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THE EXISTENCE OF TRAVELING WAVE FRONTS FOR A REACTION-DIFFUSION SYSTEM MODELLING THE ACIDIC NITRATE-FERROIN REACTION

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Abstract. We investigate the existence of traveling wave solutions to the one-dimensional reaction-diffusion system $u_t = \delta u_{xx} - 2uv/(\beta + u)$, $v_t = v_{xx} + uv/(\beta + u)$, which describes the acidic nitrate-ferroin reaction. Here β is a positive constant, u and v represent the concentrations of the ferroin and acidic nitrate respectively, and δ denotes the ratio of the diffusion rates. We show that this system has a unique, up to translation, traveling wave solution with speed c iff $c \geq 2/\sqrt{\beta + 1}$.

1. Introduction. Traveling waves of liquid-phase chemical reactions have received a lot of attention due to their thermodynamical and biological importance; see, for example, [1,3,8]. One of the examples comes from the acidic nitrate-ferroin reaction [6,9,10]. In the investigation of traveling waves, one of the important issues is how wave velocity depends on the initial concentrations of the reactants in the acidic nitrate-ferroin reaction. For this, Pota et al. [6, 9, 10] have established a two-variable reaction-diffusion model, from which a simple formula for wave velocity is derived. In addition, the theoretical values obtained from this formula are in good agreement with the experimental ones.

Our aim in this paper is to provide a rigorous proof for the existence of traveling wave fronts to the reaction-diffusion system proposed in [6, 9, 10]. Let u and v represent the concentrations of the ferroin and acidic nitrate respectively, and δ denote the ratio of the diffusion rates of the ferroin and acidic nitrate. Then a dimensionless reaction-diffusion system describing the acidic nitrate-ferroin reaction reads [7, 9]

$$\begin{aligned} u_t &= \delta u_{xx} - \frac{2uv}{\beta + u}, \\ v_t &= v_{xx} + \frac{uv}{\beta + u}, \end{aligned} \tag{1.1}$$

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where β is a positive constant.

By a traveling wave solution of system (1.1), we mean a solution of system (1.1) of the form

$$(u(x,t), v(x,t)) = (U(z), V(z)),$$

with $z = x - ct$, which connects the stable steady state $(0, 1/2)$ at $z = -\infty$ to the unstable steady state $(1, 0)$ at $z = \infty$ (see [4, 7]). Here the wave speed c is a constant to be determined and the wave profile (U, V) is a pair of nonnegative functions satisfying the system

$$\delta U'' + cU' - \frac{2UV}{\beta + U} = 0, \tag{1.2a}$$

$$V'' + cV' + \frac{UV}{\beta + U} = 0, \tag{1.2b}$$

on $\mathbb{R}$, together with the boundary conditions

$$(U, V)(-\infty) = (0, 1/2), \quad (U, V)(+\infty) = (1, 0), \tag{1.3}$$

where the prime indicates differentiation with respect to z. Consequently, to show the existence and uniqueness of a traveling wave solution to system (1.1) is equivalent to showing the existence and uniqueness of a nonnegative solution to system (1.2)-(1.3).

The existence and uniqueness of traveling wave solution to system (1.1) has been studied by the author in [4]. For the case $0 < \delta \leq 1$, it was shown that there exists $c_{min} > 0$ such that system (1.1) has a unique, up to translation, traveling wave solution with speed c iff $c \geq c_{min}$. Moreover, if $\delta = 1$, then $c_{min} = 2/\sqrt{\beta + 1}$, and if $0 < \delta < 1$, then an estimate for c_{min} is given by

$$2/\sqrt{\beta + 1} \leq c_{min} \leq (2/\sqrt{\beta})[(\sqrt{2} + 1)/\sqrt{\delta + 2(\sqrt{2} + 1)}].$$

For the case $\delta > 1$, it was shown that system (1.1) has a unique, up to translation, traveling wave solution with speed c if $c \geq 2/\sqrt{\beta}$. On the other hand, system (1.1) has no traveling wave solution with speed c if $c < 2/\sqrt{\beta + 1}$.

Since the above result is incomplete, the author in the present paper continues to study this problem. Employing the idea proposed by [2], we acquire an optimal result, which is stated in the following theorem.

Theorem 1.1. For $\delta > 0$, the system (1.1) has a unique, up to translation, traveling wave solution with speed c iff $c \geq c^* := 2/\sqrt{\beta + 1}$.

In view of [4], we know that the nonnegative solution of system (1.2)-(1.3), if it exists, is unique up to a translation. Besides, system (1.2)-(1.3) has no nonnegative solution if $c < c^*$, and the admissible speed set,

$$S := \{c \in \mathbb{R} | \text{ system (1.2)-(1.3) admits a nonnegative solution}\},$$

is closed. Therefore, to prove Theorem 1.1, it suffices to show that $(c^*, \infty) \subset S$. Hereafter, let $c > c^*$ be given. We will show that system (1.2)-(1.3) has a nonnegative solution.

We make an outline of our approach. By imposing appropriate boundary conditions, we first consider system (1.2) in a finite interval $[-l, l]$, and show that there exists a solution, denoted by (U_l, V_l), to the boundary value problem via the Schauder fixed point theorem. With the help of super- and sub-solutions, we get an upper bound and lower bound for this solution. Besides, applying an a priori estimate for a derivative of solutions to the second-order linear equations, we show that, for a given $l_0 > 0$, the derivatives of U_l and V_l are uniformly bounded in $[-l_0, l_0]$ for all $l > l_0$. Then we can use the Ascoli-Arzela theorem and a diagonal process to get the solution of system (1.2)-(1.3) by passing to the limit $l \to \infty$. We remark that system (1.2) has no maximum principle. Here we only need to apply the maximum principle for a single equation to compare the super/sub-solutions with the solutions.

The remaining parts of this paper are organized as follows. In Section 2, we construct the super- and sub-solutions. In Section 3, we derive some a priori estimates for solutions to the second-order linear equation which will be used in Section 4. In Section 4, we consider system (1.2) in a finite interval $[-l, l]$. Then Section 5 is devoted to the proof of our main result by passing to the limit $l \to \infty$.

2. Construction of super- and sub-solutions. In this section, we construct super- and sub-solutions which will be used in Section 4. For simplicity, we set

$$p(s) := s^2 - cs + \frac{1}{\beta + 1}.$$

Since $c > c^*$, the equation $p(s) = 0$ has two positive roots λ and $\lambda + d$, where

$$\lambda = \frac{1}{2} \cdot \left(c - \sqrt{c^2 - \frac{4}{\beta + 1}} \right) \quad \text{and} \quad d = \sqrt{c^2 - \frac{4}{\beta + 1}}.$$

In addition, $p(s) < 0$ when $s \in (\lambda, \lambda + d)$.

LEMMA 2.1. The function $V^+(z) := e^{-\lambda z}$ satisfies the equation

$$(V^+)'' + c(V^+)' + \frac{1}{\beta + 1} V^+ = 0, \tag{2.1}$$

for all $z \in \mathbb{R}$.

Proof. Since $p(\lambda) = 0$, it follows that

$$(V^+)'' + c(V^+)' + \frac{1}{\beta + 1} V^+ = p(\lambda) V^+ = 0, \forall z \in \mathbb{R}.$$

□

Select $0 < \gamma < \min\{c/\delta, \lambda\}$. Then $c - \delta\gamma > 0$ and $\gamma - \lambda < 0$. Since $e^{(\gamma - \lambda)z} \to 0$ as $z \to \infty$, there exists $z_0 > 0$ such that

$$e^{(\gamma - \lambda)z} \leq \frac{\beta + 1}{2} \gamma (c - \delta\gamma), \forall z \geq z_0,$$

which yields

$$(c - \delta\gamma)\gamma e^{-\gamma z} \geq \frac{2}{\beta + 1} V^+(z), \forall z \geq z_0. \tag{2.2}$$

Set $M = e^{\gamma z_0}$. Then $M > 1$ since $\gamma, z_0 > 0$.

LEMMA 2.2. The function $U^-(z) := \max\{0, 1 - Me^{-\gamma z}\}$ satisfies the inequality

$$\delta(U^-)'' + c(U^-)' - \frac{2U^-V^+}{\beta + U^-} \geq 0, \tag{2.3}$$

for all $z \neq z_0$.

Proof. For $z < z_0$, the inequality (2.3) holds immediately since $U^- \equiv 0$ in $(-\infty, z_0)$. For $z > z_0$, $U^-(z) = 1 - Me^{-\gamma z}$ and $0 < U^- < 1$. So we have

$$\frac{1}{\beta + 1} \geq \frac{U^-}{\beta + U^-}. \tag{2.4}$$

A simple computation, together with (2.2), (2.4), and the fact that $M > 1$, yields

$$\delta(U^-)'' + c(U^-)' = M\gamma(c - \delta\gamma)e^{-\gamma z} \geq \frac{2}{\beta + 1}V^+(z) \geq \frac{2U^-V^+}{\beta + U^-}.$$

Hence (2.3) holds. □

Choose $0 < \eta < \min\{\gamma, d\}$. Then $0 < \eta < \gamma$ and $p(\lambda + \eta) < 0$. Select

$$K > \max\{M, -M/[(\beta + 1)p(\lambda + \eta)]\}.$$

Set $z_1 = \ln K/\eta$. Then $z_1 > z_0 > 0$ since $z_0 = \ln M/\gamma$, $K > M > 1$, and $\eta < \gamma$.

LEMMA 2.3. The function $V^-(z) := \max\{0, V^+(z) - Ke^{-(\lambda+\eta)z}\}$ satisfies the inequality

$$(V^-)'' + c(V^-)' + \frac{U^-V^-}{\beta + U^-} \geq 0, \tag{2.5}$$

for all $z \neq z_1$.

Proof. For $z < z_1$, the inequality (2.5) holds immediately since $V^- \equiv 0$ in $(-\infty, z_1)$. For $z > z_1$, $V^- = V^+ - Ke^{-(\lambda+\eta)z}$ and $U^- = 1 - Me^{-\gamma z}$. Then a simple computation gives

$$(V^-)' = (V^+)' + K(\lambda + \eta)e^{-(\lambda+\eta)z}$$

and

$$(V^-)'' = (V^+)'' - K(\lambda + \eta)^2 e^{-(\lambda+\eta)z}.$$

Noting that

$$\frac{U^-}{\beta + U^-} = \frac{1}{\beta + 1} - \frac{\beta Me^{-\gamma z}}{(\beta + 1)(\beta + 1 - Me^{-\gamma z})},$$

it follows that

$$\begin{aligned}
\frac{U^-V^-}{\beta + U^-} &= \left(\frac{1}{\beta + 1} - \frac{\beta Me^{-\gamma z}}{(\beta + 1)(\beta + 1 - Me^{-\gamma z})}\right)\left(V^+ - Ke^{-(\lambda+\eta)z}\right) \\
&= \frac{1}{\beta + 1}V^+ - \frac{1}{\beta + 1}Ke^{-(\lambda+\eta)z} \\
&- \frac{\beta Me^{-(\lambda+\gamma)z}}{(\beta + 1)(\beta + 1 - Me^{-\gamma z})} + \frac{K\beta Me^{-(\lambda+\gamma+\eta)z}}{(\beta + 1)(\beta + 1 - Me^{-\gamma z})} \\
&\geq \frac{1}{\beta + 1}V^+ - \frac{1}{\beta + 1}Ke^{-(\lambda+\eta)z} - \frac{\beta Me^{-(\lambda+\gamma)z}}{(\beta + 1)(\beta + 1 - Me^{-\gamma z})}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& (V^-)'' + c(V^-)' + \frac{U^-V^-}{\beta + U^-} \\
\geq \ & (V^+)'' - K(\lambda+\eta)^2 e^{-(\lambda+\eta)z} + c(V^+)' + cK(\lambda+\eta)e^{-(\lambda+\eta)z} \\
+ \ & \frac{1}{\beta+1}V^+ - \frac{1}{\beta+1}Ke^{-(\lambda+\eta)z} - \frac{\beta M e^{-(\lambda+\gamma)z}}{(\beta+1)(\beta+1-Me^{-\gamma z})} \\
= \ & -Ke^{-(\lambda+\eta)z}p(\lambda+\eta) - \frac{\beta M e^{-(\lambda+\gamma)z}}{(\beta+1)(\beta+1-Me^{-\gamma z})} \\
& \quad \text{(by (2.1) and the definition of } p) \\
\geq \ & \frac{M}{\beta+1}e^{-(\lambda+\eta)z} - \frac{M}{\beta+1}e^{-(\lambda+\gamma)z} \\
& \quad \text{(since } K > -M/[(\beta+1)p(\lambda+\eta)] \text{ and } 1 - Me^{-\gamma z} > 0) \\
= \ & \frac{M}{\beta+1}e^{-(\lambda+\eta)z}\left[1 - e^{-(\gamma-\eta)z}\right] \\
\geq \ & 0 \quad \text{(since } \gamma - \eta > 0 \text{ and } z > z_1 > 0).
\end{aligned}$$

The proof of this lemma is therefore completed. □

3. Some auxiliary lemmas. In this section, we will establish some a priori estimates for solutions of the inhomogeneous linear equation

$$w''(z) + Aw'(z) + g(z)w(z) = h(z). \tag{3.1}$$

For this, we need the following lemma.

LEMMA 3.1. Let A be a positive constant and let g be a continuous function on $[a,b]$. Suppose that ϕ_1 and ϕ_2 are the unique solutions of the second-order linear equation

$$\mathcal{L}[y] := y'' - Ay' + g(z)y = 0$$

on $[a,b]$ such that

$$\phi_1(a) = 0, \quad \phi_1'(a) = 1 \tag{3.2}$$

and

$$\phi_2(b) = 0, \quad \phi_2'(b) = -1. \tag{3.3}$$

Then we have the following estimates for ϕ_1 and ϕ_2:

$$|\phi_1(z)| + |\phi_1'(z)| \leq e^{(K_1+A+1)(b-a)} \tag{3.4}$$

and

$$|\phi_2(z)| + |\phi_2'(z)| \leq e^{(K_1+A+1)(b-a)}, \tag{3.5}$$

for all $z \in (a,b)$, where $K_1 = \|g\|_{C[a,b]}$. If, in addition, $g \leq 0$, then the Wronskian of ϕ_1 and ϕ_2, denoted by $W(\phi_1,\phi_2)$, can be estimated by

$$|W(\phi_1,\phi_2)(z)| \geq \frac{1}{A}(e^{A(b-a)} - 1) > 0, \tag{3.6}$$

for all $z \in (a,b)$.

Proof. To prove (3.4) and (3.5), we rewrite the equation $\mathcal{L}[y] = 0$ as a first-order system

$$Y' = B(z)Y, \tag{3.7}$$

where

$$Y = \begin{bmatrix} y \\ y' \end{bmatrix} \text{ and } B(z) = \begin{bmatrix} 0 & 1 \\ -g(z) & A \end{bmatrix}.$$

Consider $z \in (a, b)$. It is easy to see that any solution Y of (3.7) satisfies the integral equation

$$Y(z) = Y(a) + \int_a^z B(\tau)Y(\tau)d\tau, \tag{3.8}$$

and therefore

$$\|Y(z)\| \leq \|Y(a)\| + (K_1 + A + 1)\int_a^z \|Y(\tau)\|d\tau,$$

where $\|\cdot\|$ denotes the absolute norm and we have use the fact that $\|B(\cdot)\| = \max\{|g(\cdot)|, A+1\} \leq K_1 + A + 1$. Hence one can easily deduce that

$$\|Y(z)\| \leq \|Y(a)\|e^{(K_1+A+1)(z-a)}. \tag{3.9}$$

Replacing a by b in (3.8) and arguing as above, we also get that

$$\|Y(z)\| \leq \|Y(b)\|e^{(K_1+A+1)(b-z)}. \tag{3.10}$$

Now if we choose $Y = [\phi_1\ \phi_1']^T$, then $\|Y(z)\| = |\phi_1(z)| + |\phi_1'(z)|$ and $\|Y(a)\| = 1$. Therefore (3.4) follows immediately from (3.9). In a similar way, we get (3.5) by selecting $Y = [\phi_2\ \phi_2']^T$ in (3.10).

Now we prove (3.6). Applying Abel's formula and noting that $W(\phi_1, \phi_2)(b) = -\phi_1(b)$, we get that

$$W(\phi_1, \phi_2)(z) = -\phi_1(b)e^{A(b-z)}. \tag{3.11}$$

To estimate $\phi_1(b)$, we introduce the function

$$\rho(z) = \frac{1}{A}(e^{A(z-a)} - 1),$$

which is the unique solution of the second-order linear equation $\rho'' - A\rho' = 0$ on $[a, b]$ such that

$$\rho(a) = 0, \quad \rho'(a) = 1.$$

For $z \in (a, b)$, noting that $\rho \geq 0$ and $g \leq 0$, we see that the function $\psi := \phi_1 - \rho$ satisfies

$$\psi'' - A\psi' + g\psi = -g\rho \geq 0$$

and

$$\psi(a) = \psi'(a) = 0.$$

Then it follows from [11, Theorem 13 of Chapter 1] that $\psi \geq 0$ and hence $\phi_1 \geq \rho$ on $[a, b]$. In particular,

$$\phi_1(b) \geq \rho(b) = \frac{1}{A}(e^{A(b-a)} - 1) > 0. \tag{3.12}$$

Combining (3.11) and (3.12), we finally obtain that

$$|W(\phi_1, \phi_2)(z)| = \phi_1(b)e^{A(b-z)} \geq \frac{1}{A}(e^{A(b-a)} - 1) > 0.$$

The proof of this lemma is therefore completed. $\square$

Indeed, we get the following two lemmas for a priori estimates.

Lemma 3.2. Let A be a positive constant and let g and h be continuous functions on $[a,b]$. Suppose that $w \in C([a,b]) \cap C^2((a,b))$ satisfies the differential equation (3.1) in (a,b) and $w(a) = w(b) = 0$. If

$$-K_1 \le g \le 0 \text{ and } |h| \le K_2 \text{ on } [a,b],$$

for some constants K_1, K_2, then there exists a positive constant K_3, depending only on A, K_1, and the length of the interval $[a,b]$, such that

$$\|w\|_{C([a,b])} \le K_2 K_3. \tag{3.13}$$

Proof. First, we note that both $w'(a+) := \lim_{z\to a+} w'(z)$ and $w'(b-) := \lim_{z \to b-} w'(z)$ exist. To see this, let $\bar{z} \in (a,b)$ be fixed. Integrating equation (3.1) from $\bar{z}$ to z and rearranging the resulting equation, we obtain that

$$w'(z) = w'(\bar{z}) - A(w(z) - w(\bar{z})) - \int_{\bar{z}}^{z} g(\tau)w(\tau)d\tau + \int_{\bar{z}}^{z} h(\tau)d\tau,$$

for any $z \in (a,b)$. Since w, g, and h are right-continuous at point a, the right-hand limit on the right-hand side of the above equation exists at a. Hence $w'(a+)$ exists. Arguing as above, we get that $w'(b-)$ exists.

Let ϕ_1 and ϕ_2 be as in Lemma 3.1. Consider any point $z \in (a,b)$. Multiplying (3.1) by ϕ_1, integrating the resulting inequality from a to z, and then using integration by parts, we get that

$$\phi_1(z)w'(z) - (\phi_1'(z) - A\phi_1(z))w(z) + w(a) = \int_a^z \phi_1(\tau)h(\tau)d\tau, \tag{3.14}$$

where we have used (3.2) and the fact that $\mathcal{L}[\phi_1] = 0$ and $w'(a+)$ exists. Similarly, multiplying (3.1) by ϕ_2, integrating the resulting inequality from z to b, and then using integration by parts, we deduce that

$$-\phi_2(z)w'(z) + (\phi_2'(z) - A\phi_2(z))w(z) + w(b) = \int_z^b \phi_2(\tau)h(\tau)d\tau, \tag{3.15}$$

where we have used (3.3) and the fact that $\mathcal{L}[\phi_2] = 0$ and $w'(b-)$ exists. Note that $w(a) = w(b) = 0$. Multiplying (3.14) and (3.15) by ϕ_2 and ϕ_1 respectively and then summing up, we finally get that

$$W(\phi_1,\phi_2)(z)w(z) = \phi_2(z)\int_a^z \phi_1(\tau)h(\tau)d\tau + \phi_1(z)\int_z^b \phi_2(\tau)h(\tau)d\tau,$$

which gives that

$$w(z) = \frac{\phi_2(z)\int_a^z \phi_1(\tau)h(\tau)d\tau + \phi_1(z)\int_z^b \phi_2(\tau)h(\tau)d\tau}{W(\phi_1,\phi_2)(z)}.$$

This, together with the assumption that $|h| \le K_2$, implies that

$$|w(z)| \le K_2 \cdot \frac{|\phi_2(z)|\int_a^b |\phi_1(\tau)|d\tau + |\phi_1(z)|\int_a^b |\phi_2(\tau)|d\tau}{|W(\phi_1,\phi_2)(z)|}.$$

Finally, using (3.4), (3.5), and (3.6), one can easily deduce from the above inequality that (3.13) holds. ☐

LEMMA 3.3. Let A, g, and h be as in Lemma 3.2. Suppose that $w \in C([a,b]) \cap C^2((a,b))$ satisfies (3.1) in (a,b). If $\|w\|_{C([a,b])} \leq K_0$ for some constant K_0, then there exists a positive constant K_4, depending only on A, K_0, K_1, K_2, and the length of the interval $[a,b]$, such that

$$\|w'\|_{C([a,b])} \leq K_4. \tag{3.16}$$

Proof. Let ϕ_1 and ϕ_2 be as in Lemma 3.1. Consider any point $z \in (a,b)$. Multiplying (3.14) and (3.15) by ϕ_2' and ϕ_1' respectively and then summing up, we deduce that

$$\begin{aligned} & W(\phi_1,\phi_2)(z)(w'(z)+Aw(z)) + \phi_2'(z)w(a) + \phi_1'(z)w(b) \\ = \quad & \phi_2'(z)\int_a^z \phi_1(\tau)h(\tau)d\tau + \phi_1'(z)\int_z^b \phi_2(\tau)h(\tau)d\tau, \end{aligned}$$

which yields

$$w'(z) = \frac{\phi_2'(z)[\int_a^z \phi_1(\tau)h(\tau)d\tau - w(a)] + \phi_1'(z)[\int_z^b \phi_2(\tau)h(\tau)d\tau - w(b)]}{W(\phi_1,\phi_2)(z)} - Aw(z).$$

This, together with the assumptions that $|h| \leq K_2$ and $|w| \leq K_0$, implies that

$$|w'(z)| \leq \frac{|\phi_2'(z)|[K_2\int_a^b \phi_1(\tau)|d\tau + K_0] + |\phi_1'(z)|[K_2\int_a^b |\phi_2(\tau)|d\tau + K_0]}{|W(\phi_1,\phi_2)(z)|} + AK_0.$$

Using (3.4), (3.5), and (3.6), one can easily deduce from the above inequality that (3.16) holds. □

4. The system in a finite interval $[-l,l]$**.** In this section, we consider the system

$$\delta U'' + cU' - \frac{2UV}{\beta+U} = 0 \quad \text{in } (-l,l), \tag{4.1a}$$

$$V'' + cV' + \frac{UV}{\beta+U} = 0 \quad \text{in } (-l,l), \tag{4.1b}$$

together with the boundary conditions

$$(U,V)(-l) = (U^-,V^-)(-l), \quad (U,V)(l) = (U^-,V^-)(l). \tag{4.2}$$

We will apply the Schauder fixed point theorem to show the existence of solutions of (4.1)-(4.2). For the reader's convenience, we state the theorem in the following.

LEMMA 4.1. Let E be a closed convex set in a Banach space and let $T: E \to E$ be a continuous mapping such that TE is precompact. Then T has a fixed point.

Let $l > z_1$. For convenience, we set $I_l := [-l,l]$, $X := C(I_l) \times C(I_l)$, and

$$E := \{(U,V) \in X |\, U^- \leq U \leq U^+ \equiv 1 \text{ and } V^- \leq V \leq V^+ \text{ in } I_l\}.$$

It is easy to verify that E is a closed convex set in the Banach space X equipped with the norm $\|(f_1,f_2)\|_X = \|f_1\|_{C(I_l)} + \|f_2\|_{C(I_l)}$. Since U^- and V^- are nonnegative, it follows that $U \geq 0$ and $V \geq 0$ for any $(U,V) \in E$.

LEMMA 4.2. For a given $(U_0, V_0) \in E$, there exists a unique solution to the boundary value problem

$$\delta U'' + cU' - \frac{2UV_0}{\beta + U} = 0 \quad \text{in } (-l, l), \tag{4.3a}$$

$$V'' + cV' + \frac{U_0 V_0}{\beta + U_0} = 0 \quad \text{in } (-l, l), \tag{4.3b}$$

$$(U, V)(-l) = (U^-, V^-)(-l), \quad (U, V)(l) = (U^-, V^-)(l). \tag{4.3c}$$

Moreover, this solution (U, V) satisfies $U > 0$, $V > 0$, and $U' > 0$ in $(-l, l)$.

Proof. Note that system (4.3) is not a coupled system, so we can deal with the existence and uniqueness of U and V separately. Since $l > z_1 > z_0 > 0 > -l$, the definition of U^- and V^- implies that $U^-(-l) = V^-(-l) = 0$, $U^-(l) > 0$, and $V^-(l) > 0$.

Since the equation for V is an inhomogeneous linear equation, the existence and uniqueness of V can be easily obtained by [5, Theorem 3.1 of Chapter 12]. Moreover, since $V'' + cV' \le 0$ in $(-l, l)$ and $V(\pm l) \ge 0$, it follows from the maximum principle that $V > 0$ in $(-l, l)$.

Now we treat the existence and uniqueness of U. For this, we first consider the initial value problem

$$\delta U'' + cU' - \frac{2UV_0}{\beta + U} = 0, \tag{4.4a}$$

$$U(-l) = (U^-)(-l), \quad U'(-l) = m, \tag{4.4b}$$

where m is a constant. By the existence and uniqueness theorem, for each m the initial value problem (4.4) has a unique local solution $U(z, m)$, and this solution can be continued as long as $U + \beta > 0$. When $m = 0$, $U(z, 0) \equiv 0$ due to the uniqueness. For any fixed $m < 0$, since $U(-l, m) = (U^-)(-l) = 0$ and $U'(-l, m) = m < 0$, it follows that there exists $\delta > 0$ such that $U(z, m) < 0$ for all $z \in (-l, -l + \delta]$. On the other hand, one can easily deduce from (4.4a) that

$$\left(e^{cz/\delta} U'\right)' = \frac{2}{\delta} \frac{UV_0}{\beta + U} e^{cz/\delta},$$

where the prime denotes differentiation with respect to z. Then an integration of the above equation gives

$$e^{cz/\delta} U'(z, m) = m e^{-cl/\delta} + \int_{-l}^{z} \frac{2}{\delta} \frac{U(\tau, m) V_0(\tau)}{\beta + U(\tau, m)} e^{c\tau/\delta} d\tau, \tag{4.5}$$

which implies that $U(z, m) < 0$ and $U'(z, m) < 0$ as long as $U(z, m)$ exists for $z > 0$. For any fixed $m > 0$, we can use a similar argument as that for $m < 0$ to discover that $U'(z, m) > 0$ and $U(z, m) > 0$ as long as U exists for $z > 0$, so that the solution can be continued to the interval I_l. Note that $U^-(l) > 0$ due to $l > z_1 > z_0$ and the definition of U^-. From the above discussion, we see that $U(l, m) = U^-(l)$ unless $m > 0$.

Next, we show that there exists $m^* > 0$ such that $U(l, m^*) = U^-(l)$ by using the shooting method. To this end, we consider $m > 0$. For $z \in (-l, l]$, recalling that $U(z, m) > 0$ and $V_0(z) \ge 0$ in $(-l, l]$, we deduce from (4.5) that

$$U'(z, m) \ge m e^{-c(z+l)/\delta}.$$

Then an integration of the above equation from $-l$ to l gives

$$U(l,m) \geq \frac{m\delta}{c}(1-e^{-2cl/\delta}) > U^-(l)$$

if m is sufficiently large. Note that $U(l,0) < U^-(l)$ since $U(z,0) \equiv 0$ and $U^-(l) > 0$. Since $U(z,m)$ is a continuous function of m for $m \geq 0$, there exists $m^* > 0$ such that $U(l,m^*) = U^-(l)$.

Finally, set $U(z) := U(z,m^*)$. Then U is a solution of (4.3a) with $U(-l) = (U^-)(-l)$ and $U(l) = (U^-)(l)$. This gives the existence of U. In addition, we see from the above discussion that $U > 0$ and $U' > 0$ in $(-l,l)$. Applying the maximum principle, we can easily get the uniqueness of U. Hence we complete the proof of this lemma. □

Now we define the mapping $T : E \to X$ by

$$T(U_0,V_0) = (U,V), \quad \forall (U_0,V_0) \in E,$$

where (U,V) is the unique solution of the boundary value problem (4.3). It is obvious that any fixed point of T is a solution of the problem (4.1)-(4.2).

LEMMA 4.3. *$TE \subseteq E$.*

Proof. For a given $(U_0,V_0) \in E$, let

$$(U,V) := T(U_0,V_0).$$

We claim that $V^- \leq V \leq V^+$ on I_l. Observing that $0 \leq U^- \leq U_0 \leq U^+ \equiv 1$ and $0 \leq V^- \leq V_0 \leq V^+$, we get that

$$\frac{U^-V^-}{\beta+U^-} \leq \frac{U_0V_0}{\beta+U_0} \leq \frac{V^+}{\beta+1},$$

so that

$$V'' + cV' + \frac{U^-V^-}{\beta+U^-} \leq 0 \tag{4.6}$$

and

$$V'' + cV' + \frac{V^+}{\beta+1} \geq 0, \tag{4.7}$$

for all z in $(-l,l)$. Set $w_1 = V - V^-$. Since $V^- = 0$ and $V \geq 0$ in $[-l,z_1]$, it follows that

$$w \geq 0, \forall z \in [-l,z_1]. \tag{4.8}$$

From (4.3c), we know that $w_1(l) = 0$. In addition, (2.5) and (4.6) give that $w_1''(z) + cw_1'(z) \leq 0$ for all $z \in (z_1,l)$. It follows from the maximum principle that $w_1 \geq 0$ in $[z_1,l]$. This, together with (4.8), implies that $V^- \leq V$ in I_l. With a similar argument, we also get that $V \leq V^+$ in I_l.

Now we show that $U^- \leq U$ in I_l. Since $U^- \equiv 0$ in $[-l,z_0]$ and $U \geq 0$ in $[-l,z_0]$, it follows that

$$U \geq U^- \text{ in } [-l,z_0]. \tag{4.9}$$

Hence it remains to show that $U \geq U^-$ in $(z_0,l]$. Since $V_0 \leq V^+$, it follows that

$$\frac{2UV_0}{\beta+U} \leq \frac{2UV^+}{\beta+U},$$

so that

$$\delta U'' + cU' - \frac{2UV^+}{\beta + U} \le 0 \text{ in } (z_0, l). \tag{4.10}$$

For simplicity, we set $\psi(\xi) = \xi/(\beta + \xi)$. Then (2.3) and (4.10) imply that the function $w_2 := U - U^-$ satisfies $\delta w_2'' + cw_2' - q(z)w_2 \le 0$ in (z_0, l), where

$$q(z) = \begin{cases} 2V^+(z) \cdot \dfrac{\psi(U(z)) - \psi(U^-(z))}{U(z) - U^-(z)}, & \text{if } U(z) \ne U^-(z), \\ 2V^+(z) \cdot \psi'(U(z)), & \text{if } U(z) = U^-(z). \end{cases}$$

By means of the mean-value theorem, we see that q is nonnegative in (z_0, l). In addition, from (4.9) and (4.3c), we know that $w_2(z_0) \ge 0$ and $w_2(l) = 0$. Hence the maximum principle asserts that $w_2 \ge 0$ in $[z_0, l]$. Hence $U^- \le U$ in $[z_0, l]$.

Finally, we show that $U \le U^+$ in I_l. Since $U^+ \equiv 1$ and $V_0 \ge 0$, we see that U^+ satisfies

$$\delta(U^+)'' + c(U^+)' - \frac{2U^+V_0}{\beta + U^+} \le 0 \quad \text{in } (-l, l),$$

and $U^+(\pm l) = 1 \ge U^-(\pm l) = U(\pm l)$. By a similar argument as in the proof for $U^- \le U$ in $[z_0, l]$, we get that $U \le U^+$ in I_l. □

Lemma 4.4. T is a continuous mapping.

Proof. For given (U_0, V_0) and $(\tilde{U}_0, \tilde{V}_0)$ in E, let

$$(U, V) = T(U_0, V_0) \text{ and } (\tilde{U}, \tilde{V}) = T(\tilde{U}_0, \tilde{V}_0). \tag{4.11}$$

It is easy to see that $w_1 := U - \tilde{U}$ satisfies $w_1(-l) = w_1(l) = 0$ and

$$w_1'' + \frac{c}{\delta} w_1' + g(z)w_1 = h_1(z).$$

Here,

$$g(z) = \begin{cases} -\dfrac{2V_0(z)}{\delta} \cdot \dfrac{\psi(U(z)) - \psi(\tilde{U}(z))}{U(z) - \tilde{U}(z)}, & \text{if } U(z) \ne \tilde{U}(z), \\ -\dfrac{2V_0(z)}{\delta} \cdot \psi'(U(z)), & \text{if } U(z) = \tilde{U}(z) \end{cases}$$

and

$$h_1(z) = -\frac{2}{\delta} \cdot \psi(\tilde{U}(z))(V_0(z) - \tilde{V}_0(z)),$$

where ψ is given in the proof of Lemma 4.3. Noting that $0 \le U, \tilde{U} \le 1$ and $0 \le \psi'(\xi) \le 1/\beta$ for $0 \le \xi \le 1$, we can apply the mean-value theorem to deduce that

$$0 \le \frac{\psi(U(z)) - \psi(\tilde{U}(z))}{U(z) - \tilde{U}(z)} \le \frac{1}{\beta}, \text{ for } U(z) \ne \tilde{U}(z). \tag{4.12}$$

Together with the fact that $\delta > 0$, $0 \le V_0 \le V^+$, and $0 \le \psi'(U(z)) \le 1/\beta$, we find that $-K_1 \le g \le 0$ with

$$K_1 = \frac{2}{\delta\beta} \cdot \|V^+\|_{C(I_l)}.$$

In addition, since $0 < \psi(\tilde{U}) \le 1$, it is easy to see that

$$|h_1| \le \frac{2}{\delta} \cdot \|V_0 - \tilde{V}_0\|_{C(I_l)}.$$

Then Lemma 3.2 asserts that there exists a positive constant C_1, depending only on δ, c, K_1 and l, such that

$$\|w_1\|_{C(I_l)} \leq \frac{2C_1}{\delta} \cdot \|V_0 - \tilde{V_0}\|_{C(I_l)},$$

which, together with definition of w_1, implies that

$$\|U - \tilde{U}\|_{C(I_l)} \leq \frac{2C_1}{\delta} \cdot \|V_0 - \tilde{V_0}\|_{C(I_l)}. \tag{4.13}$$

Set $w_2 = V - \tilde{V}$. Then w_2 satisfies $w_2(-l) = w_2(l) = 0$ and

$$w_2'' + cw_2' = h_2(z),$$

where

$$h_2 = \psi(\tilde{U_0})\tilde{V_0} - \psi(U_0)V_0.$$

It is easy to see that

$$h_2 = \tilde{V_0}\left(\psi(\tilde{U_0}) - \psi(U_0)\right) + \psi(U_0)(\tilde{V_0} - V_0). \tag{4.14}$$

Since $0 \leq U_0, \tilde{U_0} \leq 1$, we can apply the mean-value theorem to get that

$$|\psi(\tilde{U_0}) - \psi(U_0)| \leq \frac{1}{\beta}|\tilde{U_0} - U_0|,$$

and therefore

$$\left|\psi(\tilde{U_0}) - \psi(U_0)\right| \leq \frac{1}{\beta}\|\tilde{U_0} - U_0\|_{C(I_l)}.$$

Together with the fact that

$$|\tilde{V_0}| \leq \|V^+\|_{C(I_l)},\ |\psi(U_0)| \leq 1,\ \text{and}\ |V_0 - \tilde{V_0}| \leq \|V_0 - \tilde{V_0}\|_{C(I_l)},$$

we deduce from (4.14) that

$$|h_2| \leq \frac{1}{\beta} \cdot \|V^+\|_{C(I_l)}\|U_0 - \tilde{U_0}\|_{C(I_l)} + \|V_0 - \tilde{V_0}\|_{C(I_l)}.$$

Then Lemma 3.2 asserts that there exists a positive constant C_2, depending only on c, and l, such that

$$\|w_2\|_{C(I_l)} \leq \frac{C_2}{\beta} \cdot \|V^+\|_{C(I_l)}\|U_0 - \tilde{U_0}\|_{C(I_l)} + C_2\|V_0 - \tilde{V_0}\|_{C(I_l)},$$

which, together with the definition of w_2, implies that

$$\|V - \tilde{V}\|_{C(I_l)} \leq \frac{C_2}{\beta} \cdot \|V^+\|_{C(I_l)}\|U_0 - \tilde{U_0}\|_{C(I_l)} + C_2\|V_0 - \tilde{V_0}\|_{C(I_l)}. \tag{4.15}$$

Using (4.11), (4.13), (4.15), and the definition of the norm $\|\cdot\|_X$, we obtain that

$$\begin{aligned}
& \|T(U_0, V_0) - T(\tilde{U_0}, \tilde{V_0})\|_X \\
= \ & \|(U, V) - (\tilde{U}, \tilde{V})\|_X \\
= \ & \|U - \tilde{U}\|_{C(I_l)} + \|V - \tilde{V}\|_{C(I_l)} \\
\leq \ & \frac{C_2}{\beta} \cdot \|V^+\|_{C(I_l)}\|U_0 - \tilde{U_0}\|_{C(I_l)} + \left(\frac{2C_1}{\delta} + C_2\right) \cdot \|V_0 - \tilde{V_0}\|_{C(I_l)} \\
\leq \ & C_3(\|U_0 - \tilde{U_0}\|_{C(I_l)} + \|V_0 - \tilde{V_0}\|_{C(I_l)}) \\
= \ & C_3\|(U_0, V_0) - (\tilde{U_0}, \tilde{V_0})\|_X,
\end{aligned} \tag{4.16}$$

where $C_3 = \dfrac{C_2}{\beta} \cdot \|V^+\|_{C(I_l)} + \dfrac{2C_1}{\delta} + C_2$.

For a given $\epsilon > 0$, we choose $0 < \delta < \epsilon / C_3$. Then, by (4.16), we have

$$\|T(U_0, V_0) - T(\tilde{U_0}, \tilde{V_0})\|_X < \epsilon,$$

for any (U_0, V_0), $(\tilde{U_0}, \tilde{V_0}) \in E$ such that $\|(U_0, V_0) - (\tilde{U_0}, \tilde{V_0})\|_X < \delta$. This shows that T is a continuous mapping. Hence the proof of this lemma is completed. □

LEMMA 4.5. T is precompact.

Proof. For a given sequence $\{(U_{0,n}, V_{0,n}\}_{n\in\mathbb{N}}$ in E, let $(U_n, V_n) = T(U_{0,n}, V_{0,n})$. Since U^- and U^+ are bounded in I_l, and $U^- \geq 0$, we can easily see from definition of the set E and Lemma 4.3 that the sequences

$$\{U_{0,n}\}, \{V_{0,n}\}, \{U_n\}, \{V_n\}, \left\{\frac{2U_n V_{0,n}}{\beta + U_n}\right\}, \text{ and } \left\{\frac{2U_{0,n} V_{0,n}}{\beta + U_{0,n}}\right\}$$

are uniformly bounded in I_l. Then, by Lemma 3.3, it follows that the sequences

$$\{U_n'\} \text{ and } \{V_n'\}$$

are also uniformly bounded in I_l. Therefore, we can use the Arzela-Ascoli theorem to get a subsequence $\{(U_{n_j}, V_{n_j})\}$ of $\{(U_n, V_n\}$ such that

$$(U_{n_j}, V_{n_j}) \to (U, V),$$

uniformly in I_l as $j \to \infty$, for some $(U, V) \in E$. Hence the set $\overline{T(E)}$ is compact in E. So T is precompact. □

In Lemma 4.3, Lemma 4.4, and Lemma 4.5, we have proved that the mapping T satisfies all the assumptions of Lemma 4.1. Hence T has a fixed point, which is a nonnegative solution of system (4.1)-(4.2). So we have the following theorem.

LEMMA 4.6. System (4.1)-(4.2) admits a solution (U, V) on I_l. Moreover,

$$0 \leq U^- \leq U \leq 1 \text{ and } 0 \leq V^- \leq V \leq V^+ \tag{4.17}$$

on I_l.

5. The proof of the main result. Now we are in a position to prove the main results.

Proof of Theorem 1.1. Let $\{l_n\}_{n\in\mathbb{N}}$ be an increasing sequence in (z_1, ∞) such that $l_n \to \infty$ as $n \to \infty$ and let (U_n, V_n), $n \in \mathbb{N}$, be a solution of system (4.1)-(4.2) with $l = l_n$. For any fixed $N \in \mathbb{N}$, since the function V^+ is bounded above in $[-l_N, l_N]$, it follows from (4.17) that the sequences

$$\{U_n\}_{n\geq N}, \{V_n\}_{n\geq N}, \text{ and } \left\{\frac{U_n V_n}{\beta + U_n}\right\}_{n\geq N}$$

are uniformly bounded in $[-l_N, l_N]$. Then we can use Lemma 3.3 to infer that the sequences

$$\{U_n'\}_{n\geq N} \text{ and } \{V_n'\}_{n\geq N}$$

are also uniformly bounded in $[-l_N, l_N]$. Using (4.1), we can express U_n'' and V_n'' in terms of U_n, V_n, U_n' and V_n'. Differentiating (4.1), we can use the resulting equations to express U_n''' and V_n''' in terms of U_n, V_n, U_n', V_n', U_n'' and V_n''. Consequently, the sequences

$$\{U_n''\}_{n\geq N}, \{V_n''\}_{n\geq N}, \{U_n'''\}_{n\geq N} \text{ and } \{V_n'''\}_{n\geq N}$$

are uniformly bounded in $[-l_N, l_N]$. With the aid of the Arzela-Ascoli theorem, we can use a diagonal process to get a subsequence $\{(U_{n_j}, V_{n_j})\}$ of $\{(U_n, V_n)\}$ such that

$$U_{n_j} \to U, U_{n_j}' \to U', U_{n_j}'' \to U''$$

and

$$V_{n_j} \to V, V_{n_j}' \to V', V_{n_j}'' \to V'',$$

uniformly in any compact interval of $\mathbb{R}$ as $n \to \infty$, for some functions U and V in $C^2(\mathbb{R})$. Then it is easy to see that (U, V) is a nonnegative solution of system (1.2) and satisfies (4.17) and $U' \geq 0$ over $\mathbb{R}$. From the definitions of U^- and V^+, we see that $U^-(z) \to 1$ and $V^+(z) \to 0$ as $z \to \infty$. This, together with (4.17), implies that

$$(U, V)(+\infty) = (1, 0). \tag{5.1}$$

Now it remains to show that $(U, V)(-\infty) = (0, 1/2)$. We divide the proof into several steps:

STEP 1. We claim that

$$(U', V')(+\infty) = (0, 0). \tag{5.2}$$

Integrating equation (1.2a) from 0 to z gives that

$$\delta[U'(z) - U'(0)] + c[U(z) - U(0)] = \int_0^z \frac{U(\tau)V(\tau)}{\beta + U(\tau)} d\tau. \tag{5.3}$$

Since $U(+\infty)$ exists, we see from (5.3) that $U'(\infty)$ exists iff the improper integral

$$\int_0^\infty \frac{U(\tau)V(\tau)}{\beta + U(\tau)} d\tau \tag{5.4}$$

converges. Indeed, the improper integral (5.4) is convergent, since otherwise it diverges to ∞. Then (5.3) gives that $U'(\infty) = \infty$ and therefore $U(\infty) = \infty$, a contradiction to the fact that $U(\infty)$ exists. Hence $U'(\infty)$ exists. Moreover, one can easily verify that $U'(\infty) = 0$ due to $U(+\infty) = 1$. Similarly, integrating equation (1.2b) from 0 to z and arguing as above, we also get $V'(\infty) = 0$.

STEP 2. We claim that $(U, V)(-\infty)$ exists and $1 > U(-\infty) \geq 0$, $V(-\infty) \geq 0$. Since U is nondecreasing and $0 \leq U \leq 1$, it follows that $U(-\infty)$ exists and $0 \leq U(-\infty) \leq 1$. Note that $U(-\infty) \neq 1$. Otherwise, the monotonicity of U implies that $U \equiv 1$, which, together with (1.2a), gives that $V \equiv 0$, a contradiction to the fact that $V \geq V^- > 0$ in (z_1, ∞).

To show the existence of $V(-\infty)$, we need to claim that $V \leq 1/2$ on $\mathbb{R}$. From (1.2), we deduce that

$$\delta U' + 2V' + c(U + 2V) \equiv K,$$

for some constant K. Letting $z \to \infty$ in the above equation and using (5.1) and (5.2), we discover that $K = c$. Therefore,

$$\delta U' + 2V' + c(U + 2V - 1) \equiv 0. \tag{5.5}$$

Set $W_1 := U + 2V - 1$. Since $U \leq 1$, we can use (4.17) to get that

$$W_1(z) \leq 2V \leq 2V^+ \leq 2e^{-\lambda z}, \forall z \in \mathbb{R}. \tag{5.6}$$

In addition, since $U' \geq 0$, we can use (5.5) to deduce that

$$W_1' + cW_1 = (1-\delta)U' \leq 0 \tag{5.7}$$

on $\mathbb{R}$, if $\delta \geq 1$. Multiplying (5.7) by the integrating factor e^{cz}, one can easily deduce that $[e^{cz}W_1(z)]' \leq 0$, which, together with (5.6), implies that

$$e^{cz}W_1(z) \leq e^{cz^*}W_1(z^*) \leq 2e^{(c-\lambda)z^*},$$

for any $-\infty < z^* < z < \infty$. Letting $z^* \to -\infty$ in the above inequality and noting that $e^{(c-\lambda)z^*} \to 0$ due to $\lambda < c$, we get $W_1(z) \leq 0$ and therefore $U + 2V \leq 1$ on $\mathbb{R}$. This, together with the fact that $U \geq 0$, implies that $V \leq 1/2$ on $\mathbb{R}$. Now we consider the case $\delta < 1$. Set $W_2 := \delta U + 2V - 1$. Since $U \leq 1$ and $\delta < 1$, it follows from (4.17) that

$$W_2(z) \leq 2V \leq 2V^+ \leq 2e^{-\lambda z}, \forall z \in \mathbb{R}.$$

In addition, since $c > 0$, $\delta < 1$, and $U \geq 0$, we can use (5.5) to deduce that

$$W_2' + cW_2 = c(\delta - 1)U \leq 0$$

on $\mathbb{R}$. Arguing as the proof for $W_1 \leq 0$, we can easily get $W_2 \leq 0$ and therefore $\delta U + 2V \leq 1$ on $\mathbb{R}$. This, together with the fact that $\delta U \geq 0$, implies that $V \leq 1/2$ on $\mathbb{R}$.

Now we claim that $V(-\infty)$ exists and $V(-\infty) \geq 0$. Since $V(\infty) = 0$ and $V(z_1+1) \geq V^-(z_1+1) > 0$, we can use the mean-value theorem to infer that there exists $\xi_1 > z_1 + 1$ such that $V'(\xi_1) < 0$. Multiplying (1.2b) by the integrating factor e^{cz}, one can easily deduce that

$$[e^{cz}V'(z)]' = -e^{cz}\frac{UV}{\beta + U} \leq 0,$$

which implies that $e^{cz}V'(z)$ is nonincreasing. Therefore, for $z > \xi_1$, $e^{cz}V'(z) \leq e^{c\xi_1}V'(\xi_1) < 0$. Thus $V' < 0$ in $[\xi_1, \infty)$. Let $\xi_2 := \inf\{z |\ V' < 0 \text{ in } [z, \infty)\}$. Then $\xi_2 = -\infty$ or a finite number. If $\xi_2 = -\infty$, then $V' < 0$ over $\mathbb{R}$. This, together with the fact that $0 \leq V \leq 1/2$, implies that $V(-\infty)$ exists and $V(-\infty) \geq 0$. If ξ_2 is a finite number, then $V'(\xi_2) = 0$, which together the monotonicity of $e^{cz}V'(z)$, implies that $e^{cz}V'(z) \geq e^{c\xi_2}V'(\xi_2) = 0$ for $z \leq \xi_2$. Hence $V' \geq 0$ in $(-\infty, \xi_2]$. This, together with the fact that $V \geq 0$, implies that $V(-\infty)$ exists and $V(-\infty) \geq 0$.

Step 3. We claim that

$$(U', V')(-\infty) = (0, 0). \tag{5.8}$$

Integrating equation (1.2a) from z to ∞ and recalling that $U(\infty) = 1$ and $U'(\infty) = 0$ gives that

$$-\delta U'(z) + c[1 - U(z)] = 2\int_z^\infty \frac{U(\tau)V(\tau)}{\beta + U(\tau)}d\tau. \tag{5.9}$$

Since $U \geq 0$ and $U' \geq 0$, equation (5.9) implies that

$$\int_z^\infty \frac{U(\tau)V(\tau)}{\beta + U(\tau)}d\tau \leq \frac{c}{2},$$

so that the improper integral

$$\int_{-\infty}^{\infty} \frac{U(\tau)V(\tau)}{\beta + U(\tau)} d\tau \tag{5.10}$$

converges. Letting $z \to -\infty$ in (5.9) and recalling the fact that $U(-\infty)$ exists, we infer that $U'(-\infty)$ exists. Furthermore, since $U' \geq 0$, it follows that $U'(-\infty) \geq 0$. Indeed, $U'(-\infty) = 0$. Otherwise, $U'(-\infty) > 0$, which implies $U(-\infty) = -\infty$, a contradiction to the fact that $U(-\infty)$ exists. By a similar argument, we also get $V'(-\infty) = 0$.

STEP 4. We claim that $(U, V)(-\infty) = (0, 1/2)$. Since both $U(-\infty)$ and $V(-\infty)$ exist, the convergence of the improper integral (5.10) implies that

$$U(-\infty)V(-\infty) = 0. \tag{5.11}$$

On the other hand, letting $z \to -\infty$ in (5.5) and using (5.8), we get that

$$U(-\infty) + 2V(-\infty) = 1. \tag{5.12}$$

Recall that $U(-\infty) \neq 1$. Then (5.11) and (5.12) yield $(U, V)(-\infty) = (0, 1/2)$. This completes the proof of this theorem. $\square$

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