n-1}} \left[\int_{\mathbb{R}^{n-1}} \exp\left[-\frac{\|y-\eta\|^2}{4(t-\tau)}\right] F(V(\eta,\tau))d\eta\right] d\tau \tag{3.1}$$

where

$$u_0(x,y,t) = \int_D G_1(x,y,t;\xi,\eta,0)h(\xi,\eta)d\xi d\eta, \tag{3.2}$$

and the function V defined by $V(y,t) = u_x(0,y,t)$ for $y \in \mathbb{R}^{n-1}$ and $t > 0$ (heat flux on the surface $x = 0$) satisfies the following Volterra integral equation:

$$V(y,t) = V_0(y,t) - 2\int_0^t \frac{1}{(2\sqrt{\pi(t-\tau)})^n} \left[\int_{\mathbb{R}^{n-1}} \exp\left[-\frac{\|y-\eta\|^2}{4(t-\tau)}\right] F(V(\eta,\tau))d\eta\right] d\tau \tag{3.3}$$

in the variable $t > 0$, with $y \in \mathbb{R}^{n-1}$ being a parameter where

$$V_0(y,t) = \int_D G_{1,x}(0,y,t;\xi,\eta,0)h(\xi,\eta)d\xi d\eta. \tag{3.4}$$

Proof. As the boundary condition in Problem (1.1) is homogeneous, we have from [9]

$$\begin{aligned} u(x,y,t) &= \int_D G_1(x,y,t;\xi,\eta,0)h(\xi,\eta)d\xi d\eta \\ &\quad + \int_0^t \int_D G_1(x,y,t;\xi,\eta,\tau)[-F(V(\eta,\tau))]d\xi d\eta d\tau, \end{aligned} \tag{3.5}$$

and therefore

$$\begin{aligned} u_x(x,y,t) &= \int_D G_{1,x}(x,y,t;\xi,\eta,0)h(\xi,\eta)d\xi d\eta \\ &\quad + \int_0^t \int_D G_{1,x}(x,y,t;\xi,\eta,\tau)[-F(V(\eta,\tau))]d\xi d\eta d\tau. \end{aligned} \tag{3.6}$$

Using (2.29) we obtain

$$\begin{aligned} &\int_D G_{1,x}(0,y,t;\xi,\eta,\tau)F(V(\eta,\tau))d\xi d\eta \\ &= \int_{\mathbb{R}^{n-1}} \frac{F(V(\eta,\tau))e^{-\frac{\|y-\eta\|^2}{4(t-\tau)}}}{(t-\tau)^{\frac{n+2}{2}}(2\sqrt{\pi})^n} \left(\int_0^{+\infty} \xi e^{-\frac{\xi^2}{4(t-\tau)}} d\xi\right) d\eta \\ &= \frac{2}{(2\sqrt{\pi(t-\tau)})^n} \int_{\mathbb{R}^{n-1}} F(V(\eta,\tau))e^{-\frac{\|y-\eta\|^2}{4(t-\tau)}} d\eta. \end{aligned} \tag{3.7}$$

Thus taking $x=0$ in (3.6) with (3.7), we get (3.3).

By (2.9) (the definition of G_1), we obtain

$$\begin{aligned}
&\int_D G_1(x,y,t;\xi,\eta,\tau)F(V(\eta,\tau))d\xi d\eta \\
&\quad = \frac{1}{2(\sqrt{\pi(t-\tau)})^n}\int_D e^{\frac{-\|y-\eta\|^2}{4(t-\tau)}}\left[e^{-\frac{(x-\xi)^2}{4(t-\tau)}}-e^{-\frac{(x+\xi)^2}{4(t-\tau)}}\right]F(V(\eta,\tau))d\xi d\eta \\
&\quad = \frac{1}{2(\sqrt{\pi(t-\tau)})^n}\int_{\mathbb{R}^+}\left[e^{-\frac{(x-\xi)^2}{4(t-\tau)}}-e^{-\frac{(x+\xi)^2}{4(t-\tau)}}\right]d\xi\int_{\mathbb{R}^{n-1}}e^{\frac{-\|y-\eta\|^2}{4(t-\tau)}}F(V(\eta,\tau))d\eta,
\end{aligned}$$

and using (2.26)-(2.27), we get

$$\int_D G_1(x,y,t;\xi,\eta,\tau)F(V(\eta,\tau))d\xi d\eta = \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right)}{(2\sqrt{\pi(t-\tau)})^{n-1}}\int_{\mathbb{R}^{n-1}}e^{-\frac{\|y-\eta\|^2}{4(t-\tau)}}F(V(\eta,\tau))d\eta.\quad(3.8)$$

Taking this formula in (3.5), we obtain (3.1). □

LEMMA 3.2. The simplified form of the Volterra integral equation (3.3) is

$$\begin{aligned}
V(y,t) \;&=\; \frac{1}{t(2\sqrt{\pi}\,t)^n}\int_{\mathbb{R}^+}\xi e^{-\frac{\xi^2}{4t}}\left(\int_{\mathbb{R}^{n-1}}e^{-\frac{\|y-\eta\|^2}{4t}}h(\xi,\eta)d\eta\right)d\xi \\
&\quad -\frac{2}{(2\sqrt{\pi})^n}\int_0^t\int_{\mathbb{R}^{n-1}}\frac{F(V(\eta,\tau))}{(t-\tau)^{n/2}}e^{-\frac{\|y-\eta\|^2}{4(t-\tau)}}d\eta d\tau.
\end{aligned}\tag{3.9}$$

Proof. Using (2.29) with $\tau=0$ in the Volterra integral equation (3.3), we obtain (3.9). □

THEOREM 3.3. Assume that $h\in\mathcal{C}(D)$, $F\in\mathcal{C}(\mathbb{R})$ and locally Lipschitz in $\mathbb{R}$; then there exists a unique solution of the problem 1.1 locally in time which can be extended globally in time.

Proof. We know from Theorem (3.1) that, to prove the existence and uniqueness of the solution (3.1) of Problem (1.1), it is enough to solve the Volterra integral equation (3.9). So we rewrite it as follows:

$$V(y,t)=f(y,t)+\int_0^t g(y,\tau,V(y,\tau))d\tau \tag{3.10}$$

with

$$f(y,t)=\frac{1}{t(2\sqrt{\pi}\,t)^n}\int_{\mathbb{R}^+}\xi e^{-\frac{\xi^2}{4t}}\left(\int_{\mathbb{R}^{n-1}}e^{-\frac{\|y-\eta\|^2}{4t}}h(\xi,\eta)d\eta\right)d\xi \tag{3.11}$$

and

$$g(t,\tau,y,V(y,\tau))=-\frac{2}{(2\sqrt{\pi})^n}\int_{\mathbb{R}^{n-1}}\frac{F(V(\eta,\tau))}{(t-\tau)^{n/2}}e^{-\frac{\|y-\eta\|^2}{4(t-\tau)}}d\eta. \tag{3.12}$$

We have to check the conditions $H1$ to $H4$ in Theorem 1.1 page 87, and $H5$ and $H6$ in Theorem 1.2 page 91 in [15].

• The function f is defined and continuous for all $(y,t)\in\mathbb{R}^{n-1}\times\mathbb{R}^+$, so $H1$ holds.

• The function g is measurable in (y,t,τ,x) for $0\le\tau\le t<+\infty$, $x\in\mathbb{R}$, $y\in\mathbb{R}^{n-1}$, and continuous in x for all $(y,t,\tau)\in\mathbb{R}^{n-1}\times\mathbb{R}^+\times\mathbb{R}^+$, $g(y,t,\tau,x)=0$ if $\tau>t$, so $H2$ holds.

• For all $k > 0$ and all bounded sets B in $\mathbb{R}$, we have

$$\begin{aligned}|g(y,t,\tau,X)| &\leq \frac{2}{(2\sqrt{\pi})^n}\sup_{X\in B}|F(X)|(t-\tau)^{-n/2}\int_{\mathbb{R}^{n-1}} e^{-\|y-\eta\|^{\frac{2}{4(t-\tau)}}}d\eta \\ &\leq \frac{2}{(2\sqrt{\pi})^n}\sup_{X\in B}|F(X)|(t-\tau)^{-n/2}(2\sqrt{\pi(t-\tau)})^{n-1} \\ &= \frac{1}{\sqrt{\pi}}\sup_{X\in B}|F(X)|\frac{1}{\sqrt{(t-\tau)}};\end{aligned}$$

thus there exists a measurable function m given by

$$m(t,\tau) = \frac{1}{\sqrt{\pi}}\sup_{X\in B}|F(X)|\frac{1}{\sqrt{(t-\tau)}} \tag{3.13}$$

such that

$$|g(y,t,\tau,X)| \leq m(t,\tau) \quad \forall 0 \leq \tau \leq t \leq k, \quad X \in B \tag{3.14}$$

and satisfies

$$\begin{aligned}\sup_{t\in[0,K]}\int_0^t m(t,\tau)d\tau &= \frac{1}{\sqrt{\pi}}\sup_{X\in B}|F(X)|\sup_{t\in[0,k]}\int_0^t \frac{1}{\sqrt{t-\tau}}d\tau \\ &= \frac{1}{\pi}\sup_{X\in B}|F(X)|\sup_{t\in[0,k]}\left(-2\sqrt{(t-\tau)}|_0^t\right) \\ &= \frac{1}{\pi}\sup_{X\in B}|F(X)|\sup_{t\in[0,k]}\sqrt{t} \leq \frac{2\sqrt{k}}{\pi}\sup_{X\in B}|F(X)| < \infty,\end{aligned}$$

so $H3$ holds.

• Moreover, we also have

$$\lim_{t\to 0^+}\int_0^t m(t,\tau)d\tau = \frac{1}{\sqrt{\pi}}\sup_{X\in B}|F(X)|\lim_{t\to 0^+}\int_0^t \frac{d\tau}{\sqrt{t-\tau}} = \frac{1}{\sqrt{\pi}}\sup_{X\in B}|F(X)|\lim_{t\to 0^+}(2\sqrt{t}) = 0, \tag{3.15}$$

and

$$\lim_{t\to 0^+}\int_T^{T+t} m(t,\tau)d\tau = \frac{1}{\sqrt{\pi}}\sup_{X\in B}|F(X)|\lim_{t\to 0^+}(2\sqrt{t}) = 0. \tag{3.16}$$

• For each compact subinterval J of $\mathbb{R}^+$, each bounded set B in $\mathbb{R}^{n-1}$, and each $t_0 \in \mathbb{R}^+$, we set

$$\mathcal{A}(t,y,V(\eta)) = |g(t,\tau;y,V(\eta,\tau)) - g(t_0,\tau;y,V(\eta,\tau))|.$$

$$\mathcal{A}(t,y,V(\eta)) = \frac{2}{(2\sqrt{\pi})^n}\int_J\left|\int_{\mathbb{R}^{n-1}} e^{-\frac{\|y-\eta\|^2}{4(t-\tau)}}\frac{F(V(\eta,\tau))}{(t-\tau)^{-n/2}} - e^{-\frac{\|y-\eta\|^2}{4(t_0-\tau)}}\frac{F(V(\eta,\tau))}{(t_0-\tau)^{-n/2}}d\eta\right|d\tau$$

as the function $\tau \mapsto V(\eta,\tau)$ is continuous and is in the compact $B \subset \mathbb{R}$ for all $\eta \in \mathbb{R}^{n-1}$. So by the continuity of F we get $F(V(\eta,\tau)) \subset F(B)$; that is, there exists $M > 0$ such that $|F(V(\eta,\tau))| \leq M$ for all $(\eta,\tau) \in \mathbb{R}^{n-1}\times\mathbb{R}^+$. So

$$\sup_{V(\eta)\in\mathcal{C}(J,B)}\mathcal{A}(t,y,V(\eta)) \leq \frac{2M}{(2\sqrt{\pi})^n}\sup_{V(\eta)\in\mathcal{C}(J,B)}\left|\int_{\mathbb{R}^{n-1}}\frac{e^{-\frac{\|y-\eta\|^2}{4(t-\tau)}}}{\sqrt{(t-\tau)}^n}d\eta - \int_{\mathbb{R}^{n-1}}\frac{e^{-\frac{\|y-\eta\|^2}{4(t_0-\tau)}}}{\sqrt{(t_0-\tau)}^n}d\eta\right|.$$

Using (2.15), we obtain

$$\sup_{V(\eta)\in\mathcal{C}(J,B)} \mathcal{A}(t,y,V(\eta)) \leq \frac{2M}{(2\sqrt{\pi})^n} \sup_{V(\eta)\in\mathcal{C}(J,B)} \left| \frac{(2\sqrt{\pi(t-\tau)})^{n-1}}{(\sqrt{t-\tau})^n} - \frac{(2\sqrt{\pi(t_0-\tau)})^{n-1}}{(\sqrt{t_0-\tau})^n} \right|,$$

and thus

$$\sup_{V(\eta)\in\mathcal{C}(J,B)} \mathcal{A}(t,y,V(\eta)) \leq \frac{M}{\sqrt{\pi}} \sup_{V(\eta)\in\mathcal{C}(J,B)} \left| \frac{\sqrt{t_0-\tau}-\sqrt{t-\tau}}{\sqrt{(t-\tau)(t_0-\tau)}} \right|.$$

Thus we deduce that

$$\lim_{t\to t_0} \int_J \sup_{V(\eta)\in\mathcal{C}(J,B)} \mathcal{A}(t,y,V(\eta))d\tau = 0.$$

So $H4$ holds.

• For all compact $I \subset \mathbb{R}^+$, for all functions $\psi \in \mathcal{C}(I,\mathbb{R}^n)$, and all $t_0 > 0$,

$$|g(t,\tau;\psi(\tau)) - g(t_0,\tau,\psi(\tau))| = \frac{2}{(2\sqrt{\pi})^n} \left| \int_{\mathbb{R}^{n-1}} F(\psi(\tau)) \left(\frac{e^{-\frac{\|y-\eta\|^2}{4(t-\tau)}}}{(t-\tau)^{n/2}} - \frac{e^{-\frac{\|y-\eta\|^2}{4(t_0-\tau)}}}{(t_0-\tau)^{n/2}} \right) d\tau \right|$$

as $F \in \mathcal{C}(\mathbb{R})$ and $\psi \in \mathcal{C}(I,\mathbb{R}^n)$; then there exists a constant $M > 0$ such that $|F(\psi(\tau))| \leq M$ for all $\tau \in I$. Then we obtain, as for H4, that

$$\lim_{t\to t_0} \int_I |g(t,\tau;\psi(\tau)) - g(t_0,\tau,\psi(\tau))|d\tau = 0.$$

So H5 holds.

• Now for each constant $K > 0$ and each bounded set $B \subset \mathbb{R}^{n-1}$, there exists a measurable function φ such that

$$|g(y,t,\tau,x) - g(y,t,\tau,X)| \leq \varphi(t,\tau)|x-X|$$

whenever $0 \leq \tau \leq t \leq K$ and both x and X are in B. Indeed, as F is assumed to be a locally Lipschitz function in $\mathbb{R}$, there exists constant $L > 0$ such that

$$|F(x) - F(X)| \leq L|x-X| \quad \forall (x,X) \in B^2.$$

Then we have

$$\begin{aligned} |g(y,t,\tau,x) - g(y,t,\tau,X)| &= \frac{2}{(2\sqrt{\pi})^n} \left| \int_{\mathbb{R}^{n-1}} (t-\tau)^{-n/2} e^{-\frac{\|y-\eta\|^2}{4(t-\tau)}} (F(x)-F(X))d\eta \right| \\ &\leq \frac{2}{(2\sqrt{\pi})^n} \left(\int_{\mathbb{R}^{n-1}} e^{-\frac{\|y-\eta\|^2}{4(t-\tau)}} d\eta \right) (t-\tau)^{-n/2} L|x-X| \\ &\leq \frac{L}{\sqrt{\pi(t-\tau)}} |x-X|, \end{aligned}$$

and thus $\varphi(t,\tau) = \frac{L}{\sqrt{\pi(t-\tau)}}$. We also have for each $t \in [0,k]$ the function $\varphi \in L^1(0,t)$ as a function of τ and

$$\int_t^{t+l} \varphi(t+l,\tau)d\tau = \frac{L}{\sqrt{\pi}} \int_t^{t+l} \frac{d\tau}{\sqrt{t+l-\tau}} = \frac{L}{\sqrt{\pi}}(2\sqrt{l}) \to 0 \quad \text{with } l \to 0.$$

So H6 holds. All the conditions H1 to H6 are satisfied with (3.15) and (3.16).

Thus from [15] (Theorem 1.1 page 87, Theorem 1.2 page 91 and Theorem 2.3 page 97) there exists a unique local-in-time solution of the Volterra integral equation (3.3) which can be extended globally in time. Then the proof of this theorem is complete. $\square$

Now, we will make several observations concerning the forcing function $V_0(y,t)$ of the Volterra integral equation (3.3) with respect to the initial temperature $h(x,y)$.

LEMMA 3.4. Let h be the initial temperature, and let V_0 be defined by (3.2).

If $h \in L^\infty(D)$, then we have

$$\sqrt{\pi t}\,|V_0(y,t)| \leq \|h\|_\infty, \qquad \forall y \in \mathbb{R}^{n-1}, \quad t>0. \tag{3.17}$$

If h verifies the following assumptions:

(1) $h \in C^0(D)$,

(2) there exist positive constants A and α such that

$$|h(x,y)-h(0,y)| \leq Ax^\alpha, \qquad \forall x \geq 0, \quad y \in \mathbb{R}^{n-1}, \tag{3.18}$$

then we have

$$\lim_{t\to 0} \sqrt{\pi t} V_0(y,t) = h(0,y), \qquad \forall y \in \mathbb{R}^{n-1}. \tag{3.19}$$

Proof. By using (3.2) and formulas (2.15) and (2.30), we have

$$\begin{aligned} |V_0(y,t)| &\leq \frac{\|h\|_\infty}{(2\sqrt{\pi t})^n t} \int_0^{+\infty} \xi e^{-\frac{\xi^2}{4t}} d\xi \int_{\mathbb{R}^{n-1}} e^{-\frac{\|y-\eta\|^2}{4t}} d\eta \\ &= \frac{\|h\|_\infty}{(2\sqrt{\pi t})^n t} 2t \left(2\sqrt{\pi(t-\tau)}\right)^{n-1} = \frac{\|h\|_\infty}{\sqrt{\pi t}}, \qquad \forall y \in \mathbb{R}^n \quad t>0, \end{aligned}$$

and therefore the inequality (3.17) holds.

By making the change of variable $\xi = 2\sqrt{tz}$ in (3.2), we obtain

$$\sqrt{\pi t} V_0(y,t) = \frac{1}{\left(2\sqrt{\pi t}\right)^{n-1}} \int_{\mathbb{R}^{n-1}} \left[\int_0^{+\infty} h(2\sqrt{tz},\eta) e^{-z} dz\right] e^{-\frac{\|y-\eta\|^2}{4t}} d\eta, \tag{3.20}$$

and therefore by using the hypothesis (b) we derive

$$\begin{aligned} |h(2\sqrt{tz},\eta)| &\leq |h(2\sqrt{tz},\eta)-h(0,\eta)| + |h(0,\eta)| \\ &\leq |h(0,\eta)| + A\left(2\sqrt{tz}\right)^\alpha = |h(0,\eta)| + A2^\alpha (tz)^{\frac{\alpha}{2}} \end{aligned}$$

and

$$\sqrt{\pi t} V_0(y,t) = I_1(y,t) + I_2(y,t) \tag{3.21}$$

where

$$I_1(y,t) = \frac{1}{\left(2\sqrt{\pi t}\right)^{n-1}} \int_{\mathbb{R}^{n-1}} h(0,\eta) e^{-\frac{\|y-\eta\|^2}{4t}} d\eta \tag{3.22}$$

$$I_2(y,t) = \frac{1}{\left(2\sqrt{\pi t}\right)^{n-1}} \int_{\mathbb{R}^n} \left[\int_0^{+\infty} \left[h(2\sqrt{tz},\eta)-h(0,\eta)\right] e^{-z} dz\right] e^{-\frac{\|y-\eta\|^2}{4t}} d\eta. \tag{3.23}$$

Taking into account (2.15) and the Dirac delta, we have

$$\lim_{t\to 0^+} I_1(y,t) = h(0,y), \qquad \forall y \in \mathbb{R}^{n-1}. \tag{3.24}$$

Moreover, we have

$$\begin{aligned}|I_2(y,t)| &\leq \frac{A2^{\alpha}t^{\frac{\alpha}{2}}}{\left(2\sqrt{\pi t}\right)^{n-1}}\int_{\mathbb{R}^{n-1}} e^{-\frac{\|y-\eta\|^2}{4t}}\left(\int_0^{+\infty} z^{\frac{\alpha}{2}}e^{-z}dz\right)d\eta \\ &\leq B_0(\alpha)t^{\frac{\alpha}{2}} \end{aligned} \tag{3.25}$$

where B_0 is a positive constant which depends only on parameter $\alpha > 0$. Therefore (3.19) holds. □

4. Particular cases. In this section we consider some particular cases with physically interesting phenomena and give their relations to the considered Problem 1.1.

(1) If the initial temperature is given by

$$h(x,y) = h_0(x) \qquad \forall y \in \mathbb{R}^{n-1}, \tag{4.1}$$

from (3.2) with $h = h_0$, we have

$$\begin{aligned}u_0(x,y,t) &= \int_D G_1(x,y,t,\xi,\eta,0)h(\xi,\eta)d\xi d\eta \\ &= \int_0^{+\infty} h_0(\xi)\left[\int_{\mathbb{R}^{n-1}} G_1(x,y,t,\xi,\eta,0)d\eta\right]d\xi.\end{aligned}$$

Using the formula (2.17), we get

$$u_0(x,y,t) = \int_0^{+\infty} G(x,t,\xi,0)h_0(\xi)d\xi := u_0(x,t), \qquad \forall y \in \mathbb{R}^{n-1}. \tag{4.2}$$

From (3.11) with $h = h_0$, we have

$$\begin{aligned}f(y,t) &= \frac{1}{(2\sqrt{\pi t})^n t}\int_{\mathbb{R}^{n-1}}\left[\int_0^{+\infty}\xi e^{-\frac{\xi^2}{4t}}h_0(\xi)d\xi\right]e^{-\frac{\|y-\eta\|^2}{4t}}d\eta \\ &= \frac{1}{(2\sqrt{\pi t})^n t}\int_0^{+\infty}\xi e^{-\frac{\xi^2}{4t}}h_0(\xi)d\xi\int_{\mathbb{R}^{n-1}} e^{-\frac{\|y-\eta\|^2}{4t}}d\eta,\end{aligned}$$

and by using the formula (2.15), we get

$$f(y,t) = \frac{1}{2\sqrt{\pi}t^{\frac{3}{2}}}\int_0^{+\infty}\xi e^{-\frac{\xi^2}{4t}}h_0(\xi)d\xi := f(t), \qquad \forall y \in \mathbb{R}^{n-1}. \tag{4.3}$$

(2) If the initial temperature is given by (4.1) and the solution of the integral equation (3.3) is independent of $y \in \mathbb{R}^{n-1}$, i.e.,

$$V(y,t) = V(t), \qquad \forall y \in \mathbb{R}^{n-1}, \tag{4.4}$$

from (3.1), and using (4.1) we have

$$u(x,y,t) = u_0(x,t) - \int_0^t \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right)}{\left(2\sqrt{\pi(t-\tau)}\right)^{n-1}}F(V(\tau))\int_{\mathbb{R}^{n-1}} e^{-\frac{\|y-\eta\|^2}{4(t-\tau)}}d\eta d\tau,$$

and by (2.15), we get

$$\begin{aligned}u(x,y,t) &= u_0(x,t) - \int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right)F(V(\tau))d\tau \\ &= u(x,t), \qquad x>0, \quad t>0, \qquad \forall y \in \mathbb{R}^{n-1}\end{aligned} \tag{4.5}$$

where $u_0(x,t)$ is given in (4.2), and (3.12) becomes

$$\begin{aligned} g(y,t,\tau,V(y,\tau)) &= -\frac{2}{(2\sqrt{\pi})^n}\frac{F(V(\tau))}{(t-\tau)^{\frac{n}{2}}}\int_{\mathbb{R}^{n-1}} e^{-\frac{\|y-\eta\|^2}{4(t-\tau)}}d\eta \\ &= -\frac{F(V(\tau))}{\sqrt{\pi(t-\tau)}} := g\left(t,\tau,V(\tau)\right) \qquad \forall y\in\mathbb{R}^{n-1}. \end{aligned} \tag{4.6}$$

This particular case corresponds to the one-dimensional non-classical heat equation which was studied in [1], [18].

(3) If the initial temperature is constant, i.e.,

$$h(x,y) = h_0, \qquad \forall x>0, \quad \forall y\in\mathbb{R}^{n-1}, \tag{4.7}$$

and the solution of the integral equation (3.3) is independent of $y\in\mathbb{R}^{n-1}$, i.e., satisfying (4.4), then by (2.31), the temperature is given by the following expression:

$$u(x,t) = h_0\mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right) - \int_0^t \mathrm{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right)F(V(\tau))d\tau, \qquad x>0, \quad t>0, \tag{4.8}$$

and its heat flux $V(t)$ at $x=0$ is given by the solution of the Volterra integral equation

$$V(t) = \frac{h_0}{\sqrt{\pi t}} - \int_0^t \frac{F(V(\tau))}{\sqrt{\pi(t-\tau)}}d\tau, \qquad t>0. \tag{4.9}$$

This particular case also corresponds to the one-dimensional non-classical heat equation which was studied in [1].

REMARK 4.1. The Stefan problem for the non-classical one-dimensional heat equation was studied in [3]-[5], [17].

5. Conclusion. In this study we have considered the non-classical heat problem in a semi-n-dimensional space domain D for which the internal energy depends on the heat flux on the boundary S of the domain D. In Section 2, we have recalled and discussed the integral representation of solutions of some classical problems of heat distribution; see Lemma 2.1 and Lemma 2.2. We end this section with a technical Lemma 2.4, which contains mathematical formulas useful for the study of our Problem 1.1.

In Section 3, we gave first in Theorem 3.1 the integral representation (3.1) of the solution of our Problem 1.1, but it depends on the heat flow on the boundary S, which satisfies the Volterra integral equation (3.3) with initial condition (3.2). Then we proved, in Theorem 3.3, under some assumption on the data, that there exists a unique solution of the Problem 1.1 locally in time which can be extended globally in time.

We also made in Lemma 3.4 several observations concerning the forcing function $V_0(y,t)$ given by (3.2) with respect to the initial temperature $h(x,y)$.

In Section 4, we have considered some particular cases with given physical interesting phenomena and their relations to the considered Problem 1.1.

References

[1] L. R. Berrone, D. A. Tarzia, and L. T. Villa, *Asymptotic behaviour of a non-classical heat conduction problem for a semi-infinite material*, Math. Methods Appl. Sci. **23** (2000), no. 13, 1161–1177, DOI 10.1002/1099-1476(20000910)23:13<1161::AID-MMA157>3.0.CO;2-Y. MR1779386 (2001h:35091)

[2] G. Bluman, P. Broadbridge, J. R. King, and M. J. Ward, *Similarity: generalizations, applications and open problems*, J. Engrg. Math. **66** (2010), no. 1-3, 1–9, DOI 10.1007/s10665-009-9330-y. MR2585810

[3] Adriana C. Briozzo and Domingo A. Tarzia, *Existence and uniqueness for one-phase Stefan problems of non-classical heat equations with temperature boundary condition at a fixed face*, Electron. J. Differential Equations (2006), No. 21, 16. MR2198934 (2006j:35243)

[4] Adriana C. Briozzo and Domingo A. Tarzia, *Exact solutions for nonclassical Stefan problems*, Int. J. Differ. Equ. (2010), Art. ID 868059, 19. MR2720039 (2011h:35324)

[5] Adriana C. Briozzo and Domingo A. Tarzia, *A Stefan problem for a non-classical heat equation with a convective condition*, Appl. Math. Comput. **217** (2010), no. 8, 4051–4060, DOI 10.1016/j.amc.2010.10.015. MR2739647 (2011j:80009)

[6] John Rozier Cannon, *The one-dimensional heat equation*, Encyclopedia of Mathematics and its Applications, vol. 23, Addison-Wesley Publishing Company Advanced Book Program, Reading, MA, 1984. With a foreword by Felix E. Browder. MR747979 (86b:35073)

[7] John R. Cannon and Hong-Ming Yin, *A class of nonlinear nonclassical parabolic equations*, J. Differential Equations **79** (1989), no. 2, 266–288, DOI 10.1016/0022-0396(89)90103-4. MR1000690 (90f:35104)

[8] H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Oxford, at the Clarendon Press, 1947. MR0022294 (9,188a)

[9] Avner Friedman, *Partial differential equations of parabolic type*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1964. MR0181836 (31 #6062)

[10] K. Glashoff and J. Sprekels, *An application of Glicksberg's theorem to set-valued integral equations arising in the theory of thermostats*, SIAM J. Math. Anal. **12** (1981), no. 3, 477–486, DOI 10.1137/0512041. MR613326 (82f:45008)

[11] K. Glashoff and J. Sprekels, *The regulation of temperature by thermostats and set-valued integral equations*, J. Integral Equations **4** (1982), no. 2, 95–112. MR654076 (85d:45013)

[12] N. Kenmochi, *Heat conduction with a class of automatic heat source controls*, Free boundary problems: theory and applications, Vol. II (Irsee, 1987), Pitman Res. Notes Math. Ser., vol. 186, Longman Sci. Tech., Harlow, 1990, pp. 471–474. MR1081741

[13] Nobuyuki Kenmochi and Mario Primicerio, *One-dimensional heat conduction with a class of automatic heat-source controls*, IMA J. Appl. Math. **40** (1988), no. 3, 205–216, DOI 10.1093/imamat/40.3.205. MR974674 (89m:49008)

[14] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva, *Linear and quasilinear equations of parabolic type* (Russian), Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1968. MR0241822 (39 #3159b)

[15] Richard K. Miller, *Nonlinear Volterra integral equations*, W. A. Benjamin, Inc., Menlo Park, Calif., 1971. Mathematics Lecture Note Series. MR0511193 (58 #23394)

[16] Natalia Nieves Salva, Domingo Alberto Tarzia, and Luis Tadeo Villa, *An initial-boundary value problem for the one-dimensional non-classical heat equation in a slab*, Bound. Value Probl. (2011), 2011:4, 17. MR2821483 (2012f:35309)

[17] Domingo A. Tarzia, *A Stefan problem for a non-classical heat equation* (English, with English and Spanish summaries), 1 (Spanish) (Rosario, 1998), MAT Ser. A Conf. Semin. Trab. Mat., vol. 3, Univ. Austral, Rosario, 2001, pp. 21–26. MR1941700 (2003j:35337)

[18] Domingo A. Tarzia and Luis T. Villa, *Some nonlinear heat conduction problems for a semi-infinite strip with a non-uniform heat source*, Rev. Un. Mat. Argentina **41** (1998), no. 1, 99–114. Dedicated to the memory of Julio E. Bouillet. MR1682205 (2000i:35103)

[19] L.T. Villa, Problemas de control para una ecuación unidimensional no homogénea del calor, Rev. Unión Mat. Argentina, 32 (1986), pp. 163-169.