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ON THE BEHAVIOR OF THE SOLUTION OF A VISCOPLASTIC CONTACT PROBLEM

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Abstract. We consider a mathematical model which describes the frictionless contact between a viscoplastic body and an obstacle, the so-called foundation. The process is quasistatic and the contact is modeled with normal compliance and unilateral constraint. We provide a mixed variational formulation of the model which involves a dual Lagrange multiplier, and then we prove its unique weak solvability. We also prove an estimate which allows us to deduce the continuous dependence of the weak solution with respect to both the normal compliance function and the penetration bound. Finally, we provide a numerical validation of this convergence result.

1. Introduction. The aim of this paper is to study the behaviour of the weak solution of a contact problem for viscoplastic materials with a constitutive law of the form

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})). \tag{1.1}$$

In (1.1) and everywhere below $\boldsymbol{u}$ represents the displacement vector, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denote the stress tensor and the linearized stress tensor, respectively, $\mathcal{E}$ is the elasticity tensor and $\mathcal{G}$ is a given constitutive function. Moreover, the dot above represents the derivative with respect to the time variable.

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Various examples and mechanical interpretations in the study of viscoplastic materials of the form (1.1) can be found in [4, 12] and the references therein. Displacement-traction boundary value problems were considered in [12], both in the dynamic and quasistatic case. Quasistatic frictionless and frictional contact problems for such kinds of materials were studied in various works; see for instance [3, 9, 18, 19] and the references therein. In [18] a number of models of contact were stated and their variational analysis, including existence and uniqueness results, was provided. The numerical analysis of part of these models can be found in [9]; there, semi-discrete and fully discrete schemes were considered, error estimates were obtained and convergence results were proved. In [19], the contact was modeled with the Signorini condition in a form with a zero gap function; an evolutionary mixed variational formulation to the problem involving a Lagrange multiplier was derived, and the unique solvability of the model was obtained by using arguments on saddle point theory and fixed point. In contrast, in [3] the contact was modeled with normal compliance and unilateral constraint. This condition was introduced in [13] and then used in a large number of papers; see the references in [20]. It contains as a particular case both the normal compliance contact condition, the Signorini condition without gap and the Signorini condition with a gap function, as well. The results in [3] concern the unique solvability of the model and were obtained by using arguments of history dependent variational inequalities; the behavior of the solution with respect to the stiffness coefficient of the foundation was also studied and a convergence result was proved; finally, a numerical validation of this convergence result was provided.

The present paper represents a continuation of [3]. Here, we consider the frictionless contact problem with normal compliance and unilateral constraint studied in [3] and we investigate the behavior of the weak solution with respect to the normal compliance function and the penetration bound. This study was left open in [3] and requires a new variational formulation, different from that considered in [3]. For this reason, after the description of the contact problem, we derive a new variational formulation, similar to that in [19], which involves a Lagrange multiplier. Then we provide the unique weak solvability of the problem, which represents the first trait of novelty of this paper. The second trait of novelty consists in the fact that we prove the continuous dependence of the weak solution with respect to the normal compliance function and the penetration bound. We also provide a numerical validation of this continuous dependence result, which represents the third trait of novelty of this paper.

The rest of the paper is structured as follows. In Section 2 we introduce the notation and some preliminary material. In Section 3 we describe the mechanical problem, list the assumptions on the data and derive the mixed variational formulation of the problem. Then, we state our main existence and uniqueness result, Theorem 3.1. The proof of the theorem is given in Section 4. In Section 5 we prove a general estimate result, Theorem 5.1, which allows us to obtain the continuous dependence of the weak solution with respect to both the normal compliance function and the penetration bound. Finally, in Section 6 we present numerical simulation in the two-dimensional case, which validates our continuous dependence result.

2. Notation and preliminaries. Everywhere in this paper we denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\begin{aligned} &\boldsymbol{u}\cdot\boldsymbol{v}=u_iv_i\ , \qquad \|\boldsymbol{v}\|=(\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}} \qquad \forall\,\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^d,\\ &\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sigma_{ij}\tau_{ij}\ , \qquad \|\boldsymbol{\tau}\|=(\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\,\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^d. \end{aligned}$$

Here and below the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used.

Let $\Omega\subset\mathbb{R}^d$ ($d=1,2,3$) be a bounded domain with the boundary $\partial\Omega=\Gamma$, assumed to be Lipschitz continuous. We use the notation $\boldsymbol{x}=(x_i)$ for a typical point in $\Omega\cup\Gamma$ and we denote by $\boldsymbol{\nu}=(\nu_i)$ the outward unit normal at Γ. Moreover, an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j}=\partial u_i/\partial x_j$. Everywhere in this paper we use the standard notation for Sobolev and Lebesgue spaces associated to Ω and Γ and we send the reader to $[1,8,10]$ for more details on this topic. In addition, we consider the following spaces:

$$\begin{aligned} &Q=\{\ \boldsymbol{\sigma}=(\sigma_{ij})\ :\ \sigma_{ij}=\sigma_{ji}\in L^2(\Omega)\ \},\\ &H_1=\{\ \boldsymbol{u}=(u_i)\ :\ \boldsymbol{\varepsilon}(\boldsymbol{u})\in Q\ \},\\ &Q_1=\{\ \boldsymbol{\sigma}\in Q\ :\ \operatorname{Div}\boldsymbol{\sigma}\in L^2(\Omega)^d\ \}. \end{aligned}$$

Here and below $\boldsymbol{\varepsilon}$ and Div are the deformation and the divergence operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u})=(\varepsilon_{ij}(\boldsymbol{u})),\quad \varepsilon_{ij}(\boldsymbol{u})=\frac{1}{2}(u_{i,j}+u_{j,i}),\quad \operatorname{Div}\boldsymbol{\sigma}=(\sigma_{ij,j}).$$

The spaces Q, H_1 and Q_1 are real Hilbert spaces endowed with the canonical inner products given by

$$\begin{aligned} &(\boldsymbol{\sigma},\boldsymbol{\tau})_Q=\int_\Omega\sigma_{ij}\tau_{ij}dx,\\ &(\boldsymbol{u},\boldsymbol{v})_{H_1}=(\boldsymbol{u},\boldsymbol{v})_{L^2(\Omega)^d}+(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q,\\ &(\boldsymbol{\sigma},\boldsymbol{\tau})_{Q_1}=(\boldsymbol{\sigma},\boldsymbol{\tau})_Q+(\operatorname{Div}\boldsymbol{\sigma},\operatorname{Div}\boldsymbol{\tau})_{L^2(\Omega)^d}. \end{aligned}$$

The associated norms on these spaces are denoted by $\|\cdot\|_Q$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{Q_1}$, respectively. Also, recall that $H_1=H^1(\Omega)^d$ algebraically and topologically.

For an element $\boldsymbol{v}\in H_1$ we still write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$, and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ given by $v_\nu=\boldsymbol{v}\cdot\boldsymbol{\nu}$, $\boldsymbol{v}_\tau=\boldsymbol{v}-v_\nu\boldsymbol{\nu}$. Also, for a regular stress function $\boldsymbol{\sigma}$ we use the notation σ_ν and $\boldsymbol{\sigma}_\tau$ for the normal and the tangential traces, i.e. $\sigma_\nu=(\boldsymbol{\sigma\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau=\boldsymbol{\sigma\nu}-\sigma_\nu\boldsymbol{\nu}$. Moreover, we recall the following Green's formula:

$$\int_\Omega\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx+\int_\Omega\operatorname{Div}\boldsymbol{\sigma}\cdot\boldsymbol{v}\,dx=\int_\Gamma\boldsymbol{\sigma\nu}\cdot\boldsymbol{v}\,da\qquad\text{for all }\boldsymbol{v}\in H_1. \tag{2.1}$$

Let Γ_1 be a measurable part of Γ such that $meas\,(\Gamma_1)>0$ and consider the space

$$V=\{\,\boldsymbol{v}\in H^1(\Omega)^d:\boldsymbol{v}=\boldsymbol{0}\text{ on }\Gamma_1\,\}.$$

It is well known that V is a real Hilbert space endowed with the inner product

$$(\boldsymbol{u},\boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx$$

and the associated norm $\|\cdot\|_V$, respectively. Completeness of the space $(V,\|\cdot\|_V)$ follows from the assumption $meas(\Gamma_1)>0$, which allows the use of Korn's inequality. We also recall that there exists $c_0>0$ which depends on Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le c_0\|\boldsymbol{v}\|_V \quad \text{for all } \ \boldsymbol{v}\in V. \tag{2.2}$$

Inequality (2.2) represents a consequence of the Sobolev trace theorem.

Let Γ_3 be a measurable part of Γ. We consider the Hilbert space

$$S = \{\, \boldsymbol{w} = \boldsymbol{v}_{|\Gamma_3} : \boldsymbol{v}\in V\,\},$$

where $\boldsymbol{v}_{|\Gamma_3}$ denotes the restriction of the trace of the element $\boldsymbol{v}\in V$ to Γ_3. Thus, $S\subset H^{1/2}(\Gamma_3;\mathbb{R}^d)$, where $H^{1/2}(\Gamma_3;\mathbb{R}^d)$ is the space of the restrictions on Γ_3 of traces on Γ of functions of $H^1(\Omega)^d$. The dual of the space S will be denoted by D and the duality pairing between D and S will be denoted by $\langle\cdot,\cdot\rangle_{\Gamma_3}$. For more details on trace operators and trace spaces we refer to [1, 10], for instance.

Let T be a positive real number. For every Banach space $(X,\|\cdot\|_X)$ we use the notation $C([0,T];X)$ for the space of continuous functions defined on $[0,T]$ with values on X. For a subset $K\subset X$ we still use the symbol $C([0,T];K)$ for the set of continuous functions defined on $[0,T]$ with values on K. It is well known that $C([0,T];X)$ is a real Banach space with the norm

$$\|v\|_{C([0,T];X)} = \max_{t\in[0,T]} \|v(t)\|_X. \tag{2.3}$$

Moreover, it is easy to check that for each $\zeta>0$ the map $v\mapsto\|v\|_\zeta$ given by

$$\|v\|_\zeta = \sup_{t\in[0,T]} \|v(t)\|_X\, e^{-\zeta\, t} \tag{2.4}$$

represents a norm on the space $C([0,T];X)$, which is equivalent to the canonical norm $\|\cdot\|_{C([0,T];X)}$, defined by (2.3). The norm $\|\cdot\|_\zeta$ is called the Bielecki norm. We conclude from above that the space $C([0,T];X)$ endowed with the Bielecki norm $\|\cdot\|_\zeta$ is a Banach space, for each $\zeta>0$.

We end this section with an abstract existence and uniqueness result.

Let $(X,\,(\cdot,\,\cdot)_X,\,\|\cdot\|_X)$ and $(Y,\,(\cdot,\,\cdot)_Y,\,\|\cdot\|_Y)$ be two Hilbert spaces and consider a nonlinear operator $A: X\to X$, a bilinear form $b: X\times Y\to\mathbb{R}$ and a set $\Lambda\subset Y$ which satisfy the following conditions:

$$\begin{aligned}&\text{there exists } m_A>0 \text{ such that}\\ &\quad (Au-Av,\, u-v)_X \ge m_A\|u-v\|_X^2 \ \text{ for all } u,v\in X;\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\text{there exists } L_A>0 \text{ such that}\\ &\quad \|Au-Av\|_X \le L_A\,\|u-v\|_X \ \text{ for all } u,v\in X;\end{aligned} \tag{2.6}$$

$$\begin{aligned}&\text{there exists } M_b>0 \text{ such that}\\ &\quad |b(v,\mu)| \le M_b\|v\|_X\|\mu\|_Y \ \text{ for all } v\in X,\ \mu\in Y;\end{aligned} \tag{2.7}$$

there exists $\alpha > 0$ such that (2.8)

$$\inf_{\mu\in Y,\mu\neq 0_Y} \sup_{v\in X,v\neq 0_X} \frac{b(v,\mu)}{\|v\|_X\|\mu\|_Y} \geq \alpha;$$

Λ is a closed convex unbounded subset of Y such that $0_Y \in \Lambda$. (2.9)

Note that conditions (2.5)–(2.6) show that A is a strongly monotone and Lipschitz continuous operator. Moreover, conditions (2.7) and (2.8) show that the bilinear form b is continuous and satisfies the so-called "inf-sup" condition, respectively.

With the data above we consider the following mixed variational problem.

Problem $\mathcal{P}_0$. For given f, $h \in X$, find $u \in X$ and $\lambda \in \Lambda$ such that

$$(Au, v)_X + b(v,\lambda) = (f, v)_X \qquad \text{for all } v \in X, \tag{2.10}$$

$$b(u, \mu-\lambda) \leq b(h, \mu-\lambda) \qquad \text{for all } \mu \in \Lambda. \tag{2.11}$$

In the study of problem (2.10)–(2.11) we have the following result.

Theorem 2.1. Assume (2.5)–(2.9). Then, there exists a unique solution $(u, \lambda) \in X \times \Lambda$ to Problem $\mathcal{P}_0$.

A proof of Theorem 2.1 can be found in [17]; see Theorem 5.2 therein. That proof is based on arguments on saddle point theory which can be found in [5–7, 11], combined with a fixed point technique. We shall use Theorem 2.1 in Section 4 of this paper in order to prove the unique weak solvability of the viscoplastic frictionless contact problem we introduce in the next section.

3. The model. We consider a viscoplastic body that occupies the bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$, with the boundary $\partial\Omega = \Gamma$ partitioned into three disjoint measurable parts, Γ_1, Γ_2 and Γ_3, such that *meas* $\Gamma_1 > 0$. We assume that the boundary Γ is Lipschitz continuous and we denote by $\boldsymbol{\nu}$ its unit outward normal, defined almost everywhere. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0, T)$, and therefore the displacement field vanishes there. A volume force of density $\boldsymbol{f}_0$ acts in $\Omega \times (0, T)$, surface tractions of density $\boldsymbol{f}_2$ act on $\Gamma_2 \times (0, T)$ and, finally, we assume that the body is in contact with a deformable foundation on $\Gamma_3 \times (0, T)$. The contact is frictionless and we model it with normal compliance and unilateral constraint. Then, the classical formulation of the contact problem is the following.

Problem $\mathcal{P}$. Find a displacement field $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$ such that

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in} \quad \Omega \times (0, T), \tag{3.1}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in} \quad \Omega \times (0, T), \tag{3.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_1 \times (0, T), \tag{3.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on} \quad \Gamma_2 \times (0, T), \tag{3.4}$$

$$\left.\begin{array}{l} u_\nu \le g,\ \sigma_\nu + p(u_\nu) \le 0, \\ (u_\nu - g)(\sigma_\nu + p(u_\nu)) = 0 \end{array}\right\} \quad \text{on} \quad \Gamma_3 \times (0,T), \tag{3.5}$$

$$\boldsymbol{\sigma}_\tau = \mathbf{0} \quad \text{on} \quad \Gamma_3 \times (0,T), \tag{3.6}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in} \quad \Omega. \tag{3.7}$$

Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $\boldsymbol{x}$ or t. Equation (3.1) represents the viscoplastic constitutive law of the material, already introduced in Section 1. Equation (3.2) is the equilibrium equation, and we use it here since the process is assumed to be quasistatic. Conditions (3.3) and (3.4) are the displacement and traction boundary conditions, respectively, and condition (3.5) represents the normal compliance condition with unilateral constraint, introduced in [13]. Recall that here $g \ge 0$ is a given bound for the penetration and p represents a given normal compliance function. Condition (3.6) shows that the tangential stress on the contact surface, denoted $\boldsymbol{\sigma}_\tau$, vanishes. We use it here since we assume that the contact process is frictionless. Finally, (3.7) represents the initial conditions in which $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ denote the initial displacement and the initial stress field, respectively.

In the study of the mechanical problem (3.1)–(3.7) we assume that the elasticity tensor $\mathcal{E}$, the nonlinear constitutive function $\mathcal{G}$ and the normal compliance function satisfy the following conditions:

$$\left\{\begin{array}{l} \text{(a) } \mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) } \mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^\infty(\Omega),\ 1 \le i,j,k,l \le d. \\ \text{(c) There exists } m_{\mathcal{E}} > 0 \text{ such that} \\ \qquad \mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \ge m_{\mathcal{E}} \|\boldsymbol{\tau}\|^2 \text{ for all } \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. in } \Omega, \end{array}\right. \tag{3.8}$$

$$\left\{\begin{array}{l} \text{(a) } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{G}} > 0 \text{ such that} \\ \qquad \|\mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)\| \\ \qquad\quad \le L_{\mathcal{G}} \left(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|\right) \\ \qquad\qquad \text{for all } \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\ \qquad \text{for all } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \mathbf{0}, \mathbf{0}) \text{ belongs to } Q, \end{array}\right. \tag{3.9}$$

$$\left\{\begin{array}{l} \text{(a) } p : \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } L_p > 0 \text{ such that} \\ \qquad |p(r_1) - p(r_2)| \le L_p |r_1 - r_2| \quad \text{for all } r_1,\, r_2 \in \mathbb{R}. \\ \text{(c) } (p(r_1) - p(r_2))(r_1 - r_2) \ge 0 \quad \text{for all } r_1,\, r_2 \in \mathbb{R}. \\ \text{(d) } p(r) = 0 \text{ for all } r < 0. \end{array}\right. \tag{3.10}$$

We also assume that the body forces and tractions densities have the regularity

$$\boldsymbol{f}_0 \in C([0,T]; L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in C([0,T]; L^2(\Gamma_2)^d), \tag{3.11}$$

and the initial data satisfy

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in Q. \tag{3.12}$$

Finally, we assume that

$$\text{there exists } \ \widetilde{\boldsymbol{\theta}} \in V \ \text{ such that } \ \widetilde{\theta}_\nu = 1 \ \text{ a.e. on } \ \Gamma_3 \tag{3.13}$$

where, recall, $\widetilde{\theta}_\nu = \widetilde{\boldsymbol{\theta}} \cdot \boldsymbol{\nu}$.

We now turn to the variational formulation of Problem $\mathcal{P}$. To this end, we use Riesz's representation theorem to define the operators $L : V \to V$, $P : V \to V$ and the function $\boldsymbol{f} : [0,T] \to V$ by equalities

$$(L\boldsymbol{u},\, \boldsymbol{v})_V = \int_\Omega \mathcal{E}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \tag{3.14}$$

$$(P\boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p(u_\nu) v_\nu\, da, \tag{3.15}$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, da \tag{3.16}$$

for all $\boldsymbol{u}$, $\boldsymbol{v} \in V$ and $t \in [0,T]$. Also, let $b : V \times D \to \mathbb{R}$ denote the bilinear form defined by

$$b(\boldsymbol{v},\, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{v} \rangle_{\Gamma_3} \tag{3.17}$$

for all $\boldsymbol{v} \in V$ and $\boldsymbol{\mu} \in D$ and consider the sets

$$K = \{\, \boldsymbol{v} \in V \ : \ v_\nu \le 0 \quad \text{a.e. on } \Gamma_3 \,\}, \tag{3.18}$$

$$\Lambda = \{\, \boldsymbol{\mu} \in D \ : \ \langle \boldsymbol{\mu},\, \boldsymbol{v} \rangle_{\Gamma_3} \le 0 \quad \forall\, \boldsymbol{v} \in K \,\}. \tag{3.19}$$

Using (3.11) it is easy to see that

$$\boldsymbol{f} \in C([0,T]; V). \tag{3.20}$$

Moreover, by standard arguments it follows that the bilinear form $b(\cdot,\cdot)$ is continuous and satisfies the "inf-sup" condition, i.e. there exists $\alpha > 0$ such that

$$\inf_{\boldsymbol{\mu} \in D,\ \boldsymbol{\mu} \neq \boldsymbol{0}_D} \ \sup_{\boldsymbol{v} \in V,\ \boldsymbol{v} \neq \boldsymbol{0}_V} \frac{b(\boldsymbol{v}, \boldsymbol{\mu})}{\|\boldsymbol{v}\|_V \|\boldsymbol{\mu}\|_D} \ge \alpha. \tag{3.21}$$

As a consequence of (3.21) we obtain that

$$\sup_{\boldsymbol{v} \in V,\ \boldsymbol{v} \neq \boldsymbol{0}_V} \frac{b(\boldsymbol{v}, \boldsymbol{\mu})}{\|\boldsymbol{v}\|_V} \ge \alpha\, \|\boldsymbol{\mu}\|_D \qquad \text{for all } \ \boldsymbol{\mu} \in D. \tag{3.22}$$

Assume now that $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ are regular functions which verify (3.1)–(3.7) and let $t \in [0,T]$, $\boldsymbol{v} \in V$ and $\boldsymbol{\mu} \in \Lambda$. Using Green's formula (2.1) and (3.2) we have

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q = (\boldsymbol{f}_0(t), \boldsymbol{v})_{L^2(\Omega)^d} + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot \boldsymbol{v}\, da.$$

Then, using (3.3), (3.4) and (3.16) we obtain

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q = (\boldsymbol{f}(t), \boldsymbol{v})_V + \int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot \boldsymbol{v}\, da,$$

and, therefore, (3.6) yields

$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q=(\boldsymbol{f}(t),\boldsymbol{v})_V+\int_{\Gamma_3}\sigma_\nu(t)v_\nu\,da. \tag{3.23}$$

Denote by $\boldsymbol{\beta}(t)$ and $\boldsymbol{\lambda}(t)$ the viscoplastic stress and the Lagrange multiplier given by

$$\boldsymbol{\beta}(t)=\boldsymbol{\sigma}(t)-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \tag{3.24}$$

$$\langle\boldsymbol{\lambda}(t),\boldsymbol{v}\rangle_{\Gamma_3}=-\int_{\Gamma_3}(\sigma_\nu(t)+p(u_\nu(t)))v_\nu\,da\quad\text{ for all }\boldsymbol{v}\in V. \tag{3.25}$$

Then, combining (3.23)–(3.25) and using notation (3.14), (3.15) and (3.17), we obtain that

$$\begin{aligned}(L\boldsymbol{u}(t),\boldsymbol{v})_V+(P\boldsymbol{u}(t),\boldsymbol{v})_V+(\boldsymbol{\beta}(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q&\\ +b(\boldsymbol{v},\boldsymbol{\lambda}(t))=(\boldsymbol{f}(t),\boldsymbol{v})_V.&\end{aligned} \tag{3.26}$$

On the other hand, using (3.5), (3.18) and (3.19) we deduce that $\boldsymbol{\lambda}(t)\in\Lambda$ for all $t\in[0,T]$. Let $\lambda\in\Lambda$. Then, using assumption (3.13) and the definition (3.17) of the bilinear form b, it is easy to see that

$$\begin{aligned}b(\boldsymbol{u}(t),\boldsymbol{\mu}-\boldsymbol{\lambda}(t))&=b(\boldsymbol{u}(t)-g\widetilde{\boldsymbol{\theta}},\boldsymbol{\mu}-\boldsymbol{\lambda}(t))+b(g\widetilde{\boldsymbol{\theta}},\boldsymbol{\mu}-\boldsymbol{\lambda}(t))\\ &=\langle\boldsymbol{\mu}-\boldsymbol{\lambda}(t),\boldsymbol{u}(t)-g\widetilde{\boldsymbol{\theta}}\rangle_{\Gamma_3}+b(g\widetilde{\boldsymbol{\theta}},\boldsymbol{\mu}-\boldsymbol{\lambda}(t)),\end{aligned}$$

and, therefore,

$$b(\boldsymbol{u}(t),\boldsymbol{\mu}-\boldsymbol{\lambda}(t))=\langle\boldsymbol{\mu},\boldsymbol{u}(t)-g\widetilde{\boldsymbol{\theta}}\rangle_{\Gamma_3}-\langle\boldsymbol{\lambda}(t),\boldsymbol{u}(t)-g\widetilde{\boldsymbol{\theta}}\rangle_{\Gamma_3}+b(g\widetilde{\boldsymbol{\theta}},\boldsymbol{\mu}-\boldsymbol{\lambda}(t)). \tag{3.27}$$

In addition, (3.5) and (3.13) imply that

$$\boldsymbol{u}(t)-g\widetilde{\boldsymbol{\theta}}\in K,\qquad\langle\boldsymbol{\lambda}(t),\boldsymbol{u}(t)\rangle_{\Gamma_3}=\langle\boldsymbol{\lambda}(t),g\widetilde{\boldsymbol{\theta}}\rangle_{\Gamma_3},$$

which show that

$$\langle\boldsymbol{\mu},\boldsymbol{u}(t)-g\widetilde{\boldsymbol{\theta}}\rangle_{\Gamma_3}\le 0,\qquad\langle\boldsymbol{\lambda}(t),\boldsymbol{u}(t)-g\widetilde{\boldsymbol{\theta}}\rangle_{\Gamma_3}=0. \tag{3.28}$$

We now combine (3.27) and (3.28) to deduce that

$$b(\boldsymbol{u}(t),\boldsymbol{\mu}-\boldsymbol{\lambda}(t))\le b(g\widetilde{\boldsymbol{\theta}},\boldsymbol{\mu}-\boldsymbol{\lambda}(t)). \tag{3.29}$$

Finally, we integrate (3.1) with the initial condition (3.7) and use (3.24) to find that

$$\boldsymbol{\beta}(t)=\int_0^t\mathcal{G}(\mathcal{E}\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s))+\boldsymbol{\beta}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds+\boldsymbol{\sigma}_0-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0). \tag{3.30}$$

We now gather equalities (3.26), (3.30) and inequality (3.29) to obtain the following variational formulation of the mechanical problem $\mathcal{P}$.

PROBLEM $\mathcal{P}_V$. Find a displacement field $\boldsymbol{u}:[0,T]\to V$, a viscoplastic stress field $\boldsymbol{\beta}:[0,T]\to Q$ and a Lagrange multiplier $\boldsymbol{\lambda}:[0,T]\to\Lambda$ such that, for all $t\in[0,T]$,

$$\begin{aligned}(L\boldsymbol{u}(t),\boldsymbol{v})_V+(\boldsymbol{\beta}(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q+(P\boldsymbol{u}(t),\boldsymbol{v})_V&\\ +b(\boldsymbol{v},\boldsymbol{\lambda}(t))=(\boldsymbol{f}(t),\boldsymbol{v})_V\quad\text{for all }\boldsymbol{v}\in V,&\end{aligned} \tag{3.31}$$

$$b(\boldsymbol{u}(t),\boldsymbol{\mu}-\boldsymbol{\lambda}(t))\le b(g\widetilde{\boldsymbol{\theta}},\boldsymbol{\mu}-\boldsymbol{\lambda}(t))\quad\text{for all }\boldsymbol{\mu}\in\Lambda, \tag{3.32}$$

$$\boldsymbol{\beta}(t) = \int_0^t \mathcal{G}(\mathcal{E}\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s)) + \boldsymbol{\beta}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0). \tag{3.33}$$

The unique solvability of Problem $\mathcal{P}_V$ is given by the following result, that we state here and prove in the next section.

THEOREM 3.1. Assume (3.8) – (3.13). Then Problem $\mathcal{P}_V$ has a unique solution $(\boldsymbol{u}, \boldsymbol{\beta}, \boldsymbol{\lambda})$ which satisfies

$$\boldsymbol{u} \in C([0,T];V), \quad \boldsymbol{\beta} \in C([0,T];Q), \quad \boldsymbol{\lambda} \in C([0,T];\Lambda). \tag{3.34}$$

A triple of functions $(\boldsymbol{u}, \boldsymbol{\beta}, \boldsymbol{\lambda})$ which satisfies (3.31)–(3.33) is called a *weak solution* to Problem $\mathcal{P}$. We conclude from Theorem 3.1 that, under the assumptions (3.8) – (3.13), Problem $\mathcal{P}$ has a unique weak solution with regularity (3.34). Moreover, we note that, once the weak solution is known, the stress field $\boldsymbol{\sigma}$ can be easily computed by using equality (3.24). Also, using standard arguments, it can be shown that $\boldsymbol{\sigma} \in C([0,T];Q_1)$.

4. Proof of Theorem 3.1. The proof of Theorem 3.1 will be carried out in several steps. To present it, throughout this section we assume that (3.8)–(3.13) hold. Moreover, we denote by c a positive generic constant which may depend on the data but is independent on the time variable and whose value may change from place to place.

We start by solving the contact problem in the particular case when the viscoplastic stress is known. To this end let $\boldsymbol{\eta}$ be an arbitrary element of the space $C([0,T];V)$ and consider the following auxiliary problem.

PROBLEM $\mathcal{P}^1_\eta$. Find a displacement field $\boldsymbol{u}_\eta : [0,T] \to V$ and a Lagrange multiplier $\boldsymbol{\lambda}_\eta : [0,T] \to \Lambda$ such that, for all $t \in [0,T]$,

$$\begin{aligned}(L\boldsymbol{u}_\eta(t), \boldsymbol{v})_V + (P\boldsymbol{u}_\eta(t), \boldsymbol{v})_V + b(\boldsymbol{v}, \boldsymbol{\lambda}_\eta(t))& \\ = (\boldsymbol{f}(t) - \boldsymbol{\eta}(t), \boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in V,&\end{aligned} \tag{4.1}$$

$$b(\boldsymbol{u}_\eta(t), \boldsymbol{\mu} - \boldsymbol{\lambda}_\eta(t)) \le b(g\widetilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}_\eta(t)) \quad \text{for all } \boldsymbol{\mu} \in \Lambda. \tag{4.2}$$

In the study of Problem $\mathcal{P}^1_\eta$ we have the following result.

LEMMA 4.1. There exists a unique solution $(\boldsymbol{u}_\eta, \boldsymbol{\lambda}_\eta)$ of Problem $\mathcal{P}^1_\eta$ which satisfies

$$\boldsymbol{u}_\eta \in C([0,T];V), \quad \boldsymbol{\lambda}_\eta \in C([0,T];\Lambda). \tag{4.3}$$

Moreover, if $(\boldsymbol{u}_i, \boldsymbol{\lambda}_i)$ represents the solution of Problem $\mathcal{P}^1_\eta$ for $\boldsymbol{\eta} = \boldsymbol{\eta}_i \in C([0,T];V)$, $i = 1, 2$, then there exists $c > 0$ such that

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V + \|\boldsymbol{\lambda}_1(t) - \boldsymbol{\lambda}_2(t)\|_D \le c\,\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_V \quad \text{for all } t \in [0,T]. \tag{4.4}$$

Proof. We use Theorem 2.1 with $X = V$, $Y = D$ and Λ given by (3.19). To this end we consider the operator $A : V \to V$ defined by

$$(A\boldsymbol{u}, \boldsymbol{v})_V = (L\boldsymbol{u}, \boldsymbol{v})_V + (P\boldsymbol{u}, \boldsymbol{v})_V \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V. \tag{4.5}$$

Using assumptions (3.8) and (3.10) as well as the trace inequality (2.2) it is easy to see that A is a strongly monotone and Lipschitz continuous operator, i.e. it satisfies conditions (2.5) and (2.6). On the other hand, we recall that the form $b(\cdot,\cdot)$ given by (3.17) verifies conditions (2.7) and (2.8) and, clearly, condition (2.9) holds. We now use

(4.5) and Theorem 2.1 to see that, at each moment $t \in [0, T]$, there exists a unique pair $(\boldsymbol{u}_\eta(t), \boldsymbol{\lambda}_\eta(t)) \in X \times \Lambda$ which satisfies (4.1)–(4.2).

In order to prove the regularity (4.3) of the solution $(\boldsymbol{u}_\eta, \boldsymbol{\lambda}_\eta)$, consider two elements t_1, $t_2 \in [0, T]$. We have

$$(A\boldsymbol{u}_\eta(t_1), \boldsymbol{v})_V + b(\boldsymbol{v}, \boldsymbol{\lambda}_\eta(t_1)) = (\boldsymbol{f}(t_1) - \boldsymbol{\eta}(t_1), \boldsymbol{v})_V, \tag{4.6}$$

$$b(\boldsymbol{u}_\eta(t_1), \boldsymbol{\mu} - \boldsymbol{\lambda}_\eta(t_1)) \le b(g\widetilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}_\eta(t_1)), \tag{4.7}$$

$$(A\boldsymbol{u}_\eta(t_2), \boldsymbol{v})_V + b(\boldsymbol{v}, \boldsymbol{\lambda}_\eta(t_2)) = (\boldsymbol{f}(t_2) - \boldsymbol{\eta}(t_2), \boldsymbol{v})_V, \tag{4.8}$$

$$b(\boldsymbol{u}_\eta(t_2), \boldsymbol{\mu} - \boldsymbol{\lambda}_\eta(t_2)) \le b(g\widetilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}_\eta(t_2)), \tag{4.9}$$

for all $\boldsymbol{v} \in V$ and $\boldsymbol{\mu} \in \Lambda$. We take $\boldsymbol{v} = \boldsymbol{u}_\eta(t_2) - \boldsymbol{u}_\eta(t_1)$ in (4.6), $\boldsymbol{v} = \boldsymbol{u}_\eta(t_1) - \boldsymbol{u}_\eta(t_2)$ in (4.8) and add the corresponding equalities to obtain

$$\begin{aligned} &(A\boldsymbol{u}_\eta(t_1) - A\boldsymbol{u}_\eta(t_2), \boldsymbol{u}_\eta(t_2) - \boldsymbol{u}_\eta(t_1))_V \\ &\quad + b(\boldsymbol{u}_\eta(t_1) - \boldsymbol{u}_\eta(t_2), \boldsymbol{\lambda}_\eta(t_2) - \boldsymbol{\lambda}_\eta(t_1)) \\ &\qquad = (\boldsymbol{f}(t_1) - \boldsymbol{f}(t_2), \boldsymbol{u}_\eta(t_2) - \boldsymbol{u}_\eta(t_1))_V \\ &\qquad\quad + (\boldsymbol{\eta}(t_1) - \boldsymbol{\eta}(t_2), \boldsymbol{u}_\eta(t_1) - \boldsymbol{u}_\eta(t_2))_V. \end{aligned} \tag{4.10}$$

Next, we take $\boldsymbol{\mu} = \boldsymbol{\lambda}_\eta(t_2)$ in (4.7), $\boldsymbol{\mu} = \boldsymbol{\lambda}_\eta(t_1)$ in (4.9) and add the corresponding inequalities to find

$$b(\boldsymbol{u}_\eta(t_1) - \boldsymbol{u}_\eta(t_2), \boldsymbol{\lambda}_\eta(t_2) - \boldsymbol{\lambda}_\eta(t_1)) \le 0. \tag{4.11}$$

We now combine (4.10) and (4.11), and then use the strong monotonicity of the operator A to see that

$$\|\boldsymbol{u}_\eta(t_1) - \boldsymbol{u}_\eta(t_2)\|_V \le c\,(\|\boldsymbol{f}(t_1) - \boldsymbol{f}(t_2)\|_V + \|\boldsymbol{\eta}(t_1) - \boldsymbol{\eta}(t_2)\|_V). \tag{4.12}$$

Moreover, we use (4.6), (4.8), the Lipschitz continuity of the operator A and (4.12) to deduce that

$$\begin{aligned} &b(\boldsymbol{v}, \boldsymbol{\lambda}_\eta(t_1) - \boldsymbol{\lambda}_\eta(t_2)) \\ &\qquad \le c\,(\|\boldsymbol{f}(t_1) - \boldsymbol{f}(t_2)\|_V + \|\boldsymbol{\eta}(t_1) - \boldsymbol{\eta}(t_2)\|_V)\|\boldsymbol{v}\|_V \quad \text{for all } \boldsymbol{v} \in V. \end{aligned}$$

This inequality combined with (3.22) yields

$$\|\boldsymbol{\lambda}_\eta(t_1) - \boldsymbol{\lambda}_\eta(t_2)\|_D \le c\,(\|\boldsymbol{f}(t_1) - \boldsymbol{f}(t_2)\|_V + \|\boldsymbol{\eta}(t_1) - \boldsymbol{\eta}(t_2)\|_V). \tag{4.13}$$

The regularity (4.3) is now a consequence of the inequalities (4.12) and (4.13), combined with the regularity (3.20) of $\boldsymbol{f}$ and the assumption $\boldsymbol{\eta} \in C([0, T]; V)$. We conclude from above the existence of a solution to Problem $\mathcal{P}^1_\eta$ which satisfies (4.3). The uniqueness of the solution follows from the unique solvability of the system (4.1)–(4.2), at each time moment $t \in [0, T]$, guaranteed by Theorem 2.1.

Now consider $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in C([0, T]; V)$ and denote by $(\boldsymbol{u}_i, \boldsymbol{\lambda}_i)$ the solution of Problem $\mathcal{P}^1_{\eta_i}$ for $i = 1, 2$. Arguments similar to those used in the proof of (4.12) and (4.13) yield to the inequalities

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \le c\,\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_V \quad \text{for all } t \in [0, T], \tag{4.14}$$

$$\|\boldsymbol{\lambda}_1(t) - \boldsymbol{\lambda}_2(t)\|_D \le c\,\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_V \quad \text{for all } t \in [0, T]. \tag{4.15}$$

Inequalities (4.14) and (4.15) imply (4.4), which concludes the proof. □

In the next step we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 4.1 to construct the following auxiliary problem for the viscoplastic stress field.

PROBLEM $\mathcal{P}_\eta^2$. Find a viscoplastic stress field $\boldsymbol{\beta}_\eta : [0,T] \to Q$ such that

$$\boldsymbol{\beta}_\eta(t) = \int_0^t \mathcal{G}(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)) + \boldsymbol{\beta}_\eta(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)))\,ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \tag{4.16}$$

for all $t \in [0,T]$.

In the study of this problem we have the following result.

LEMMA 4.2. There exists a unique solution of Problem $\mathcal{P}_\eta^2$ which satisfies

$$\boldsymbol{\beta}_\eta \in C([0,T];Q). \tag{4.17}$$

Moreover, if $\boldsymbol{\beta}_i$ represents the solution of Problem $\mathcal{P}_{\eta_i}^2$ for $\boldsymbol{\eta} = \boldsymbol{\eta}_i \in C([0,T];V)$, $i = 1,2$, then there exists $c > 0$ such that

$$\|\boldsymbol{\beta}_1(t) - \boldsymbol{\beta}_2(t)\|_Q \le c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_V\,ds \qquad \text{for all } t \in [0,T]. \tag{4.18}$$

Proof. Let $\Theta_\eta : C([0,T];Q) \to C([0,T];Q)$ be the operator given by

$$\Theta_\eta\,\boldsymbol{\beta}(t) = \int_0^t \mathcal{G}(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)) + \boldsymbol{\beta}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)))ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \tag{4.19}$$

for all $\boldsymbol{\beta} \in C([0,T];Q)$ and $t \in [0,T]$. Let $t \in [0,T]$ be given and consider two elements $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2 \in C([0,T];Q)$. Then, using the definition (4.19) and assumption (3.9) we obtain

$$\|\Theta_\eta\,\boldsymbol{\beta}_1(t) - \Theta_\eta\,\boldsymbol{\beta}_2(t)\|_Q \le L_{\mathcal{G}} \int_0^t \|\boldsymbol{\beta}_1(s) - \boldsymbol{\beta}_2(s)\|_Q\,ds. \tag{4.20}$$

Let $\zeta > 0$. We multiply the previous inequality by $e^{-\zeta\, t}$. Then, after some elementary calculus, we find that

$$\|\Theta_\eta\,\boldsymbol{\beta}_1 - \Theta_\eta\,\boldsymbol{\beta}_2\|_\zeta \le \frac{L_{\mathcal{G}}}{\zeta}\|\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2\|_\zeta,$$

where, recall, $\|\cdot\|_\zeta$ represents the Bielecki norm on the space $C([0,T];Q)$; see (2.4). Now choosing $\zeta > L_{\mathcal{G}}$ it follows that Θ_η is a contraction on the Banach space $C([0,T];Q)$. Consequently, there exists a unique element $\boldsymbol{\beta}_\eta \in C([0,T];Q)$ such that $\Theta_\eta\boldsymbol{\beta}_\eta = \boldsymbol{\beta}_\eta$ and, moreover, $\boldsymbol{\beta}_\eta$ is the unique solution of Problem $\mathcal{P}_\eta^2$.

Now consider $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in C([0,T];V)$ and, for $i = 1,2$, denote $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\boldsymbol{\beta}_{\eta_i} = \boldsymbol{\beta}_i$. Let $t \in [0,T]$. Using (4.16) we have

$$\boldsymbol{\beta}_1(t) = \int_0^t \mathcal{G}(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_1(s)) + \boldsymbol{\beta}_1(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_1(s)))\,ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0),$$

$$\boldsymbol{\beta}_2(t) = \int_0^t \mathcal{G}(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_2(s)) + \boldsymbol{\beta}_2(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_2(s)))\,ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0).$$

These equalities combined with assumptions (3.8) and (3.9) imply that

$$\|\boldsymbol{\beta}_1(t) - \boldsymbol{\beta}_2(t)\|_Q \le c\Big(\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\,ds + \int_0^t \|\boldsymbol{\beta}_1(s) - \boldsymbol{\beta}_2(s)\|_Q\,ds\Big)$$

and, taking into account (4.4), yield

$$\begin{aligned}&\|\boldsymbol{\beta}_1(t)-\boldsymbol{\beta}_2(t)\|_Q\\&\quad\le c\Big(\int_0^t\|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_V\,ds+\int_0^t\|\boldsymbol{\beta}_1(s)-\boldsymbol{\beta}_2(s)\|_Q\,ds\Big).\end{aligned}$$

Using now a Gronwall inequality we deduce that (4.18) holds, which concludes the proof of the lemma. ☐

We now introduce the operator $\Theta : C([0,T];V) \to C([0,T];V)$ which maps every element $\boldsymbol{\eta} \in C([0,T];V)$ to the element $\Theta\boldsymbol{\eta} \in C([0,T];V)$ defined as follows: for each $\boldsymbol{\eta} \in C([0,T];V)$ and for each moment $t \in [0,T]$, $\Theta\boldsymbol{\eta}(t)$ is the unique element in V which satisfies the equality

$$(\Theta\boldsymbol{\eta}(t),\boldsymbol{v})_V=(\boldsymbol{\beta}_\eta(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q\qquad\text{for all }\boldsymbol{v}\in V.\tag{4.21}$$

Recall that here $\boldsymbol{\beta}_\eta$ represents the viscoplastic stress obtained in Lemma 4.2.

We proceed with the following property of the operator Θ.

LEMMA 4.3. The operator Θ has a unique fixed point $\boldsymbol{\eta}^* \in C([0,T];V)$.

Proof. Let $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in C([0,T];V)$, denote $\boldsymbol{\beta}_i = \boldsymbol{\beta}_{\eta_i}$, $i = 1,2$, and let $t \in [0,T]$. Using (4.21) we have

$$(\Theta\boldsymbol{\eta}_1(t)-\Theta\boldsymbol{\eta}_2(t),\boldsymbol{v})_V=(\boldsymbol{\beta}_1(t)-\boldsymbol{\beta}_2(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q\quad\text{for all }\boldsymbol{v}\in V$$

which shows that

$$\|\Theta\boldsymbol{\eta}_1(t)-\Theta\boldsymbol{\eta}_2(t)\|_V\le\|\boldsymbol{\beta}_1(t)-\boldsymbol{\beta}_2(t)\|_Q.$$

Next, using estimate (4.18) we deduce that

$$\|\Theta\boldsymbol{\eta}_1(t)-\Theta\boldsymbol{\eta}_2(t)\|_V\le c\int_0^t\|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_V\,ds.\tag{4.22}$$

Finally, we use inequality (4.22) and an argument similar to that used in the proof of Lemma 4.2, based on the Bielecki norm, to conclude the proof. ☐

We are now in a position to provide the proof of Theorem 3.1.

Proof. Let $\boldsymbol{\eta}^*$ be the fixed point of the operator Θ introduced in (4.21) and denote $\boldsymbol{u}^* = \boldsymbol{u}_{\eta^*}$, $\boldsymbol{\lambda}^* = \boldsymbol{\lambda}_{\eta^*}$, $\boldsymbol{\beta}^* = \boldsymbol{\beta}_{\eta^*}$. We shall prove that the triple $(\boldsymbol{u}^*, \boldsymbol{\beta}^*, \boldsymbol{\lambda}^*)$ satisfies the system (3.31)–(3.33), for all $t \in [0,T]$.

Let $t \in [0,T]$. First, we use (4.1) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ to write

$$(L\boldsymbol{u}^*(t),\boldsymbol{v})_V+(P\boldsymbol{u}^*(t),\boldsymbol{v})_V+(\boldsymbol{\eta}^*(t),\boldsymbol{v})_V+b(\boldsymbol{v},\boldsymbol{\lambda}^*(t))=(\boldsymbol{f}(t),\boldsymbol{v})_V\ \text{ for all }\boldsymbol{v}\in V,$$

and, since

$$(\boldsymbol{\eta}^*(t),\boldsymbol{v})_V=(\Theta\boldsymbol{\eta}^*(t),\boldsymbol{v})_V=(\boldsymbol{\beta}^*(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q\quad\text{ for all }\boldsymbol{v}\in V,$$

we obtain

$$\begin{aligned}&(L\boldsymbol{u}^*(t),\boldsymbol{v})_V+(P\boldsymbol{u}^*(t),\boldsymbol{v})_V+(\boldsymbol{\beta}^*(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q\\&\qquad+b(\boldsymbol{v},\boldsymbol{\lambda}^*(t))=(\boldsymbol{f}(t),\boldsymbol{v})_V\text{ for all }\boldsymbol{v}\in V,\end{aligned}$$

which shows that (3.31) holds. Now taking $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ in (4.2) we obtain that (3.32) holds and, since $\boldsymbol{\beta}^*$ is a solution of Problem $\mathcal{P}^2_{\eta^*}$, we deduce that (3.33) is satisfied, too. Consequently, the triple $(\boldsymbol{u}^*, \boldsymbol{\beta}^*, \boldsymbol{\lambda}^*)$ is a solution of Problem $\mathcal{P}_V$. Finally, the regularity

(3.34) follows from Lemmas 4.1–4.2 and, therefore, we conclude the existence part of the theorem.

To prove the uniqueness part we consider two solutions $(\boldsymbol{u}_i, \boldsymbol{\beta}_i, \boldsymbol{\lambda}_i)$ of Problem $\mathcal{P}_V$ which satisfy (3.34) for $i = 1, 2$. Let $t \in [0, T]$; we use (3.31), (3.32) and arguments similar to those used in the proof of the inequalities (4.12) and (4.13) to obtain

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \le c\, \|\boldsymbol{\beta}_1(t) - \boldsymbol{\beta}_2(t)\|_Q, \tag{4.23}$$

$$\|\boldsymbol{\lambda}_1(t) - \boldsymbol{\lambda}_2(t)\|_D \le c\, \|\boldsymbol{\beta}_1(t) - \boldsymbol{\beta}_2(t)\|_Q. \tag{4.24}$$

On the other hand, from (3.33), (3.9) and (3.8) we find that

$$\begin{aligned} &\|\boldsymbol{\beta}_1(t) - \boldsymbol{\beta}_2(t)\|_Q \\ &\quad \le c\Big(\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V \, ds + \int_0^t \|\boldsymbol{\beta}_1(s) - \boldsymbol{\beta}_2(s)\|_Q \, ds\Big). \end{aligned} \tag{4.25}$$

We now substitute (4.23) in (4.25) to deduce that

$$\|\boldsymbol{\beta}_1(t) - \boldsymbol{\beta}_2(t)\|_Q \le c \int_0^t \|\boldsymbol{\beta}_1(s) - \boldsymbol{\beta}_2(s)\|_Q \, ds,$$

and, using a Gronwall argument, we find that $\boldsymbol{\beta}_1(t) = \boldsymbol{\beta}_2(t)$. The uniqueness part of the theorem is now a straight consequence of the inequalities (4.23) and (4.24), which concludes the proof. □

5. A convergence result. In this section we study the behavior of the solution with respect to a perturbation of the normal compliance function p and the bound g. To this end, we assume in what follows that (3.8)–(3.13) hold and we denote by $(\boldsymbol{u}, \boldsymbol{\beta}, \boldsymbol{\lambda})$ the solution of Problem $\mathcal{P}_V$ obtained in Theorem 3.1. Also, for each $\rho > 0$ let $g^\rho \ge 0$ and consider a function p^ρ which satisfies

$$\left\{ \begin{array}{l} \text{(a) } p^\rho : \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } L_p^\rho > 0 \text{ such that} \\ \qquad |p^\rho(r_1) - p^\rho(r_2)| \le L_p^\rho |r_1 - r_2| \quad \text{for all } r_1,\, r_2 \in \mathbb{R}. \\ \text{(c) } (p^\rho(r_1) - p^\rho(r_2))(r_1 - r_2) \ge 0 \quad \text{for all } r_1,\, r_2 \in \mathbb{R}. \\ \text{(d) } p^\rho(r) = 0 \ \ \text{for all } r < 0. \end{array} \right. \tag{5.1}$$

We use the Riesz representation theorem to define the operator $P^\rho : V \to V$ by equality

$$(P^\rho \boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p^\rho(u_\nu) v_\nu \, d\Gamma \qquad \text{for all } \boldsymbol{u},\ \boldsymbol{v} \in V. \tag{5.2}$$

Then, we consider the following perturbation of the variational problem $\mathcal{P}_V$.

PROBLEM $\mathcal{P}_V^\rho$ 1. Find a displacement field $\boldsymbol{u}^\rho : [0, T] \to V$, a viscoplastic stress field $\boldsymbol{\beta}^\rho : [0, T] \to Q$ and a Lagrange multiplier $\boldsymbol{\lambda}^\rho : [0, T] \to \Lambda$ such that, for all $t \in [0, T]$,

$$\begin{aligned} &(L\boldsymbol{u}^\rho(t), \boldsymbol{v})_V + (\boldsymbol{\beta}^\rho(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P^\rho \boldsymbol{u}^\rho(t), \boldsymbol{v})_V \\ &\qquad + b(\boldsymbol{v}, \boldsymbol{\lambda}^\rho(t)) = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in V, \end{aligned} \tag{5.3}$$

$$b(\boldsymbol{u}^\rho(t), \boldsymbol{\mu} - \boldsymbol{\lambda}^\rho(t)) \le b(g^\rho \widetilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}^\rho(t)) \quad \text{for all } \boldsymbol{\mu} \in \Lambda, \tag{5.4}$$

$$\boldsymbol{\beta}^\rho(t) = \int_0^t \mathcal{G}(\mathcal{E}\,\boldsymbol{\varepsilon}(\boldsymbol{u}^\rho(s)) + \boldsymbol{\beta}^\rho(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\rho(s)))\,ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0). \tag{5.5}$$

It follows from Theorem 3.1 that Problem $\mathcal{P}_V^\rho$ has a unique solution $(\boldsymbol{u}^\rho, \boldsymbol{\beta}^\rho, \boldsymbol{\lambda}^\rho)$ with the regularity expressed in (3.34). Now consider the following assumption on the normal compliance functions p^ρ and p:

$$\begin{gathered}\text{there exists } G : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that}\\ |p^\rho(r) - p(r)| \le G(\rho)(|r| + 1) \quad \text{for all } r \in \mathbb{R} \text{ and } \rho > 0.\end{gathered} \tag{5.6}$$

Then, we have the following estimate, which represents the main result in this section.

THEOREM 5.1. Assume (3.8)–(3.13), (5.1) and (5.6). Then there exists $c > 0$ which depends on Ω, Γ_1, Γ_3, $\mathcal{E}$, $\mathcal{G}$, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, g, p, $\boldsymbol{u}_0$, $\boldsymbol{\sigma}_0$ and T, but does not depend on ρ, such that

$$\begin{aligned}&\|\boldsymbol{u}^\rho - \boldsymbol{u}\|_{C([0,T];V)} + \|\boldsymbol{\beta}^\rho - \boldsymbol{\beta}\|_{C([0,T];Q)} + \|\boldsymbol{\lambda}^\rho - \boldsymbol{\lambda}\|_{C([0,T];D)}\\ &\qquad \le c\,(G(\rho) + 1)\big[(G(\rho) + 1)|g^\rho - g| + G(\rho)\big].\end{aligned} \tag{5.7}$$

Proof. Let $t \in [0, T]$. Below we denote by c a positive generic constant that may depend on Ω, Γ_1, Γ_3, $\mathcal{E}$, $\mathcal{G}$, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, g, p, $\boldsymbol{u}_0$, $\boldsymbol{\sigma}_0$ and T, but does not depend on ρ and t, and whose value may change from line to line.

First, we test in (3.31) and (5.3) with $\boldsymbol{v} = \boldsymbol{u}^\rho(t) - \boldsymbol{u}(t)$. Then we subtract the resulting equalities to obtain

$$\begin{aligned}&(L\boldsymbol{u}^\rho(t) - L\boldsymbol{u}(t), \boldsymbol{u}^\rho(t) - \boldsymbol{u}(t))_V = b(\boldsymbol{u}^\rho(t) - \boldsymbol{u}(t), \boldsymbol{\lambda}(t) - \boldsymbol{\lambda}^\rho(t))\\ &\quad + (\boldsymbol{\beta}^\rho(t) - \boldsymbol{\beta}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}^\rho(t)))_Q + (P^\rho\boldsymbol{u}^\rho(t) - P\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}^\rho(t))_V.\end{aligned} \tag{5.8}$$

A similar argument based on inequalities (3.32), (5.4) and the definition (3.17) yields

$$b(\boldsymbol{u}^\rho(t) - \boldsymbol{u}(t), \boldsymbol{\lambda}(t) - \boldsymbol{\lambda}^\rho(t)) \le \langle \boldsymbol{\lambda}(t) - \boldsymbol{\lambda}^\rho(t), (g^\rho - g)\widetilde{\boldsymbol{\theta}}\rangle_{\Gamma_3}. \tag{5.9}$$

We now combine (5.8) and (5.9) and use the monotonicity of the operator P^ρ to see that

$$\begin{aligned}&(L\boldsymbol{u}^\rho(t) - L\boldsymbol{u}(t), \boldsymbol{u}^\rho(t) - \boldsymbol{u}(t))_V \le \langle \boldsymbol{\lambda}(t) - \boldsymbol{\lambda}^\rho(t), (g^\rho - g)\widetilde{\boldsymbol{\theta}}\rangle_{\Gamma_3}\\ &\quad + (\boldsymbol{\beta}^\rho(t) - \boldsymbol{\beta}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}^\rho(t)))_Q + (P^\rho\boldsymbol{u}(t) - P\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}^\rho(t))_V.\end{aligned} \tag{5.10}$$

Next, taking into account assumption (5.6), we find that

$$|(P^\rho\boldsymbol{u}(t) - P\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}^\rho(t))_V| \le \int_{\Gamma_3} G(\rho)(|u_\nu(t)| + 1)|u_\nu^\rho(t) - u_\nu(t)|\,da.$$

Therefore, by using (2.2), after some elementary calculus we obtain that

$$|(P^\rho\boldsymbol{u}(t) - P\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}^\rho(t))_V| \le c\,G(\rho)(\|\boldsymbol{u}(t)\|_V + 1)\|\boldsymbol{u}^\rho(t) - \boldsymbol{u}(t)\|_V.$$

This inequality shows that

$$|(P^\rho\boldsymbol{u}(t) - P\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}^\rho(t))_V| \le c\,G(\rho)\|\boldsymbol{u}^\rho(t) - \boldsymbol{u}(t)\|_V, \tag{5.11}$$

where c depends on $\boldsymbol{u}$ and, therefore, on the data of Problem $\mathcal{P}$. Consequently, combining (5.10) and (5.11) we find that

$$(L\boldsymbol{u}^\rho(t) - L\boldsymbol{u}(t), \boldsymbol{u}^\rho(t) - \boldsymbol{u}(t))_V \le c\big[\,\|\boldsymbol{\lambda}^\rho(t) - \boldsymbol{\lambda}(t)\|_D\,|g^\rho - g| \tag{5.12}$$

$$+\|\boldsymbol{\beta}^\rho(t)-\boldsymbol{\beta}(t)\|_Q\|\boldsymbol{u}^\rho(t)-\boldsymbol{u}(t)\|_V+G(\rho)\|\boldsymbol{u}^\rho(t)-\boldsymbol{u}(t)\|_V\big].$$

Also, subtracting (5.3) and (3.31) yields

$$\begin{aligned}b(\boldsymbol{v},\boldsymbol{\lambda}^\rho(t)-\boldsymbol{\lambda}(t))+(L\boldsymbol{u}^\rho(t)-L\boldsymbol{u}(t),\boldsymbol{v})_V+(\boldsymbol{\beta}^\rho(t)-\boldsymbol{\beta}(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q\\ +(P^\rho\boldsymbol{u}^\rho(t)-P\boldsymbol{u}(t),\boldsymbol{v})_V=0 \quad \text{ for all } \boldsymbol{v}\in V\end{aligned}$$

and, therefore,

$$\begin{aligned}b(\boldsymbol{v},\boldsymbol{\lambda}^\rho(t)-\boldsymbol{\lambda}(t))\le c\,\big(\|\boldsymbol{u}^\rho(t)-\boldsymbol{u}(t)\|_V\|\boldsymbol{v}\|_V+\|\boldsymbol{\beta}^\rho(t)-\boldsymbol{\beta}(t)\|_Q\|\boldsymbol{v}\|_V\\ +\|P^\rho\boldsymbol{u}^\rho(t)-P\boldsymbol{u}(t)\|_V\|\boldsymbol{v}\|_V\big)\quad \text{ for all } \boldsymbol{v}\in V.\end{aligned}\tag{5.13}$$

Note that the definition of the operators P^ρ and P, assumption (5.6) and the continuity of the trace operator, (2.2), show that

$$|(P^\rho\boldsymbol{u}^\rho(t)-P\boldsymbol{u}^\rho(t),\boldsymbol{v})_V\|\le c\,G(\rho)(\|\boldsymbol{u}^\rho(t)\|_V+1)\|\boldsymbol{v}\|_V\quad \text{ for all } \boldsymbol{v}\in V,$$

which imples that

$$\|P^\rho\boldsymbol{u}^\rho(t)-P\boldsymbol{u}^\rho(t)\|_V\le c\,G(\rho)(\|\boldsymbol{u}^\rho(t)\|_V+1).$$

Using this inequality and the Lipschitz continuity of the operator P we deduce that

$$\begin{aligned}\|P^\rho\boldsymbol{u}^\rho(t)-P\boldsymbol{u}(t)\|_V&\le\|P^\rho\boldsymbol{u}^\rho(t)-P\boldsymbol{u}^\rho(t)\|_V+\|P\boldsymbol{u}^\rho(t)-P\boldsymbol{u}(t)\|_V\\ &\le c\,G(\rho)(\|\boldsymbol{u}^\rho(t)\|_V+1)+c\,\|\boldsymbol{u}^\rho(t)-\boldsymbol{u}(t)\|_V\\ &\le c\,G(\rho)(\|\boldsymbol{u}^\rho(t)-\boldsymbol{u}(t)\|_V+\|\boldsymbol{u}(t)\|_V+1)+c\,\|\boldsymbol{u}^\rho(t)-\boldsymbol{u}(t)\|_V.\end{aligned}$$

Then, since $\|\boldsymbol{u}(t)\|_V\le\|\boldsymbol{u}\|_{C(0,T];V)}\le c$, we find that

$$\|P^\rho\boldsymbol{u}^\rho(t)-P\boldsymbol{u}(t)\|_V\le c\big[G(\rho)\|\boldsymbol{u}^\rho(t)-\boldsymbol{u}(t)\|_V+G(\rho)+\|\boldsymbol{u}^\rho(t)-\boldsymbol{u}(t)\|_V\big].\tag{5.14}$$

Combining (5.13) and (5.14) we obtain that

$$\begin{aligned}b(\boldsymbol{v},\boldsymbol{\lambda}^\rho(t)-\boldsymbol{\lambda}(t))\le c\big[\,\|\boldsymbol{u}^\rho(t)-\boldsymbol{u}(t)\|_V+\|\boldsymbol{\beta}^\rho(t)-\boldsymbol{\beta}(t)\|_Q\big]\,\|\boldsymbol{v}\|_V\\ +c\big[G(\rho)\|\boldsymbol{u}^\rho(t)-\boldsymbol{u}(t)\|_V+G(\rho)+\|\boldsymbol{u}^\rho(t)-\boldsymbol{u}(t)\|_V\big]\|\boldsymbol{v}\|_V\quad \text{ for all } \boldsymbol{v}\in V.\end{aligned}$$

Now using (3.22) we find that

$$\|\boldsymbol{\lambda}^\rho(t)-\boldsymbol{\lambda}(t)\|_D\le c\big[(G(\rho)+1)\|\boldsymbol{u}^\rho(t)-\boldsymbol{u}(t)\|_V+\|\boldsymbol{\beta}^\rho(t)-\boldsymbol{\beta}(t)\|_Q+G(\rho)\big]\tag{5.15}$$

and, moreover,

$$\begin{aligned}\|\boldsymbol{\lambda}^\rho-\boldsymbol{\lambda}\|_{C([0,T];D)}\le c\,[(G(\rho)+1)\|\boldsymbol{u}^\rho-\boldsymbol{u}\|_{C([0,T];V)}\\ +\|\boldsymbol{\beta}^\rho-\boldsymbol{\beta}\|_{C([0,T];Q)}+G(\rho)\big].\end{aligned}\tag{5.16}$$

Next, we use (3.8) and (3.14) to write

$$(L\boldsymbol{u}^\rho(t)-L\boldsymbol{u}(t),\boldsymbol{u}^\rho(t)-\boldsymbol{u}(t))_V\ge m_{\mathcal{E}}\|\boldsymbol{u}^\rho(t)-\boldsymbol{u}(t)\|_V^2,$$

and we combine this inequality with (5.12) and (5.15) to see that

$$\begin{aligned}\|\boldsymbol{u}^\rho(t)-\boldsymbol{u}(t)\|_V^2\le c\big[(G(\rho)+1)\|\boldsymbol{u}^\rho(t)-\boldsymbol{u}(t)\|_V|g^\rho-g|\\ +\|\boldsymbol{\beta}^\rho(t)-\boldsymbol{\beta}(t)\|_Q|g^\rho-g|+G(\rho)|g^\rho-g|\\ +\|\boldsymbol{\beta}^\rho(t)-\boldsymbol{\beta}(t)\|_Q\|\boldsymbol{u}^\rho(t)-\boldsymbol{u}(t)\|_V+G(\rho)\|\boldsymbol{u}^\rho(t)-\boldsymbol{u}(t)\|_V\big].\end{aligned}$$

Now using the elementary inequality

$$ab \leq \frac{a^2}{2\alpha} + \frac{\alpha b^2}{2} \qquad \text{for all} \quad \alpha > 0,\ a,\ b \in \mathbb{R}$$

with a convenient choice of α, we obtain that

$$\begin{aligned} &\|\boldsymbol{u}^\rho(t) - \boldsymbol{u}(t)\|_V^2 \\ &\quad \leq c\big[\,|g^\rho - g|^2 + \|\boldsymbol{\beta}^\rho(t) - \boldsymbol{\beta}(t)\|_Q^2 + G^2(\rho)|g^\rho - g|^2 + G^2(\rho)\big]. \end{aligned} \tag{5.17}$$

On the other hand, using equalities (3.33) and (5.5) together with assumptions (3.9) and (3.8), we deduce that

$$\|\boldsymbol{\beta}^\rho(t) - \boldsymbol{\beta}(t)\|_Q \leq c \int_0^t \big(\,\|\boldsymbol{u}^\rho(s) - \boldsymbol{u}(s)\|_V + \|\boldsymbol{\beta}^\rho(s) - \boldsymbol{\beta}(s)\|_Q\big)\, ds,$$

and, therefore, using the Gronwall argument we obtain that

$$\|\boldsymbol{\beta}^\rho(t) - \boldsymbol{\beta}(t)\|_Q \leq c \int_0^t \|\boldsymbol{u}^\rho(s) - \boldsymbol{u}(s)\|_V\, ds. \tag{5.18}$$

This inequality also shows that

$$\|\boldsymbol{\beta}^\rho(t) - \boldsymbol{\beta}(t)\|_Q^2 \leq c \int_0^t \|\boldsymbol{u}^\rho(s) - \boldsymbol{u}(s)\|_V^2\, ds. \tag{5.19}$$

Combining (5.17) with (5.19) we have

$$\begin{aligned} \|\boldsymbol{u}^\rho(t) - \boldsymbol{u}(t)\|_V^2 &\leq c\,\big[\,|g^\rho - g|^2 + G^2(\rho)|g^\rho - g|^2 + G^2(\rho)\big] \\ &+ c\int_0^t \|\boldsymbol{u}^\rho(s) - \boldsymbol{u}(s)\|_V^2\, ds. \end{aligned}$$

Again using the Gronwall inequality yields

$$\|\boldsymbol{u}^\rho(t) - \boldsymbol{u}(t)\|_V^2 \leq c\big[\,|g^\rho - g|^2 + G^2(\rho)|g^\rho - g|^2 + G^2(\rho)\big],$$

and, using the elementary inequality

$$\sqrt{a^2 + b^2 + c^2} \leq a + b + c \qquad \text{for all} \;\; a,\ b,\ c \in \mathbb{R}_+$$

we find that

$$\|\boldsymbol{u}^\rho(t) - \boldsymbol{u}(t)\|_V \leq c\big[\,|g^\rho - g| + G(\rho)|g^\rho - g| + G(\rho)\big]. \tag{5.20}$$

Since t was arbitrarily fixed in $[0, T]$ inequality (5.20) shows that

$$\|\boldsymbol{u}^\rho - \boldsymbol{u}\|_{C([0,T];V)} \leq c\big[(G(\rho) + 1)|g^\rho - g| + G(\rho)\big]. \tag{5.21}$$

Next, using (5.20) and (5.18) we deduce that

$$\|\boldsymbol{\beta}^\rho - \boldsymbol{\beta}\|_{C([0,T];Q)} \leq c\big[(G(\rho) + 1)|g^\rho - g| + G(\rho)\big]. \tag{5.22}$$

Finally, substituting (5.21) and (5.22) in (5.14) we obtain that

$$\|\boldsymbol{\lambda}^\rho - \boldsymbol{\lambda}\|_{C([0,T];D)} \leq c\,(G(\rho) + 1)[(G(\rho) + 1)|g^\rho - g| + G(\rho)]. \tag{5.23}$$

Inequality (5.7) is now a consequence of inequalities (5.21)–(5.23). $\square$

As a direct consequence of Theorem 5.1 we have the following convergence result.

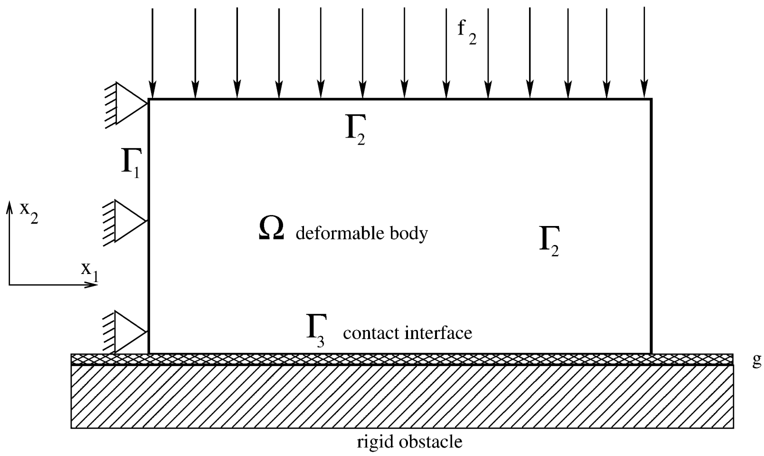

FIG. 1. Physical setting.

COROLLARY 5.2. Assume (3.8)–(3.13), (5.1) and (5.6) and, moreover, assume that

$$g^\rho \to g, \quad G(\rho) \to 0 \quad \text{as} \quad \rho \to 0. \tag{5.24}$$

Then the solution $(\boldsymbol{u}^\rho, \boldsymbol{\lambda}^\rho, \boldsymbol{\beta}^\rho)$ of Problem $\mathcal{P}_V^\rho$ converges to the solution $(\boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{\beta})$ of Problem $\mathcal{P}_V$, i.e.

$$\boldsymbol{u}^\rho \to \boldsymbol{u} \text{ in } C([0,T];V), \quad \boldsymbol{\beta}^\rho \to \boldsymbol{\beta} \text{ in } C([0,T];Q), \quad \boldsymbol{\lambda}^\rho \to \boldsymbol{\lambda} \text{ in } C([0,T];D),$$

as $\rho \to 0$.

In addition to the mathematical interest, the convergence result in Corollary 5.2 is important from the mechanical point of view, since it shows that the weak solution of the viscoplastic contact problem $\mathcal{P}$ depends continuously on both the normal compliance function and the penetration bound.

6. Numerical validation. This section is devoted to the numerical validation of the convergence result obtained in Corollary 5.2. Details on the numerical approximation of Problems $\mathcal{P}_V$ and $\mathcal{P}_V^\rho$ can be found in [3]. Here we restrict ourselves to recall that for the numerical treatment of the contact conditions we use the penalized method for the compliance contact combined with the augmented Lagrangean approach for the unilateral constraint. To this end, we consider additional fictitious nodes for the Lagrange multiplier in the initial mesh. The construction of these nodes depends on the contact element used for the geometrical discretization of the interface Γ_3. In the case of the numerical example presented below, the discretization is based on a "node-to-rigid" contact element, which is composed by one node of Γ_3 and one Lagrange multiplier node. More details about the discretization step and the numerical method can be found in [2, 14–16, 21].

For the numerical simulations we consider the physical setting depicted in Figure 1. There, $\Omega = (0, L_1) \times (0, L_2) \subset \mathbb{R}^2$ with $L_1,\, L_2 > 0$ and

$$\Gamma_1 = \{0\} \times [0, L_2],\ \Gamma_2 = (\{L_1\} \times [0, L_2]) \cup ([0, L_1] \times \{L_2\}),\ \Gamma_3 = [0, L_1] \times \{0\}.$$

The domain Ω represents the cross-section of a three-dimensional deformable body subjected to the action of tractions in such a way that a plane stress hypothesis is assumed. On the part $\Gamma_1 = \{0\} \times [0, L_2]$ the body is clamped and, therefore, the displacement field vanishes there. Vertical tractions act on the part $[0, L_1] \times \{L_2\}$ of the boundary, and the part $\{L_1\} \times [0, L_2]$ is traction-free. No body forces are assumed to act on the body during the process. The body is in frictionless contact with an obstacle on the part $\Gamma_3 = [0, L_1] \times \{0\}$ of the boundary.

We model the material's behavior with a constitutive law of the form (3.1) in which the elasticity tensor $\mathcal{E}$ satisfies

$$(\mathcal{E}\boldsymbol{\tau})_{\alpha\beta} = \frac{E\kappa}{1-\kappa^2}(\tau_{11} + \tau_{22})\delta_{\alpha\beta} + \frac{E}{1+\kappa}\tau_{\alpha\beta} \quad \text{for all } \boldsymbol{\tau} = (\tau_{\alpha\beta}), \quad 1 \le \alpha, \beta \le 2, \tag{6.1}$$

where E is the Young modulus, κ the Poisson ratio of the material and $\delta_{\alpha\beta}$ denotes the Kronecker symbol. Moreover, in order to facilitate the numerical implementation, we assume that $\mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})$, where the tensor $\mathcal{C}$ satisfies

$$(\mathcal{C}\boldsymbol{\tau})_{\alpha\beta} = \gamma_1(\tau_{11} + \tau_{22})\delta_{\alpha\beta} + \gamma_2\tau_{\alpha\beta} \quad \text{for all } \boldsymbol{\tau} = (\tau_{\alpha\beta}), \quad 1 \le \alpha, \beta \le 2. \tag{6.2}$$

For the computations below we use the following data:

$$\begin{aligned}
&L_1 = 2m, \quad L_2 = 1m, \quad T = 1s,\\
&E = 1000N/m^2, \quad \kappa = 0.3, \quad \gamma_1 = 1N/m^2, \quad \gamma_2 = 2N/m^2,\\
&\boldsymbol{f}_0 = (0,0)N/m^2, \quad \boldsymbol{f}_2 = \begin{cases} (0,0)\,N/m & \text{on } \{2\} \times [0,1],\\ (0,-300)\,N/m & \text{on } [0,2] \times \{1\},\end{cases}\\
&p(s) = c_\nu s_+, \quad g = 0.04m, \quad c_\nu = 110N/m^2,\\
&p^\rho(s) = (c_\nu - \rho)\, s_+, \quad g^\rho = g + \rho.
\end{aligned}$$

Here s_+ represents the positive part of s, i.e. $s_+ = \max\{s, 0\}$. Note that, obviously, conditions (5.6) and (5.24) are satisfied. The problem is discretized in 3728 elastic finite elements and 128 contact elements. The total number of degrees of freedom is equal to 4192 and the duration of the time step is $k = 0.01s$.

Our results are presented in Figures 2–6 and are described in what follows.

First, the deformed configuration as well as the contact interface forces corresponding to Problem $\mathcal{P}_V$, at the final time $t = 1s$, are plotted in Figure 2. We recall that the contact follows a normal compliance condition as far as the penetration is less than the bound $g = 0.04\,m$ and, when this bound is reached, it follows a unilateral condition. In the zoom depicted in Figure 2 we can see that a large proportion of the contact nodes are in the status of unilateral contact, since the complete flattening of the asperities of size $g = 0.04\,m$ was reached. Note also that the contact nodes on the left extremity of the boundary Γ_3 remain in the status of contact with normal compliance.

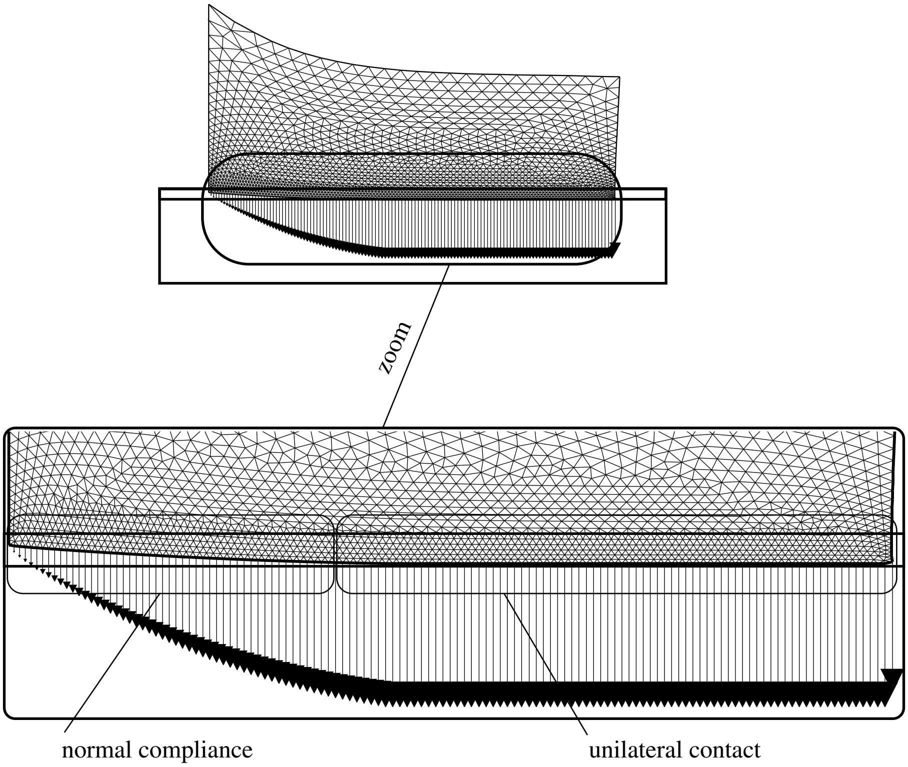

FIG. 2. Deformed mesh and contact interface forces at $t = 1s$.

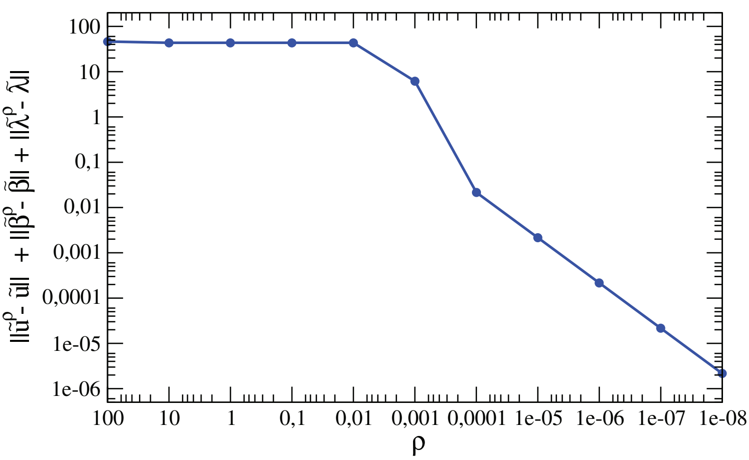

FIG. 3. First numerical validation of the convergence result in Corollary 5.2.

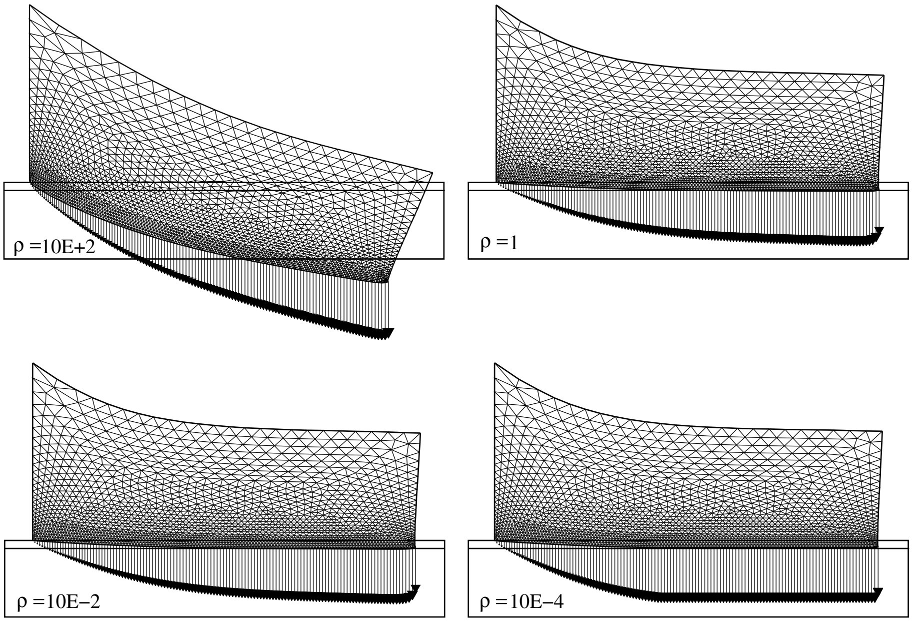

FIG. 4. Deformed meshes and contact interface forces at $t = 1s$ for $\rho = 10^2$, $\rho = 1$, $\rho = 10^{-2}$ and $\rho = 10^{-4}$.

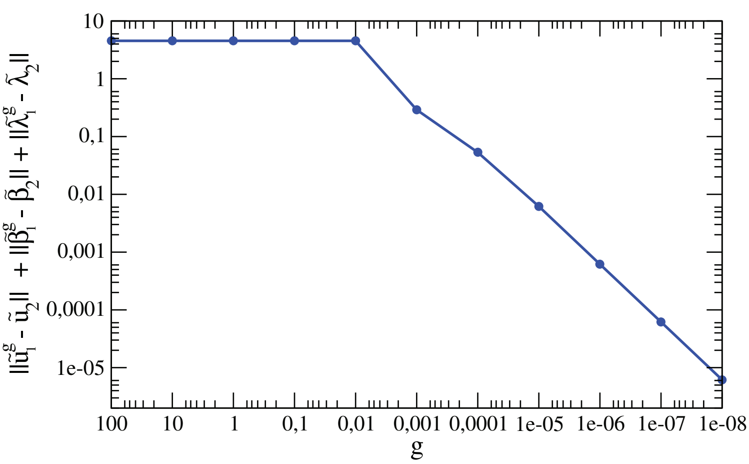

FIG. 5. Second numerical validation of the convergence result in Corollary 5.2.

Next, we denote by $(\widetilde{\boldsymbol{u}}^\rho, \widetilde{\boldsymbol{\beta}}^\rho, \widetilde{\boldsymbol{\lambda}}^\rho)$ and $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\beta}}, \widetilde{\boldsymbol{\lambda}})$ the discrete solutions of the contact Problems $\mathcal{P}_V^\rho$ and $\mathcal{P}_V$, respectively. We compute a sequence of numerical solutions corresponding to Problem $\mathcal{P}_V^\rho$ for eleven successive values of ρ from 100 to 10^{-8}. Then, in

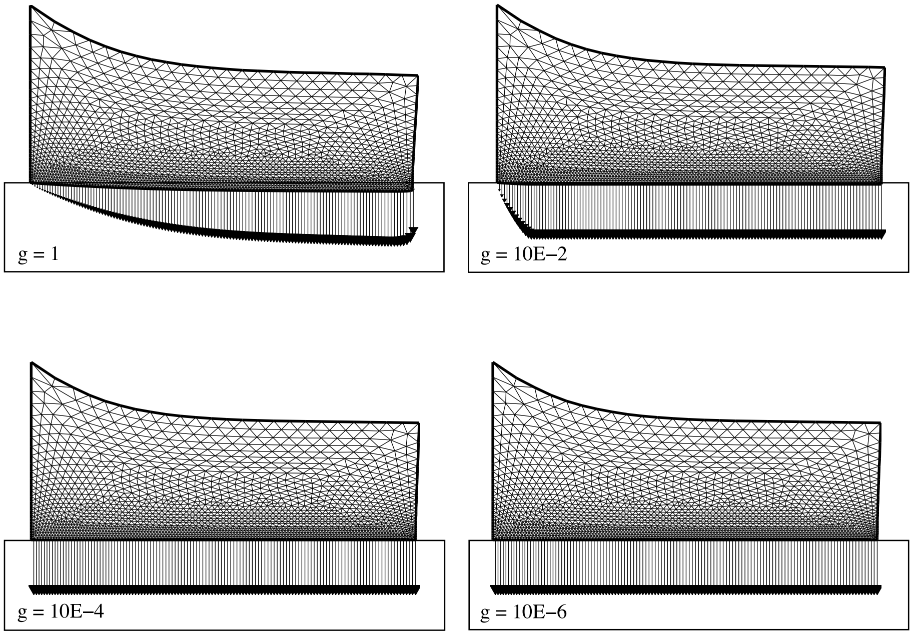

FIG. 6. Deformed meshes and contact interface forces at $t = 1s$ for $g = 1$, $g = 10^{-2}$, $g = 10^{-4}$ and $g = 10^{-6}$.

Figure 3 we present the numerical estimations of the difference

$$\|\widetilde{\boldsymbol{u}}^{\rho} - \boldsymbol{u}\| + \|\widetilde{\boldsymbol{\beta}}^{\rho} - \widetilde{\boldsymbol{\beta}}\| + \|\widetilde{\boldsymbol{\lambda}}^{\rho} - \widetilde{\boldsymbol{\lambda}}\|$$

at the time $t = 1\,s$, for various values of the parameter ρ. Here $\|\cdot\|$ represents the corresponding discrete L^2-norm. It results from Figure 3 that this difference converges to zero when ρ tends to zero, which represents a first numerical validation of the convergence result obtained in Corollary 5.2. To highlight this study, in Figure 4 we plot four deformed meshes and the associated contact forces at $t = 1s$, corresponding to Problem $\mathcal{P}_V^{\rho}$ for $\rho = 10^2,\ 1,\ 10^{-2}$ and 10^{-4}, respectively. One can see that for $\rho = 10^2$ all the contact nodes are in contact with normal compliance contact, whereas at $\rho = 10^{-4}$ two-third of the contact nodes are in unilateral contact, since the complete flattening of the asperities of size $g = 0.04\,m$ was reached.

Next, we denote by $(\widetilde{\boldsymbol{u}}_1^g, \widetilde{\boldsymbol{\beta}}_1^g, \widetilde{\boldsymbol{\lambda}}_1^g)$ the discrete solution of the contact Problem $\mathcal{P}_V$ with $g > 0$ and let $(\widetilde{\boldsymbol{u}}_2, \widetilde{\boldsymbol{\beta}}_2, \widetilde{\boldsymbol{\lambda}}_2)$ be the discrete solution of the contact Problem $\mathcal{P}_V$ with $g = 0$. We compute a sequence of numerical solutions corresponding to Problem $\mathcal{P}_V$ for eleven positive values of g from 100 to 10^{-8}. Then, in Figure 5 we present the numerical estimations of the difference

$$\|\widetilde{\boldsymbol{u}}_1^g - \widetilde{\boldsymbol{u}}_2\| + \|\widetilde{\boldsymbol{\beta}}_1^g - \widetilde{\boldsymbol{\beta}}_2\| + \|\widetilde{\boldsymbol{\lambda}}_1^g - \widetilde{\boldsymbol{\lambda}}_2\|$$

at the time $t = 1\,s$, for various values of the parameter $g > 0$. It results from here that this difference converges to zero when g tends to zero, which represents a second validation of the convergence result obtained in Corollary 5.2. To highlight this study, in Figure 6 we plot four deformed meshes and the associated contact forces at $t = 1s$, for $g = 1,\ 10^{-2},\ 10^{-4}$ and 10^{-6}, respectively. One can see that for $g = 1$ all the contact nodes are in contact with normal compliance, whereas at $g = 10^{-6}$ all the contact nodes are in the Signorini unilateral contact without gap, i.e. with $g = 0$.

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